QUASI-GALOIS EXTENSIONS OF SIMPLE RINGS

Dedicated to Professor Mikao Moriya on his 60th birthday

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Recently, in [3], H. Tominaga has obtained many important results for two kinds of Galois extensions: One is called a "q-Galois extension (or quasi-Galois extension)", and the other is called an "h-q-Galois extension".

One purpose of this paper is to present some useful q-Galois conditions to unify the above extensions under q-Galois extension. The other purpose is to develop extension theorems in a somewhat different way to obtain more explicit results.

Throughout the present paper, $R=\sum De_{ij}$ will be a simple ring, where e_{ij}'s are matrix units and D is the centralizer of $\{e_{ij}\}$ in R which is a division ring. For any subset A of R, we denote by $V_R(A)$ the centralizer of A in R, and we write $V_R^2(A)$ for $V_R(V_R(A))$. We shall be concerned with a fixed subring S of R which is a simple ring containing the identity element 1 of R. We write $V_R(S)=V$, $V_R^2(S)=H$. If H is a simple ring, we set $H=\sum Kd_{hk}$, where d_{hk}'s are matrix units of H and $K=V_H(\{d_{hk}\})$ is a division ring. If A is a subring of R containing S then A contains a linearly independent left (right) basis over S. By $[A:S]_l$ ($[A:S]_r$) we denote the left (right) dimension. In case $[A:S]_l=[A:S]_r$ they are denoted by $[A:S]$. For any simple subring A of R, we denote by $[A|A]$ the uniquely determined number of irreducible direct summands of A-module A. If A contains the identity element 1 of R, then one can see that $[A|A]\leqq[R|R]$. For any subsets $A_1, \cdots, A_n$ of R, we denote by $[A_1, \cdots, A_n]$ a subring of R generated by $A_1, \cdots, A_n$. If, for each finite subset F of R, $[[S, F]:S]_l<\infty$ then we say that R is *left locally finite* over S. Given rings A, B, we shall understand by an A-B-module a two-sided module which is treated as a left A-module and right B-module. A subring A of R is said to be *regular* if A and $V_R(A)$ are both simple. By $\mathfrak{R}$ we denote the set of all regular subrings of R containing S, and we shall use the following notations:

$\mathfrak{R}^0=\{S'\in\mathfrak{R};\ [S'|S']=[R|R]\}$,

$\mathfrak{R}_{l.f}=\{S'\in\mathfrak{R};\ [S':S]_l<\infty\}$ ($\mathfrak{R}_{r.f}=\{S'\in\mathfrak{R};\ [S':S]_r<\infty\}$),

$\mathfrak{R}/A=\{S'\in\mathfrak{R};\ S'\supset A\}$, where A is a subset of R, and

we set $\mathfrak{R}_{l.f}^0=\mathfrak{R}^0\cap\mathfrak{R}_{l.f}$, $\mathfrak{R}^0/A=\mathfrak{R}^0\cap\mathfrak{R}/A$, $\mathfrak{R}_{l.f}/A=\mathfrak{R}_{l.f}\cap\mathfrak{R}/A$, $\mathfrak{R}_{l.f}^0/A=\mathfrak{R}_{l.f}^0\cap\mathfrak{R}/A$. For any subring A of R containing S, we mean by a S-ring homomorphism of A into R a ring homomorphism which leaves S element-wise invariant. Now, let $S'\in\mathfrak{R}$, and A, B subsets of R such that

$A \supset S' \supset B$. Then, we shall use the following notations:

$\mathfrak{G}(S'/S, R)$ = the set of all S-ring isomorphisms of S' into R whose images belong to $\mathfrak{R}$. If in particular $S' \in \mathfrak{R}^0$ then $\mathfrak{G}(S'/S, R)$ coincides with the set of all S-ring isomorphisms of S' into R.

$\mathfrak{G}(S'/S, A) = \{\sigma \in \mathfrak{G}(S'/S, R)\,;\ S'\sigma \subset A\}$.

$\mathfrak{G}(S, S') = \{\sigma \in \mathfrak{G}(S'/S, R)\,;\ S'\sigma = S'\}$, which coincides with the set of all S-ring automorphisms of S'.

$\mathrm{Hom}_{S_l}(S', R)$ = the set of all left S-module homomorphisms of S' into R, (in general, this notation is used for any left S-modules). Clearly $\mathfrak{G}(S, S') \subset \mathfrak{G}(S'/S, S') \subset \mathfrak{G}(S'/S, A) \subset \mathfrak{G}(S'/S, R) \subset \mathfrak{G}(S'/S, R)R_r \subset \mathrm{Hom}_{S_l}(S', R)$, where R_r is the set of right multiplications determined by elements of R. For any subset $\mathfrak{G}'$ of $\mathrm{Hom}_{S_l}(S', R)$, we set

$J(\mathfrak{G}', S') = \{x \in S'\,;\ x\sigma = x$ for every $\sigma \in \mathfrak{G}'\}$,

$\mathfrak{G}' | B$ = the restriction of $\mathfrak{G}'$ to B,

$\sharp \mathfrak{G}' | B$ = the cardinal number of $\mathfrak{G}' | B$.

For any regular element a of R, we shall denote by $\langle a \rangle$ an inner automorphism $a_l \cdot a_r^{-1}$ of R, and for any subset A of R, we denote by $\langle A \rangle$ the set of inner automorphisms of R determined by regular elements of A. Clearly $\langle V \rangle | S' \subset \mathfrak{G}(S'/S, R)$. Finally, we shall make a remark on $\mathfrak{R}$ which is frequently used in our study. If $S' \in \mathfrak{R}$ then $(V \cap S')\mathfrak{G}(S'/S, R) \subset V$, (and by notation, $S'\sigma \in \mathfrak{R}$ for every $\sigma \in \mathfrak{G}(S'/S, R)$). If $T \in \mathfrak{R}^0$ then $T\sigma \in \mathfrak{R}^0$ for every $\sigma \in \mathfrak{G}(T/S, R)$. If moreover $T \in \mathfrak{R}^0_{l.f}$ then $T\sigma \in \mathfrak{R}^0_{l.f}$ for every $\sigma \in \mathfrak{G}(T/S, R)$. Let R be left locally finite over S. Then, $\mathfrak{R}_{l.f}$ is a direct set such that $\bigcup_{S' \in \mathfrak{R}_{l.f}} S' = R$, and $\mathfrak{R}^0_{l.f}$ is a direct subset of $\mathfrak{R}_{l.f}$ such that $[T, F] \in \mathfrak{R}^0_{l.f}$ for every $T \in \mathfrak{R}^0_{l.f}$ and for every finite subset F of R. Clearly $[S, \{e_{ij}\}] \in \mathfrak{R}^0_{l.f}$. Hence $\mathfrak{R}^0_{l.f}/[S', F]$ is non-empty for every $S' \in \mathfrak{R}_{l.f}$ and for every finite subset F of R.

1. On q-Galois extensions

In the consideration of Galois theory of simple rings, the modules

$$\mathfrak{G}(S'/S, R)R_r, \quad S' \in \mathfrak{R}_{l.f}$$

will play important roles. In fact, if $\mathfrak{G}(S'/S, R)R_r = \mathrm{Hom}_{S_l}(S', R)$ for all $S' \in \mathfrak{R}_{l.f}$ and R is left locally finite over S, we can discuss the structure of the extension R/S adequately. In this section, the main theme of our discussion will concern the condition

$$(q) \qquad \mathfrak{G}(S'/S, R)R_r = \mathrm{Hom}_{S_l}(S', \mathrm{R}) \text{ for all } S' \in \mathfrak{R}_{l.f}\,.$$

To say more precisely, our study is about the following: *the investigation of structural conditions of R/S for which* (q) *is satisfied ; the characterization of the system* $\{\mathfrak{G}(S'/S, R)\,;\ S' \in \mathfrak{R}_{l.f}\}$ $(\supset \{\mathfrak{G}(T/S, R)\,;\ T \in \mathfrak{R}^0_{l.f}\})$

with (q); *the heredity of* (q) *to the extensions* R/S', $S'\in\mathfrak{R}_{l.f}$; *the characterization of regular subrings* (*i. e.*, *elements of* $\mathfrak{R}$) *under* (q).

Now, let $S'\in\mathfrak{R}$. As is easily seen, we have $\mathfrak{G}(S'/S, R)\langle V\rangle\subset\mathfrak{G}(S'/S, R)$. If $\mathfrak{G}'$ is a subset of $\mathfrak{G}(S'/S, R)$ such that $\mathfrak{G}'\langle V\rangle\subset\mathfrak{G}'$ then we say that $\mathfrak{G}'$ is a $\langle V\rangle$-*subset* of $\mathfrak{G}(S'/S, R)$. Clearly, from $\mathfrak{G}'\langle V\rangle\subset\mathfrak{G}'$, it follows that $\mathfrak{G}'\langle V\rangle=\mathfrak{G}'$. For any number of $\langle V\rangle$-subsets $\{\mathfrak{G}_i\}$ of $\mathfrak{G}(S'/S, R)$, $\bigcup\mathfrak{G}_i$ and $\bigcap\mathfrak{G}_i$ are also $\langle V\rangle$-subsets of $\mathfrak{G}(S'/S, R)$. Now, if, for each $T\in\mathfrak{R}^0_{l.f}$, there exists a $\langle V\rangle$-subset $\hat{\mathfrak{G}}(T/S, R)$ of $\mathfrak{G}(T/S, R)$ associated with T, and the system $\hat{\mathfrak{G}}=\{\hat{\mathfrak{G}}(T/S, R);\ T\in\mathfrak{R}^0_{l.f}\}$ satisfies the following conditions (α)—(δ) then we say that $\hat{\mathfrak{G}}$ is a *left q-system.*

- (α) $J(\hat{\mathfrak{G}}(T/S, R), T)=S$ *for every* $T\in\mathfrak{R}^0_{l.f}$, *and*
- (β) $\hat{\mathfrak{G}}(T_1/S, R)\,|\,T_2=\hat{\mathfrak{G}}(T_2/S, R)$ *for every* $T_1, T_2\in\mathfrak{R}^0_{l.f}$ *such that* $T_1\supset T_2$, *and*
- (γ) $\sigma\hat{\mathfrak{G}}(T\sigma/S, R)\subset\hat{\mathfrak{G}}(T/S, R)$ *for every* $T\in\mathfrak{R}^0_{l.f}$ *and for every* $\sigma\in\hat{\mathfrak{G}}(T/S, R)$, *and*
- (δ) $(H\cap T)\hat{\mathfrak{G}}(T/S, R)\subset H$ *for every* $T\in\mathfrak{R}^0_{l.f}$.

The notion of left q-system is a generalization of Galois groups. In fact, if R is Galois over S (i. e., $J(\mathfrak{G}(S, R), R)=S$, $S\in\mathfrak{R}$) and $\hat{\mathfrak{G}}$ is a subgroup of $\mathfrak{G}(S, R)$ such that $\hat{\mathfrak{G}}\langle V\rangle\subset\hat{\mathfrak{G}}$ and $J(\hat{\mathfrak{G}}, R)=S$, then $\{\hat{\mathfrak{G}}\,|\,T;\ T\in\mathfrak{R}^0_{l.f}\}$ is a left q-system. Moreover, the system $\{\hat{\mathfrak{G}}\,|\,T;\ T\in\mathfrak{R}^0_{l.f}\}$ plays important roles in the consideration of Galois theory.

Now, we shall consider the following conditions which correspond to the conditions $(A_l)-(E_r)$ given in [4].

(I_l): (i) $S\in\mathfrak{R}$ *and* $\mathrm{Hom}_{S_l}(S', R)=\mathfrak{G}(S'/S, R)R_r$ *for every* $S'\in\mathfrak{R}_{l.f}$ (i. e., the condition (q)), *and*
(ii) *R is left locally finite over S.*

(II_l): (i) $S\in\mathfrak{R}$ *and* $\mathrm{Hom}_{S_l}(T, R)=\mathfrak{G}(T/S, R)R_r$ *for every* $T\in\mathfrak{R}^0_{l.f}$, *and*
(ii) *R is left locally finite over S.*

(III_l): (i) *There exists a left q-system, and*
(ii) *R is S·V-R-irreducible, and*
(iii) *R is left locally finite over S.*

(III'_l): $(III_l$: i, iii) *plus the condition that R is R-S·V-irreducible.*

(III''_l): $(III_l$: i, iii) *plus the condition that* $S, H\in\mathfrak{R}$ *and* $[V^2_R(T): H]_l=[V: V_R(T)]_r$ *for every* $T\in\mathfrak{R}^0_{l.f}$.

We shall also consider conditions $(I_r)-(III''_r)$ obtained by replacing left by right in conditions $(I_l)-(III''_l)$.

In [3], H. Tominaga has defined q-Galois extensions and h-q-Galois extensions as follows: If the condition $(I_l$: i) is satisfied and $\mathfrak{G}(S_1/S, R)\,|$

$S_2 \subset \mathfrak{G}(S_2/S, R)$ for every $S_1, S_2 \in \mathfrak{R}_{l.f}$ such that $S_1 \supset S_2$ then R is said to be q-Galois over S; if $S \in \mathfrak{R}$, and for every $S' \in \mathfrak{R}_{l.f}$, R is q-Galois over S' then R is said to h-q-Galois over S. However, we shall prove that the condition (I_l) contains that $\mathfrak{G}(S_1/S, R)|S_2 = \mathfrak{G}(S_2/S, R)$ for every $S_1, S_2 \in \mathfrak{R}_{l.f}$ such that $S_1 \supset S_2$ (Theorem 2). Moreover, we shall prove that $(\mathrm{I}_l)-(\mathrm{III}''_r)$ are all equivalent (Theorem 3). Hence, R is q-Galois and left locally finite over S if and only if one of $(\mathrm{I}_l)-(\mathrm{III}''_r)$ is satisfied. For the proof of our results, several lemmas will be needed.

Lemma 1. *Let S' be a subring of R containing S, and set $V' = V_R(S')$. Then*

(i) *If R is $S \cdot V$-R-irreducible then*

(a) *V and H are both simple;*

(b) *if V' is simple and $[V: V']_l < \infty$ then $[V: V']_l \geqq [V_R(V'): V_R(V)]_r = [V_R^2(S'): H]_r$.*

(ii) *If R is $S' \cdot V'$-R-irreducible then*

(a) *R is S'-R-irreducible if and only if V' is a division ring;*

(b) *V' and $V_R(V')$ $(= V_R^2(S'))$ are both simple;*

(c) *if $[S': S]_l < \infty$ then $[S': S]_l \geqq [V: V']_r$.*

(iii) *If R is S'-R-irreducible then*

(a) *R is $S' \cdot V'$-R-irreducible and V' is a division ring;*

(b) *if S'' is a simple subring of R containing S' then S'' is S'-S''-irreducible;*

(c) *if W is a subset of V such that W is linearly right-independent over V' and W consists of regular elements, then $\langle W \rangle | S'$ is linearly right-independent over R_r.*

(iv) *If R is left locally finite over S then*

(a) *if R is $S' \cdot V'$-R-irreducible and $[S': S]_l < \infty$ then $[S': S]_l \geqq [V: V']_r$, $[V: V']_l$;*

(b) *if $S' \in \mathfrak{R}_{l.f}^0$ then $[S': S]_l \geqq [V: V']_r$, $[V: V']_l$.*

Proof. (i)–(iii, a) are direct consequences of [4, Lemma 1]. We shall prove (iii, b). Let a be a non-zero element of S''. Then $S'aS'' = eS''$ for some idempotent $e \in S''$. Hence we may derive $R = S'aR = S'aS''R = eS''R = eR$. Thus e is regular; this means that $S'aS'' = eS'' = S''$. Therefore S'' is S'-S''-irreducible. (iii, c) is proved by the same method as in the proof of [6, Lemma 1.4]. (iv, a): By (ii, b), V' is simple. Let g_{pq}'s be matrix units of V' such that $V_{V'}(\{g_{pq}\})$ is a division ring, and set $S^* = [S', \{g_{pq}\}]$. Then, by (ii, a), R is S^*-R-irreducible. For an arbitrary finite subset F of V, we choose a $T \in \mathfrak{R}_{l.f}^0 / [S^*, F]$. Then, by (iii, b), T is S^*-T-irreducible. Since $S^* \subset S' \cdot V_T(S')$, T is $S' \cdot V_T(S')$-T-irreducible. From this and (ii, c), we have $[S': S]_l \geqq [V_T(S): V_T(S')]_r$. Moreover, we have $[T:S]_l \geqq [V_T(S):V_T(T)]_r$. Noting that $V_T(T)$ is the center of T, one can see that $[V_T(S):V_T(T)]_r = [V_T(S):V_T(T)]_l$ and $[V_T(S'):V_T(T)]_r = [V_T(S'):V_T(T)]_l$. Therefore $[S':S]_l \geqq [V_T(S):V_T(S')]_r = [V_T(S):V_T(S')]_l$. Then, from $V_T(S) \subset V$ and $V_T(S') \subset V'$, we obtain our assertion. (iv, b) is a special case of (iv, a).

Lemma 2. *Let* $T \in \mathfrak{R}^0$. *Then*
(i) $\mathfrak{G}(T'/S, R)|T \subset \mathfrak{G}(T/S, R)$ *for every* $T' \in \mathfrak{R}^0/T$.
(ii) *For each* $\sigma \in \mathfrak{G}(T/S, R)$, *$R$ is $T\sigma$-R-irreducible, and σR_r is T_r-R_r-irreducible, and* $\sigma R_r \cong R$ *(module isomorphism).*
(iii) *Let* $\sigma_1, \sigma_2 \in \mathfrak{G}(T/S, R)$. *Then,* $\sigma_1 R_r \cong \sigma_2 R_r$ *(T_r-R_r-isomorphism) if and only if* $\sigma_2 = \sigma_1 \langle v \rangle$ *for some regular* $v \in V$.
(iv) *If τ is an S-(ring) homomorphism of T into R such that* $\tau \in \mathfrak{G}(T/S, R)R_r$ *then* $\tau \in \mathfrak{G}(T/S, R)$.

