PRIMARY IDEAL REPRESENTATIONS IN NON-COMMUTATIVE RINGS

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Introduction. In his paper [4], H. Tominaga has given a necessary and sufficient condition that every ideal in a (non-commutative) ring be represented as the intersection of a finite number of s-right and s- left primary ideals[1]. It is the purpose of this paper to present a condition that every ideal in a ring be represented as a finite intersection of s-right primary ideals. After several definitions (§ 1), we shall prove in § 2 the uniqueness theorem for s-right primary representations: in any two s-right primary short representations of an ideal, the number of s-right primary components are the same and their radicals coincide in some order. In § 3, we shall give a necessary and sufficient condition that every ideal have a representation as a finite intersection of s-right primary ideals, which is analogous to that in [4]. In case the maximum condition is satisfied for ideals, the first half of our condition can be excluded (§ 4).

1. Definitions. Let R be a (non-commutative) ring. The term "ideal" in R will always mean "two-sided ideal".

Definition 1. A and B are ideals in R, the ideal consisting of all elements x of R such that $xRB \subseteq A$ is called the *right ideal quotient* of A by B and is denoted by AB^{-1}. Similary, $B^{-1}A$ consists of all x in R such that $BRx \subseteq A$.

The following properties of quotients are verified:

(1) $(AB^{-1})C^{-1} = A(CRB)^{-1}$,

(2) $(\bigcap_\alpha A_\alpha)B^{-1} = \bigcap_\alpha A_\alpha B^{-1}$,

(3) $A(\sum_\alpha B_\alpha)^{-1} = \bigcap_\alpha AB_\alpha^{-1}$, where A, B, C, A_α and B_α are ideals in R.

Definition 2. An element a is *right non-prime to* an ideal A if there exists an element b not in A such that $bRa \subseteq A$. An ideal B is *right non-prime* to A if $AB^{-1} \supset A$.

For positive integers n we define inductively $AB^{-n} = (AB^{-(n-1)})B^{-1}$. If $AB^{-k} = AB^{-(k+1)}$ for some positive integer k then we say that AB^{-k} is the *right limit ideal* of A by B. The left limit ideal $B^{-k}A$ can be defined in the same way. An ideal P in R is *prime* if $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, where A and B are ideals in R. It has been shown by McCoy [2] that an ideal P is prime if and only if $aRb \subseteq P(a,\ b \in R)$ implies that

1) "s-right primary" means "strongly right primary".

either a or b belongs to P. The *radical* of an ideal A is understood in the sense of McCoy [2] and denoted by $r(A)$. It has been shown by McCoy [2] that $r(A)$ is the intersection of all minimal prime divisors of A.

Definition 3. An ideal Q is said to be *right primary* if $aRb \subseteq Q$ and $a \notin Q$ imply $b \in r(Q)$, and a right primary ideal Q is defined to be *s-right primary* if $r(Q)$ is nilpotent modulo Q.

One will easily see that an ideal Q is s-right primary if and only if it is s-right primary in Tominaga's sense, and so the radical of an s-right primary ideal is prime by Theorem 1 of [4].

Definition 4. If a prime ideal P is the radical of an s-right primary ideal Q, we say that Q *belongs to* P and also that Q is *P-s-right primary.* A prime ideal P is called a *prime ideal associated with* an ideal A if there exists an s-right primary ideal Q belonging to P such that $Q = B^{-1}A$ for some ideal B not contained in A.

2. Uniqueuess theorem for s-right primary representations.

A representation $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ of an ideal A as the intersection of s-right primary ideals $Q_1, Q_2, \cdots, Q_n$ will be called *irredundant* if no one of the Q_i contains the intersection of the remaining ones.

Theorem 1. *Let $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ be an irredundant representation of $A\subset R$, where Q_i is P_i-s-right primary $(1\leqslant i\leqslant n)$. Then an element x is right non-prime to A if and only if $x\in P_j$ for some j, namely, $x\in P_1\cup P_2\cup\cdots\cup P_n$.*

Proof. If x is right non-prime to A then $bRx\subseteq A$ for some b not in A. But this implies $bRx\subseteq Q_i (1\leqslant i\leqslant n)$, while $b\notin Q_j$ for some j. Since Q_j is P_j-s-right primary, we obtain $x\in P_j$. Conversely, suppose that x is in P_1. Since the representation $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ is irredundant, we can choose an element b which is contained in $Q_2\cap\cdots\cap Q_n$ but not in Q_1. Noting that $(RP_1)^k\subseteq Q_1$ for some k, we have then $b(RP_1)^k\subseteq A$. Accordingly, there exists the least positive integer k_1 such that $b(RP_1)^{k_1}\subseteq A$. If $k_1=1$ then $bRP_1\subseteq A$. Hence $bRx\subseteq A$. Thus, x is right non-prime to A. If $k_1>1$ then the product $b(RP_1)^{k_1-1}$ contains an element b_1 not in A. Since $b_1Rx\subseteq A$, x is right non-prime to A.

