ALGEBRAIC EXTENSIONS OF SIMPLE RINGS II

TAKASI NAGAHARA and HISAO TOMINAGA

This paper is the natural sequel to "Algebraic extensions of simple rings I" [9]. The notation and terminology employed there will be used here. In [9], A. Nakajima and present authors introduced the notion of QG-1 extensions of simple rings, and considered the problem of local finiteness for QG-1 extensions of division rings. In this paper, we discuss this theme for QG-1 extensions of simple rings. Moreover, some consideration is also given to that of h-Galois extensions of simple rings.

1. Preliminaries

Let R/S be a unital ring extension. We consider here the following properties:

(I) For every finitely generated right $V_R(S)$-submodule N of R there exist a countably infinite number of elements $s_1, s_2, \cdots \in S^{\cdot}$ such that $\sum_1^\infty Ns_i = \oplus_1^\infty Ns_i$.

(I′) For every finitely generated right $V_S(S)$-submodule N of R there exist a countably infinite number of elments $s_1, s_2, \cdots \in S^{\cdot}$ such that $\sum_1^\infty Ns_i = \oplus_1^\infty Ns_i$.

(II) For every S-S-submodule X of R left finite over S there exists an element x such that $X = SxS$.

If R/S possesses the property (II) and is left algebraic then for an arbitrary finite subset F of R we can find an element f such that $S[F] = S[f]$. In particular, if A/B is left algebraic and of bounded degree then $[A:B]_L < \infty$, provided A/B possesses the property (II).

By the proof of [5, Lemma 6.6], we can easily see the following:

Proposition 1. *If S is a division ring and $[S:V_S(S)] = \infty$, any unital ring extension R/S possesses the property* (I′).

Now, let A/B be a left QG-1 extension. If $B = \sum B_o c_{ij}$ with the system of matrix units $\{c_{ij}\}$ such that $B_o = V_B(\{c_{ij}\})$ is a division ring, then one will easily see that A/B_o is a left QG-1 extension. By the aid of this remark, we can reduce several problems concerning left QG-1 extensions of simple rings to those concerning such extensions of division rings.

Proposition 2. *Let B be a division ring. If A/B is a left QG-1 ex-*

tension then (I) *implies* (II).

Proof. Let $\mathfrak{H}$ be a left Galois semigroup containing $\tilde{V}$ which belongs to the left QG-1 extension A/B, and let X be a B-B-submodule of A such that $X=Bx_1+\cdots+Bx_m$. Then $N=\sum_1^m x_i\mathfrak{H}V_R$ is a finitely generated right V-submodule of A ([9, Prop. 6]). Hence, by the property (I), there exist a countably infinite number of non-zero elements $b_1, b_2, \cdots\in B$ such that $\sum_1^\infty Nb_i=\bigoplus_1^\infty Nb_i$. If we set $x=\sum_1^m x_ib_i\,(\in X)$, then one will easily see that $\alpha|X\longrightarrow x\alpha$ $(\alpha\in\mathfrak{H}V_R)$ is a V-isomorphism. It follows therefore $[x\mathfrak{H}V_R : V]_R=[(\mathfrak{H}|X)V_R : V_R]_R=[X:B]_L$ ([9, Prop. 6]). Now, by [9, Prop. 6], we obtain $X=BxB$.

Proposition 3. *Assume that $\mathfrak{G}A_R$ is dense in $V_{\mathfrak{A}}(B_L)$ and $[B:Z]=\infty$.*

(a) *If A/B is left algebraic then it is left locally finite. If B is regnlar in addition, it is locally finite.*

(b) *If A/B is left algebraic and of bounded degree then $[A:B]_L<\infty$.*

Proof. Our assumption $[B:Z]=\infty$ implies $B\not\subset C$. Then, by [9, Prop. 1], there exists some $B'=\sum_1^n D'e'_{ij}\in\mathfrak{R}_{l,f}^0$ such that $V_A(\{e'_{ij}\})/D'$ is left algebraic (and of bounded degree in (b)). Now, by a slight modification of the proof of [7, Th. 1], we can prove that $J(\mathfrak{G}(B'), A)=B'$. Hence, $V_A(\{e_{ij}\})$ is Galois over the division ring D' with $[D': V_{D'}(D')]=\infty$ ([10, Lemma]).

(a) By [9, Th. 6], $V_A(\{e'_{ij}\})/D'$ is left locally finite, Accordingly, A/B' and hence A/B is left locally finite. The latter assertion is now evident by [3, Th. 1] or [11, Th. 8].