Proof. For each S-(ring) isomorphism σ of T into R, we have $T\sigma \in \mathfrak{R}^0$. From this, (i) and (ii) follow immediately. (iii): If $\sigma_1 R_r \cong \sigma_2 R_r$ $(\sigma_1 \to \sigma_2 a_r)$ then $\sigma_1 R_r \cong \sigma_2 a_r R_r = \sigma_2 R_r$. From (ii), we have $aR = R$; whence a is regular. Then, one will easily see that $\sigma_2 = \sigma_1 \langle a \rangle$. Since $\sigma_1 | S = \sigma_2 | S = 1$, it follows that $\langle a \rangle | S = 1$, and so $a \in V_R(S) = V$. Conversely, if $\sigma_2 = \sigma_1 \langle v \rangle$ for a regular element $v \in V$ then $\sigma_1 R_r$ is T_r-R_r-isomorphic to $\sigma_2 v_r R_r = \sigma_2 R_r$. (iv): Since $[R|R] < \infty$, there exists a minimal $T\tau$-R-submodule A of R. Noting here $T\tau \supset S \ni 1$, we have $A = T\tau xR$ for every non-zero $x \in A$. Hence τA_r is a minimal T_r-R_r-submodule of $\mathfrak{G}(T/S, R)R_r$. Since $\mathfrak{G}(T/S, R)R_r = \sum_\oplus \tau_i R_r$ for some subset $\{\tau_i\}$ of $\mathfrak{G}(T/S, R)$, and $\tau_i R_r$ is T_r-R_r-irreducible for every i, τA_r is T_r-R_r-isomorphic to some $\tau_{i'} R_r$. If $\tau_{i'} \leftrightarrow \tau a_r$ $(a \in A)$ under the above isomorphism, then $\tau_{i'} R_r \cong \tau a_r R_r$. Hence, it follows from (ii) that $R \cong aR$, so that $aR = R$. Thus a is regular, whence R is $T\tau$-R-irreducible, and $\tau R_r \cong R$ (module isomorphism). Then, by making use of the same method as in the proof of (iii), we obtain $\tau = \tau_{i'} \langle a \rangle$ and $a \in V$; whence $\tau \in \mathfrak{G}(T/S, R)$.

Remark 1. Let $T \in \mathfrak{R}^0$. If $\sigma_1, \sigma_2 \in \mathfrak{G}(T/S, R)$ and $\sigma_1 = \sigma_2 \langle v \rangle$ for some $\langle v \rangle \in \langle V \rangle$ then we write $\sigma_1 \sim \sigma_2$. Since $\langle V \rangle$ is a group of automorphisms of R, the relation $\sim$ induces an equivalence relation in $\mathfrak{G}(T/S, R)$. Then, for any $\langle V \rangle$-subset $\mathfrak{G}'$ of $\mathfrak{G}(T/S, R)$, we denote by $(\mathfrak{G}' : \langle V \rangle)$ the uniquely determined number of equivalence classes of $\mathfrak{G}'$ with respect to $\sim$.

From Lemma 2 (ii, iii) and Remark 1, we have the following

Corollary 1. *Let* $T \in \mathfrak{R}^0$. *If* $\mathfrak{G}_1$ *is a* $\langle V \rangle$*-subset of* $\mathfrak{G}(T/S, R)$ *then* $\mathfrak{G}_1 = \bigcup_{i \in I} \sigma_i \langle V \rangle$ *(direct union) for some subset* $\{\sigma_i\,;\, i \in I\}$ *of* $\mathfrak{G}_1$, *and* $\mathfrak{G}_1 R_r = \sum_{\oplus i \in I} \sigma_i \langle V \rangle R_r$ *($\sum_\oplus$ means a direct sum). If moreover* $\mathfrak{G}_2$ *is a* $\langle V \rangle$*-subset of* $\mathfrak{G}_1$ *then* $\mathfrak{G}_2 = \bigcup_{j \in J} \sigma_j \langle V \rangle$ *for some subset* $\{\sigma_j\,;\, j \in J\}$ *of*

$\{\sigma_i ; i \in I\}$, *and* $\mathfrak{G}_2 R_r = \sum_{\oplus j \in J} \sigma_j \langle V \rangle R_r$.

An another consequence of Lemma 2 and Remark 1 is the next

Corollary 2. *Let* $T \in \mathfrak{R}^0_{l.f}$. *Then*

(i) $(\mathfrak{G}(T/S, R) : \langle V \rangle) \leqq [T : S]_l < \infty$ *and* $\mathfrak{G}(T/S, R)$ *satisfies minimum condition for* $\langle V \rangle$*-subsets.*

(ii) *If* $\mathfrak{G}_1, \mathfrak{G}_2$ *are* $\langle V \rangle$*-subsets of* $\mathfrak{G}(T/S, R)$ *such that* $\mathfrak{G}_1 \supset \mathfrak{G}_2$ *then* $(\mathfrak{G}_1 : \langle V \rangle) \geqq (\mathfrak{G}_2 : \langle V \rangle)$.

(iii) *For each* $T^* \in \mathfrak{R}^0_{l.f}/T$ *and for every* $\langle V \rangle$*-subset* $\mathfrak{G}^*$ *of* $\mathfrak{G}(T^*/S, R)$, $\mathfrak{G}^* | T$ *is a* $\langle V \rangle$*-subset of* $\mathfrak{G}(T/S, R)$ *and* $(\mathfrak{G}^* : \langle V \rangle) \geqq (\mathfrak{G}^* | T : \langle V \rangle)$.

(iv) *For each* $\sigma \in \mathfrak{G}(T/S, R)$ *and for every* $\langle V \rangle$*-subset* $\mathfrak{G}'$ *of* $\mathfrak{G}(T\sigma/S, R)$, $\sigma\mathfrak{G}'$ *is a* $\langle V \rangle$*-subset of* $\mathfrak{G}(T/S, R)$, *and* $(\sigma\mathfrak{G}' : \langle V \rangle) = (\mathfrak{G}' : \langle V \rangle)$.

Theorem 1. *Let* $T \in \mathfrak{R}^0_{l.f}$, *and* $\mathfrak{G}_1, \mathfrak{G}_2$ $\langle V \rangle$*-subsets of* $\mathfrak{G}(T/S, R)$. *Then,* $\mathfrak{G}_1 = \mathfrak{G}_2$ *if and only if* $\mathfrak{G}_1 R_r = \mathfrak{G}_2 R_r$. *Moreover, under the assumption that* $\mathfrak{G}_1 \supset \mathfrak{G}_2$, *the following conditions are all equivalent:*

(a) $\mathfrak{G}_1 = \mathfrak{G}_2$. (b) $\mathfrak{G}_1 R_r = \mathfrak{G}_2 R_r$. (c) $(\mathfrak{G}_1 : \langle V \rangle) = (\mathfrak{G}_2 : \langle V \rangle)$.

Proof. If $\mathfrak{G}_1 \supset \mathfrak{G}_2$ then our assertion follows immediately from Remark 1 and Corollary 1. If $\mathfrak{G}_1 R_r = \mathfrak{G}_2 R_r$ then $\mathfrak{G}_1 R_r = (\mathfrak{G}_1 \cup \mathfrak{G}_2) R_r = \mathfrak{G}_2 R_r$. Hence, it follows that $\mathfrak{G}_1 = \mathfrak{G}_1 \cup \mathfrak{G}_2 = \mathfrak{G}_2$. The converse is clear.

Lemma 3. *Let* R *be left locally finite over* S, *and let* $\mathfrak{G}(T_1/S, R) | T_2 = \mathfrak{G}(T_2/S, R)$ *for every* $T_1, T_2 \in \mathfrak{R}^0_{l.f}$ *such that* $T_1 \supset T_2$. *Let* A *be a subring of* R *such that* $A \supset S$, $[A : S]_l < \infty$ *and* R A*-*R*-irreducible. Let* $T \in \mathfrak{R}^0_{l.f}/A$. *Then*

(i) *For each* $\sigma \in \mathfrak{G}(T/S, R)$, R *is* $A\sigma$*-*R*-irreducible and* $(\sigma | A)R_r$ *is* A_r*-*R_r*-irreducible, and* $(\sigma | A)R_r \cong R$ *(module isomorphism).*

(ii) *Let* $\sigma_1, \sigma_2 \in \mathfrak{G}(T/S, R)$. *Then,* $(\sigma_1 | A)R_r \cong (\sigma_2 | A)R_r$ *(*A_r*-*R_r*-isomorphism) if and only if* $(\sigma_2 | A) = (\sigma_1 | A)\langle v \rangle$ *for some regular* $v \in V$.

(iii) $\mathfrak{G}(T/S, R) | A = \cup (\sigma_i | A)\langle V \rangle$ *(direct union) for some finite subset* $\{\sigma_i\}$ *of* $\mathfrak{G}(T/S, R)$, *and* $(\mathfrak{G}(T/S, R) | A)R_r = \sum_\oplus (\sigma_i | A)\langle V \rangle R_r$.

(iv) *If* τ *is an* S*-(ring) homomorphism of* A *into* R *such that* $\tau \in (\mathfrak{G}(T/S, R) | A)R_r$ *then* $\tau \in \mathfrak{G}(T/S, R) | A$.

(v) *Let* $S \in \mathfrak{R}$, $\sigma \in \mathfrak{G}(T/S, R)$, *and* $\{v_i\}$ *a linearly independent* $V_R(A)$*-right basis of* V *which consists of regular elements* ($[V : V_R(A)]_r \leqq [A : S]_l < \infty$ by Lemma 1(iv)). *Let* $T' \in \mathfrak{R}^0_{l.f}/[T, \{v_i\}]$. *Then* $(\sigma | A)\langle V \rangle R_r = \sum_\oplus (\sigma | A)\langle v_i \sigma' \rangle R_r$ *for each* $\sigma' \in \mathfrak{G}(T'/S, R)$ *such that* $\sigma' | T = \sigma$, *whence* $[(\sigma | A)\langle V \rangle R_r : R_r]_r = [V : V_R(A)]_r$.

Proof. Let $a \neq 0 \in R$. If we choose a $T_1 \in \mathfrak{R}^0_{l.f}/[T\sigma, a]$ then $\sigma^{-1} = \sigma_1 | T$ for some $\sigma_1 \in \mathfrak{G}(T_1/S, R)$. Clearly $a\sigma_1 \neq 0$, whence $A(a\sigma_1)R = R$ by our assumption. Therefore $\sum a_i(a\sigma_1)b_i = 1$ for some a_i's $\subset A$ and for some

b_i's$\subset R$. Moreover, we shall choose a $T_2 \in \mathfrak{R}^0_{l.f}/[T_1\sigma_1, \{b_i\}]$. Then $\sigma_1^{-1} = \sigma_2 | T_1\sigma_1$ for some $\sigma_2 \in \mathfrak{G}(T_2/S, R)$. From $T_2 \supset T_1\sigma_1 \supset [T, a\sigma_1]$ and $\sigma_2 | T = \sigma$, it follows that $1 = (\sum a_i(a\sigma_1)b_i)\sigma_2 = \sum (a_i\sigma_2)(a\sigma_1\sigma_2)(b_i\sigma_2) = \sum (a_i\sigma)a(b_i\sigma_2)$. Hence $(A\sigma)aR = R$; i. e., R is $A\sigma$-R-irreducible. The proofs of (ii)—(iv) are similar to those of Lemma 2 and Corollary 1. (v): Let v be a regular element of V. If we choose a $T_1 \in \mathfrak{R}^0_{l.f}/[T'\sigma', v]$, then $\sigma'^{-1} = \sigma_1 | T'\sigma'$ for some $\sigma_1 \in \mathfrak{G}(T_1/S, R)$.
From this, we have

(1) $(\sigma | A)\langle v\rangle = \sigma\langle v\rangle | A = \sigma_1^{-1}\langle v\rangle | A = \langle v\sigma_1\rangle\sigma_1^{-1} | A = (v\sigma_1)_l \cdot (v\sigma_1)_r^{-1}\sigma_1^{-1} | A.$

Now, we can write $v\sigma_1 = \sum v_i c_i$ for some c_i's$\subset V_R(A)$. If we choose a $T_2 \in \mathfrak{R}^0_{l.f}/[T_1\sigma_1, \{c_i\}]$, then $\sigma_1^{-1} = \sigma_2 | T_1\sigma_1$ for some $\sigma_2 \in \mathfrak{G}(T_2/S, R)$. Hence, we have

(2) $$(v\sigma_1)_l \cdot (v\sigma_1)_r^{-1}\sigma_1^{-1} | A = (\sum v_i c_i)_l \cdot (v\sigma_1)_r^{-1}\sigma_2 | A$$
$$= \sum \langle v_i\rangle d_{ir}\sigma_2 | A \quad (d_i = v_i c_i (v\sigma_1)^{-1})$$
$$= \sum \sigma_2\langle v_i\sigma_2\rangle (d_i\sigma_2)_r | A = \sum \sigma\langle v_i\sigma'\rangle (d_i\sigma_2)_r | A.$$

Combining (1) with (2), we obtain $(\sigma | A)\langle v\rangle \in \sum (\sigma | A)\langle v_i\sigma'\rangle R_r$.
Therefore we have

(3) $(\sigma | A)\langle V\rangle R_r = \sum (\sigma | A)\langle v_i\sigma'\rangle R_r$.

Finally, we shall prove that the right-hand side of (3) is a direct sum. We suppose that

(4) $\sum (\sigma | A)\langle v_i\sigma'\rangle a_{ir} = 0$

for some a_i's$\subset R$. If we choose a $T'' \in \mathfrak{R}^0_{l.f}/[T'\sigma', \{a_i\}]$, then $\sigma'^{-1} = \sigma'' | T'\sigma'$ for some $\sigma'' \in \mathfrak{G}(T''/S, R)$. Operating on (4), we obtain $0 = (\sum (\sigma | A)\langle v_i\sigma'\rangle \cdot a_{ir})\sigma'' = \sum (\langle v_i\rangle | A)(a_i\sigma'')_r$. Hence, it follows from Lemma 1(iii) that $a_i\sigma'' = 0$, and so $a_i = 0$ for every i. This completes the proof.

Lemma 4. *Let R be left locally finite over S, and let* $\mathrm{Hom}_{S_l}(T, R) = \mathfrak{G}(T/S, R)R_r$ *for every $T \in \mathfrak{R}^0_{l.f}$. Then*
(i) $\mathfrak{G}(T_1/S, R) | T_2 = \mathfrak{G}(T_2/S, R)$ *for every $T_1, T_2 \in \mathfrak{R}^0_{l.f}$ such that $T_1 \supset T_2$.*
(ii) *Let A is a subring of R such that $A \supset S$ and $[A:S]_l < \infty$, and τ an S-(ring) homomorphism of A into R. If R is A-R-irreducible or $A\tau$-R-irreducible then $\tau \in \mathfrak{G}(T/S, R) | A$ for every $T \in \mathfrak{R}^0_{l.f}/A$, (and so τ is an isomorphism).*

Proof. (i): From Lemma 2 (i), we have $\mathfrak{G}(T_1/S, R) | T_2 \subset \mathfrak{G}(T_2/S, R)$, ($T_1, T_2 \in \mathfrak{R}^0_{l.f}$, $T_1 \supset T_2$). Moreover, we have $\mathfrak{G}(T_2/S, R)R_r = \mathrm{Hom}_{S_l}(T_2, R) = \mathrm{Hom}_{S_l}(T_1, R) | T_2 = \mathfrak{G}(T_1/S, R)R_r | T_2 = (\mathfrak{G}(T_1/S, R) | T_2)R_r$. Then, noting that $\mathfrak{G}(T_1/S, R) | T_2$ is a $\langle V\rangle$-subset of $\mathfrak{G}(T_2/S, R)$, we obtain (i) by Theorem 1. (ii): Let $T \in \mathfrak{R}^0_{l.f}/A$. Then $\tau \in \mathrm{Hom}_{S_l}(A, R) = \mathrm{Hom}_{S_l}(T, R) | A = \mathfrak{G}(T/S, R) \cdot R_r | A = (\mathfrak{G}(T/S, R) | A)R_r$. Hence, if R is A-R-irreducible then $\tau \in \mathfrak{G}(T/S, R) | A$ by Lemma 3 (iv). In the other case, it suffices to prove that if R is $A\tau$-R-irreducible then R is A-R-irreducible. Noting $[T:S]_l < \infty$, we can write $\mathfrak{G}(T/S, R)R_r=\sum\sigma_i R_r$ for some finite subset $\{\sigma_i\}$ of $\mathfrak{G}(T/S, R)$. Since $\tau\in \mathrm{Hom}_{S_l}(A, R)=\mathrm{Hom}_{S_l}(T, R)|A=\mathfrak{G}(T/S, R)R_r|A=\sum(\sigma_i|A)R_r$, we can find a minimal subset $\{\tau_i\}$ of $\{\sigma_i\}$ such that $\tau\in\sum(\tau_i|A)R_r$. Let $a\neq 0\in R$. If we write $\tau=\sum(\tau_i|A)c_{ir}$ $(c_i\in R)$ then $\tau a_r=\sum(\tau_i|A)(c_i a)_r$. Noting here $A\tau aR=R$, we have $\tau\in A_r\tau a_r R_r=A_r(\sum(\tau_i|A)(c_i a)_r)R_r\subset\sum_{i\in\{i\,;\,c_i a\neq 0\}}(\tau_i|A)R_r\subset\sum(\tau_i|A)R_r$. By the minimality of $\{\tau_i\}$, we obtain $c_i a\neq 0$ for every i, (and for every $a\neq 0\in R$). Thus c_i is regular for every i. We consider $\tau(c_1^{-1}a)_r=(\tau_1|A)a_r+\sum_{2\leq i}(\tau_i|A)\,(c_i c_1^{-1}a)_r$. Clearly $c_1{}^{-1}a\neq 0$. Then, by similar methods, we have $\tau=(\tau_1|A)c_1{}'_r+\sum_{2\leq i}(\tau_i|A)c_i{}'_r$ for some $c_1{}'\in A\tau_1 aR$ and for some $c_i{}'$s$\subset R$, and all $c_i{}'$s are regular. Hence $A\tau_1 aR=R$. Therefore R is $A\tau_1$-R-irreducible. Then, noting $\tau_1^{-1}\in\mathfrak{G}(T\tau_1/S, R)$, R is A-R-irreducible by Lemma 3 (i).