Lemma 1. *If $Q_1, Q_2, \cdots, Q_n$ are P-s-right primary ideals then $Q=Q_1\cap Q_2\cap\cdots\cap Q_n$ is also a P-s-right primary ideal.*

Proof. Let k_i be the nilpotency index of P modulo $Q_i (1\leqslant i\leqslant n)$. Then, $P^{k_1+\cdots+k_n}\subseteq Q$. If P_1 is any prime divisor of Q, we have $P^{k_1+\cdots+k_n}\subseteq P_1$, whence it follows $P\subseteq P_1$. Hence, P is a unique minimal prime divisor of Q and therefore $P=r(Q)$. Moreover, if $aRb\subseteq Q$ and $a\notin Q$ then $aRb\subseteq Q_i (1\leqslant i\leqslant n)$, while $a\notin Q_j$ for some j. Since Q_j is P-s-right primary, this implies that $b\in P=r(Q_j)$. Hence, Q is P-s-right primary.

By the same argument as in Theorem 14 of [3] we have

Lemma 2. *If* $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ *is an irredundant representation of* A, *where* Q_i *is* P_i*-s-right-primary* $(1\leqslant i\leqslant n)$ *and* $P_j\neq P_k$ *for some* $j\neq k$, *then* A *is not s-right primary.*

Definition 5. An irredundant representation $A=Q_1\cap Q_2\cap\cdots Q_n$ will be called a *short* representation if none of the intersections of two or more of the ideals $Q_1, Q_2, \cdots, Q_n$ are s-right primary.

In view of Lemmas 1 and 2, an irredundant representation $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ is a short representation if and only if any two of the radicals of $Q_1, Q_2, \cdots, Q_n$ are distinct.

Let M be a non-empty m-system in R. For any ideal A in R the right upper and lower isolated M-components of A (in the sense of [3]) will be denoted by $U(A, M)$ and $L(A, M)$, respectively. If P is a prime ideal $(\neq R)$ and $M=C(P)$ is its complement in R then $U(A, M)$ will be denoted by $U(A, P)$.

Theorem 2. *Let* $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ *be an irredundant representation of* A, *where* Q_i *is* P_i*-s-right primary* $(1\leqslant i\leqslant n)$. *If* $M(\subseteq R)$ *is a non-empty m-system which does not meet* $P_1, \cdots, P_r$ *but meets* $P_{r+1}, \cdots, P_n$, *then* $U(A, M)=L(A, M)=Q_1\cap Q_2\cap\cdots\cap Q_r$. *If* M *meets every* P_i *then* $U(A, M)=L(A, M)=R$.

Proof. By the same argument as in Theorem 15 of [3], we can easily see that if M does not meet $P_1, \cdots, P_r$ but meets $P_{r+1}, \cdots, P_n$ then $U(A, M)=Q_1\cap Q_2\cap\cdots\cap Q_r$ and that if M meets every P_i then $U(A, M)=R$.

We assume first that M does not meet $P_1, \cdots, P_r$ but meets $P_{r+1}, \cdots, P_n$. Let b be an element of $L(A, M)$. Then we have $bRm\subseteq A$ for some $m\in M$ and thus $bRm\subseteq Q_i (1\leqslant i\leqslant r)$. However, m is not in any $P_i (1\leqslant i\leqslant r)$. Hence $b\in Q_i (1\leqslant i\leqslant r)$ and thus $L(A, M)\subseteq Q_1\cap Q_2\cap\cdots\cap Q_r$. We shall prove now the converse inclusion. *If* $r=n$ then this is trivial by $A\subseteq L(A, M)$. In case $r<n$, since M meets P_j for $j>r$, it follows that M meets Q_j for $j>r$. Hence there exist $m_1, m_2, \cdots, m_{n-r}$ such that $m_i\in Q_{r+i}\cap M (1\leqslant i\leqslant n-r)$. Now, since every m_i is in M, there exist $x_1, x_2, \cdots, x_{n-r-1}$ such that $m=m_1x_1m_2x_2\cdots x_{n-r-1}m_{n-r}$ is contained in M. Since it is clear that $m\in Q_{r+1}\cap Q_{r+2}\cap\cdots\cap Q_n$, $qRm\subseteq A$ for every element $q\in Q_1\cap Q_2\cap\cdots\cap Q_r$. Thus q is in $L(A, M)$.

