

ALGEBRAIC EXTENSIONS OF SIMPLE RINGS I

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In this paper we study the local finiteness of algebraic extensions of simple rings. Our study starts with the preliminary section §1, which contains some matrix calculations of independent interest. In §2, we consider simple algebras of finite rank, and show under what conditions they are locally finite, of bounded degree over simple subrings. In §3, we study the equality of left and right dimensionalities for some types of simple ring extensions. In §4, we consider QG-1 extensions of simple rings which are generalized Galois extensions, and in §5, we study the local finiteness of QG-1 division ring extensions of infinite rank. For QG-1 extensions of simple rings, the situations will be studied in a continuation of the present title, II.

Throughout the present paper, $A = \sum_{i,j}^n D e_{ij}$ will represent a simple ring where $E = \{e_{ij}\}$ is a system of matrix units and D is the centralizer of E in A which is a division ring, and B a simple subring of A containing the identity 1 of A . For any subset S of A , we denote by $V_A(S)$ the centralizer of S in A . Specially we use the following convention: V means $V_A(B)$. Moreover we write $C = V_A(A)$, $Z = V_B(B)$, $C_0 = V_V(V)$ the centers of A , B , and V respectively. If, for every finite subset F of A , $[B[F]: B]_L < \infty$ (resp. $[B[F]: B]_R < \infty$) then we say that A is left (resp. right) locally finite over B . Given rings R , S , we shall understand by an R - S -module a two-sided module which is treated as a left R - and right S -module. For any simple subring T of A , we denote by $[T|T]$ the uniquely determined number of irreducible direct summands of T -module T . If $T \supset E$ then one can see that $[T|T] = [A|A]$. A subring T of A is said to be regular if T and $V_A(T)$ are both simple. By $\mathcal{R}$ we denote the set of all regular subrings of A containing B , and we shall use the following notations:

$$\mathcal{R}^0 = \{B' \in \mathcal{R}; [B'|B'] = [A|A]\},$$

$$\mathcal{R}_{l,r} = \{B' \in \mathcal{R}; [B': B]_L < \infty\} (\mathcal{R}_{r,l} = \{B' \in \mathcal{R}; [B': B]_R < \infty\}),$$

$\mathcal{R}/S = \{B' \in \mathcal{R}; B' \supset S\}$, where S is a subset of A , and we set $\mathcal{R}_{l,r}^0/S = \mathcal{R}^0 \cap \mathcal{R}_{l,r}/S$, $\mathcal{R}^0/S = \mathcal{R}^0 \cap \mathcal{R}/S$, $\mathcal{R}_{l,r}/S = \mathcal{R}_{l,r} \cap \mathcal{R}/S$, $\mathcal{R}_{l,r}^0/S = \mathcal{R}_{l,r}^0 \cap \mathcal{R}/S$. As to other notations and terminologies used in this paper, we follow [3—11].

1. Preliminaries

We consider the following

$$u(E, d) = de_{21} + \sum_{i=3}^n e_{ii-1}$$

with non-zero $d \in D$, which plays an important role in our subsequent consideration. First we shall prove the following

Lemma 1. *Let $n > 1$, and T a left Artinian unital subring of A . If T contains $u(E, d)$ and $a = \sum x_{ij}e_{ij}$ ($x_{ij} \in D$) with $x_{1n} = 1$ and $x_{in} = 0$ for every $i > 1$, then T contains E , d , and x_{ij} 's.*

Proof. Since A is T - A -irreducible by [7, Lemma 8], T is in $\mathcal{R}$ (cf. [11, p. 69]). We set $u = u(E, d)$. Then $u^{k-1}au^{n-1} = d^2e_{k1}$ is a non-zero element of $T \cap e_{kk}A$ ($k = 1, \dots, n$), and so by [7, Lemma 5] T contains $e_{11}, e_{22}, \dots, e_{nn}$. It follows therefore $e_{1n} = e_{11}ae_{nn} \in T$ and $d = (u + e_{1n})^n \in T$, so that $e_{21} = d^{-2}(d^2e_{21}) \in T$. It follows then $\sum_{i=3}^n e_{ii-1} = (1-d)e_{21} + u \in T$, and therefore our assertion is a direct consequence of [7, Lemma 6].

The next contains evidently [1, Satz].