Remark 2. Let R be left locally finite over S, and let $\mathrm{Hom}_{S_l}(S', R)=\mathfrak{G}(S'/S, R)R_r$ for every $S'\in\mathfrak{R}_{l.f}$. Then, in [3], H. Tominaga has proved the following results which are useful in our study:

(i) For every $S'\in\mathfrak{R}_{l.f}$, R is $S'\cdot V_R(S')$-R-irreducible ([3, Theorem 1]).

(ii) For each $R'\in\mathfrak{R}$ and for an arbitrary finite subset F of R', there exists some $S'\in\mathfrak{R}_{l.f}$ such that $R'\supset S'\supset F$ ([3, Lemma 5]).

Now, we shall prove the following theorem which contains [3, Theorem 3].

Theorem 2. *Let R be left locally finite over S. Then, the following conditions are equivalent:*

(a) $\mathrm{Hom}_{S_l}(T, R)=\mathfrak{G}(T/S, R)R_r$ *for every* $T\in\mathfrak{R}^0_{l.f}$.

(b) $\mathrm{Hom}_{S_l}(S', R)=\mathfrak{G}(S'/S, R)R_r$ *for every* $S'\in\mathfrak{R}_{l.f}$.

Moreover, if one of these is satisfied then the following holds:

(i) $[V_R(S')|V_R(S')]=[V_R(S'\sigma')|V_R(S'\sigma')]$ *for each* $S'\in\mathfrak{R}_{l.f}$ *and for every* $\sigma'\in\mathfrak{G}(S'/S, R)$,

(ii) $\mathfrak{G}(S_1/S, R)|S_2=\mathfrak{G}(S_2/S, R)$ *for every* S_1, $S_2\in\mathfrak{R}_{l.f}$ *such that* $S_1\supset S_2$.

Proof. (b) $\Longrightarrow$ (a) is obvious. Let $S'\in\mathfrak{R}_{l.f}$, and $\{g_{pq}\,;\,1\leq p, q\leq m\}$ a system of matrix units of $V_R(S')$ such that the centralizer of $\{g_{pq}\}$ in $V_R(S')$ is a division ring, $(m=[V_R(S')|V_R(S')])$.

(a) $\Longrightarrow$ (b): First, we shall prove that

(1) $\mathfrak{G}(S'/S, R)\supset\mathfrak{G}(T/S, R)|S'$

for every $T\in\mathfrak{R}^0_{l.f}/S'$. We set $S_1=[S', \{g_{pq}\}]$ and $T_1=[T, \{g_{pq}\}]$ $(T\in\mathfrak{R}^0_{l.f})$. Then $S_1\in\mathfrak{R}_{l.f}$ and $T_1\in\mathfrak{R}^0_{l.f}/T$. Let $\sigma\in\mathfrak{G}(T/S, R)$. Then, by Lemma 4, we can find some $\sigma_1\in\mathfrak{G}(T_1/S, R)$ such that $\sigma_1|T=\sigma$. Since $S_1\sigma_1=[S'\sigma_1, \{g_{pq}\}\sigma_1]=[S'\sigma, \{g_{pq}\}\sigma_1]$ and $V_R(S'\sigma)\supset\{g_{pq}\}\sigma_1$, it suffices to prove that $V_R(S_1\sigma_1)$ is a division ring. Let $a\neq 0\in V_R(S_1\sigma_1)$, and set $T_2=[T_1\sigma_1, a]$. Then $T_1\sigma_1$, $T_2\in\mathfrak{R}^0_{l.f}$, whence $\sigma_1^{-1}=\sigma_2|T_1\sigma_1$ for some $\sigma_2\in\mathfrak{G}(T_2/S, R)$ (Lemma 4). Since $xa=ax$ for every $x\in S_1\sigma_1$, we have $(x\sigma_2)(a\sigma_2)=(a\sigma_2)(x\sigma_2)$ for every

$x\in S_1\sigma_1$. From $\sigma_1\sigma_2|S_1=1$, it follows that $x(a\sigma_2)=(a\sigma_2)x$ for every $x\in S_1$; whence $a\sigma_2\in V_R(S_1)$ (a division ring), and so $a\sigma_2$ is regular. Then, one will easily see that a is regular, so that $V_R(S_1\sigma_1)$ is a division ring. Now, for any $T\in\mathfrak{R}^0_{l.f}/S'$, it follows from (1) that $\mathrm{Hom}_{S_l}(S', R)=\mathrm{Hom}_{S_l}(T, R)|S'=\mathfrak{G}(T/S, R)R_r|S'=(\mathfrak{G}(T/S, R)|S')R_r\subset\mathfrak{G}(S'/S, R)R_r\subset\mathrm{Hom}_{S_l}(S', R)$, so that $\mathrm{Hom}_{S_l}(S', R)=\mathfrak{G}(S'/S, R)R_r$. This proves our assertion (a)$\Longrightarrow$(b). (i): Let $\sigma'\in\mathfrak{G}(S'/S, R)$, $\{h_{uv}\,;\,1\leqq u, v\leqq n\}$ a system of matrix units of $V_R(S'\sigma')$ such that the centralizer of $\{h_{uv}\}$ in $V_R(S'\sigma')$ is a division ring, $(n=[V_R(S'\sigma')|V_R(S'\sigma')])$. If $m\geqq n$ then there exists an S-(ring) homomorphism τ of a subring $\sum_{p,q\leqq n}S'g_{pq}+S'(\sum_{n<t}g_{tt})$ onto a subring $\sum S'\sigma'h_{uv}$ of R given by

$$(\textstyle\sum_{p,q\leqq n}s'_{pq}g_{pq}+s'(\sum_{n<t}g_{tt}))\tau=\sum_{p,q\leqq n}(s'_{pq}\sigma')h_{pq},$$

where $s'_{pq}\in S'$ for every p, q, and $s'\in S'$. Since $\sum S'\sigma'h_{uv}\in\mathfrak{R}_{l.f}$ and $V_R(\sum S'\sigma'h_{uv})$ is a division ring, R is $(\sum S'\sigma'h_{uv})$-R-irreducible by Remark 2-(i) and Lemma 1 (ii, a). Hence τ is an isomorphism by Lemma 4 (ii). Therefore $m=n$. On the other hand, if $m\leqq n$ then the same arguement holds for σ'^{-1}; therefore $m=n$. (ii): First, we shall prove that

(2) $\mathfrak{G}(S'/S, R)=\mathfrak{G}(T/S, R)|S'$

for every $T\in\mathfrak{R}^0_{l.f}/S'$. Let $\sigma'\in\mathfrak{G}(S'/S, R)$. Then $[V_R(S')|V_R(S')]=[V_R(S'\sigma')|V_R(S'\sigma')]$ by (i). Set $T_1=[T,\{g_{pq}\}]$ $(T\in\mathfrak{R}^0_{l.f}/S')$. Then, by making use of the same method as in the proof of (i), we can find some $\sigma_1\in\mathfrak{G}(T_1/S, R)$ such that $\sigma_1|S'=\sigma'$. Hence $\sigma'=\sigma_1|S'=(\sigma_1|T)|S'\in\mathfrak{G}(T/S, R)|S'$ by Lemma 4 (i). Therefore $\mathfrak{G}(S'/S, R)\subset\mathfrak{G}(T/S, R)|S'$. Combining this with (1), we obtain (2). Now, for each S_1, $S_2\in\mathfrak{R}_{l.f}$ such that $S_1\supset S_2$, we choose a $T\in\mathfrak{R}^0_{l.f}/S_1$. Then, by (2), we have $\mathfrak{G}(T/S, R)|S_1=\mathfrak{G}(S_1/S, R)$ and $\mathfrak{G}(T/S, R)|S_2=\mathfrak{G}(S_2/S, R)$. From this, it follows that $\mathfrak{G}(S_2/S, R)=\mathfrak{G}(T/S, R)|S_2=(\mathfrak{G}(T/S, R)|S_1)|S_2=\mathfrak{G}(S_1/S, R)|S_2$. This completes the proof.

Corollary 3. *Let* $\mathrm{Hom}_{S_l}(T, R)=\mathfrak{G}(T/S, R)R_r$ *for every* $T\in\mathfrak{R}^0_{l.f}$. *Then*

(i) $J(\mathfrak{G}(T/S, R), T)=S$ *for every* $T\in\mathfrak{R}^0_{l.f}$.

(ii) *If* R *is left locally finite over* S *then* $J(\mathfrak{G}(S'/S, R), S')=S$ *for every* $S'\in\mathfrak{R}_{l.f}$.

Proof. (i): Let $T\in\mathfrak{R}^0_{l.f}$, and set $A=J(\mathfrak{G}(T/S, R), T)$. Then $T\supset A\supset S$ and $\mathrm{Hom}_{S_l}(A, R)=\mathrm{Hom}_{S_l}(T, R)|A=\mathfrak{G}(T/S, R)R_r|A=(\mathfrak{G}(T/S, R)|A)R_r=(1|A)R_r$. Hence we have $[A:S]_l=[\mathrm{Hom}_{S_l}(A, R):R_r]_r=[(1|A)R_r:R_r]_r=1$. Therefore $A=S$. (ii): Let $S'\in\mathfrak{R}_{l.f}$, and choose a $T\in\mathfrak{R}^0_{l.f}/S'$. Then $\mathfrak{G}(T/S, R)|S'=\mathfrak{G}(S'/S, R)$ by Theorem 2 (ii). Hence, by (i), we have $S=J(\mathfrak{G}(T/S, R), T)=J(\mathfrak{G}(T/S, R)|S', S')=J(\mathfrak{G}(S'/S, R), S')$.

Lemma 5. *For the system* $\mathfrak{G}=\{\mathfrak{G}(T/S, R)\,;\ T\in\mathfrak{R}^0_{l.f}\}$, *the following hold*:

(i) $\sigma\mathfrak{G}(T\sigma/S, R)=\mathfrak{G}(T/S, R)$ *for every* $T\in\mathfrak{R}^0_{l.f}$ *and for every* $\sigma\in\mathfrak{G}(T/S, R)$.

(ii) *If* R *is left locally finite over* S *and* $\mathfrak{G}(T_1/S, R)|T_2=\mathfrak{G}(T_2/S, R)$ *for every* $T_1, T_2\in\mathfrak{R}^0_{l.f}$ *such that* $T_1\supset T_2$ *then* $(H\cap T)\mathfrak{G}(T/S, R)\subset H$ *for every* $T\in\mathfrak{R}^0_{l.f}$.

Proof. (i): Let $T\in\mathfrak{R}^0_{l.f}$, and $\sigma\in\mathfrak{G}(T/S, R)$. Then, for every $\tau\in\mathfrak{G}(T/S, R)$, we have that $\tau=(\sigma\sigma^{-1})\tau=\sigma(\sigma^{-1}\tau)$ and $\sigma^{-1}\tau\in\mathfrak{G}(T\sigma/S, R)$; whence $\tau\in\sigma\mathfrak{G}(T\sigma/S, R)$. This implies that $\mathfrak{G}(T/S, R)\subset\sigma\mathfrak{G}(T\sigma/S, R)$. On the other hand, it is easily seen that $\mathfrak{G}(T/S, R)\supset\sigma\mathfrak{G}(T\sigma/S, R)$. Therefore we obtain $\mathfrak{G}(T/S, R)=\sigma\mathfrak{G}(T\sigma/S, R)$. (ii): Let $T\in\mathfrak{R}^0_{l.f}$, and assume that $h\sigma\notin H$ for some $h\in H\cap T$ and for some $\sigma\in\mathfrak{G}(T/S, R)$. Then, there exists some $v\in V$ such that $(h\sigma)v\neq v(h\sigma)$. We choose a $T_1\in\mathfrak{R}^0_{l.f}/[T\sigma, v]$. Then $\sigma^{-1}=\sigma_1|T\sigma$ for some $\sigma_1\in\mathfrak{G}(T_1/S, R)$. If σ_1 operates on $(h\sigma)v\neq v(h\sigma)$ then $(h\sigma\sigma_1)(v\sigma_1)\neq(v\sigma_1)(h\sigma\sigma_1)$, i. e., $h(v\sigma_1)\neq(v\sigma_1)h$. However, this is a contradiction, (noting $v\sigma_1\in V$). Therefore we obtain $(H\cap T)\mathfrak{G}(T/S, R)\subset H$ for every $T\in\mathfrak{R}^0_{l.f}$.

From Lemma 5, we have the next

Corollary 4. *Let* R *be left locally finite over* S. *If* $J(\mathfrak{G}(T/S, R), T)=S$ *for every* $T\in\mathfrak{R}^0_{l.f}$, *and* $\mathfrak{G}(T_1/S, R)|T_2=\mathfrak{G}(T_2/S, R)$ *for every* $T_1, T_2\in\mathfrak{R}^0_{l.f}$ *such that* $T_1\supset T_2$, *then* $\mathfrak{G}=\{\mathfrak{G}(T/S, R);\ T\in\mathfrak{R}^0_{l.f}\}$ *is a left q-system.*

Combining Corollary 4 with Lemma 4 (i) and Corollary 3 (i), we have the following

Corollary 5. *Let* R *be left locally finite over* S. *If* $\mathrm{Hom}_{S_l}(T, R)=\mathfrak{G}(T/S, R)R_r$ *for every* $T\in\mathfrak{R}^0_{l.f}$ *then* $\mathfrak{G}=\{\mathfrak{G}(T/S, R);\ T\in\mathfrak{R}^0_{l.f}\}$ *is a left q-system.*

Now, by Theorem 2, Corollary 5 and Remark 2 (i), we obtain the following

Proposition 1. $(\mathrm{I}_l)\Longleftrightarrow(\mathrm{II}_l)\Longrightarrow(\mathrm{III}_l)$. *Similarly* $(\mathrm{I}_r)\Longleftrightarrow(\mathrm{II}_r)\Longrightarrow(\mathrm{III}_r)$.

From now, our study will concern left q-systems.

Lemma 6. *Let* $\hat{\mathfrak{G}}=\{\hat{\mathfrak{G}}(T/S, R);\ T\in\mathfrak{R}^0_{l.f}\}$ *be a left q-system. Then*

(i) $\sharp\hat{\mathfrak{G}}(T/S, R)|H\cap T\leqq(\hat{\mathfrak{G}}(T/S, R):\langle V\rangle)<\infty$ *for every* $T\in\mathfrak{R}^0_{l.f}$.

(ii) *For each* $T\in\mathfrak{R}^0_{l.f}$, *there exists some* $\rho\in\hat{\mathfrak{G}}(T/S, R)$ *such that* $\hat{\mathfrak{G}}(T\rho/S, R)=\tau\hat{\mathfrak{G}}(T\rho\tau/S, R)$ *for every* $\tau\in\hat{\mathfrak{G}}(T\rho/S, R)$.

(iii) *If* $T^*\in\mathfrak{R}^0_{l.f}$ *and* $\hat{\mathfrak{G}}(T^*/S, R)=\tau\hat{\mathfrak{G}}(T^*\tau/S, R)$ *for every* $\tau\in\hat{\mathfrak{G}}(T^*/S, R)$ *then* $\hat{\mathfrak{G}}(T^*\tau/S, R)\ni 1$ *(identity map) and* $\hat{\mathfrak{G}}(T^*\tau/S, R)=\tau'\hat{\mathfrak{G}}(T^*\tau\tau'/S, R)$ *for every* $\tau\in\hat{\mathfrak{G}}(T^*/S, R)$ *and for every* $\tau'\in\hat{\mathfrak{G}}(T^*\tau/S, R)$.

Proof. (i): Let $T\in\mathfrak{R}^0_{l.f}$. By Remark 1 and Corollary 2 (i, ii), we have $\hat{\mathfrak{G}}(T/S, R)=\cup\sigma_i\langle V\rangle$ (direct union) for some finite subset $\{\sigma_i\}$ of

$\hat{\mathfrak{G}}(T/S, R)$. Then, noting that $(H\cap T)\hat{\mathfrak{G}}(T/S, R)\subset H$ (Condition (δ)), we obtain $\hat{\mathfrak{G}}(T/S, R)|H\cap T=\{\sigma_i\}|H\cap T$. This proves $\sharp\hat{\mathfrak{G}}(T/S, R)|H\cap T\leqq(\hat{\mathfrak{G}}(T/S, R):\langle V\rangle)<\infty$. (ii): Let $T\in\mathfrak{R}^0_{l.f}$. By Corollary 2 (i, ii), we have $(\hat{\mathfrak{G}}(T\sigma/S, R):\langle V\rangle)\leqq[T\sigma:S]_l=[T:S]_l<\infty$ for every $\sigma\in\hat{\mathfrak{G}}(T/S, R)$. Hence, we can find some $\rho\in\hat{\mathfrak{G}}(T/S, R)$ such that $(\hat{\mathfrak{G}}(T\rho/S, R):\langle V\rangle)\leqq(\hat{\mathfrak{G}}(T\sigma/S, R):\langle V\rangle)$ for every $\sigma\in\hat{\mathfrak{G}}(T/S, R)$. Let $\tau\in\hat{\mathfrak{G}}(T\rho/S, R)$. Then, by Corollary 2 (iv) and Condition (γ), we obtain $(\tau\hat{\mathfrak{G}}(T\rho\tau/S, R):\langle V\rangle)=(\hat{\mathfrak{G}}(T\rho\tau/S, R):\langle V\rangle)\geqq(\hat{\mathfrak{G}}(T\rho/S, R):\langle V\rangle)$. On the other hand, since $\hat{\mathfrak{G}}(T\rho/S, R)\supset\tau\hat{\mathfrak{G}}(T\rho\tau/S, R)$ (Condition (γ)), we have $(\hat{\mathfrak{G}}(T\rho/S, R):\langle V\rangle)\geqq(\tau\hat{\mathfrak{G}}(T\rho\tau/S, R):\langle V\rangle)$ by Corollary 2 (ii). Hence $(\hat{\mathfrak{G}}(T\rho/S, R):\langle V\rangle)=(\tau\hat{\mathfrak{G}}(T\rho\tau/S, R):\langle V\rangle)$. Therefore we conclude $\hat{\mathfrak{G}}(T\rho/S, R)=\tau\hat{\mathfrak{G}}(T\rho\tau/S, R)$ by Theorem 1. (iii): Let $\tau\in\hat{\mathfrak{G}}(T^*/S, R)$. Then $\hat{\mathfrak{G}}(T^*/S, R)=\tau\hat{\mathfrak{G}}(T^*\tau/S, R)$, and so $\tau=\tau\varepsilon$ for some $\varepsilon\in\hat{\mathfrak{G}}(T^*\tau/S, R)$. Hence $1=\varepsilon\in\hat{\mathfrak{G}}(T^*\tau/S, R)$. Let $\tau'\in\hat{\mathfrak{G}}(T^*\tau/S, R)$. Then $\tau\tau'\in\hat{\mathfrak{G}}(T^*/S, R)$ and $\hat{\mathfrak{G}}(T^*\tau/S, R)\supset\tau'\hat{\mathfrak{G}}(T^*\tau\tau'/S, R)$ (Condition (γ)). Hence, by Corollary 2 (ii, iv), we have $(\hat{\mathfrak{G}}(T^*/S, R):\langle V\rangle)=(\tau\hat{\mathfrak{G}}(T^*\tau/S,R):\langle V\rangle)=(\hat{\mathfrak{G}}(T^*\tau/S,R):\langle V\rangle)\geqq(\tau'\hat{\mathfrak{G}}(T^*\tau\tau'/S,R):\langle V\rangle)=(\hat{\mathfrak{G}}(T^*\tau\tau'/S, R):\langle V\rangle)=(\tau\tau'\hat{\mathfrak{G}}(T^*\tau\tau'/S, R):\langle V\rangle)=(\hat{\mathfrak{G}}(T^*/S, R):\langle V\rangle)$. This implies that $(\hat{\mathfrak{G}}(T^*\tau/S, R):\langle V\rangle)=(\tau'\hat{\mathfrak{G}}(T^*\tau\tau'/S, R):\langle V\rangle)$. Therefore, it follows from Theorem 1 that $\hat{\mathfrak{G}}(T^*\tau/S, R)=\tau'\hat{\mathfrak{G}}(T^*\tau\tau'/S, R)$.