If M meets every P_i then the last part of the above proof shows that there is an element $m\in M$ such that $m\in Q_1\cap Q_2\cap\cdots\cap Q_n=A$. Hence $rRm\subseteq A$ for every $r\in R$, that is, $R=L(A, M)$.

Theorem 3. *Let* $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ *be an irredundant representation of* A, *where* Q_i *is* P_i*-s-right primary* $(1\leqslant i\leqslant n)$. *Then the minimal prime divisors of* A *are exactly those primes which are minimal in the set* $\{P_1, P_2, \cdots, P_n\}$.

Proof. This is immediate.

Theorem 4. *Let $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ be a short representation of $A\subset R$, where Q_i is P_i-s-right primary $(1\leqslant i\leqslant n)$. A prime divisor $P(\neq R)$ of A is one of P_i if and only if every element of P is right non-prime to $U(A, P)$. The ring R is itself one of the P_i if and only if every element of R is right non-prime to A.*

Proof. Let $P(\neq R)$ be a prime divisor of A. If P coincides with one of P_i, then by Theorem 2 $U(A, P)=Q_{i_1}\cap Q_{i_2}\cap\cdots\cap Q_{i_r}$ is a short representation of $U(A, P)$, where $P_{i_1}, P_{i_2}, \cdots, P_{i_r}$ are those primes among $\{P_i\}$ which are contained in P(and so P is maximal among them). Hence, by Theorem 1, every element of P is right non-prime to $U(A, P)$. Conversely, assume that every element of P is right non-prime to $U(A, P)$. By Theorem 3, P contains at least one of P_i. Suppose that P contains $P_1, \cdots, P_r$ but does not contain $P_{r+1}, \cdots, P_n$. Then, again by Theorem 2, $U(A, P)=Q_1\cap Q_2\cap\cdots\cap Q_r$ is a short representation of $U(A, P)$. Hence, $P\subseteq P_1\cup P_2\cup\cdots\cup P_r$ by Theorem 1, and then by Theorem 5 of [1] there exists some i such that $P\subseteq P_i$, namely, $P=P_i$. The latter assertion is also an easy consequence of Theorem 5 of [1] and Theorem 1.

As an immediate consequence of Theorem 4, we obtain the following:

Theorem 5. *Let $A=Q_1\cap Q_2\cap\cdots\cap Q_n=Q'_1\cap Q'_2\cap\cdots\cap Q'_m$ be two short representations of A, where Q_i is P_i-s-right primary and Q'_j is P'_j-s-right primary. Then, $m=n$ and it is possible to number the components in such a way that $P_i=P'_i$ $(1\leqslant i\leqslant m=n)$.*

Let $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ be a short representation of A, where Q_i is P_i-s-right primary $(1\leqslant i\leqslant n)$. These uniquely determined prime ideals $P_1, P_2, \cdots, P_n$ will be called the *prime ideals belonging to* A (cf. Theorem 5). A subset $\{P_{i_1}, P_{i_2}, \cdots, P_{i_r}\}$ of these prime ideals is called an *isolated set* of prime ideals belonging to A if every P_j contained in one of the primes $P_{i_1}, P_{i_2}, \cdots, P_{i_r}$ is necessarily a member of the subset.

Now, by Theorem 2, one will readily obtain the following:

Theorem 6. *Let $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ be a short representation of A, where Q_i is P_i-s-right primary $(1\leqslant i\leqslant n)$. If $\{P_{i_1}, P_{i_2}, \cdots, P_{i_r}\}$ is an isolated set of prime ideals belonging to A then $Q_{i_1}\cap Q_{i_2}\cap\cdots\cap Q_{i_r}$ depends only on $\{P_{i_1}, P_{i_2}, \cdots, P_{i_r}\}$ and not on the particular short representation considered.*

3. A necessary and sufficient condition that every ideal be represented as a finite intersection of s-right primary ideals.