(b) Since $[V_A(\{e'_{ij}\}):D']<\infty$ by [1, Th. 4], we readily obtain $[A:B]_L<\infty$.

Corollary 1. *Assume that A/B is a left algebraic h-Galois extension. If T is a regular intermediate ring of A/B such that $V_T(B)$ is simple then $[T:B]_L=[T:B]_R$, provided we do not distinguish between two infinite dimensions.*

Proof. In case $[B:Z]=\infty$, A/B is locally finite by Prop. 3. Hence, by [3, Th. 1] or [11, Th. 8], $[T:B]_L=[T:B]_R$. On the other hand, if $[B:Z]<\infty$, our assertion is obvious by [9, Prop. 4].

2. Almost trivial extensions

If $[B:Z]<\infty$ and Z is contained in C then, as is well known, $A=B\otimes_Z V$ and V is a simple ring. In general, a simple ring extension A/B

is called *almost trivial* if B is regular and $A=B\cdot V(=B\otimes_Z V)$. For example, if W is a central simple algebra of finite rank whose center contains the center Z of B then $B\otimes_Z W$ is a simple ring and $(B\otimes_Z W)/B$ is an almost trivial extension. If A/B is almost trivial then C_0 coincides with C. Throughout the rest of this note, if V is a simple ring then we write $V=\sum Ug_{pq}$, where $\Gamma=\{g_{pq}\}$ is a system of matrix units and $U=V_V(\Gamma)$ is a division ring.

Proposition 4. *If A/B is almost trivial then it is a QG-1 extension.*

Proof. We set $\mathfrak{H}=V_{\mathfrak{A}}(B_L\cdot B_R)$. Evidently, $A_R\mathfrak{H}=B_R\cdot V_R\mathfrak{H}=B_R\mathfrak{H}=\mathfrak{H}B_R=\mathfrak{H}A_R$ and $\mathfrak{H}|V\subset \mathrm{Hom}_Z(V, V)$. If τ is in $\mathrm{Hom}_Z(V, V)$ then $1\otimes\tau$ is evidently in $\mathfrak{H}$. Hence, $\mathfrak{H}|V=\mathrm{Hom}_Z(V, V)$. Now, we shall prove that $V_{\mathfrak{A}}(\mathfrak{H}A_R)=B_L$. Let $\{b_i\,;\,i\in I\}$ be a Z-basis of B. Then, $A=\bigoplus_{i\in I}b_iV$. If $a_L\in V_{\mathfrak{A}}(\mathfrak{H}A_R)$ then $a=\sum b_iv_i$ $(v_i\in V)$, where $v_i=0$ for almost every i. Since $(av)\sigma=a\cdot v\sigma$ for every $\sigma\in\mathfrak{H}$ and $v\in V$, we obtain $(v_iv)\sigma=v_i\cdot v\sigma$, namely, $v_i\in Z$ by $\mathfrak{H}|V=\mathrm{Hom}_Z(V, V)$. Hence, we have $a\in B$.

Proposition 5. *Let A/B be almost trivial, and $[B:Z]=\infty$. If A/B is left algebraic then it is locally finite.*

Proof. Evidently $A/B\cdot C$ is a left algebraic Galois extension with $\tilde{V}$ as a Galois group (cf. [4, Remark 1]). Since $(B\cdot C)\cdot V_A(B\cdot C)=B\cdot V=A$, $\mathfrak{G}(A/B\cdot C)A_R$ is dense in $V_{\mathfrak{A}}((B\cdot C)_L)$ by [8, Cor. 2 (b)]. Hence, by Prop. 3, $A/B\cdot C$ is locally finite, which implies that V/C is locally finite. Combining this with the fact that C/Z is locally finite, we see that V/Z is locally finite. Hence, so is A/B.

Corollary 2. *Let A/B be almost trivial, $[B:Z]=\infty$, and $[V:C]=\infty$. If A/B and A/V are left algebraic then A/Z is locally finite.*

Theorem 1. *Let A/B be almost trivial, and $[B:Z]=\infty$. If A/B is left algebraic then A/B possesses the property* (II).