Corollary 1. *Let $\phi(\subset C)$ be a perfect field. If A/ϕ is 3-algebraic then A/ϕ is locally finite. In case $n > 1$, if A/ϕ is 2-algebraic then A/ϕ is locally finite.*

Proof. Firstly, assume that A is a division ring and 3-algebraic over ϕ . If a_1, a_2, a_3 are in A then $\phi[a_1, a_2, a_3] = \phi[a'_1, a'_2]$ with some $a'_1, a'_2 \in A$ ([7, Th. 1]). Hence, an easy induction will prove that A/ϕ is locally finite. Next, assume that $n > 1$ and A/ϕ is 2-algebraic. Let d_1, d_2, d_3 be non-zero elements of D . If $a_1 = \sum x_{ij}e_{ij}$ where $x_{11} = d_1, x_{21} = d_2, x_{1n} = 1$ and each of other x_{ij} 's is 0, and $a_3 = u(E, d_3)$, then $\phi[a_1, a_2]$ is finite over ϕ and contains $\phi[d_1, d_2, d_3, E]$ (Lemma 1), which means that D/ϕ is 3-algebraic. Hence, A/ϕ is locally finite by the first assertion proved above.

Proposition 1. *Let A be left algebraic (resp. left algebraic and of bounded degree) over B . If $B \not\subset C$ then there exists some $B' = \sum_{i=1}^n D'e'_{ij} \in \mathcal{R}_{i,j}^0$ such that $V_A(\{e'_{ij}\})/D'$ is left algebraic (resp. left algebraic and of bounded degree).*

Proof. It suffices to prove the proposition for the case $n > 1$. By [7, Lemma 7], there exists an element $r \in A^*$ such that $B\bar{r}$ contains an element $a = \sum x_{ij}e_{ij}$ ($x_{ij} \in D$) with $x_{1n} = 1$ and $x_{in} = 0$ for every $i > 1$. Given an arbitrary non-zero element $d \in D$, we set $u = u(E, d)$. Then, $(B\bar{r})[u]$ is left finite over $B\bar{r}$ and contains $\{E, d\}$ (Lemma 1). If we set $B^* = (B\bar{r})[E] = \sum D^*e_{ij}$ with the division ring $D^* = V_{B^*}(E)$, then D/D^* is left

algebraic. Hence, $B^* \tilde{r}^{-1}$ can be taken as B' requested.

Proposition 2. *Let $[B : Z] < \infty$ (or $[A : C] < \infty$). If $[A : B]_L < \infty$ then $[A : B]_R < \infty$.*

Proof. By [9, Lemma], we obtain $[A : C] < \infty : A = \sum_{i=1}^n a_i C$. Since $C \cdot B = \sum_{j=1}^n B c_j$ ($c_j \in C$), it follows $\sum_{i,j} a_i c_j B = \sum_i a_i C \cdot B = A$, which means $[A : B]_R < \infty$.

Corollary 2. *Let $[B : Z] < \infty$. If A/B is left locally finite then it is locally finite.*

2. Algebraic extensions of a simple algebra of finite rank

The present section is devoted exclusively to the treaty of algebraic extensions of a simple algebra of finite rank.

Proposition 3. *Let A be left algebraic over B .*

(a) *In order that $[A : C] < \infty$, it is necessary and sufficient that $[V : C] < \infty$ and $[B : Z] < \infty$.*

(b) *If $[A : C] < \infty$ then A/B is locally finite.*

Proof. (a) By the validity of [10, Lemma], it suffices to prove the sufficiency. In case B is contained in C , our assertion is trivial. If B is not contained in C then there exists some $B' = \sum_{i,j} D' e'_{ij} \in \mathcal{R}_{B',f}^0$ such that $V_A(\{e'_{ij}\})$ is left algebraic over D' (Prop. 1). Since $[B' : V_{B'}(B')] < \infty$ again by [10, Lemma], we may assume from the beginning that A is a division ring. Then, noting that $B \cdot C = B \otimes_Z Z \cdot C$, there holds $[B \cdot C : C] = [B \cdot C : Z \cdot C] \cdot [Z \cdot C : C] \leq [B : Z] \cdot [V : C] < \infty$. Hence, $B \cdot C = V_A^2(B \cdot C) = V_A(V)$, and so $[A : C] = [A : B \cdot C] \cdot [B \cdot C : C] = [V : C] \cdot [B \cdot C : C] < \infty$.