Lemma 7. *Let R be left locally finite over S. If $\hat{\mathfrak{G}}=\{\hat{\mathfrak{G}}(T/S, R); T\in\mathfrak{R}^0_{l.f}\}$ is a left q-system then $(H\cap T)\sigma=H\cap T\sigma$ and $[H\cap T:S]_l=[H\cap T\sigma:S]_l$ for each $T\in\mathfrak{R}^0_{l.f}$ and for every $\sigma\in\hat{\mathfrak{G}}(T/S, R)$.*

Proof. Let $T\in\mathfrak{R}^0_{l.f}$, and $\sigma\in\hat{\mathfrak{G}}(T/S, R)$. Then $(H\cap T)\sigma\subset H$ by Condition (δ). Hence, it follows that $(H\cap T)\sigma\subset H\cap T\sigma$. On the other hand, if $h\in H\cap T\sigma$ then we can find some $a\in T$ such that $a\sigma=h$. We assume that $a\notin H\cap T$. Then, there exists some $v\in V$ such that $av\neq va$. If we choose a $T_1\in\mathfrak{R}^0_{l.f}/[T, v]$, then $\sigma=\sigma_1|T$ for some $\sigma_1\in\hat{\mathfrak{G}}(T_1/S, R)$ (Condition (β)). Since $av\neq va$, we have $(a\sigma_1)(v\sigma_1)\neq(v\sigma_1)(a\sigma_1)$. Noting here $a\sigma_1=a\sigma=h$, it follows that $h(v\sigma_1)\neq(v\sigma_1)h$. However, this is a contradiction, (noting $v\sigma_1\in V$). Therefore $a\in H\cap T$. This proves $(H\cap T)\sigma=H\cap T\sigma$. Now, the rest of the proof follows from the fact that σ is an S-(ring) isomorphism of T into R.

Now, let A be a simple subring of R containing S. If S is a regular subring of A (i. e , S and $V_A(S)$ are both simple) and $J(\mathfrak{G}(S, A), A)=S$ then we say that A is *Galois* over S. If moreover $V_A(S)=V_A(A)$ (the center of A) then we say that A is *outer Galois* over S. If for each finite subset F of A there exists a simple subring N containing $[S, F]$ such that $[N:S]_l$

$<\infty$ and N is Galois over S then we say that A is *locally Galois* over S. Now, we shall make a remark on outer Galois theory which is frequently used in our study. If $\mathfrak{G}'$ is a finite group of outer automorphisms of A then $J(\mathfrak{G}', A)$ is simple; A is outer Galois and finite over $J(\mathfrak{G}', A)$; $\mathfrak{G}(J(\mathfrak{G}', A), A)=\mathfrak{G}'$. If A is locally Galois over S and $V_A(S)=V_A(A)$ (the center of A) then A is outer Galois and two-sided locally finite over S. Conversely, if A is outer Galois and left locally finite (or right locally finite) over S then A is locally Galois over S (and $V_A(S)=V_A(A)$). Let A be outer Galois and locally finite over S. If B is a subring of A containing S then B is simple; A is outer Galois and locally finite over B; $\mathfrak{G}(S, A)|B=\mathfrak{G}(B/S, A)$; if B is Galois over S then this is also outer Galois over S and $\mathfrak{G}(S, A)|B=\mathfrak{G}(S, B)$; if $[B:S]_l<\infty$ and $\mathfrak{G}^*$ is a subgroup of $\mathfrak{G}(S, A)$ such that $J(\mathfrak{G}^*, A)=S$, then $\mathfrak{G}^*|B=\mathfrak{G}(S, A)|B$, $J(\mathfrak{G}(B, A)\cap\mathfrak{G}^*, A)=B$, and $[B:S]_l=[B:S]_r=\sharp\mathfrak{G}^*|B=\sharp\mathfrak{G}(S, A)|B=\sharp\mathfrak{G}(B/S, A)$. Moreover, there exists an 1—1 dual correspondence between closed subgroups of $\mathfrak{G}(S, A)$ and subrings of A containing S, in the usual sense of Galois theory (cf. [2], [5, Theorem 1.1, Corollary 1.4, and Lemma 4.2], and [6, Lemma 1.8]).

Lemma 8. *Let R be left locally finite over S, and H simple. If there exists a left q-system $\hat{\mathfrak{G}}=\{\hat{\mathfrak{G}}(T/S, R)\,;\ T\in\mathfrak{R}^0_{l.f}\}$ then the following hold:*

(i) *H is outer Galois over S.*

(ii) *$\hat{\mathfrak{G}}(T/S, R)|H\cap T=\mathfrak{G}(S, H)|H\cap T=\mathfrak{G}(H\cap T/S, H)$ and $\sharp\hat{\mathfrak{G}}(T/S, R)|H\cap T=[H\cap T:S]$ for every $T\in\mathfrak{R}^0_{l.f}$.*

Proof. Let $T\in\mathfrak{R}^0_{l.f}$, and choose a $T_1\in\mathfrak{R}^0_{l.f}/[T, \{d_{hk}\}]$, (as in the introduction, $\{d_{hk}\}$ is a system of matrix units of H such that $V_H(\{d_{hk}\})$ is a division ring; $H=\sum Kd_{hk}$, $K=V_H(\{d_{hk}\})$). Then, by Lemma 6 (i), we have $\hat{\mathfrak{G}}(T_1/S, R)|\,H\cap T_1=\{\sigma_1|H\cap T_1, \cdots, \sigma_n|H\cap T_1\}$ for some finite subset $\{\sigma_i\}$ of $\hat{\mathfrak{G}}(T_1/S, R)$. We set $T_2=[T_1, T_1\sigma_1, T_1\sigma_2, \cdots, T_1\sigma_n]$ and $H_2=[H\cap T_1, (H\cap T_1)\sigma_1, \cdots, (H\cap T_1)\sigma_n]$. Then, by Conditions ($\beta, \gamma$), we have $\hat{\mathfrak{G}}(T_2/S, R)|T_1\sigma_i=\hat{\mathfrak{G}}(T_1\sigma_i/S, R)$ and $\sigma_i\hat{\mathfrak{G}}(T_1\sigma_i/S, R)\subset\hat{\mathfrak{G}}(T_1/S, R)$ $(i=1,\cdots, n)$. Hence $H_2\hat{\mathfrak{G}}(T_2/S, R)=[(H\cap T_1)\hat{\mathfrak{G}}(T_2/S, R), (H\cap T_1)\sigma_1\hat{\mathfrak{G}}(T_2/S, R), \cdots, (H\cap T_1)\sigma_n\hat{\mathfrak{G}}(T_2/S, R)]=[(H\cap T_1)(\hat{\mathfrak{G}}(T_2/S, R)|T_1), (H\cap T_1)\sigma_1(\hat{\mathfrak{G}}(T_2/S, R)|T_1\sigma_1), \cdots, (H\cap T_1)\sigma_n(\hat{\mathfrak{G}}(T_2/S, R)|T_1\sigma_n)]=[(H\cap T_1)\hat{\mathfrak{G}}(T_1/S, R), (H\cap T_1)\sigma_1\hat{\mathfrak{G}}(T_1\sigma_1/S, R), \cdots, (H\cap T_1)\sigma_n\hat{\mathfrak{G}}(T_1\sigma_n/S, R)]\subset(H\cap T_1)\hat{\mathfrak{G}}(T_1/S, R)=[(H\cap T_1)\sigma_1, \cdots, (H\cap T_1)\sigma_n]\subset H_2$. From this and $[H_2:S]_l<\infty$, $\hat{\mathfrak{G}}(T_2/S, R)|H_2$ is a set of S-(ring) automorphisms of H_2. Since $H\supset H_2\supset H\cap T_1\supset\{d_{hk}\}$, H_2 is a simple subring of H. Hence, noting that $J(\hat{\mathfrak{G}}(T_2/S, R)|H_2, H_2)=S$ (Condition (α)) and $V_{H_2}(S)=V_{H_2}(H_2)$, H_2 is outer Galois over S. Therefore since $H_2\supset H\cap T_1\supset$

$H\cap T$, H is locally Galois over S. Thus H is outer Galois over S by outer Galois theory. Now, let $\sigma\in\hat{\mathfrak{G}}(T_2/S, R)$, and choose a $T_3\in\mathfrak{R}^0_{l.f}/[T_2, T_2\sigma]$. Then $H_2=H_2\sigma\subset T_2\cap T_2\sigma\subset T_3$. Hence, we have $\hat{\mathfrak{G}}(T_2/S, R)|H_2=(\hat{\mathfrak{G}}(T_3/S, R)|T_2)|H_2=\hat{\mathfrak{G}}(T_3/S, R)|H_2=(\hat{\mathfrak{G}}(T_3/S, R)|T_2\sigma)|H_2=\hat{\mathfrak{G}}(T_2\sigma/S, R)|H_2$. From this, it follows that $(\hat{\mathfrak{G}}(T_2/S, R)\mid H_2)(\hat{\mathfrak{G}}(T_2/S, R)\mid H_2)=(\bigcup_{\sigma\in\hat{\mathfrak{G}}(T_2/S,R)}\sigma\hat{\mathfrak{G}}(T_2\sigma/S, R))|H_2\subset\hat{\mathfrak{G}}(T_2/S, R)|H_2$, so that $\hat{\mathfrak{G}}(T_2/S, R)|H_2$ is closed under the product. Noting that $\mathfrak{G}(S, H_2)\supset\hat{\mathfrak{G}}(T_2/S, R)|H_2$ and $\mathfrak{G}(S, H_2)$ is a finite group, $\hat{\mathfrak{G}}(T_2/S, R)|H_2$ is a subgroup of $\mathfrak{G}(S, H_2)$. Since $J(\hat{\mathfrak{G}}(T_2/S, R)|H_2, H_2)=S$, we have $\hat{\mathfrak{G}}(T_2/S, R)|H_2=\mathfrak{G}(S, H_2)$ by outer Galois theory. Therefore we obtain $\hat{\mathfrak{G}}(T/S, R)|H\cap T=(\hat{\mathfrak{G}}(T_2/S, R)|T)|H\cap T=\hat{\mathfrak{G}}(T_2/S, R)|H\cap T=(\hat{\mathfrak{G}}(T_2/S, R)|H_2)|H\cap T=\mathfrak{G}(S, H_2)|H\cap T=\mathfrak{G}(S, H)|H\cap T$, i. e., $\hat{\mathfrak{G}}(T/S, R)|H\cap T=\mathfrak{G}(S, H)|H\cap T$, as required. Now, the rest of the proof follows from outer Galois theory.

Remark 3. If $\hat{\mathfrak{G}}=\{\hat{\mathfrak{G}}(T/S, R)\,;\,T\in\mathfrak{R}^0_{l.f}\}$ is a left q-system then for each $T\in\mathfrak{R}^0_{l.f}$ and for every subring A of T containing S, we write $\hat{\mathfrak{G}}(T/A, R)$ for $\{\sigma\in\hat{\mathfrak{G}}(T/S, R)\,;\,\sigma|A=1\}$. Now, if two subrings A, B of R satisfy the following conditions then we say that A is left linearly disjoint from B over $A\cap B$: A contains a linearly independent left basis over $A\cap B$, and for any linearly independent left basis $\{d_i\,;\,i\in I\}$ of A over $A\cap B$, the sum $\sum_{i\in I}Bd_i$ is direct and every Bd_i is left B-isomorphic to B. If A is left linearly disjoint and right linearly disjoint from B over $A\cap B$ then we say that A is linearly disjoint from B over $A\cap B$.

Lemma 9. *Let R be left locally finite over S, and H simple. If there exists a left q-system $\hat{\mathfrak{G}}=\{\hat{\mathfrak{G}}(T/S, R)\,;\,T\in\mathfrak{R}^0_{l.f}\}$ then the following hold:*

(i) *If $T^*\in\mathfrak{R}^0_{l.f}/\{d_{hk}\}$ and $H\cap J(\hat{\mathfrak{G}}(T'/T^*, R), T')=H\cap T^*$ for every $T'\in\mathfrak{R}^0_{l.f}/T^*$ then T^* is linearly disjoint from H over $H\cap T^*$.*

(ii) *If $T^*\in\mathfrak{R}^0_{l.f}$ and T^* is linearly disjoint from H over $H\cap T^*$ then for every $\sigma\in\hat{\mathfrak{G}}(T^*/S, R)$, $T^*\sigma$ is linearly disjoint from H over $H\cap T^*\sigma$.*

Proof. (i): We shall prove that T^* is right linearly disjoint from H over $H\cap T^*$, (the proof of left lineary disjointness is similar to that of right linearly disjointness). Set $H^*=H\cap T^*$, and choose a linearly independent right basis $\{d_i\,;\,i\in I\}$ of T^* over H^*. Then, we have

$$\sum d_iH=\sum_{i,h,k}d_id_{hk}K,\quad and\ H^*=\sum d_{hk}(K\cap T^*),\ (H=\sum d_{hk}K).$$

Hence, it will suffice to prove that $\{d_id_{hk}\}$ is linearly right-independent over K. If not, without loss of generality we may assume that

$$d_1d_{11}=\sum d_id_{hk}a_{ihk}\qquad(a_{ihk}\in K)$$

is a non-trivial relation of the shortest length. We choose a $T'\in\mathfrak{R}^0_{l.f}/[T^*, \{a_{ihk}\}]$. Then, we have $H\cap J(\hat{\mathfrak{G}}(T'/T^*, R), T')=H\cap T^*=H^*=\sum d_{hk}\cdot(K\cap T^*)$ and $(K\cap T')\hat{\mathfrak{G}}(T'/T^*, R)\subset H\cap V_R(\{d_{hk}\})=K$. As there exists some $a_{i'h'k'}$ not contained in $K\cap T^*$, we can find some $\sigma\in\hat{\mathfrak{G}}(T'/T^*, R)$ such that $a_{i'h'k'}\neq a_{i'h'k'}\sigma$ $(\in K)$. Hence, we have a non-trivial relation of shorter length:

$$0=d_1d_{11}-(d_1d_{11})\sigma=\sum d_id_{hk}(a_{ihk}-a_{ihk}\sigma).$$

This contradiction implies that $\{d_id_{hk}\}$ is linearly right-independent over K. (ii): Let $\sigma\in\hat{\mathfrak{G}}(T^*/S, R)$ and $T^*\sigma=\sum_{\oplus}d_i'(H\cap T^*\sigma)$. Then, we can find some $\{d_i^*\}\subset T^*$ such that $d_i^*\sigma=d_i'$ for every i. Since $(H\cap T^*)\sigma=H\cap T^*\sigma$ (Lemma 7), we have $(\sum d_i^*(H\cap T^*))\sigma=\sum d_i'(H\cap T^*\sigma)=T^*\sigma$. Therefore $T^*=\sum_{\oplus}d_i^*(H\cap T^*)$. We suppose that $\sum d_i'h_i=0$ for some finite subset $\{h_i\}\subset H$. By Lemma 8 (i) and outer Galois theory, there exists a subring N of H containing $[S, \{h_i\}]$ which is Galois and finite over S. We choose a $T_1\in\mathfrak{R}^0_{l.f}/[T^*, N]$. Then $\sigma=\sigma_1|T^*$ for some $\sigma_1\in\hat{\mathfrak{G}}(T_1/S, R)$ (Condition (β)). By Lemma 8 and outer Galois theory, we have $N\sigma_1=N$, and so, we can find some $\{g_i\}\subset N$ such that $g_i\sigma_1=h_i$ for every i. If σ_1 operates on $\sum d_i^*g_i$ then $(\sum d_i^*g_i)\sigma_1=\sum d_i^*\sigma_1g_i\sigma_1=\sum d_i^*\sigma h_i=\sum d_i'h_i=0$; this implies $\sum d_i^*g_i=0$. Since T^* is linearly disjoint from H over $H\cap T^*$, it follows that $g_i=0$, and so $h_i=0$ for every i. This proves our assertion.