Proof. By the remark stated just before Prop. 2, we may assume that B is a division ring. Moreover, by Prop. 5, V/Z is locally finite. If we set $B'=\sum Bg_{pq}$ then $V_{B'}(B')=Z$ and $A=B'\otimes_Z U$. Now, let N be a finitely generated right V-submodule of A. Since $[V:U]<\infty$, we obtain $N=\bigoplus_1^m t_jU$. Then, there exists an intermediate ring U' of U/Z such that $[U':Z]<\infty$ and $t_1,\cdots,t_m\in B'\otimes_Z U'$. Obviously, $N'=\bigoplus_1^m t_jU'$ is right finite over Z, so that there exist a countably infinite number of non-zero elements $b_1, b_2,\cdots$ in B such that $\sum_1^\infty N'b_i=\bigoplus_1^\infty N'b_i=\bigoplus_{i,j}t_jb_iU'$ (Prop. 1). Recalling here that every t_jb_i is in $B'\otimes_Z U'$ and $(B'\otimes_Z U')\cdot U=B'\otimes_Z U$, one will reabily see that $\sum_{i,j}t_jb_iU=\bigoplus_{i,j}t_jb_iU=\bigoplus_i(\bigoplus_j t_jU)b_i=\bigoplus_i Nb_i$. Hence, our conclusion is a consequence of Prop. 2.

Let A/B be almost trivial. If $[B:Z]<\infty$ then [9, Th. 2] proves that in order A/B to be left algebraic and of bounded degree it is necessary and sufficient that $[A:C]<\infty$ and C/Z be of bounded degree. While, in case $[B:Z]=\infty$, we obtain the following that will be needed in the next section.

Corollary 3. *Let A/B be almost trivial, and $[B:Z]=\infty$. If A/B is left algebraic and of bounded degree then $[A:B]<\infty$ (or $[V:Z]<\infty$).*

Proof. This is an easy consequence of Th. 1.

3. Left QG-1 extensions

Theorem 2. *Let A/B be a left QG-1 extension, and $[B:Z]=\infty$. If $[V:Z]<\infty$ then A/B possesses the property* (II).

Proof. Without loss of generality, we may assume that B is a division ring. Then, by Prop. 1, A/B has the property (I′). Since $[V:Z]<\infty$, (I′) coincides with (I). Hence, our theorem is obvious by Prop. 2.

Corollary 4. *Let A/B be a left QG-1 extension, and $[B:Z]=\infty$.*

(a) *If A/B is left algebraic and $[V:Z]<\infty$ then A/B is left locally finite.*

(b) *If $[A:B]_L<\infty$ then for an arbitrary B-B-submodule X of A there exists an element x such that $X=BxB$.*

Theorem 3. *Let A/B be a left QG-1 extension, and $[B:Z]=\infty$. If A/B is left algebraic and of bounded degree then $[A:B]_L<\infty$.*

Proof. Since V/Z is algebraic and of bounded degree, $[V:C_0]<\infty$ by [1, Lemma 4]. Hence, V/Z is locally finite. Noting that $B\cdot V$ is a simple ring (cf. [4, Remark 1]), we see that $B\cdot V/B$ is almost trivial. Corollary 3 proves therefore $[V:Z]<\infty$. Now, by the validity of Th. 2, we obtain $[A:B]_L<\infty$.

4. h-Galois extensions

Lemma 1. *Assume that A/B is h-Galois. Let $T\in\mathfrak{R}^0_{l.f}/\Gamma$, and $W=T\cap U$. If a subset of T is right linearly independent over W, then so is over U.*

Proof. Since $J(\mathfrak{G}(T),A)=T$ by [7, Th. 1], $\mathfrak{G}(T)|U$ is a Galois group of U/W. Now, assume that $\{t_1,\cdots,t_m\}(\subset T)$ is right linearly dependent over U. Then, without loss of generality, we may assume that $t_1=\sum_2^m t_ju_j\,(u_j\in U)$ is a non-trivial relation of the shortest length. Since

$0=t_1-t_1\sigma=\sum_2^m t_j(u_j-u_j\sigma)$ for every $\sigma\in\mathfrak{G}(T)$, it follows then $u_2,\cdots,u_m\in W$, namely, $\{t_1,\cdots,t_m\}$ is right linearly dependent over W.

