ON ISOMETRIES IN AN AFFINE SYMMETRIC SPACE

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In this paper we deal with the group of affine transformations in an affine symmetric space with an indefinite metric and an affine connection. An affine transformation is defined to be a transformation which keeps the connection invariant in the sense of *K. Nomizu* [2]*. We prove in the paper that every affine transformation is isometric and the group of all isometric transformations has a finite number of components. Our results are applicable to the case of non-compact Riemannian symmetric space, since our case is non-compact and pseudo-Riemannian.

§**1.** We explain some preliminary concepts and prove a lemma necessary for the later. Let G/H be an affine symmetric space i. e. there exists an involutive automorphism Σ of G whose fixed point set H_Σ contains H and its connected component of the identity $(H_\Sigma)_0$ is contained in H. We always assume that G is a connected semi-simple Lie group and each element of G acts on G/H left and effectively. We denote by $\mathfrak{G}$ and $\mathfrak{H}$ the Lie algebra of G and H respectively and by σ the differential of Σ. According to the eigenvalues $+1$ and -1 of σ, $\mathfrak{G}$ is decomposed into $\mathfrak{G}=\mathfrak{H}+\mathfrak{M}$, and as usually $\mathfrak{M}$ is identified to the tangent space of G/H at the origin p_0 $(=H)$. Let μ be the Killing form of $\mathfrak{G}$. Then its restriction to $\mathfrak{M}$ gives rise to an invariant indefinite metric of G/H. We denote this metric by μ for the simplicity of the notation.

According to *K. Nomizu* [2] and *A. Lichnerovichz* [3], if G/H is effective, then the connected component of the identity of the affine transformation group of G/H coincides with G. Next we prove the following

Lemma. *In the decomposition $\mathfrak{G}=\mathfrak{H}+\mathfrak{M}$, the number of involutive automorphisms, such that their restrictions to $\mathfrak{H}$ are identity, is finite.*

Proof. Let A be an involutive automorphism as said in the lemma. Since $\mathfrak{M}$ is the orthogonal complement of $\mathfrak{H}$ with respect to the Killing form μ and $\mathfrak{H}$ is invariant under A, $\mathfrak{M}$ is also invariant under A. By virtue of the assumption we have a decomposition $\mathfrak{M}=\mathfrak{M}_{+1}+\mathfrak{M}_{-1}$ according to eigenvales $+1$ and -1 of A. We then have

$$[\mathfrak{H}, \mathfrak{M}_{+1}]\subset\mathfrak{M}_{+1},\ [\mathfrak{H}, \mathfrak{M}_{-1}]\subset\mathfrak{M}_{-1},\ [\mathfrak{M}_{+1}, \mathfrak{M}_{-1}]=0.$$

From this relations it is easily seen that $[\mathfrak{M}_{+1}, \mathfrak{M}_{+1}]+\mathfrak{M}_{+1}$ and $[\mathfrak{M}_{-1},$

*) The numbers of bracyets refer to the references at the end of the paper.

$\mathfrak{M}_{-1}]+\mathfrak{M}_{-1}$ are ideals of $\mathfrak{G}$ and $([\mathfrak{M}_{+1}, \mathfrak{M}_{+1}]+\mathfrak{M}_{+1})\cap([\mathfrak{M}_{-1}, \mathfrak{M}_{-1}]+\mathfrak{M}_{-1})$ is an ideals of $\mathfrak{G}$ contained in $\mathfrak{H}$. On the other hand, by the assumption G/H is effective. Hence this ideal is zero. As we see in [2], $[\mathfrak{M}, \mathfrak{M}]=\mathfrak{H}$, we thus get

$$\mathfrak{G}=([\mathfrak{M}_{+1}, \mathfrak{M}_{+1}]+\mathfrak{M}_{+1})\oplus([\mathfrak{M}_{-1}, \mathfrak{M}_{-1}]+\mathfrak{M}_{-1}).$$

The direct decomposition of $\mathfrak{G}$ into σ-invariant ideals is finite in number. Since $\mathfrak{G}$ is semi-simple, each of the above two ideals is expressed as a direct sum of simple ideals and further since the automorphism A is determined by such decomposition, the lemma is proved.

As we stated before the purpose of the paper is to prove the following

Theorem. *Let G be a connected semi-simple Lie group and G/H an effective affine symmetric space with a pseudo-Riemannian metric and the connection induced from its metric. Then the group of all affine transformations of G/H coincides with that of all isometries. Moreover the number of components of this group is finite.*

We conveniently divide the proof into two parts. The first part is proved in § 2 and the later in § 3.