Lemma 10. *Let R be left locally finite over S, and H simple. If there exists a left q-system $\hat{\mathfrak{G}}=\{\hat{\mathfrak{G}}(T/S, R);\ T\in\mathfrak{R}^0_{l.f}\}$ then for each $T\in\mathfrak{R}^0_{l.f}$, there exists some $T^*\in\mathfrak{R}^0_{l.f}$ such that*

(i) *$T\rho\subset T^*$ for some $\rho\in\hat{\mathfrak{G}}(T/S, R)$;*

(ii) *T^* is linearly disjoint from H over $H\cap T^*$.*

Proof. Let $T\in\mathfrak{R}^0_{l.f}$, and choose a $T_1\in\mathfrak{R}^0_{l.f}/[T, \{d_{hk}\}]$. By Lemma 6 (ii), there exists some $\rho_1\in\hat{\mathfrak{G}}(T_1/S, R)$ such that $\tau\hat{\mathfrak{G}}(T_1\rho_1\tau/S, R)=\hat{\mathfrak{G}}(T\rho_1/S, R)$ for every $\tau\in\hat{\mathfrak{G}}(T_1\rho_1/S, R)$. For any $\rho_2\in\hat{\mathfrak{G}}(T_1\rho_1/S, R)$, we set $T_2=T_1\rho_1\rho_2$. Then, by Lemma 6 (iii), we have

(1) $\tau\hat{\mathfrak{G}}(T_2\tau/S, R)=\hat{\mathfrak{G}}(T_2/S, R)$ for every $\tau\in\hat{\mathfrak{G}}(T_2/S, R)$, and $\hat{\mathfrak{G}}(T_2/S, R)\ni 1$.

By Corollary 2 (i, ii), we have $(\hat{\mathfrak{G}}(T_2/S, R):\langle V\rangle)<\infty$. We set $\mathfrak{F}=\{T'\in\mathfrak{R}^0_{l.f}/T_2;\ (\hat{\mathfrak{G}}(T'/S, R):\langle V\rangle)=(\hat{\mathfrak{G}}(T_2/S, R):\langle V\rangle)\}$ $(\ni T_2)$. Then, by Lemma 8 (ii) and Lemma 6 (i), we have $[H\cap T':S]=\sharp\hat{\mathfrak{G}}(T'/S, R)|H\cap T'\leqq(\hat{\mathfrak{G}}(T'/S, R):\langle V\rangle)=(\hat{\mathfrak{G}}(T_2/S, R):\langle V\rangle)<\infty$ for every $T'\in\mathfrak{F}$. Hence we can find some $T_3\in\mathfrak{F}$ such that $[H\cap T':S]\leqq[H\cap T_3:S]$ for every $T'\in\mathfrak{F}$. For each $\tau\in\hat{\mathfrak{G}}(T_3/S, R)$, we have $\tau\hat{\mathfrak{G}}(T_3\tau/S, R)\subset\hat{\mathfrak{G}}(T_3/S, R)$ (Condition (γ)). Since $\tau\hat{\mathfrak{G}}(T_3\tau/S, R)|T_2=(\tau|T_2)(\hat{\mathfrak{G}}(T_3\tau/S, R)|T_2\tau)=(\tau|T_2)(\hat{\mathfrak{G}}(T_2\tau/S, R)$ and

$\tau | T_2 \in \hat{\mathfrak{G}}(T_2/S, R)$ (Condition (β)), it follows from (1) that $\tau\hat{\mathfrak{G}}(T_3\tau/S, R) | T_2 = \hat{\mathfrak{G}}(T_2/S, R)$. Hence, by Corollary 2 (ii, iii) and the choice of T_3, we obtain $(\hat{\mathfrak{G}}(T_2/S, R) : \langle V \rangle) = (\tau\hat{\mathfrak{G}}(T_3\tau/S, R) | T_2 : \langle V \rangle) \leqq (\tau\hat{\mathfrak{G}}(T_3\tau/S, R) : \langle V \rangle) \leqq (\hat{\mathfrak{G}}(T_3/S, R) : \langle V \rangle) = (\hat{\mathfrak{G}}(T_2/S, R) : \langle V \rangle)$. Thus $(\tau\hat{\mathfrak{G}}(T_3\tau/S, R) : \langle V \rangle) = (\hat{\mathfrak{G}}(T_3/S, R) : \langle V \rangle)$. Therefore, it follows from Theorem 1 that

(2) $\tau\hat{\mathfrak{G}}(T_3\tau/S, R) = \hat{\mathfrak{G}}(T_3/S, R)$ for every $\tau \in \hat{\mathfrak{G}}(T_3/S, R)$.

By Condition (β) and (1), we have $\hat{\mathfrak{G}}(T_3/S, R) | T_2 = \hat{\mathfrak{G}}(T_2/S, R) \ni 1$. Hence, we can find some $\delta \in \hat{\mathfrak{G}}(T_3/S, R)$ such that $\delta | T_2 = 1$. We set $T^* = T_3\delta$. Then, by Lemma 6 (iii) and (2), we have

(3) $\tau\hat{\mathfrak{G}}(T^*\tau/S, R) = \hat{\mathfrak{G}}(T^*/S, R)$ for every $\tau \in \hat{\mathfrak{G}}(T^*/S, R)$, and $\hat{\mathfrak{G}}(T^*/S, R) \ni 1$.

From $\delta | T_2 = 1$, we have $T^* \supset T_2$. By Corollary 2 (iv) and (2), we have $(\hat{\mathfrak{G}}(T^*/S, R) : \langle V \rangle) = (\hat{\mathfrak{G}}(T_3\delta/S, R) : \langle V \rangle) = (\delta\hat{\mathfrak{G}}(T_3\delta/S, R) : \langle V \rangle) = (\hat{\mathfrak{G}}(T_3/S, R) : \langle V \rangle) = (\hat{\mathfrak{G}}(T_2/S, R) : \langle V \rangle)$. Hence we obtain $T^* \in \mathfrak{T}$. Moreover, noting that $H \cap T^* = H \cap T_3\delta = (H \cap T_3)\delta$ (Lemma 7), we obtain

(4) $[H \cap T^* : S] = [(H \cap T_3)\delta : S] = [H \cap T_3 : S] \geqq [H \cap T' : S]$

for every $T' \in \mathfrak{T}$. Now, we set $\rho = \rho_1\rho_2$. Then $T\rho \subset T_1\rho = T_2 \subset T^*$ and $\rho | T \in \hat{\mathfrak{G}}(T/S, R)$ (Conditions (β, γ)). Moreover $\{d_{hk}\}\rho \subset T^*$, and $\{d_{hk}\}\rho$ is a system of matrix units of H such that $V_H(\{d_{hk}\}\rho)$ is a division ring. Hence, by Lemma 9 (i), it suffices to prove that $H \cap T^* = H \cap J(\hat{\mathfrak{G}}(T'/T^*, R), T')$ for every $T' \in \mathfrak{R}^0_{l.f}/T^*$. Let $T' \in \mathfrak{R}^0_{l.f}/T^*$, and set $T'' = J(\hat{\mathfrak{G}}(T'/T^*, R), T')$. Then $T' \supset T'' \supset T^*$, and by Condition ($\beta$), we have $\hat{\mathfrak{G}}(T^*/S, R) = \hat{\mathfrak{G}}(T'/S, R) | T^* = \cup_{1 \leqq i \leqq n}(\sigma_i | T^*)\langle V \rangle$ ($n = (\hat{\mathfrak{G}}(T^*/S, R) : \langle V \rangle) < \infty$) for some $\{\sigma_i\}$ of $\hat{\mathfrak{G}}(T'/S, R)$. Let $\sigma \in \hat{\mathfrak{G}}(T'/S, R)$. Then $\sigma | T^* = (\sigma' | T^*)\langle v \rangle$ for some $\sigma' \in \{\sigma_i\}$ and for some $\langle v \rangle \in \langle V \rangle$. By (3), we have $(\sigma | T^*)\hat{\mathfrak{G}}(T^*\sigma/S, R) = \hat{\mathfrak{G}}(T^*/S, R) \ni 1$. From this, we can find some $\sigma^* \in \hat{\mathfrak{G}}(T^*\sigma/S, R)$ such that $(\sigma | T^*)\sigma^* = 1$. We choose a $T_x \in \mathfrak{R}^0_{l.f}/[T', T'\sigma, T'\sigma', v]$. Clearly $T_x \supset T^*\sigma$. Then $\sigma^* = \sigma_x | T^*\sigma$ for some $\sigma_x \in \hat{\mathfrak{G}}(T_x/S, R)$ (Condition (β)), and then, we have $1 | T^* = (\sigma | T^*)\sigma_x = (\sigma' | T^*)\langle v \rangle \sigma_x$. By Conditions ($\beta, \gamma$), we can see that $\sigma\sigma_x, \sigma'\langle v \rangle\sigma_x \in \hat{\mathfrak{G}}(T'/S, R)$. Hence $1 | T'' = (\sigma | T'')\sigma_x = (\sigma' | T'')\langle v \rangle\sigma_x$. Thus $\sigma | T'' = (\sigma' | T'')\langle v \rangle$. From this and Condition (β), it follows that $\hat{\mathfrak{G}}(T''/S, R) = \hat{\mathfrak{G}}(T'/S, R) | T'' = \cup_{1 \leqq i \leqq n}(\sigma_i | T'')\langle V \rangle$. Therefore $(\hat{\mathfrak{G}}(T''/S, R) : \langle V \rangle) \leqq n = (\hat{\mathfrak{G}}(T^*/S, R) : \langle V \rangle) = (\hat{\mathfrak{G}}(T''/S, R) | T^* : \langle V \rangle)$. On the other hand, by Corollary 2 (iii), we have $(\hat{\mathfrak{G}}(T''/S, R) : \langle V \rangle) \geqq (\hat{\mathfrak{G}}(T''/S, R) | T^* : \langle V \rangle)$. Consequently we obtain $(\hat{\mathfrak{G}}(T''/S, R) : \langle V \rangle) = (\hat{\mathfrak{G}}(T^*/S, R) : \langle V \rangle) = (\hat{\mathfrak{G}}(T_2/S, R) : \langle V \rangle)$. This implies that $T'' \in \mathfrak{T}$ and $H \cap T'' \supset H \cap T^*$. Then, by (4), we conclude that $H \cap T'' = H \cap T^*$.

Proposition 2. (i) $(\mathrm{III}_l) \Longrightarrow (\mathrm{III}'_r)$, $(\mathrm{III}_r) \Longrightarrow (\mathrm{III}'_l)$.
(ii) $(\mathrm{III}'_l) \Longrightarrow (\mathrm{III}''_l) \Longrightarrow (\mathrm{II}_l)$, $(\mathrm{III}'_r) \Longrightarrow (\mathrm{III}''_r) \Longrightarrow (\mathrm{II}_r)$.
(iii) *If the condition* (III''_l) *is satisfied and* $\hat{\mathfrak{G}}=\{\hat{\mathfrak{G}}(T/S, R)\,;\, T\in\mathfrak{R}^0_{l.f}\}$ *is a left q-system then* $\hat{\mathfrak{G}}(T/S, R)=\mathfrak{G}(T/S, R)$ *for every* $T\in\mathfrak{R}^0_{l.f}$.

Proof. If one of the conditions (III_l), (III'_l) and (III''_l) is satisfied and $\hat{\mathfrak{G}}=\{\hat{\mathfrak{G}}(T/S, R)\,;\, T\in\mathfrak{R}^0_{l.f}\}$ is a left q-system, then H is simple by Lemma 1 (i) and a condition $H\in\mathfrak{R}$ of (III''_l), and then, by Lemma 10, for each $T\in\mathfrak{R}^0_{l.f}$, there exists some $T^*\in\mathfrak{R}^0_{l.f}$ such that
(1) $T\rho\subset T^*$ for some $\rho\in\hat{\mathfrak{G}}(T/S, R)$, and
(2) T^* is linearly disjoint from H over $H\cap T^*$ (simple).
Moreover, from Lemma 1 (iv) and (2), we have
(3) $[V : V_R(T^*)]_l,\ [V : V_R(T^*)]_r \leqq [T^* : H\cap T^*]_l \leqq [T^* : S]_l < \infty$,
(4) $[T^* : H\cap T^*]_l = [HT^* : H]_l \leqq [V_R^2(T^*) : H]_l$,
(5) $[T^* : H\cap T^*]_r = [T^*H : H]_r \leqq [V_R^2(T^*) : H]_r$.
(i): $(\mathrm{III}_l) \Longrightarrow (\mathrm{III}'_r)$: Since R is $S\cdot V$-R-irreducible, it follows from Lemma 1 (i) and (3) that
(6) $[V_R^2(T^*) : H]_r \leqq [V : V_R(T^*)]_l < \infty$.
Combining (5) with (6), we have $[T^* : H\cap T^*]_r < \infty$, so that $[T^* : S]_r = [T^* : H\cap T^*]_r [H\cap T^* : S] < \infty$. Hence, by (1), we have $[T\rho : S]_r \leqq [T^* : S]_r < \infty$, and so $[T : S]_r < \infty$. Therefore R is right locally finite over S, and moreover, $\hat{\mathfrak{G}}$ is also a right q-system. This means that $(\mathrm{III}_l) \Longrightarrow (\mathrm{III}'_r)$. Similarly, we have $(\mathrm{III}_r) \Longrightarrow (\mathrm{III}'_l)$. (ii): $(\mathrm{III}'_l) \Longrightarrow (\mathrm{III}''_l)$: By Lemma 1-(iv), we have $[V : V_R(T)]_r \leqq [T : S]_l < \infty$. Since R is R-$S\cdot V$-irreducible, V and H are both simple and $[V_R^2(T) : H]_l \leqq [V : V_R(T)]_r$ (Lemma 1 (i)). On the other hand, since R is $V_R^2(T)$-R-irreducible, we have $[V_R^2(T) : H]_l \geqq [V : V_R(T)]_r$ (setting $S=H$ and $S'=V_R^2(T)$ in Lemma 1 (ii, c)). Hence we obtain $[V_R^2(T) : H]_l = [V : V_R(T)]_r$. This implies that $(\mathrm{III}'_l) \Longrightarrow (\mathrm{III}''_l)$. $(\mathrm{III}''_l) \Longrightarrow (\mathrm{II}_l)$: Combining (3) with (4) and our condition $[V_R^2(T^*) : H]_l = [V : V_R(T^*)]_r$, we have
(7) $[T^* : H\cap T^*]_l = [V : V_R(T^*)]_r$.
If we set $\hat{\mathfrak{G}}(T^*/S, R)\,|\,H\cap T^* = \{\sigma_1|H\cap T^*, \cdots, \sigma_n|H\cap T^*\}$ for some σ_i's$\subset \hat{\mathfrak{G}}(T^*/S, R)$ ($\sigma_i|H\cap T^* \neq \sigma_j|H\cap T^*$ for every $i\neq j$) then for every $i\neq j$, $\sigma_i R_r$ is not T^*_r-R_r-isomorphic to $\sigma_j R_r$ by Lemma 2 (iii) (noting that $(H\cap T^*)\sigma_i, (H\cap T^*)\sigma_j \subset H$). Hence, we have
(8) $\sum_1^n \sigma_i\langle V\rangle R_r = \sum_{\oplus 1}^n \sigma_i\langle V\rangle R_r$.
For each i, $T^*\sigma_i$ is linearly disjoint from H over $H\cap T^*\sigma_i$ by (2) and Lemma 9 (ii). Hence, by similar methods, we obtain that
(9) $[T^*\sigma_i : H\cap T^*\sigma_i]_l = [V : V_R(T^*\sigma_i)]_r$.
Since $(H\cap T^*)\sigma_i = H\cap T^*\sigma_i$ (Lemma 7), it follows from (9) and (7) that
(10) $[T^* : H\cap T^*]_l = [T^*\sigma_i : H\cap T^*\sigma_i]_l = [V : V_R(T^*\sigma_i)]_r = [V : V_R(T^*)]_r$.

For each i, if we choose a linearly independent $V_R(T^*\sigma_i)$-right basis $\{v_{ij};\ 1\leqq j\leqq m\}$ $(m=[T^*:H\cap T^*]_l)$ of V which consists of regular elements, then by Lemma 1 (iii) and (8), we have

$$\sum_{j=1}^{m}(\langle v_{ij}\rangle \mid T^*\sigma_i)R_r=\sum\oplus_{j=1}^{m}(\langle v_{ij}\rangle \mid T^*\sigma_i)R_r, \text{ and}$$

$$\sum_{i=1}^{n}\sigma_i\langle V\rangle R_r=\sum\oplus_{i=1}^{n}\sigma_i\langle V\rangle R_r=\sum\oplus_{i=1,}^{n}{}_{j=1}^{m}\ \sigma_i\langle v_{ij}\rangle R_r\,.$$

Since $\mathrm{Hom}_{S_l}(T^*, R)\supset\sum_{i=1}^{n}\sigma_i\langle V\rangle R_r$ and $[\mathrm{Hom}_{S_l}(T^*, R):R_r]_r=[T^*:S]_l=[T^*:H\cap T^*]_l[H\cap T^*:S]=mn$ (Lemma 8 (ii)), it follows that $\mathrm{Hom}_{S_l}(T^*, R)=\sum_{i=1}^{n}\sigma_i\langle V\rangle R_r$. Hence $\mathrm{Hom}_{S_l}(T^*, R)=\hat{\mathfrak{G}}(T^*/S, R)R_r$. From this and (1), we have $\mathrm{Hom}_{S_l}(T\rho, R)=\mathrm{Hom}_{S_l}(T^*, R)\mid T\rho=\hat{\mathfrak{G}}(T^*/S, R)R_r\mid T\rho=(\hat{\mathfrak{G}}(T^*/S, R)\mid T\rho)R_r=\hat{\mathfrak{G}}(T\rho/S, R)R_r$. Therefore we obtain $\mathrm{Hom}_{S_l}(T, R)=\rho\mathrm{Hom}_{S_l}(T\rho, R)R_r=\rho\hat{\mathfrak{G}}(T\rho/S, R)R_r\subset\hat{\mathfrak{G}}(T/S, R)R_r\subset\mathfrak{G}(T/S, R)R_r\subset\mathrm{Hom}_{S_l}(T, R)$. This implies that

(11) $\mathrm{Hom}_{S_l}(T, R)=\hat{\mathfrak{G}}(T/S, R)R_r=\mathfrak{G}(T/S, R)R_r$.

From (11) we have $(\mathrm{III}''_l)\Longrightarrow(\mathrm{II}_l)$. Similarly we have $(\mathrm{III}'_r)\Longrightarrow(\mathrm{III}''_r)\Longrightarrow(\mathrm{II}_r)$. Our last assertion (iii) follows immediately from (11) and Theorem 1.

