ON EXTREMAL PROPERTIES OF CIRCULAR SLIT COVERING SURFACES

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1. Let B be a non-compact domain of finite connectivity on the z-plane. We suppose that each component C_j $(j=1, \cdots, N)$ of its boundary C is a continuum. Let z_j and ζ_k $(j=1, \cdots, \iota;\ k=1, \cdots, \kappa;\ \iota \geqq 1, \kappa \geqq 1)$ be arbitrarily preassigned $\iota+\kappa$ points in B, and m_j and n_k $(j=1, \cdots, \iota;\ k=1, \cdots, \kappa)$ be arbitrarily preassigned positive integers under the condition

$$p \equiv \sum_{j=1}^{\iota} m_j = \sum_{k=1}^{\kappa} n_k. \tag{1}$$

We shall conventionally agree to take as $\zeta_1 = \infty \in B$ through the present paper. Let $\mathfrak{F}_p$ be the class of analytic functions $w=f(z)$ on B with the following properties:

(a) f has the only zeros z_j $(j=1, \cdots, \iota)$ and the only poles ζ_k $(k=1, \cdots, \kappa)$ with their orders m_j and n_k, respectively;

(b) The rotation number of the image of each C_j $(j=1, \cdots, N)$ about $w=0$ under f is equal to zero; i. e.

$$\nu_j(f) \equiv \frac{1}{2\pi} \int_{C_j^*} d \arg f = 0 \qquad (j=1, \cdots, N),$$

where C_j^* $(j=1, \cdots, N)$ are simple analytic closed curves homotopic to C_j in $B - \sum_{j=1}^{\iota} \{z_j\} - \sum_{k=1}^{\kappa} \{\zeta_k\}$ and $\nu_j(f)$ $(j=1, \cdots, N)$ are integers not depending on a particular choice of C_j^*;

(c) $$\left| \int_C \lg |f| \, d \arg f \right| < +\infty,$$

where the line integral means $\lim_{n \to \infty} \int_{\partial B_n} \lg |f| \, d \arg f$ with an exhaustion $\{B_n\}_{n=1}^{\infty}$ of B;

(d) f satisfies the normalization condition

$$\lim_{z \to \infty} \frac{f(z)}{z^n} = 1.$$

2. Let

$$J(f) = \int_C \lg |f| \, d \arg f - 2\pi \sum_{j=1}^{\iota} m_j \lg |f^{[m_j]}(z_j)| - 2\pi \sum_{k=2}^{\kappa} n_k \lg |f^{[n_k]}(\zeta_k)| \tag{2}$$

for $f\in\mathfrak{F}_p$, where

$$f^{[m_j]}(z_j)=\lim_{z\to z_j}\frac{f(z)}{(z-z_j)^{m_j}}=\frac{1}{m_j!}f^{(m_j)}(z_j)\qquad (j=1,\cdots,\iota),$$

$$f^{[n_k]}(\zeta_k)=\lim_{z\to\zeta_k}\frac{1}{(z-\zeta_k)^{n_k}f(z)}=\frac{1}{n_k!}\left[\left(\frac{1}{f(z)}\right)^{(n_k)}\right]_{z=\zeta_k}\qquad (k=2,\cdots,\kappa).$$

Then, the theorem 1 immediately follows from the theorem of [2].

THEOREM 1. *There exists the unique element φ in $\mathfrak{F}_p$ which minimizes $J(f)$ on $\mathfrak{F}_p$. Further φ is the unique element of $\mathfrak{F}_p$ which maps B onto the p-sheeted covering surface of which the boundary consists of circular slits centred at the origin on the basic w-plane.*

3. Let $\mathfrak{F}_p'$ be the subclass of $\mathfrak{F}_p$ which consists of functions $f(z)$ of $\mathfrak{F}_p$ satisfying the condition

$$\int_C \lg|f|\,d\arg f\leqq 0. \tag{3}$$

Then $\mathfrak{F}_p'$ is not vacuous, for there exists a rational function on the z-plane with the properties. Let

$$I(f)=\prod_{j=1}^{\iota}|f^{[m_j]}(z_j)|^{m_j}\prod_{k=2}^{\kappa}|f^{[n_k]}(\zeta_k)|^{n_k} \tag{4}$$

for $f\in\mathfrak{F}_p'$. Then, we have the following theorem.