(b) In case B is contained in C , our assertion is clear by [2, Prop. 10.12.3] or by a direct computation. In what follows, we shall assume that $B \not\subset C$. Then, by [7, Th. 1], there exists an element $a \in A$ such that $A = (B \cdot C)[a]$. Now, let F be an arbitrary finite subset of A . Then, C contains a finite subset F' such that $B[F] \subset B[F', a] = B[a] \cdot B[F']$. Since we can easily see that $[B[F'] : B]_L < \infty$, our assertion is a consequence of (a) and Cor. 2.

The next theorem contains obviously Prop. 3 (b).

Theorem 1. *Let $[B : Z] < \infty$, and let A be left algebraic over B . In order that A/B be locally finite, it is necessary and sufficient that $A/B \cdot C$ be left locally finite.*

Proof. It suffices to prove the sufficiency. At first we shall prove

our assertion for the case that B is a regular subring. Since $B \cdot C = B \otimes_Z C \cdot Z$, $Z \cdot C$ is a subfield of C_0 and $B \cdot C$ is a simple ring with $[B \cdot C : Z \cdot C] = [B : Z] < \infty$. Now, let F be an arbitrary finite subset of A . Then, $A' = (B \cdot C)[E, F]$ is a simple ring with $[A' : B \cdot C]_r < \infty$. Recalling here that $[B \cdot C : Z \cdot C] < \infty$, we have then $[A' : V_{A'}(A')] < \infty$ by [10, Lemma]. Hence, A'/B is locally finite by Prop. 3 (b), which means obviously that A/B is locally finite. Next, we shall proceed into the general case. By the first step, we may restrict our attention to the case that $B \not\subset C$. Then, by Props. 1 and 2 (a), there exists some $B' = \sum_{i=1}^n D' e'_{ij} \in \mathcal{R}_{i,j} = \mathcal{R}_{r,j}^0$ such that $V_A(\{e'_{ij}\})$ is left algebraic over D' . One may remark here $[D' : V_{B'}(D')] \leq [B' : V_{B'}(B')] < \infty$ ([10, Lemma]). Since $B' \cdot C = \sum_{i=1}^n (D' \cdot C) e'_{ij}$ is left finite over $B \cdot C$, $A/B' \cdot C$ is left locally finite, or what is the same, $V_A(\{e'_{ij}\})/D' \cdot C$ is left locally finite. Hence, by the first step, $V_A(\{e'_{ij}\})/D'$ is locally finite, or equivalently A/B' is locally finite. Consequently, it follows that A/B is locally finite.

Theorem 2. *Let A be left algebraic over a simple ring B . Then, the following conditions are equivalent :*

- (1) $[B : Z] < \infty$ and A/B is of bounded degree.
- (2) $[A : C] < \infty$ and $B \cdot C/B$ is of bounded degree.
- (3) $[A : C] < \infty$ and $Z \cdot C/Z$ is of bounded degree.

Proof. Evidently, $B \cdot C = B \otimes_Z Z \cdot C$ and (2) $\Leftrightarrow$ (3).

(3) $\Rightarrow$ (1). We set $A = \sum_{i=1}^s C a_i$, where $a_i = 1$. There exists a positive integer k such that every subring of the form $B[c]$ ($c \in C$) possesses a linearly independent B -basis consisting of at most k elements of C . Accordingly, if $c_1, \dots, c_t$ are in C then $B[c_1, \dots, c_t]$ possesses a linearly independent B -basis consisting of at most k^t elements of C . In case $B \subset C$, the last yields $[B[x] : B] \leq_s k^{s^2+s}$ for every $x \in A$. On the other hand, if $B \not\subset C$ then $A = (B \cdot C)[a]$ with some a ([7, Th. 1]). There exists then a finite subset F of C such that $\{a_1, \dots, a_s\} \subset B[F, a]$. If $x = \sum_{i=1}^s c'_i a_i$ is an element of A and $F' = \{c'_1, \dots, c'_s\}$, it is obvious that $B[x] \subset B[F, F', a]$. Hence, $[B[x] : B]_r \leq [B[F, F', a] : B]_r = [B[a] \cdot B[F, F'] : B] \leq [B[a] : B] \cdot k^{sF+s}$. One may remark here $[B : Z] \leq [A : C] < \infty$ ([10, Lemma] or Prop. 3).