Now, we are in the position to prove the following

Theorem 3. *The conditions* (I_l) —— (III''_r) *are all equivalent. Moreover, if one of these is satisfied then any left q-system is also a right q-system, and this coincides with a left q-system* $\mathfrak{G}=\{\mathfrak{G}(T/S, R);\ T\in\mathfrak{R}^0_{l.f}\}$; *i. e., there exists a unique left q-system, which is a unique right q-system, and if* $\hat{\mathfrak{G}}=\{\hat{\mathfrak{G}}(T/S, R);\ T\in\mathfrak{R}^0_{l.f}\}$ *is a left q-system then* $\hat{\mathfrak{G}}(T/S, R)\mid S'=\mathfrak{G}(S'/S, R)$ *for each* $S'\in\mathfrak{R}_{l.f}$ *and for every* $T\in\mathfrak{R}^0_{l.f}/S'$.

Proof. Combining Proposition 1 and Proposition 2, we have that $(\mathrm{I}_l)\Longleftrightarrow(\mathrm{II}_l)$, $(\mathrm{I}_r)\Longleftrightarrow(\mathrm{II}_r)$ and $(\mathrm{II}_l)\Longrightarrow(\mathrm{III}_l)\Longrightarrow(\mathrm{III}'_r)\Longrightarrow(\mathrm{III}''_r)\Longrightarrow(\mathrm{II}_r)\Longrightarrow(\mathrm{III}_r)\Longrightarrow(\mathrm{III}'_l)\Longrightarrow(\mathrm{III}''_l)\Longrightarrow(\mathrm{II}_l)$. The rest of the proof is a direct consequence of Theorem 2 and Proposition 2.

Remark 4. By Definition and Theorem 2, R is q-Galois and left locally finite over S if and only if the condition (I_l) is satisfied. Hence, in the rest of this note, if one of the conditions (I_l)——(III''_r) is satisfied then we say that R is *q-Galois and locally finite* over S.

Now, from Corollary 4 and Theorem 3 we have the following

Theorem 4. *The following conditions* (IV_l) ——(IV''_r) *are all equivalent. Moreover, R is q-Galois and locally finite over S if and only if one of these is satisfied.*

(IV_l): (i) $J(\mathfrak{G}(T/S, R), T)=S$ *for every* $T\in\mathfrak{R}^0_{l.f}$, *and*

(ii) $\mathfrak{G}(T_1/S, R)\mid T_2=\mathfrak{G}(T_2/S, R)$ *for every* $T_1, T_2\in\mathfrak{R}^0_{l.f}$ *such that* $T_1\supset T_2$, *and*

(iii) *R is S·V-R-irreducible, and*

(iv) *R is left locally finite over S.*

(IV'_l): $(\mathrm{IV}_l$: i, ii, iv) *plus the condition that R is R-S·V-irreducible.*

(IV''_l): $(\mathrm{IV}_l$: i, ii, iv) *plus the condition that* $S, H\in\mathfrak{R}$ *and* $[V_R^2(T):H]_l = [V:V_R(T)]_r$ *for every* $T\in\mathfrak{R}^0_{l.f}$.

(IV_r). (IV'_r). (IV''_r). (These are symmetric to (IV_l)——(IV''_l)).

Now, for each $T\in\mathfrak{R}^0_{l.f}$, we set

$$\tilde{\mathfrak{G}}(T/S, R)=\cap_{T'}\mathfrak{G}(T'/S, R)\,|\,T,$$

where T' runs over all the elements of $\mathfrak{R}^0_{l.f}/T$. Clearly $\tilde{\mathfrak{G}}(T/S, R)\subset \mathfrak{G}(T'/S, R)\,|\,T\subset\mathfrak{G}(T/S, R)$ for every $T'\in\mathfrak{R}^0_{l.f}/T$ (Lemma 2 (i)). Then, we have the following

Lemma 11. *Let* $T\in\mathfrak{R}^0_{l.f}$. *Then*

(i) *For* $\sigma\in\mathfrak{G}(T/S, R)$, $\sigma\in\tilde{\mathfrak{G}}(T/S, R)$ *if and only if for an arbitrary* $T'\in\mathfrak{R}^0_{l.f}/T$, *there exists some* $\sigma'\in\mathfrak{G}(T'/S, R)$ *such that* $\sigma=\sigma'\,|\,T$.

(ii) $\tilde{\mathfrak{G}}(T/S, R)$ *is a* $\langle V\rangle$*-subset of* $\mathfrak{G}(T/S, R)$.

Lemma 12. *Let R be left locally finite over S, and let* $T\in\mathfrak{R}^0_{l.f}$. *Then*

(i) *There exists some* $T^*\in\mathfrak{R}^0_{l.f}/T$ such that $\mathfrak{G}(T'/S, R)\,|\,T=\tilde{\mathfrak{G}}(T/S, R)$ *for every* $T'\in\mathfrak{R}^0_{l.f}/T^*$.

(ii) $\tilde{\mathfrak{G}}(T/S, R)=\tilde{\mathfrak{G}}(T'/S, R)\,|\,T$ *for every* $T'\in\mathfrak{R}^0_{l.f}/T$.

(iii) $\sigma\tilde{\mathfrak{G}}(T\sigma/S, R)\subset\tilde{\mathfrak{G}}(T/S, R)$ *for every* $\sigma\in\tilde{\mathfrak{G}}(T/S, R)$.

Proof. (i): By Corollary 2 (i), we have

$$\tilde{\mathfrak{G}}(T/S, R)=\mathfrak{G}(T_1/S, R)\,|\,T\cap\cdots\cap\mathfrak{G}(T_n/S, R)\,|\,T$$

for some finite subfamily $\{T_i\}\subset\mathfrak{R}^0_{l.f}/T$. We set $T^*=[T_1, \cdots, T_n]$. Then $T^*\in\mathfrak{R}^0_{l.f}/T$. Let $T'\in\mathfrak{R}^0_{l.f}/T^*$. Then $\tilde{\mathfrak{G}}(T/S, R)\subset\mathfrak{G}(T'/S, R)\,|\,T$ (by Definition). On the other hand, by Lemma 2 (i), we have $\mathfrak{G}(T'/S, R)\,|\,T= (\mathfrak{G}(T'/S, R)\,|\,T_i)\,|\,T\subset\mathfrak{G}(T_i/S, R)\,|\,T$ for every i, so that $\mathfrak{G}(T'/S, R)\,|\,T\subset \cap_{1\leq i\leq n}\mathfrak{G}(T_i/S, R)\,|\,T=\tilde{\mathfrak{G}}(T/S, R)$. Therefore $\mathfrak{G}(T'/S, R)\,|\,T=\tilde{\mathfrak{G}}(T/S, R)$. (ii): Let $T'\in\mathfrak{R}^0_{l.f}/T$. Then, from (i) we can find $T''\in\mathfrak{R}^0_{l.f}/T'$ such that $\mathfrak{G}(T''/S, R)\,|\,T=\tilde{\mathfrak{G}}(T/S, R)$ and $\mathfrak{G}(T''/S, R)\,|\,T'=\tilde{\mathfrak{G}}(T'/S, R)$. Hence, we have $\tilde{\mathfrak{G}}(T'/S, R)\,|\,T=(\mathfrak{G}(T''/S, R)\,|\,T')\,|\,T=\mathfrak{G}(T''/S, R)\,|\,T=\tilde{\mathfrak{G}}(T/S, R)$. (iii): Let $\sigma\in\tilde{\mathfrak{G}}(T/S, R)$, and $T'\in\mathfrak{R}^0_{l.f}/T$. Then $\sigma=\sigma'\,|\,T$ for some $\sigma'\in\tilde{\mathfrak{G}}(T'/S, R)$ (by (ii)). Let $\tau\in\tilde{\mathfrak{G}}(T\sigma/S, R)$, and set $T''=T'\sigma'$. Clearly $T''\in\mathfrak{R}^0_{l.f}$ and $T''\supset T\sigma$. Then $\tau=\tau''\,|\,T\sigma$ for some $\tau''\in\tilde{\mathfrak{G}}(T''/S, R)$ (by (ii)), and then $\sigma'\tau''\,|\,T=\sigma(\tau''\,|\,T\sigma)=\sigma\tau$, $\sigma'\tau''\in\mathfrak{G}(T'/S, R)$. Hence, it follows from Lemma 11 (i) that $\sigma\tau\in\tilde{\mathfrak{G}}(T/S, R)$.

Lemma 13. *Let R be left locally finite over S, and $J(\mathfrak{G}(T/S, R), T) = S$ for every $T \in \mathfrak{R}^0_{l.f}$. Then $J(\tilde{\mathfrak{G}}(T/S, R), T) = S$ for every $T \in \mathfrak{R}^0_{l.f}$.*

Proof. Let $T \in \mathfrak{R}^0_{l.f}$, and let T^* be as in Lemma 12 (i). Then $\mathfrak{G}(T^*/S, R) \mid T = \tilde{\mathfrak{G}}(T/S, R)$. Since $S = J(\mathfrak{G}(T^*/S, R), T^*)$, we have $S = J(\mathfrak{G}(T^*/S, R) \mid T, T) = J(\tilde{\mathfrak{G}}(T/S, R), T)$.

From Lemma 11 (ii), Lemma 12 (ii, iii) and Lemma 13, we have the following

Lemma 14. *Let R be left locally finite over S. If $J(\mathfrak{G}(T/S, R), T) = S$ and $(H \cap T)\tilde{\mathfrak{G}}(T/S, R) \subset H$ for every $T \in \mathfrak{R}^0_{l.f}$ then $\tilde{\mathfrak{G}} = \{\tilde{\mathfrak{G}}(T/S, R); T \in \mathfrak{R}^0_{l.f}\}$ is a left q-system.*

Combining Theorem 3 with Lemma 14, we obtain the following

Theorem 5. *The following conditions* (V_l) —— (V''_r) *are all equivalent. Moreover, R is q-Galois and locally finite over S if and only if one of these is satisfied.*

(V_l): (i) *$J(\mathfrak{G}(T/S, R), T) = S$ for every $T \in \mathfrak{R}^0_{l.f}$, and*

(ii) *$(H \cap T)\tilde{\mathfrak{G}}(T/S, R) \subset H$ (or $(H \cap T)\mathfrak{G}(T/S, R) \subset H$) for every $T \in \mathfrak{R}^0_{l.f}$, and*

(iii) *R is $S \cdot V$-R-irreducible, and*

(iv) *R is left locally finite over S.*

(V'_l): (V_l : i, ii, iv) *plus the condition that R is R-$S \cdot V$-irreducible.*

(V''_l): (V_l : i, ii, iv) *plus the condition that S, $H \in \mathfrak{R}$ and $[V_R^2(T) : H]_l = [V : V_R(T)]_r$ for every $T \in \mathfrak{R}^0_{l.f}$.*

(V_r). (V'_r). (V''_r). (These are symmetric to (V_l) —— (V''_l)).

Now, by C_0 we denote the center of V, and we shall prove the next

Theorem 6. *Let R be left locally finite over S, $S \in \mathfrak{R}$, and $[V : C_0] < \infty$. If $J(\mathfrak{G}(T/S, R), T) = S$ for every $T \in \mathfrak{R}^0_{l.f}$ then R is q-Galois and locally finite over S.*

Proof. Let $\{v_i\}$ be a C_0-basis of V which is a finite subset of V, and let $T \in \mathfrak{R}^0_{l.f}$. We choose a $T' \in \mathfrak{R}^0_{l.f}/[T, \{v_i\}]$. Then we have $[V : V_R(T')]_r \leqq [T' : S]_l < \infty$ by Lemma 1 (iv). Moreover, we can see that $V_R^2(T') \supset V$ and $V_R(T')$ is the center of $V_R^2(T')$. Hence, by the fundamental theorem of simple rings, H is simple and $[V_R^2(T') : H] = [V : V_R(T')]$. Moreover, from $V \supset V_R(T) \supset V_R(T')$, we have $[V_R^2(T') : V_R^2(T)] = [V_R(T) : V_R(T')]$. Hence, we obtain $[V_R^2(T) : H] = [V : V_R(T)]$. Therefore by Theorem 5 (V''_l), it suffices to prove that $(H \cap T)\tilde{\mathfrak{G}}(T/S, R) \subset H$ for every $T \in \mathfrak{R}^0_{l.f}$. Since $S \cdot V \cong S \times_{S \cap V} V$ (tensor product) and $S \cdot V$ is left locally finite over S, V is locally finite over $S \cap V$ ($S \cap V \subset C_0$). Let $\{g_{pq}\}$ be a system of matrix units of V such that $V_V(\{g_{pq}\})$ is a division ring, and set

$V^*=[S\cap V, \{v_i\}, \{g_{pq}\}]$. Then $V^*\supset V^*\cap C_0\supset S\cap V$, $[V^*:S\cap V]<\infty$, $V=V^*\cdot C_0\cong V^*\times_{V^*\cap C_0}C_0$ (tensor product), and $V^*\cap C_0$ is the center of V^*. Let $T\in\mathfrak{R}^0_{l.f}$, and we assume that $h\sigma\notin H$ for some $h\in H\cap T$ and for some $\sigma\in\tilde{\mathfrak{G}}(T/S, R)$. Then, there exists some $v\in V$ such that $(h\sigma)v\neq v(h\sigma)$. Now, we choose a $T_1\in\mathfrak{R}^0_{l.f}/[T, V^*]$. Then $\sigma=\sigma_1|T$ for some $\sigma_1\in\tilde{\mathfrak{G}}(T_1/S, R)$ (Lemma 12 (ii)). Clearly $V^*\sigma_1\subset V$, $V_V(V^*\sigma_1)\supset C_0$ and $V^*\sigma_1\cap V_V(V^*\sigma_1)$ $=$the center of $V^*\sigma_1=$(the center of $V^*)\sigma_1=(V^*\cap C_0)\sigma_1$. Hence $[V: C_0]\geqq [V^*\sigma_1\cdot V_V(V^*\sigma_1): V_V(V^*\sigma_1)]=[V^*\sigma_1:(V^*\cap C_0)\sigma_1]=[V^*: V^*\cap C_0]=[V^*\cdot C_0: C_0]=[V: C_0]$. This implies that

$$V_V(V^*\sigma_1)=C_0,\ \text{and}\ V^*\sigma_1\cdot C_0=V.$$

Since $V_{T_1}(V^*)\ni h$, we have $V_{T_1\sigma_1}(V^*\sigma_1)\ni h\sigma_1=h\sigma$. Hence, from $V^*\sigma_1\cdot C_0=V\ni v$ and $(h\sigma)v\neq v(h\sigma)$, there exists some $c\in C_0$ such that $(h\sigma)c\neq c(h\sigma)$. By field theory, we can find a subring N of C_0 such that $N\supset[V^*\cap C_0, c]$, $[N: V\cap S]<\infty$ and $\mathfrak{G}(N/V\cap S, C_0)=\mathfrak{G}(N/V\cap S, N)=\mathfrak{G}(V\cap S, N)$. We choose a $T_2\in\mathfrak{R}^0_{l.f}/[T_1, N]$. Then $\sigma_1=\sigma_2|T_1$ for some $\sigma_2\in\tilde{\mathfrak{G}}(T_2/S, R)$ (Lemma 12 (ii)), and then $N\sigma_2\subset(C_0\cap T_2)\sigma_2\subset V\cap V_{T_2\sigma_2}(V^*\sigma_2)=V_V(V^*\sigma_2)=V_V(V^*\sigma_1)=C_0$. Hence $\sigma_2|N\in\mathfrak{G}(V\cap S, N)$. Therefore, noting that $c\in N$, there exists some $c'\in N$ such that $c'\sigma_2=c$. Since $(h\sigma)c\neq c(h\sigma)$ and $h\sigma=h\sigma_1=h\sigma_2$, it follows that $(h\sigma_2)(c'\sigma_2)\neq(c'\sigma_2)(h\sigma_2)$; i. e., $(hc')\sigma_2\neq(c'h)\sigma_2$, which implies that $hc'\neq c'h$. However, this is a contradiction.

Remark 5. Let R be as in Theorem 6; i. e., let R be left locally finite over S, $S\in\mathfrak{R}$, $[V: C_0]<\infty$, and $J(\mathfrak{G}(T/S, R), T)=S$ for every $T\in\mathfrak{R}^0_{l.f}$. Then R is locally Galois over S by Corollary 15 which will be proved later.

Remark 6. If R is Galois over S then for every $T\in\mathfrak{R}^0_{l.f}$, $\mathfrak{G}(S, R)|T$ is a $\langle V\rangle$-subset of $\mathfrak{G}(T/S, R)$, and $\{\mathfrak{G}(S, R)|T; T\in\mathfrak{R}^0_{l.f}\}$ satisfies the conditions (α)—(δ) (the condition (δ) follows the fact that $V\sigma=V$ and $H\sigma\subset H$ for every $\sigma\in\mathfrak{G}(S, R)$). Hence $\{\mathfrak{G}(S, R)|T;\ T\in\mathfrak{R}^0_{l.f}\}$ is a left q-system.

Remark 7. We set $\mathfrak{G}=\mathfrak{G}(S, R)$, and we shall consider the following conditions (A_l)——(B''_r) as in [4].

(A_l): (i) *$S\in\mathfrak{R}$ and $\mathfrak{G}R_r$ is dense in* $\mathrm{Hom}_{S_l}(R, R)$ *in the finite topology, and*

(ii) *R is left locally finite over S.*

(B_l): (i) *R is Galois over S, and*

(ii) *R is $S\cdot V$-R-irreducible, and*

(iii) *R is left locally finite over S.*

(B'_l): (B_l: i, iii) *plus the condition that R is $S\cdot V$-R-irreducible.*

(B''_l): (B_l : i, iii) *plus the condition that S, $H\in\mathfrak{R}$ and* $[V^2_R(T):H]_l=$

$[V: V_R(T)]_r$ *for every* $T\in\mathfrak{R}^0_{l.f}$.

(A_r). (B_r). (B'_r). (B''_r). (These are symmetric to (A_l) —— (B''_l)).