Theorem 4. *Let A/B be h-Galois, and $[B:Z]=\infty$. If A/B is left algebraic then A/B possesses the property* (II), *in particular, for an arbitrary finite subset F of A there exists an element f such that $B[F]=B[f]$.*

Proof. By Prop. 3, A/B is locally finite. Then, taking the validity of [3, Th. 1] into the mind, we may assume from the beginning that B is a division ring. Now, let N be a finitely generated right V-submodule of A. Then, $N=\oplus_1^m t_jU$. Take an arbitrary $T\in\mathfrak{R}_{l,f}^0/\{\Gamma,t_1,\cdots,t_m\}$, and set $W=T\cap U(\subset V_T(B))$ and $N'=\oplus_1^m t_jW$. Obviously, $[W:Z]\leqslant[V_T(B):Z]\leqslant[T:B]_L<\infty$. Accordingly, by Prop. 1, there exist a countably infinite number of non-zero elements $b_1, b_2,\cdots$ in B such that $\sum_1^\infty N'b_i=\oplus_1^\infty N'b_i=\oplus_{i,j}t_jb_iW$. Recalling here that every t_jb_i is contained in T, Lemma 1 implies that $\sum_{i,j}t_jb_iU=\oplus_{i,j}t_jb_iU=\oplus_i Nb_i$. We have seen thus A/B has the property (I). Now, our assertion is contained in Prop. 2.

Lemma 2. *Let $H=V_A(V)$. If A is $H\cdot V$-A-irreducible and H is B-H-irreducible then A is $B\cdot V$-A-irreducible, and conversely.*

Proof. Let a be an arbitrary non-zero element of A, and $M=BVaA$. Since A is V-A-completely reducible ([8, Lemma 2]), we obtain $A=M\oplus M'$ with some V-A-submodule M'. If $1=e+e'$ $(e\in M,\ e'\in M')$, then it is evident that e is a non-zero element of H. Hence, $M\supset BeH=H\ni 1$, which means $M=A$. The converse will be easily seen (cf. the proof of [8, Cor. 2]).

Theorem 5. *Let A be $H\cdot V$-A-irreducible, and Galois over B.*

(a) *If H/B is left algebraic then A/B is h-Galois.*

(b) *If A/B is left algebraic and $[B:Z]=\infty$ then A/B is locally finite.*

Proof. (b) is a consequence of (a) and Prop. 3. Now, we shall prove (a). Since H is B-H-irreducible by [7, Lemma 4] and [3, Th. 1], A is $B\cdot V$-A-irreducible by Lemma 2. Hence, by [8, Cor. 2], A/B is h-Galois.

In [7], the result of the following corollary has been shown. The proof given there is somewhat complicated. Now, we shall present an alternative proof of this result as an application of Prop. 3, Th. 5 and [9, Prop 5].

Corollary 5. *Let A be Galois over B, and $[V:C]<\infty$. If A/B is left algebraic then it is locally finite and h-Galois.*

Proof. Since A is Galois and finite over H, A is $H\cdot V$-A-irreducible by [3, Th. 1]. Hence, A/B is h-Galois by Th. 5 (a). If $[B:Z]=\infty$ then

A/B is locally finite by Prop. 3. While, in case $[B:Z]<\infty$, the local finiteness of A/B is given in [9, Prop. 5].

Remark. In the proof of Th. 5, we used the fact that if an algebraic extension A/B is Galois and outer (i. e., $V=C$) then A/B is locally finite. This result has been shown in the previous papers [2] and [7]. For this result, we shall present a simple proof as an application of matrices $u(E, d)$ considered in [9, §1]. If B is contained in C then $A=V$ is commutative, so that A/B is obviously locally finite. Henceforth, we shall consider the case $B \not\subset C$. By [9, Prop. 1] there exists then some $B'=\sum_1^n D'e'_{ij} \in \mathfrak{R}^0_{l.f}$ such that $V_A(\{e'_{ij}\})/D'$ is left algebraic (note that [9, Prop. 1] was proved by use of matrices $u(E, d)$). Given $d \in V_A(\{e'_{ij}\})$, we set $B''=B'[d] (\in \mathfrak{R}^0_{l.f})$. Then, by [6, Lemma 1.3], there holds for $\mathfrak{G}=\mathfrak{G}(A/B)$ that $\mathfrak{G}\,|\,B'' = \bigcup_1^s (\sigma_i\,|\,B'')$ with some finite $\sigma_i \in \mathfrak{G}$. From this, it follows at once $\sharp\{(de'_{ij})\mathfrak{G}\} \leqq \sharp(\mathfrak{G}\,|\,B'')<\infty$, which means the local finiteness of $\mathfrak{G}$, and hence A/B is locally finite.

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DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

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