(1) $\Rightarrow$ (2). If $B \subset C$ then A/C is of bounded degree, and then $[A : C] < \infty$ by [4, Lemma 4]. Next, assume that $B \not\subset C$. By Prop. 1, there exists some $B' = \sum_{i=1}^n D' e'_{ij} \in \mathcal{R}_{i,j}^0$ such that $V_A(\{e'_{ij}\})/D'$ is left algebraic and of bounded degree. Since $[D' : V_{B'}(D')] \leq [B' : V_{B'}(B')] < \infty$ by [10, Lemma], $V_A(\{e'_{ij}\})$ is left algebraic and of bounded degree over the field $V_{B'}(B') \cdot C$. Hence, again by [4, Lemma 4], $[V_A(\{e'_{ij}\}) : C] < \infty$, namely, $[A : C] < \infty$.

3. Left and right dimensionalities

Proposition 4. *Let $[B:Z] < \infty$ (or $[A:C] < \infty$). If B is a regular subring then $[A:B]_L = [A:B]_R$, provided we do not distinguish between two infinite dimensions.*

Proof. Assume that $[A:B]_L < \infty$. Since $B \cdot V = B \otimes_Z V$ is a simple intermediate ring of A/C and $[A:C] < \infty$ by [10, Lemma], we obtain $B \cdot V = V_A^1(B \cdot V) = V_A(C_0)$ and $[B \cdot V : B] = [V : Z]$. Hence, $[A:B]_L = [A:B \cdot V]_L \cdot [B \cdot V : B] = [C_0 : C] \cdot [B \cdot V : B] = [A:B \cdot V]_R \cdot [B \cdot V : B] = [A:B]_R$, which proves our assertion.

As an application of Th. 2, we shall prove the following that contains [6, Th. 5.2].

Theorem 3. *Let $[B:Z] < \infty$, and let A be left algebraic and of bounded degree over B . If $Z \cdot C$ is a separable field extension of Z then $[A:B]_L = [A:B]_R < \infty$.*

Proof. By Th. 2, there holds $[A:B]_L \leq [A:C] \cdot [Z \cdot C : Z] < \infty$. Since the simple ring $V_A(Z) = V_A(Z \cdot C)$ is $B \otimes_Z V$ by Wedderburn's theorem, V is a simple ring. Now, our assertion is a consequence of Prop. 4.

Throughout the rest of this note, $\mathfrak{A}$ and $\mathfrak{G}$ will denote the absolute endomorphism ring $\text{Hom}(A, A)$ of A and the group of all B -ring automorphisms of A , respectively. We now consider some results which can be proved for Galois extensions A/B .

Proposition 5. *Let $\mathfrak{G}A_R$ and $\mathfrak{G}A_L$ be dense in $V(B_L)$ and $V_{\mathfrak{A}}(B_R)$, respectively. Let B' be an intermediate ring of A/B left (or right) finite over B such that A is B' - A -irreducible and A - B' -irreducible. In order that $[B':B]_L = [B':B]_R$, it is necessary and sufficient that $[V:V_A(B')]_R = [V:V_A(B')]_L$.*

Proof. If $\sigma|B' = \tau|B'$ for some $v \in V^*$ ($\sigma, \tau \in \mathfrak{G}$), we write $\sigma|B' \sim \tau|B'$. Evidently, the relation $\sim$ is an equivalence relation in $\mathfrak{G}|B'$, and the number of the equivalence classes w. r. t. $\sim$ denoted as $(\mathfrak{G}|B': \tilde{V})$ coincides with the number of $B'_R \cdot A_R$ -homogeneous components of $(\mathfrak{G}|B')V_L A_R$ as well as with that of $B'_L \cdot A_L$ -homogeneous components of $(\mathfrak{G}|B')V_R A_L$. Hence, by [8, Lemmas 1.3, 1.4, 1.5], we obtain $[B':B]_L = (\mathfrak{G}|B': \tilde{V}) \cdot [V:V_A(B')]_R$ and $[B':B]_R = (\mathfrak{G}|B': \tilde{V}) \cdot [V:V_A(B')]_L$. Comparing those above, we readily see our conclusion.

Combining Prop. 5 with [9, Cor. 2 (b)], we obtain the next:

Corollary 3. *Let $J(\mathfrak{G}, A) = B$, and let A be $B \cdot V$ - A -irreducible and*

A-B-V-irreducible. Let B' be an intermediate ring of A/B left (or right) finite over B such that A is B' - A -irreducible and A - B' -irreducible. In order that $[B' : B]_L = [B' : B]_R$, it is necessary and sufficient that $[V : V_A(B')]_R = [V : V_A(B')]_L$.

The next is a partial extension of [9, Th. 3 (b)].