THEOREM 2. *The function φ of the theorem 1 is the unique element in $\mathfrak{F}_p'$ which maximizes $I(f)$ on $\mathfrak{F}_p'$.*

Proof. It is immediately seen that there holds

$$\int_C \lg|\varphi|\,d\arg\varphi=0 \tag{5}$$

for φ of the theorem 1 and thus $\varphi\in\mathfrak{F}_p'$. We note that

$$J(f)=\int_C \lg|f|\,d\arg f-2\pi\lg I(f) \tag{6}$$

for any $f\in\mathfrak{F}_p'$. Then, by (5), (6) and theorem 1, there holds

$$-2\pi\lg I(\varphi)=J(\varphi)\leqq J(f)\leqq -2\pi\lg I(f)$$

and thus

$$I(\varphi)\geqq I(f) \tag{7}$$

for φ of the theorem 1 and any $f \in \mathfrak{F}_p'$. Further, by the theorem 1, the equality in (7) occurs if and only if $f(z) \equiv \varphi(z)$.

In the case $p=1$ in 1, we know that $\iota=\kappa=1$, $m_1=n_1=1$ and

$$I(f)=|f'(z_1)|. \tag{8}$$

Thus we have the following corollary of the theorem 2.

COROLLARY. *There exists a unique element φ in $\mathfrak{F}_1'$ which maximizes $|f'(z_1)|$ on $\mathfrak{F}_1'$. Further, φ is the unique element of $\mathfrak{F}_1'$ which univalently maps B onto the domain of which the boundary consists of circular slits centred at the origin.*

4. Let $\mathfrak{F}_p''$ be the subclass of $\mathfrak{F}_p$ which consists of functions $f(z)$ of $\mathfrak{F}_p$ being p-valent.

LEMMA. $\mathfrak{F}_p'' \subset \mathfrak{F}_p'$.

Proof. Let $f(z)$ be an arbitrary element of $\mathfrak{F}_p''$, let F be the image covering surface of B by the mapping $w=f(z)$ and let Γ be the boundary of F. We can take a sufficiently small positive number r such that Γ does not lie over $|w| \leqq r$ and $|w| \geqq 1/r$. Let F_r be the subset of F obtained by taking off from F the portions of F over $|w| \leqq r$ and $|w| \geqq 1/r$. Then, by the green's formula, we have

$$\begin{aligned} D_{F_r}(\lg|w|) &= \int_{\partial F_r} \lg|w|\,d \arg w \\ &= \int_\Gamma \lg|w|\,d \arg w - 4\pi p \lg r \\ &= \int_C \lg|f|\,d \arg f - 4\pi p \lg r, \end{aligned} \tag{9}$$

where $D_{F_r}(\lg|w|)$ is the Dirichlet's integral of $\lg|w|$ on F_r. On the other hand,

$$D_{F_r}(\lg|w|) \leqq p\, D_{(r<|w|<1/r)}(\lg|w|) = -4\pi p \lg r, \tag{10}$$

for $f(z)$ is p-valent. By (9) and (10), we have

$$\int_C \lg|f|\,d \arg f \leqq 0$$

and thus $f \in \mathfrak{F}_p'$.

We note that $\varphi \in \mathfrak{F}_p''$ for φ of the theorem 1. Then, by the theorem 2 and the lemma, we have immediately the following theorem.

THEOREM 3. *The function* φ *of the theorem* 1 *is the unique element in* $\mathfrak{F}_p''$ *which maximizes* $I(f)$ *on* $\mathfrak{F}_p''$.

We note that $\mathfrak{F}_1''$ consists of all univalent functions $f(z)$ on B which satisfy the conditions

$$f(z_1)=0, \quad f(\infty)=\infty, \quad f'(\infty)=1.$$

Then we have the following classical theorem as the corollary of the theorem 3 (cf. [1], [3]).

COROLLARY. *The function* φ *of the corollary of the theorem* 2 *is the unique element in* $\mathfrak{F}_1''$ *which maximizes* $|f'(z_1)|$ *on* $\mathfrak{F}_1''$.

5. Example 1. $\mathfrak{F}_p''$ is a strict subclass of $\mathfrak{F}_p'$; i. e. $\mathfrak{F}_p'' \subsetneqq \mathfrak{F}_p'$. To see this, it is sufficient to show that there exists even the function of $\mathfrak{F}_p'$ of which the valence is *not bounded.*

Let $\varDelta$ be the covering surface over the ω-plane obtained as the image of the angular domain

$$\left\{ \zeta \,\middle|\, \frac{3}{4}\pi < \arg \zeta < \frac{5}{4}\pi \right\}$$

by the mapping $\omega = e^{\zeta}$. Then, the valence of $\varDelta$ over ω-plane is not bounded. Let G be the region obtained from the strip region

$$\left\{ \omega \,\middle|\, -\frac{\pi}{2} i \leqq \Im \omega \leqq \frac{3}{2}\pi i \right\}$$

by taking off the closed disk

$$\{\omega \,|\, |\omega - \pi i| \leqq r\} \quad \left(\frac{1}{\sqrt{2\pi}} \leqq r < \frac{\pi}{2}\right).$$