§2. Let A $(=A(G/H))$ be the group of all affine transformations of G/H and A_0 $(=A_0(G/H))$ its connected component of the identity. We define a homomorphism θ from A into $Aut(A_0)$, the group of automorphisms, in such a way that $\theta(\alpha)(x)=\alpha x\alpha^{-1}$, $\alpha \in A$, $x\in A_0$. The homomorphism θ induces a homomorphism $\tilde{\theta}$:

$$A/A_0 \longrightarrow Aut\ (A_0)/Int\ (A_0).$$

As stated in § 1, we can assume $A_0=G$. To prove the theorem we study the kernel and the image of $\tilde{\theta}$.

1). The kernel of $\tilde{\theta}$.

At first we put

$$B=\{\alpha\,|\,\alpha\in A \text{ and } \alpha x=x\alpha \text{ for any } x\in A_0\}.$$

We further denote $\{\alpha(p_0)|\ \alpha\in B\}$ by $B(p_0)$ where $p_0=H$. If $\alpha(p_0)=k\cdot H$, then $\alpha(g\cdot H)=g\cdot k\cdot H$, that is, the transformation α is completely determined by the image point $\alpha(p_0)$. Therefore the set $B(p_0)$ and B is in 1-1 correspondence. Let $A(p_0)$ be the set of fixed points of the isotropy subgroup H, then we can easily see $B(p_0)\subset A(p_0)$. Let K be the normalizer of H in G. We then see $A(p_0)=K/H$. For any $k\cdot H \in A(p_0)$ we define a diffeomorphism α of G/H by $\alpha(g\cdot H)=g\cdot k\cdot H$ $(g\in G)$. α is commutative with any element of A_0, since for any element $x=\tau(l)\in A_0$, $l\in G$ we obtain

$$\alpha x(g\cdot H)=\alpha(l\cdot g\cdot H)=l\cdot g\cdot k\cdot H=x\alpha(g\cdot H).$$

Next we show that α is an isometry of G/H. In fact. from $k\in K$, it follows that $Ad(k)\mathfrak{H} = \mathfrak{H}$ and $\mathfrak{M}$ is the orthogonal complement of $\mathfrak{H}$ with respect to the Killing form μ, hence we see $Ad(k)\mathfrak{M} = \mathfrak{M}$ and

$$\begin{aligned}\mu_{k\cdot H}(d\alpha X,\ d\alpha Y) &= \mu(d\tau(k^{-1})\ d\alpha X,\ d\tau(k^{-1})d\alpha Y)\\ &= \mu(Ad(k^{-1})X,\ Ad(k^{-1})Y)\\ &= \mu(X, Y) \quad \text{for } X,\ Y \varepsilon \mathfrak{M}\end{aligned}$$

i. e. α is isometric. Since an isometric transformation is affine, $\alpha \varepsilon B$ and $k \cdot H \varepsilon B(p_o)$ and therefore $A(p_o) = B(p_o)$. From the above the correspondence: $k \cdot H \longrightarrow \alpha^{-1}$ is an isomorphism of K/H onto B. On the other hand we have

$$\theta^{-1}(Int(A_o)) = B \cdot A_o.$$

In fact, if $\alpha \varepsilon \theta^{-1}(Int(A_o))$ there exists an element α_0 of A_o such that $\theta(\alpha) = \theta(\alpha_o)$. Then $\alpha_o^{-1} \cdot \alpha \varepsilon B$ and $\alpha \varepsilon B \cdot A_o$. We thus see that $\theta^{-1}(Int(A_o))$ consists of isometries.

2). The image of $\tilde{\theta}$.

Now consider an automorphism ζ of A_o $(= G)$ such that $\zeta(H) \subset H$ and define a diffeomorphism α of G/H by $\alpha(g \cdot \mathrm{H}) = \zeta(g) \cdot H$, $g \varepsilon G$. For any element $x = \tau(l)$ $(l \varepsilon G)$ of A_o,

$$\begin{aligned}\alpha x \alpha^{-1}(g \cdot H) &= \alpha x(\zeta^{-1}(g) \cdot H) = \alpha(l \cdot \zeta^{-1}(g) \cdot H)\\ &= \zeta(l) \cdot g \cdot H = \tau(\zeta(l))g \cdot H.\end{aligned}$$

It follows from this that $\alpha x \alpha^{-1} = \tau(\zeta(l))$. We further see that α is an isometry and $\theta(\alpha) = \zeta$, in fact

$$\begin{aligned}\mu(d\alpha X,\ d\alpha Y) &= \mu(d\zeta X,\ d\zeta Y)\\ &= \mu(X,\ Y) \ \text{ for } X,\ Y \varepsilon \mathfrak{M},\end{aligned}$$

because of $d\zeta(\mathfrak{M}) = \mathfrak{M}$.