Theorem 4. *Let a division ring A be Galois over B , and $[V : C_0] < \infty$. If B' is an intermediate ring of A/B left finite over B then $[B' : B]_L = [B' : B]_R$.*

Proof. Since $[V : V_A(B')]_R \leq [B' : B]_L < \infty$, we obtain $[V : V_A(B')]_R = [V : V_A(B')]_L$ by Prop. 4. Hence, by Cor. 3, $[B' : B]_L = [B' : B]_R$.

4. The notion of QG-1 extensions

Let $\mathfrak{G}$ be a (multiplicative) sub-semigroup of $\mathfrak{A} = \text{Hom}(A, A)$.

Definition 1. $\mathfrak{G}$ is said to be a *left (resp. right) Galois semigroup* of A/B if $V_{\mathfrak{A}}(\mathfrak{G}) \cap A_L = B_L$ and $V_{\mathfrak{A}}(\mathfrak{G}) \cap A_R \supset B_R$ (resp. $V_{\mathfrak{A}}(\mathfrak{G}) \cap A_L \supset B_L$ and $V_{\mathfrak{A}}(\mathfrak{G}) \cap A_R = B_R$). If $\mathfrak{G}$ is a left and right Galois semigroup of A/B , we call $\mathfrak{G}$ a *Galois semigroup* of A/B .

Definition 2. Let $\mathfrak{G}$ be a left (resp. right) Galois semigroup of A/B . If A is $B \cdot V$ - A -irreducible and $\mathfrak{G}A_R$ is a subring of $\mathfrak{A}$ (resp. A is $A \cdot B \cdot V$ -irreducible and $\mathfrak{G}A_L$ is a subring of $\mathfrak{A}$), then we call A/B a *left (resp. right) QG-1 extension with respect to $\mathfrak{G}$* , and $\mathfrak{G}$ a *left (resp. right) Galois semigroup* belonging to the left (resp. right) QG-1 extension A/B . Moreover, if A/B is a left and right QG-1 extension with respect to a Galois semigroup $\mathfrak{G}$ then we call A/B a *QG-1 extension with respect to $\mathfrak{G}$* .

Finally we shall present a proposition which is useful in our consideration.

Proposition 6. *Assume that A/B is a left QG-1 extension w. r. t. a left Galois semigroup $\mathfrak{G}$ containing $\bar{V}$. Let M be a B - B -submodule of A possessing a linearly independent finite left B -basis.*

(a) $(\mathfrak{G}|M)V_R$ possesses a linearly independent V_R -basis that forms at the same time a linearly independent A_R -basis of $(\mathfrak{G}|M)A_R$, so that there holds $[M : B]_L = [(\mathfrak{G}|M)A_R : A_R]_R = [(\mathfrak{G}|M)V_R : V_R]_R$.

(b) Let B be a division ring, and a a non-zero element of M . The right V -module $a\mathfrak{G}V_R$ possesses a linearly independent right V -basis. In particular, if $[M : B]_L = [a\mathfrak{G}V_R : V]_R$ then $M = BaB$.

Proof. (a) is essentially [9, Lemma 6], and the argument used in

the proof of [6, Lemma 2.2] enables us to obtain the remaining.

5. Left QG-1 extensions of division rings

In the present section, we shall state without proof the extensions of [4, Th. 1] and [4, Th. 2] to left QG-1 extensions of division rings. In fact, by the validity of Prop. 6 and [6, Lemma 6.6], the respective proofs of them proceed in the same way as those of [4, Th. 1] (and [4, Cor. 2]) and [4, Th. 2] did.

Theorem 5. *Let a division ring A be a left QG-1 extension of B , and $[B:Z]=\infty$. If M is a B - B -submodule of A left finite over B then $M=BaB$ for some $a \in M$.*

Theorem 6. *Let a division ring A be a left QG-1 extension of B , and $[B:Z]=\infty$. If A/B is left algebraic then it is left locally finite.*

Remark. In [5, Th. 1], we have seen that if A/B is Galois and left algebraic and of bounded degree then $[A:B]<\infty$. However, the following example shows that [5, Th. 1] can not be extended to QG-1 extensions. Let $\phi = GF(p)$ (p a prime), and $x, y, z, \dots$ an infinite number of indeterminates. If $B = \phi(x, y, z, \dots)$ and $A = \phi(x^{1/p}, y^{1/p}, z^{1/p}, \dots)$ then A/B is not finite dimensional but algebraic and of bounded degree.

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