Let $\varDelta'$ and G' be $\varDelta$ and G slit along the segment

$$l = \left\{ \omega \,\middle|\, \Re \omega = \frac{1}{\sqrt{2}} e^{-\pi/4}, \quad -\frac{1}{2} e^{-\pi/4} \leqq \Im \omega \leqq \frac{1}{2} e^{\pi/4} \right\},$$

respectively, F_ω be the covering surface over the ω-plane obtained by the crosswise connection of $\varDelta'$ and G' along the common slit l and F' be the image covering surface of F_ω by the mapping $w = e^{\omega}$. Then we obtain the covering surface F over the w-plane from F' by the identification along both side of the boundary component over the negative imaginary axis of F'. F is the double connected planar covering surface over the w-plane. Thus we can conformally map F onto the schlicht domain B of which the boundary consists of the circular slits centred at the origin and further can take B and the mapping function $z=g(w)$ such that the conditions

$$g(0)=0, \quad g(\infty)=\infty, \quad g'(\infty)=1$$

is satisfied. The inverse function $w=f(z)\equiv g^{-1}(z)$ maps B onto F under the condition

$$f(0)=0, \quad f(\infty)=\infty, \quad f'(\infty)=1.$$

It is immediately verified that the function $f(z)$ belongs to $\mathfrak{F}_1'$. In fact,

$$\int_{C_1} \lg|f| d \arg f = 2\int_0^\infty d\theta \int_0^{e^{-\theta}} r\, dr = \frac{1}{2},$$

$$\int_{C_2} \lg|f|\, d \arg f = -\pi r^2 \leqq -\frac{1}{2}$$

and then

$$\int_C \lg|f| d \arg f \leqq 0,$$

where C_1 (or C_2) is the boundary component of B on the right (or left) half plane, respectively. However the function $f(z)$ cannot belong to $\mathfrak{F}_1''$, because the valence of $f(z)$ is not bounded.

By the analogy the present example we can infer that *any* class $\mathfrak{F}_p'$ contains the functions of which the valences are not bounded and $\mathfrak{F}_p'$ is a large class in comparison with $\mathfrak{F}_p''$. The theorem 2 asserts that φ preserves the extremality with respect to the functional $I(f)$ even on such the class $\mathfrak{F}_p'$. For instance, by the corollary of the theorem 2 there holds

$$|f'(0)|<1$$

for the function $f(z)$ of the present example, for the extremal function $\varphi(z)$ in the present case is $\varphi(z)\equiv z$.

6. Example 2. Does the function φ of the theorem 1 preserve the extremality with respect to the functional $I(f)$ on the class $\mathfrak{F}_p$? The following example gives the negative answer for this question.

Let Δ be the domain

$$\{w\,|\,e^{-\varepsilon}<|w|<e^{\varepsilon},\ -(\alpha+\varepsilon)<\arg w<\alpha+\varepsilon\} \quad (0<\alpha<\pi-\varepsilon,\ \varepsilon>0),$$

and G be the whole w-plane. Let Δ' and G' be Δ and G slit along the circular arc

$$l=\{w\,|\,|w|=1,\ -\alpha\leqq \arg w\leqq\alpha\},$$

and F be the covering surface over the w-plane obtained by the crosswise connection of Δ' and G' along the common slit l. Then F is the simple-connected planar surface. Thus we can map F onto the schlicht domain B of which the boundary is a circular slit centred at the origin and further can take B and the mapping function $z=g(w)$ such that the conditions

$$g(0)=0, \quad g(\infty)=\infty, \quad g'(\infty)=1$$

is satisfied. The inverse function $w=f(z)\equiv g^{-1}(z)$ maps B onto F under the condition

$$f(0)=0, \quad f(\infty)=\infty, \quad f'(\infty)=1.$$

It is obvious that $f(z)\in\mathfrak{F}_1$. However $f(z)\notin\mathfrak{F}_1'$, for

$$\int_{\sigma} \lg |f| d \arg f = \int_{\partial\Delta} \lg|w| d \arg w = 4\varepsilon(\alpha+\varepsilon)>0.$$

Let B^* be the image domain of G' by $g(w)$. Then we see that $\bar{B}^*\subset B$ and the restriction of $f(z)$ on B^* is the mapping function of B^* onto the domain G' of which the boundary is the circular slit l. Thus, by the corollary of the theorem 3, we have

$$|f'(0)|>1.$$

On the other hand, $\varphi(z)\equiv z$ and thus $\varphi'(0)=1$ for the present B. Consequently, we see that

$$|f'(0)|>\varphi'(0),$$

which rejects the extremality of $\varphi(z)$ with respect to $I(f)$ on the class $\mathfrak{F}_1$.

By the analogy of the present example, we can infer that the function φ of the theorem 1 does not preserve the extremality with respect to the functional $I(f)$ on any class $\mathfrak{F}_p$.

7. In the next paper, we shall concern ourselves with the conformal mappings onto other types of canonical slit covering surfaces and their extremality.

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