Theorem 7. *Let $A=Q_1\cap Q_2\cap\cdots\cap Q_n$ be a short representation of $A\subset R$, where Q_i is P_i-s-right primary $(1\leqslant i\leqslant n)$. If P is a minimal prime* *divisor of A then P is right non-prime to A.*

Proof. By Theorem 3, we can assume that P is contained in $P_1, \cdots, P_r (r \geq 1)$ but not contained in $P_{r+1}, \cdots, P_n$. Then $R = Q_i P^{-k} = Q_i P^{-(k+1)}$ for a sufficiently large positive integer $k (1 \leqslant i \leqslant r)$. On the other hand, if $r+1 \leqslant j \leqslant n$ then $Q_j = Q_j P^{-h} = Q_j P^{-(h+1)}$ for every positive integer h. Thus $AP^{-k} = AP^{-(k+1)} = Q_{r+1} \cap Q_{r+2} \cap \cdots \cap Q_n$. Since $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a short representation, we have $AP^{-k} \supset A$, and therefore $AP^{-1} \supset A$.

Lemma 3. *If Q is a P-s-right primary ideal then $B^{-1}Q$ is P-s-right primary for any ideal $B \not\subseteq Q$.*

Proof. Since $BR(B^{-1}Q) \subseteq Q$ and $B \not\subseteq Q$, we have $Q \subseteq B^{-1}Q \subseteq P$, and thus $r(B^{-1}Q) = P$. Suppose that $aRb \subseteq B^{-1}Q$ and $b \notin P$. Then we have $BRaRb \subseteq Q$. Hence, by the definition of s-right primary, $BRa \subseteq Q$, that is, $a \in B^{-1}Q$.

By the same arguments as in Theorems 4 and 6 of [4], we have the following two theorems.

Theorem 8. *Let $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ be a short representation of A. Then, for any ideal B there exists the right limit ideal of A by B, and the number of ideals which are obtained starting from A by repeating successively the procedure to make right limit ideals is finite and is uniquely determined by A.*

Theorem 9. *Let $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ be a short representation of $A \subset R$, where Q_i* is *P_i-s-right primary $(1 \leqslant i \leqslant n)$. Then, a prime divisor P of A is a prime ideal associated with A if and only if P coincides with one of P_i, and every primary component $Q_i (1 \leqslant i \leqslant n)$ has the following property: $B^{-1}A$ is not P_i-s-right primary for any ideal B such that $B \subset Q_i$ and $B \not\subseteq A$.*

Corollary 1. *Let $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ be a short representation of $A \subset R$. If P is a minimal prime divisor of A then P is a prime ideal associated with A.*

Now, we can summarize the above-mentioned results as follows:

Theorem 10. *In order that every ideal in R be represented as the intersection of a finite nnmber of s-right primary ideals, the following conditions are necessary:*

(A) *For any ideals A, B in R there exists the right limit ideal of A by B and there exist a finite number $n(A)$ of ideals which are obtained starting from A by repeating successively the procedure to make right limit ideals, where the number $n(A)$ is uniquely determined by A.*

(B) *Every ideal $A \subset R$ has a minimal prime divisor which is right non-prime to A.*

(C) *Every minimal prime divisor of an arbitrary not s-right primary* *ideal A is a prime ideal associated with A.*

(D) *If P is an arbitrary prime ideal associted with an ideal A then there exists an s-right primary ideal $Q \supseteq A$ belonging to P such that $B^{-1}A$ is not P-s-right primary for any subideal B of Q not contained in A.*

Next, we shall show that these conditions are sufficient, too.

Lemma 4. *Assume the conditions* (A) *and* (B) *in Theorem* 10. *If A is an ideal of R then $r(A)$ is nilpotent modulo A.*

Proof. Let P be a minimal prime divisor of $A(\subset R)$ which is right non-prime to A. Then $A \subset AP^{-1} \subseteq Ar(A)^{-1}$. If $Ar(A)^{-1}$ is not R itself then we have $Ar(A)^{-1} \subset Ar(A)^{-1}r(Ar(A)^{-1})^{-1} \subseteq Ar(A)^{-2}$. Continuing in this way, we obtain the right limit ideal $Ar(A)^{-k}$ of A by $r(A)$. We have then $Ar(A)^{-k} = R$, whence it follows $r(A)^{2k+1} \subseteq A$.