Then, the conditions (A_l) —— (B''_r) are all equivalent. The proof is as follows: By Remark 6 and Theorem 3, the conditions (B_l), (B'_l), (B''_l), (B_r), (B'_r) and (B''_r) are all equivalent. Moreover, we have $(B_l)\Longrightarrow(III_l)\Longrightarrow(II_l)\Longrightarrow(A_l)$. Conversely, from (A_l), we have that $(A_l)\Longrightarrow(II_l)\Longrightarrow(III_l)$ and $\mathfrak{G}\,|\,T=\mathfrak{G}(T/S, R)$ for every $T\in\mathfrak{R}^0_{l.f}$ (Theorem 1), and from this the condition (B_l) follows. Therefore we obtain that $(A_l)\Longleftrightarrow(B_l)$, and symmetrically $(A_r)\Longleftrightarrow(B_r)$. This completes the proof, (in [4], we have showed a direct proof of the above result, which is very complicated). If one of the conditions (A_l)——(B''_r) is satisfied then we say that R is *h-Galois and locally finite* over S. As in the above proof, if R is h-Galois and locally finite over S then R is q-Galois and locally finite over S, (in [3], this result has been proved directly). Moreover, the h-Galois theory has been studied in [5], [6] and [7].

Remark 8. Let R be h-Galois and locally finite over S. If $\mathfrak{G}'$ is a semi-subgroup of $\mathfrak{G}(S, R)$ such that $J(\mathfrak{G}', R)=S$ and $\mathfrak{G}'\langle V\rangle\subset\mathfrak{G}'$ then $\{\mathfrak{G}'\,|\,T\,;\,T\in\mathfrak{R}^0_{l.f}\}$ is a left q-system. Hence, by Theorem 3, we have that $\mathfrak{G}'\,|\,S'=(\mathfrak{G}'\,|\,T)\,|\,S'=\mathfrak{G}(S'/S, R)$ for every $S'\in\mathfrak{R}_{l.f}$ and for every $T\in\mathfrak{R}^0_{l.f}/S'$. This is an extension theorem.

Remark 9. Let R be q-Galois and locally finite over S. Then, from Lemma 8 and Theorem 3, it follows that H is outer Galois over S, and for each $S'\in\mathfrak{R}_{l.f}$ and for every subring H'' of $H\cap S'$ containing S, the following hold: $H''\in\mathfrak{R}_{l.f}$; $H''\mathfrak{G}(S'/S, R)\subset H$; $\mathfrak{G}(S'/S, R)\,|\,H''=\mathfrak{G}(H''/S, R)=\mathfrak{G}(H''/S, H)=\mathfrak{G}(S, H)\,|\,H''$; $[H'':S]=\#\mathfrak{G}(S'/S, R)\,|\,H''$, (on the other hand, this results follow [3, Theorem 4]).

Now, for each subring A of R containing S and for every $S'\in\mathfrak{R}/A$, we denote by $\mathfrak{G}(S'/A, R)$ the set $\{\sigma\in\mathfrak{G}(S'/S, R)\,;\,\sigma\,|\,A=1\}$. Clearly $\mathfrak{G}(S'/A, R)$ is the set of all A-(ring) isomorphisms of S' into R whose images belong to $\mathfrak{R}$.

Lemma 15. *Let R be q-Galois and locally finite over S. If A is a subring of R containing S which is left finite over S, then*

$$\mathfrak{G}(S_1/A, R)\,|\,S_2=\mathfrak{G}(S_2/A, R)$$

for every S_1, $S_2\in\mathfrak{R}_{l.f}/A$ such that $S_1\supset S_2$.

Proof. Let $S_1, S_2\in\mathfrak{R}_{l.f}/A$ and $S_1\supset S_2$. Then, by Theorem 2 (ii), we have $\mathfrak{G}(S_1/S, R)\,|\,S_2=\mathfrak{G}(S_2/S, R)$. Since $\mathfrak{G}(S_1/A, R)\subset\mathfrak{G}(S_1/S, R)$, we obtain $\mathfrak{G}(S_1/A, R)\,|\,S_2\subset\mathfrak{G}(S_1/S, R)\,|\,S_2=\mathfrak{G}(S_2/S, R)$. This implies that $\mathfrak{G}(S_1/A, R)\,|\,S_2\subset\mathfrak{G}(S_2/A, R)$. Conversely, from $\mathfrak{G}(S_2/A, R)\subset\mathfrak{G}(S_2/S, R)=\mathfrak{G}(S_1/S, R)\,|\,S_2$, we have $\mathfrak{G}(S_2/A, R)\subset\mathfrak{G}(S_1/A, R)\,|\,S_2$. Therefore, it follows that $\mathfrak{G}(S_1/A,$

$R)|S_2=\mathfrak{G}(S_2/A, R)$.

Lemma 16. *Let R be q-Galois and locally finite over S. If A is a subring of R containing S such that $[A:S]_l<\infty$ and R is A-R-irreducible, then $J(\mathfrak{G}(T/A, R), T)=A$ for every $T\in\mathfrak{R}^0_{l.f}/A$.*

Proof. Let $T\in\mathfrak{R}^0_{l.f}/A$, and set $A'=J(\mathfrak{G}(T/A, R), T)$. Then $T\supset A'\supset A\supset S$ and $V_R(A')=V_R(A)$ (a division ring). Clearly R is A'-R-irreducible. Hence, by Lemma 3 (iii, v), we have

(1) $\mathrm{Hom}_{S_l}(A', R)=\mathrm{Hom}_{S_l}(T,R)|A'=\mathfrak{G}(T/S,R)R_r|A'=(\mathfrak{G}(T/S,R)|A')R_r=$
$$=\textstyle\sum_{\oplus}(\sigma_i|A')\langle V\rangle R_r$$
for some finite subset $\{\sigma_i\}\subset\mathfrak{G}(T/S, R)$;

(2) $[(\sigma_i|A')\langle V\rangle R_r:R_r]_r=[V:V_R(A')]_r$

for every σ_i. Moreover, we have

(3) $\mathrm{Hom}_{S_l}(A, R)=\mathrm{Hom}_{S_l}(A', R)|A=\sum(\sigma_i|A)\langle V\rangle R_r$

for the same $\{\sigma_i\}$ as in (1);

(4) $[(\sigma_i|A)\langle V\rangle R_r:R_r]_r=[V:V_R(A)]_r=[V:V_R(A')]_r$

for every σ_i. We assume that $A'\neq A$. Then, we have $[\mathrm{Hom}_{S_l}(A', R):R_r]_r>[\mathrm{Hom}_{S_l}(A, R):R_r]_r$. From this and (1)——(4), the right hand side of (3) is not a direct sum. Hence, we can find some σ_j, $\sigma_k\in\{\sigma_i\}$ such that $\sigma_j\neq\sigma_k$ and $(\sigma_j|A)R_r\cong(\sigma_k|A)R_r$ (A_r-R_r-isomorphism), so that $\sigma_j|A=(\sigma_k|A)\cdot\langle v\rangle$ for some regular $v\in V$ (Lemma 3 (i, ii)). We choose a $T^*\in\mathfrak{R}^0_{l.f}/[T, T\sigma_j, T\sigma_k, v]$. Then $\sigma_j^{-1}=\sigma^*|T\sigma_j$ for some $\sigma^*\in\mathfrak{G}(T^*/S, R)$, and then, we have $1|A=(\sigma_j|A)\sigma^*=(\sigma_k|A)\langle v\rangle\sigma^*$, so that $1|A'=(\sigma_j|A')\sigma^*=(\sigma_k|A')\langle v\rangle\sigma^*$, (noting that $\sigma_j\sigma^*$, $\sigma_k\langle v\rangle\sigma^*\in\mathfrak{G}(T/A, R)$); whence $(\sigma_j|A')=(\sigma_k|A')\langle v\rangle$. However, this contradicts (1). Therefore we obtain $A'=A$.

Corollary 6. *Let R be q-Galois and locally finite over S. If $S'\in\mathfrak{R}_{l.f}$ then $J(\mathfrak{G}(T/S', R), T)=S'$ for every $T\in\mathfrak{R}^0_{l.f}/S'$.*

Proof. Let $T\in\mathfrak{R}^0_{l.f}/S'$, and set $V_R(S')=\sum Wg_{pq}$, where $\{g_{pq}\}$ is a system of matrix units of $V_R(S')$, and W is the centralizer of $\{g_{pq}\}$ in $V_R(S')$ which is a division ring. Moreover, we set $S^*=\sum S'g_{pq}$, and choose a $T^*\in\mathfrak{R}^0_{l.f}/\{T, \{g_{pq}\}]$. Then $S^*\subset T^*$, $S^*\in\mathfrak{R}_{l.f}$, and $V_R(S^*)=W$ (a division ring); whence R is S^*-R-irreducible by Remark 2 (i) and Lemma 1 (ii, a). Therefore by Lemma 16, we have $J(\mathfrak{G}(T^*/S^*, R), T^*)=S^*$. Moreover, noting that $V_{S^*}(S')=\sum V_{S'}(S')g_{pq}$ (simple) and $V^2_{S^*}(S')=S'$, we have $J(\mathfrak{G}(S^*/S', R), S^*)=S'$. Since $\mathfrak{G}(T^*/S^*, R)\subset\mathfrak{G}(T^*/S', R)$ and $\mathfrak{G}(S^*/S', R)=\mathfrak{G}(T^*/S', R)|S^*$ (Lemma 15), it follows that $J(\mathfrak{G}(T^*/S', R), T^*)=S'$, so that $J(\mathfrak{G}(T^*/S', R)|T, T)=S'$. Hence, from $\mathfrak{G}(T^*/S', R)|T=\mathfrak{G}(T/S', R)$ (Lemma 15), we obtain $J(\mathfrak{G}(T/S', R), T)=S'$.

Theorem 7. *Let R be q-Galois and locally finite over S. If $S'\in\mathfrak{R}_{l.f}$*

then $J(\mathfrak{G}(S''/S', R), S'')=S'$ *for every* $S''\in\mathfrak{R}_{l,f}/S'$.

Proof. We choose a $T\in\mathfrak{R}_{l,f}^{0}/S''$. Then $\mathfrak{G}(S''/S', R)=\mathfrak{G}(T/S', R)|S''$ (Lemma 15), and $J(\mathfrak{G}(T/S', R), T)=S'$ (Corollary 6). Hence, it follows that $J(\mathfrak{G}(S''/S', R), S'')=J(\mathfrak{G}(T/S', R)|S'', S'')=S'$.

Corollary 7. *Let R be q-Galois and locally finite over S. If* $S'\in\mathfrak{R}_{l,f}/\{d_{hk}\}$ *then* S' *is linearly disjoint from H over* $H\cap S'$.

Proof. For every $T\in\mathfrak{R}_{l,f}^{0}/S'$, we have $J(\mathfrak{G}(T/S', R), T)=S'$ (Corollary 6), so that $H\cap J(\mathfrak{G}(T/S', R), T)=H\cap S'$. Hence, the same arguement as in the proof of Lemma 9 (i) enables us to prove our assertion.

Now, combining the condition (IV_l) with Lemma 15, Corollary 6 and Remark 2 (i), we obtain the following

Theorem 8. *Let R be q-Galois and locally finite over S. Then, for every* $S'\in\mathfrak{R}_{l,f}$, *R is q-Galois and locally finite over* S'.

Lemma 17. *Let R be q-Galois and locally finite over S. Then*
(i) *If* H', N *are subrings of H containing S and if N is Galois and finite over* $H'\cap N$, *then* $\mathfrak{G}(H', H)|N=\mathfrak{G}(H'\cap N, N)$.
(ii) *If* $S'\in\mathfrak{R}_{l,f}$ *and* H^* *is a subring of H containing* $H\cap S'$ *which is finite over S, then* $[H^*, S']\in\mathfrak{R}_{l,f}$, *and we have*

(a) $[[H^*, S']:S']=[H^*:H\cap S']$,

(b) $[[H^*, S']:H^*]_l=[S':H\cap S']_l$,

(c) $H\cap[H^*, S']=H^*$,

(d) $\mathfrak{G}([H^*, S']/S', R)|H^* = \mathfrak{G}(H^*/H\cap S', R) = \mathfrak{G}(H^*/H\cap S', H)= \mathfrak{G}(H\cap S', H)|H^*$.

Proof. (i): Since H is outer Galois over S (Lemma 8, and Remark 9), H is outer Galois over H' and over $H'\cap N$. Noting that $J(\mathfrak{G}(H', H), H)=H'$ and $N\mathfrak{G}(H', H)\subset N$, we have $\mathfrak{G}(H', H)|N=\mathfrak{G}(H'\cap N, N)$ as desired (by outer Galois theory). (ii): Set $R'=V_R^2(S')$. Then R' is outer Galois over S' and $[H^*, S']\in\mathfrak{R}_{l,f}$ (Theorem 8, Lemma 8, and Remark 9). We choose a subring N of H containing H^* which is Galois and finite over $H\cap S'$. Then, from $H\mathfrak{G}(S', R')\subset H$, it follows that $N\mathfrak{G}(S', R')\subset N$ and $J(\mathfrak{G}(S', R')|N, N)=N\cap S'=H\cap S'$. Hence, we have

(1) $\mathfrak{G}(S', R')|N=\mathfrak{G}(H\cap S', N)$.

Furthermore, from $[N, S']\mathfrak{G}(S', R')\subset[N, S']$ and $J(\mathfrak{G}(S', R')|[N, S'], [N, S'])=S'$, it follows that $[N, S']$ is outer Galois over S', and

(2) $\mathfrak{G}(S', R')|[N, S']=\mathfrak{G}(S', [N, S'])$.

Therefore by (1, 2) and outer Galois theory, we have

(3) $[H^*:H\cap S']=\sharp(\mathfrak{G}(H\cap S', N)|H^*=\sharp(\mathfrak{G}(S', R')|N)|H^*=\sharp\mathfrak{G}(S', R')|H^*$

$$=\sharp\mathfrak{G}(S', R')\,|\,[H^*, S'] = \sharp(\mathfrak{G}(S', R')\,|\,[N, S'])\,|\,[H^*, S']$$
$$=\sharp\mathfrak{G}(S', [N, S'])\,|\,[H^*, S'] = [[H^*, S'] : S'].$$

Hence we obtain (a). Moreover, (b) is a direct consequence of (a). If we set $H^{*\prime}=H\cap[H^*, S']$, then $H^{*\prime}\supset H^*\supset H\cap S'$ and $[H^{*\prime}: H\cap S]<\infty$. Then, we also have

(4) $[H^{*\prime}: H\cap S'] = [[H^{*\prime}, S'] : S']$.

From $[H^*, S'] = [H^{*\prime}, S']$ and (3, 4), we obtain $H^{*\prime}=H^*$ which shows (c). Since $\mathfrak{G}([H^*, S']/S', R)=\mathfrak{G}([H^*, S']/S', R')=\mathfrak{G}(S', R')\,|\,[H^*, S']$ (Theorem 8, and Remark 9), it follows from (1) that $\mathfrak{G}([H^*, S']/S', R)\,|\,H^*=\mathfrak{G}(S', R')\,|\,H^*=\mathfrak{G}(H\cap S', N)\,|\,H^*=\mathfrak{G}(H\cap S', H)\,|\,H^*= \mathfrak{G}(H^*/H\cap S', H) = \mathfrak{G}(H^*/H\cap S', R)$.

Remark 10. Let R be q-Galois and locally finite over S. Then, by Lemma 17 (ii), we have the following: If $S'\in\mathfrak{R}_{l.f}$ and H^* is a subring of H containing $H\cap S'$, then $H\cap[H^*, S']=H^*$ and $[[H^*, S'] : H^*]_l\leqq[S': H\cap S']_l<\infty$.

In [3], the following Corollary 8, Theorem 9 and Corollary 9 have been shown under the assumption that R is h-q-Galois and left locally finite over S.

Corollary 8. *Let R be q-Galois and locally finite over S. Then, for every subring H' of H containing S, R is locally finite over H'.*

Proof. Let H' be a subring of H containing S, and F a finite subset of R. We choose a $S^*\in\mathfrak{R}_{l.f}/[S, F]$, and set $H^*=[H', H\cap S^*]$. Then, we have $[[H^*, S^*]: H^*]_l\leqq[S^*: H\cap S^*]_l<\infty$ by Remark 10. Noting that $[H^*: H']<\infty$, we obtain $[[H^*, S^*]: H']_l<\infty$; this means that $[[H', F]: H']_l<\infty$. Similarly we have $[[H', F] : H']_r<\infty$.

Theorem 9. *Let R be q-Galois and locally finite over S. If $S'\in\mathfrak{R}_{l.f}$ then $[S' : S]_l = [S': S]_r\geqq[S': H\cap S']_l = [S' : H\cap S']_r = [V: V_R(S')]_l = [V: V_R(S')]_r = [V_R^2(S') : H]_l = [V_R^2(S'): H]_r$ and $V_R^2(S') = [H, S']$.*

Proof. If we set $H^*=[H\cap S', \{d_{hk}\}]$, then by Corollary 7, $[H^*, S']$ is linearly disjoint from H over $H\cap[H^*, S']$. From this and Lemma 17(ii), we have $[[H, S']: H]_l\geqq[H\cdot[H^*, S']: H]_l = [[H^*, S'] : H\cap[H^*, S']]_l = [[H^*, S'] : H^*]_l = [S' : H\cap S']_l$. On the other hand, since R is $S'\cdot V_R(S')$-R-irreducible and R-$S\cdot V$-irreducible, it follows that $[S' : H\cap S']_l\geqq[V: V_R(S')]_r\geqq[V_R^2(S') : H]_l\geqq[[H, S'] : H]_l$ (Lemma 1). We have therefore that $V_R^2(S') = [H, S']$ and $[S': H\cap S']_l = [V: V_R(S')]_r = [V_R^2(S') : H]_l$. Similarly we have $[S' : H\cap S']_r = [V : V_R(S')]_l = [V_R^2(S') : H]_r$. Since $[H\cap S' : S]_l = [H\cap S' : S]_r$, it suffices to prove that $[S' : H\cap S']_l = [S' : H\cap S']_r$. Noting that R is $S'\cdot V_R(S')$-R-irreducible, we have $[S' : H\cap S']_l\geqq[V : V_R(S')]_l$ by Lemma 1 (iv, a). Hence $[S' : H\cap S']_l\geqq[S': H\cap S']_r$. Similarly we have

$[S' : H\cap S']_l \leqq [S' : H\cap S']_r$. Therefore we obtain $[S' : H\cap S']_l = [S' : H\cap S']_r$.