Next choose an element $\alpha \varepsilon A$, $\alpha(p_o) = p_o$ from a class of A/A_o, if we put $\theta(\alpha) = \zeta$, $\alpha x \alpha^{-1} = \tau(\zeta(l))$ where $x = \tau(l) \varepsilon A_o$, $l \varepsilon G$. For any $h \varepsilon H$ we see

$$\zeta(h) \cdot H = \alpha\tau(h)\alpha^{-1} \cdot H = \alpha \cdot \tau(h) \cdot H = \alpha \cdot H \mid H.$$

Hence we have $\zeta(H) = H$. From these facts $\tilde{\theta}(A/A_o)$ consists of the classes of $Aut(A_o)/Int(A_o)$ such that each of these classes contains an automorphism ζ, such that $\zeta(H) \supset H$. If we denote by $Aut(G : H)$ the group of automorphisms of G which preserve H. We then get

$$\mu = [Img\ \tilde{\theta} : 1] = [Aut(G : H) : Int_G(K)],$$

where $Int_G(K)$ means the subgroup of $Int(G)$ by elements of K. We can now conclude that the affine transformation group coincides with the isometric transformation group because the kernel of θ consists of isometries and an image ζ by θ is also an image of an isometry α. This proves the first part of the theorem.

§3. We now prove the later part of the theorem. By the assumption G is semi-simple and then $Aut(G)/Int(G)$ is a finite group [4]. This implies $\mu<\infty$. On the other hand we can see from the facts said in §2 that $ker\tilde{\theta}\cong\theta^{-1}(Iut(A_o))/A_o\cong B/B_o$ where $B_o=B\cap A_o$. We put $\nu=[B:B_o]$. In order to prove that the number of components of A is finite, it is sufficient to show $\nu<\infty$.

Let Z be the center of G and put $\tilde{H}=H\cdot Z$, $\tilde{H}'=H'\cdot Z$ where $H'=K\cap H_\Sigma$. On the other hand $\tilde{H}/H\cong B_o$ under the isomorphism $k\cdot H\longrightarrow \alpha^{-1}$. Thus we have

$$B/B_o\cong K/H\Big/\tilde{H}/H\cong K/\tilde{H}$$
$$\nu=[K:\tilde{H}'][\tilde{H}':\tilde{H}].$$

It is easy to see $[\tilde{H}':\tilde{H}]\leqq[H_\Sigma:H]$ and we can see from [1] $[H_\Sigma:H]<\infty$. Hence it is sufficient to prove $[K:\tilde{H}']<\infty$.

We can assume that G is an adjoint group. In fact, let $\bar{G}=G/Z$, $\bar{H}=H\cdot Z/Z=\tilde{H}/Z$, then $\bar{G}/\bar{H}$ is an effective affine symmetric space. If $\bar{K}$ is the normalizer of $\bar{H}$ in $\bar{G}$, then

$$\nu=[K:\tilde{H}]=[K/Z:\tilde{H}/Z]\leqq[K'/Z:\tilde{H}/Z]=\bar{\nu},$$

where K' is a suitable subgroup of G such that $K'/Z=\bar{K}$. Under this assumption, we put $k^{-1}\cdot\sum(k)=a$ for any $k\varepsilon K$. Since K is $\sum$-invariant, $a\varepsilon K$ and $\sum(a)=a^{-1}$ and it is clear that $Ad(a)/\mathfrak{H}=id.$ and $Ad(a)\mathfrak{M}=\mathfrak{M}$.

Operating $\sum$ on the both sides of

$$a(exptx)a^{-1}=exptAd(a)X \quad \text{for} \quad x\varepsilon\mathfrak{M},$$

we have $a^{-1}(exp\,tX)a=exp\,tAd(a)X$. Since $a^{-1})exp\,tX)a=exp\,Ad\,(a^{-1})X$, we get $Ad(a^{-1})X=Ad(a)X$ which implies $Ad(a)^2=id$. By virtue of the lemma the number of such elements a is finite. We denote these elements by $g_1, g_2, \cdots, g_l$. If the element a is identical with some g_i, we have $k'^{-1}\cdot\sum(k')=g_i$ for some element $k'\varepsilon K$, then $k'\cdot k^{-1}=\sum(k'\cdot k^{-1})$ and hence $k'\cdot k^{-1}\varepsilon H_\Sigma\cap K=H'$. Thus we see $k'\cdot H'=k\cdot H'$ and $[K:H']<l$. The last part of the theorem follows from this.

Example: For $SL(2n,R)/S_p(n,R)$ it is easy to see

if $n=even$, $\nu=2$ and $\mu=2$,
if $n=odd$, $\nu=1$ and $\mu=4$.

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