By the same argument as in Lemma 4 of [4], we have the following:

Lemma 5. *Assume the conditions* (A), (B) *and* (C) *in Theorem* 10. *Then the number of prime ideals associated with an ideal which is not s-right primary is finite.*

We assume here the conditions (A), (B), (C) and (D) in Theorem 10. Let $P_1, P_2, \cdots, P_n$ be all the prime ideals associated with an ideal A which is not s-right primary, and let $Q_1, Q_2, \cdots, Q_n$ be s-right primary divisor of A belonging to $P_1, P_2, \cdots, P_n$ with the property cited in (D), respectively (Lemma 5). We set $B = Q_1 \cap Q_2 \cap \cdots \cap Q_n$. By the condition (C), every minimal prime divisor of A is a prime ideal associated with A, and so $B \subseteq r(A)$. Since $r(A)$ is nilpotent modulo A by Lemma 4, we obtain $B^{-1}A \supset A$. We suppose now that $B \supset A$. If $B^{-1}A$ is not s-right primary then by the condition (C) we have an s-right primary ideal $C_0^{-1}B^{-1}A$ for some $C_0 \not\subseteq B^{-1}A$. So we set $C = BRC_0$. If $B^{-1}A$ is s-right primary, we set $C = B$. Thus, in either case, we have an s-right primary ideal $Q = C^{-1}A$, where $C \not\subseteq A$ and $C \subseteq B$. Since $r(Q)$ is a prime ideal associated with A, $r(Q) = P_i$ for some i. On the other hand, since $C \subseteq B \subseteq Q_i$, the ideal $Q = C^{-1}A$ is not P_i-s-right primary by the condition (D). This contradiction means $A = B$. Hence, we have the following theorem.

Theorem 11. *In order that every ideal in R be represented as the intersection of a finite number of s-right primary ideals, it is necessary and sufficient that the conditions* (A), (B), (C) *and* (D) *be satisfied.*

4. Rings with maximum condition for ideals.

Throughout the present section, R be a ring with maximum condition for ideals. Then, needless to say, for any ideals A, B of R there exists the right limit ideal of A by B.

Lemma 6. *Every ideal $A \subset R$ has a minimal prime divisor which is right non-prime to A.*

Proof. One may assume that A is not prime. By Theorem 10 of [3], we have $P_1RP_2R \cdots RP_s \subseteq A$, where $P_1, \cdots, P_s$ are minimal prime divisors of A and $s>1$. Hence we can assume that $P_1RP_2R \cdots RP_s \subseteq A$ and $P_1RP_2R \cdots RP_{s-1} \not\subseteq A$. If b is an arbitrary element of $P_1RP_2R \cdots RP_{s-1}$ not contained in A then $bRP_s \subseteq A$, and so P_s is right non-prime to A.

From the proof of Lemma 6, the following will be obvious.

Corollary 1. *If A is an ideal of R then $r(A)$ is nilpotent modulo A. In particular, every primary ideal of R is s-right primary.*

Lemma 7. *Assume the condition* (**C**). *If an ideal A is not right primary then the number of prime ideals associated with A is finite.*

Proof. Let $\{P_\alpha\}$ be the set of all prime ideals associated with A, and let $Q_\alpha = B_\alpha^{-1}A$ $(B_\alpha \not\subseteq A)$ be a P_α-right primary ideal. The set $\{P_\alpha\}$ is not empty by the condition (**C**). Let $\{P_1, P_2, \cdots, P_k\}$ be a subset in $\{P_\alpha\}$ such that $P_i \not\subseteq P_j$ for every $i>j$. We define now the ideals $B'_1, B'_2, \cdots, B'_k$ in the following way: B'_1 is the right limit ideal of A by P_1 and B'_i is the right limit ideal of B'_{i-1} by P_i $(i=2, \cdots, k)$. Then, by the analogous argument as in Lemma 4 of [4], we have an ascending chain $A \subset B'_1 \subset B'_2 \subset \cdots \subset B'_k$. From this fact, the lemma will be easily seen.

Now, by the validity of Lemmas 6, 7 and Corollary 1 to Lemma 6, the proof of the following theorem proceeds just like that of Theorem 11 did.

Theorem 12. *Let R be a ring with maximum condition for ideals. In order that every ideal in R can be represented as the intersection of a finite number of right primary ideals, it is necessary and sufficient that the conditions* (**C**) *and* (**D**) *be satisfied.*

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