Now, for $S' \in \mathfrak{R}$, if $[V : V_R(S')]_r < \infty$ then we say that S' is *f-regular*. Let R be q-Galois and locally finite over S. If $S' \in \mathfrak{R}_{l.f}$ then S' is f-regular (Theorem 9). Let $S' \in \mathfrak{R}$. If S' is left finite (or right finite) over a subring H^* of $H\cap S'$ containing S, then S' is f-regular. In fact, if $\{d_1, \cdots, d_n\}$ $(n<\infty)$ is a left H^*-basis of S' and if $\{f_{uv}\}$ is a system of matrix units of S' such that $V_{S'}(\{f_{uv}\})$ is a division ring, then $S^* = [S, \{d_i\}, \{f_{uv}\}] \in \mathfrak{R}_{l.f}$, $[H^*, S^*] = S'$, and $V_R(S^*) = V_R(S')$; this implies that S' is f-regular. Clearly, for each finite subset F of S', we have that $[S^*, F] \in \mathfrak{R}_{l.f}$, and $[H^*, [S^*, F]] = S'$. Conversely, if S' is f-regular, then there exists some $S^* \in \mathfrak{R}_{l.f}$ such that $S' \supset S^*$ and $V_R(S') = V_R(S^*)$. Then, by Theorem 8 and Remark 9, $V_R^2(S^*)$ is outer Galois over S^*. Moreover, we have $V_R^2(S^*) \supset S' \supset S^*$. Hence, for each finite subset F of S', we have $[S^*, F] \in \mathfrak{R}_{l.f}$, and by Theorem 9, we obtain $[[S^*, F] : H\cap [S^*, F]] = [V : V_R([S^*, F])] = [V : V_R(S^*)] = [S^* : H\cap S^*]$. From this, it follows that $[S' : H\cap S']_l$, $[S' : H\cap S']_r \leqq [S^* : H\cap S^*] = [V : V_R(S^*)] = [V : V_R(S')] < \infty$. Hence, we have the following

Corollary 9. *Let R be q-Galois and locally finite over S. Let $S' \in \mathfrak{R}$. If $S' \in \mathfrak{R}_{l.f}$ then S' is f-regular. In general, if S' is left finite (or right finite) over a subring H^* of $H\cap S'$ containing S, then S' is f-regular, and for each finite subset F of S', there exists some $S^* \in \mathfrak{R}_{l.f}$ such that $S^* \supset F$, and $[H^*, S^*] = S'$. Conversely, if S' is f-regular then* $[S' : H\cap S']_l$, $[S' : H\cap S']_r \leqq [V : V_R(S')] < \infty$.

Remark 11. Let A be a subring of R containing the identity element 1 of R. If there exists a direct set $\{A_i ; i \in I\}$ which consists of simple subrings A_i's of R containing 1 such that $A = \bigcup_{i\in I} A_i$, then A is a simple ring (cf. [6, Lemma 1.1]).

Now, we shall prove

Theorem 10. *Let R be q-Galois and locally finite over S. Then*
(i) *For a subring A of R containing S, the following conditions are equivalent to each other*:

(a) *R is A-R-irreducible, and $[A : S]_l < \infty$ (or $[A : S]_r < \infty$).*

(b) *$A \in \mathfrak{R}_{l.f}$, and $V_R(A)$ is a division ring.*

(ii) *The following conditions are equivalent to each other*:

(a′) *Every subring of R containing S is simple.*

(b′) *V is a division ring.*

Proof. (i): (b) $\Longrightarrow$ (a) is a direct consequence of Remark 2 (i) and Lemma 1 (ii, a). (a) $\Longrightarrow$ (b): By Lemma 1 (ii), $V_R(A)$ is a division ring,

$V_R^2(A)$ is a simple ring, and $[V: V_R(A)]_r \leqq [A:S]_l < \infty$. Since R is R-$S\cdot V$-irreducible, we have $[V_R^2(A):H]_l \leqq [V: V_R(A)]_r$ (Lemma 1 (i)). From this and $[H, A] \subset V_R^2(A)$, it follows that $[[H, A]:H]_l \leqq [V_R^2(A):H]_l \leqq [V: V_R(A)]_r$. On the other hand, since R is $[H, A]$-R-irreducible, we have $[[H, A]: H]_l \geqq [V_R(H): V_R([H, A])]_r = [V: V_R(A)]_r$ (Lemma 1 (ii)). Therefore $[H, A] = V_R^2(A)$ and this is simple. Let $\{f_{uv}\}$ be a system of matrix units of $[H, A]$ such that $V_{[H,A]}(\{f_{uv}\})$ is a division ring. Then $\{f_{uv}\} \subset [F, A]$ for some finite subset F of H, and by outer Galois theory, H contains a subring N such that $N \supset [F, S]$ and N is Galois and finite over S. We set $A' = [N, A]$. Then $A' \in \mathfrak{R}_{l.f}/A$. Now, we choose a $T \in \mathfrak{R}_{l.f}^0/A'$. Then, we have that $J(\mathfrak{G}(T/A, R), T) = A$ (Lemma 16), $\mathfrak{G}(T/A, R) | A' = \mathfrak{G}(A'/A, R)$ (Lemma 15), and $N\mathfrak{G}(T/A, R) \subset N\mathfrak{G}(T/S, R) \subset N$. Hence, it follows that $J(\mathfrak{G}(A'/A, R), A') = A$ and $A'\mathfrak{G}(A'/A, R) \subset A'$ Moreover, we have $V_{A'}(A) = V_{A'}(A')$ (the center of A'). Therefore A' is outer Galois over A, and A is simple by outer Galois theory.
(ii): (a$'$) $\Longrightarrow$ (b$'$): For every non-zero element a of V, $[S, a]$ is simple, and so, the center of $[S, a]$ is a field. Since a is contained in the center of $[S, a]$, a is a regular element (of R). Hence V is a division ring. (b$'$) $\Longrightarrow$ (a$'$): Since R is $S\cdot V$-R-irreducible and V is a division ring, R is S-R-irreducible by Lemma 1 (ii, a). Hence, for each finite subset F of R, R is $[S, F]$-R-irreducible, and so, $[S, F]$ is simple by (i). If A is a subring of R containing S, then $A = \bigcup_F [S, F]$ (F runs over all the finite subsets of A), and then, A is simple by Remark 11.

Corollary 10. *Let R be q-Galois and locally finite over S. Then, for a subring A of R containing S, the following conditions are equivalent to each other:*

(a) *R is A-R-irreducible, and $[V: V_R(A)]_r < \infty$ (or $[V: V_R(A)]_l < \infty$).*

(b) *$A \in \mathfrak{R}$, A is f-regular, and $V_R(A)$ is a division ring.*

Proof. (b) $\Longrightarrow$ (a): By Corollary 9, we can find some $S^* \in \mathfrak{R}_{l.f}$ such that $[H \cap A, S^*] = A$. Then $V_R(S^*) = V_R([H \cap A, S^*]) = V_R(A)$, and this is a division ring. Hence, by Theorem 10 (i), A is S^*-R-irreducible. Therefore R is A-R-irreducible. (a) $\Longrightarrow$ (b): Since R is A-R-irreducible, $V_R(A)$ is a division ring, and $V_R^2(A)$ is a simple ring (Lemma 1 (ii)). First, we shall prove our assertion for the case $[V: V_R(A)]_r < \infty$. Since R is R-$S\cdot V$-irreducible, we have

(1) $[[H, A]:H]_l \leqq [V_R^2(A):H]_l \leqq [V: V_R(A)]_r < \infty$

by Lemma 1 (i). Moreover, since R is $[H, A]$-R-irreducible, we have

(2) $[V: V_R(A)]_r = [V: V_R([H, A])]_r \leqq [[H, A]:H]_l$

by Lemma 1 (ii). Combining (1) with (2), we obtain $[H, A] = V_R^2(A)$, and so, $[H, A]$ is f-regular. Hence, we can find some finite subset $F_1 \subset A$

and some finite subset $F_2 \subset H$ such that $V_R([S, F_1, F_2]) = V_R(A)$, and $[S, F_1, F_2] \in \mathfrak{R}_{l.f}$. We set $S^* = [S, F_1]$. Then R is $[S^*, F_2]$-R-irreducible (Theorem 10 (i)). Now, let a be a non-zero element of R. Since R is $S \cdot V$-R-irreducible, R is a completely reducible S-R-module. Noting that S^*aR is an S-R-submodule of R, we have $S^*aR = eR$ for some element e of V. For every $x \in S^*$ and for every $y \in F_2$, we have

$$xyeR = xeyR \subset eR, \text{ and}$$
$$yxeR = y(xeR) \subset yeR = eyR \subset eR.$$

This implies that eR is an $[S^*, F_2]$-R-module. Since R is $[S^*, F_2]$-R-irreducible, we obtain $eR = R$. Hence R is S^*-R-irreducible. Therefore, it follows from Theorem 10 (i) that $S^* \in \mathfrak{R}_{l.f}$. By Theorem 8 and Remark 9, $V_R^2(S^*)$ is outer Galois over S^*. Then, since $V_R^2(S^*) \supset A \supset S^*$, A is simple by outer Galois theory. For tne case $[V : V_R(A)]_l < \infty$, it suffices to prove that $[V : V_R(A)]_r < \infty$. Since R is $S \cdot V$-R-irreducible, we have $[V_R^2(A) : H]_r \leqq [V : V_R(A)]_l < \infty$ (Lemma 1 (i)). Hence we obtain $[V_R^2(A) : H]_l < \infty$ (Corollary 8). Since R is $V_R^2(A)$-R-irreducible, it follows that $[V : V_R(A)]_r \leqq [V_R^2(A) : H]_l < \infty$.

2. Extension Theorems

If R is q-Galois and left locally finite over S, then R is inner Galois over H (i. e., $J(\langle V \rangle, R) = H$, $\mathfrak{G}(H, R) =$ the closure of $\langle V \rangle$), and H is outer Galois over S. Clearly $\mathfrak{G}(S, R) | H \subset \mathfrak{G}(S, H)$. However, the equality $\mathfrak{G}(S, R) | H = \mathfrak{G}(S, H)$ is a problem in extension theory. If there holds that $\mathfrak{G}(S, R) | H = \mathfrak{G}(S, H)$, then R is Galois over S. In view of this point, the following extensions are important:

$$\mathfrak{G}(H'/S, R) \longrightarrow \mathfrak{G}([H', R']/S, R);$$
$$\mathfrak{G}(R'/S, R) \longrightarrow \mathfrak{G}([H', R']/S, R);$$
$$\mathfrak{G}(R'/S, R) \dashrightarrow \mathfrak{G}(R''/S, R),$$

where $S \subset H' \subset H$, $R' \subset R''$, $H', R', R'' \in \mathfrak{R}$. In this section, the main theme of our discussion will concern the above extensions and Galois extensions. Moreover, our discussion will be developed in a somewhat different way which is based on the following maps:

$$\mathfrak{G}(H'/S, R) \times_{H' \cap R'} \mathfrak{G}(R'/S, R), \quad S \subset H' \subset H, H', R' \in \mathfrak{R},$$

whose notation will be defined later.

First, we shall prove a lemma which contains the result of Remark 2 (ii) ([3, Lemma 5]).

Lemma 18. *Let R be q-Galois and locally finite over S. Then, for each $R' \in \mathfrak{R}$, the following hold:*

(i) *There exists a $U \in \mathfrak{R}_{l.f}$ such that $U \subset R'$ and $[U, F] \in \mathfrak{R}_{l.f}$ for every* *finite subset* F *of* R'.

(ii) $\mathfrak{G}(R'/S, R)|S' \subset \mathfrak{G}(S'/S, R)$ *for every* $S' \in \mathfrak{R}_{l.f}$ *such that* $S' \subset R'$.

(iii) $(H \cap R')\mathfrak{G}(R'/S, R) \subset H$. *In particular, if* $R' \subset H$ *then* $\mathfrak{G}(R'/S, R) = \mathfrak{G}(R'/S, H)$.

(iv) $[R', E] \in \mathfrak{R}$ *for every subset* E *of* H.

(v) *If* H^* *is a subring of* H *containing* $H \cap R'$ *then* $H \cap [H^*, R'] = H^*$.

Proof. (i): Let $\{f_{uv}\}$ be a system of matrix units of R' such that $V_{R'}(\{f_{uv}\})$ is a division ring, and we set $\mathfrak{F} = \{S' \in \mathfrak{R}_{l.f} / \{f_{uv}\}\ ;\ S' \subset R'\}$. Then, $\mathfrak{F}$ is non-empty by Remark 2 (ii), and one can see that $[R|R] \geqq [V_R(S_\iota)|V_R(S_\iota)]$ for every $S_\iota \in \mathfrak{F}$. Hence, we can find some $U \in \mathfrak{F}$ such that $[V_R(U)|V_R(U)] \leqq [V_R(S_\iota)|V_R(S_\iota)]$ for every $S_\iota \in \mathfrak{F}$. Now, we shall show that U satisfies the condition of (i). Clearly $U \subset R'$. Let A be a subring of R' containing U which is finite over U. Then A is a simple subring of R' which is finite over S. By Remark 2 (ii), there exists a $S_{\iota'} \in \mathfrak{F}$ such that $S_{\iota'} \supset A$. By the minimality of $[V_R(U)|V_R(U)]$, we have $[V_R(U)|V_R(U)] \leqq [V_R(S_{\iota'})|V_R(S_{\iota'})]$. On the other hand, from $V_R(U) \supset V_R(S_{\iota'})$, we have $[V_R(U)|V_R(U)] \geqq [V_R(S_{\iota'})|V_R(S_{\iota'})]$. Hence, we obtain $[V_R(U)|V_R(U)] = [V_R(S_{\iota'})|V_R(S_{\iota'})]$. From this and $V_R(U) \supset V_R(A) \supset V_R(S_{\iota'})$, it follows that $V_R(A)$ is simple. Therefore $A \in \mathfrak{R}_{l.f}$. (ii): Let $\sigma \in \mathfrak{G}(R'/S, R)$. Then $R'\sigma \in \mathfrak{R}$. Hence, by (i), we can choose a $U^* \in \mathfrak{R}_{l.f}$ such that $U^* \subset R'\sigma$ and $[U^*, F^*] \in \mathfrak{R}_{l.f}$ for every finite subset F^* of $R'\sigma$. Then, for each $S' \in \mathfrak{R}_{l.f}$ such that $S' \subset R'$, we have that $[U, U^*\sigma^{-1}, S'] \subset R'$, $[U\sigma, U^*, S'\sigma] \subset R'\sigma$, and $[U, U^*\sigma^{-1}, S']$, $[U\sigma, U^*, S'\sigma] \in \mathfrak{R}_{l.f}$. Hence $\sigma|[U, U^*\sigma^{-1}, S'] \in \mathfrak{G}([U, U^*\sigma^{-1}, S']/S, R)$. Therefore, it follows that $\sigma|S' = (\sigma|[U, U^*\sigma^{-1}, S'])|S' \in \mathfrak{G}([U, U^*\sigma^{-1}, S']/S, R)|S' = \mathfrak{G}(S'/S, R)$ (Theorem 2). (iii): This is a direct consequence of (i, ii) and Remark 9. (iv): Let F be an arbitrary finite subset of R'. Then $[U, F] \in \mathfrak{R}_{l.f}$, and so, $V_R^2([U, F])$ is outer Galois over $[U, F]$ by Theorem 8 and Remark 9. Let E be a subset of H. Then, noting that $V_R^2([U, F]) \supset [U, F, E] \supset [U, F]$, $[U, F, E]$ is simple by outer Galois theory. Since $[R', E] = \cup_F [U, F, E]$ (F runs over all the finite subsets of R'), it follows from Remark 11 that $[R', E]$ is simple. Moreover, $V_R([R', E]) = V_R(R')$ is simple. Therefore $[R', E] \in \mathfrak{R}$. (v): If $a \in H \cap [H^*, R']$ then $a \in [F_1, F_2]$ for some finite subset $F_1 \subset H^*$ and for some finite subset $F_2 \subset R'$. Set here $S' = [U, F_2]$. Then $S' \in \mathfrak{R}_{l.f}$. Hence, by Lemma 17 (ii), we have that $a \in H \cap [H \cap S', F_1, S'] = [H \cap S', F_1] \subset H^*$. Therefore, it follows that $H \cap [H^*, R'] \subset H^*$; whence $H \cap [H^*, R'] = H^*$.

Corollary 11. *Let* R *be q-Galois and locally finite over* S. *If* $S' \in \mathfrak{R}$ *and* S' *is f-regular, then* $\mathfrak{G}(R'/S, R)|S' \subset \mathfrak{G}(S'/S, R)$ *for every* $R' \in$

$\mathfrak{R}/S'$.

Proof. By Corollary 9, we can find some $S^* \in \mathfrak{R}_{l.f}$ such that $[H \cap S', S^*] = S'$. Let $R' \in \mathfrak{R}/S'$, and $\sigma \in \mathfrak{G}(R'/S, R)$. Then $(H \cap S')\sigma \subset (H \cap R')\sigma \subset H$ (Lemma 18 (iii)), and $\sigma | S^* \in \mathfrak{G}(S^*/S, R)$, i. e., $S^*\sigma \in \mathfrak{R}_{l.f}$ (Lemma 18(ii)). Hence $S'\sigma = [H \cap S', S^*]\sigma = [(H \cap S')\sigma, S^*\sigma] \in \mathfrak{R}$ (Lemma 18 (iv)). Therefore, it follows that $\sigma | S' \in \mathfrak{G}(S'/S, R)$.

Now, let R be q-Galois and locally finite over S. Let H^* be a subring of H containing S, and $R' \in \mathfrak{R}$. Then, we have again $[H^*, R'] \in \mathfrak{R}$ by Lemma 18 (iv). If $\sigma \in \mathfrak{G}(H^*/S, R)$, $\tau \in \mathfrak{G}(R'/S, R)$ and $\sigma | H^* \cap R' = \tau | H^* \cap R'$, then we denote by $\sigma \times_{H^* \cap R'} \tau$ (or $\sigma \times \tau$) the map of $[H^*, R']$ to $[H^*\sigma, R'\tau]$ given by

$$\sum h_{i_1} r_{i_1} h_{i_2} r_{i_2} \cdots h_{i_n} r_{i_n} \longrightarrow \sum (h_{i_1}\sigma)(r_{i_1}\tau)(h_{i_2}\sigma)(r_{i_2}\tau) \cdots (h_{i_n}\sigma)(r_{i_n}\tau),$$

where $h_{i_j} \in H^*$, $r_{i_j} \in R'$. This is not necessarily single-valued. Denote by $\mathfrak{G}(H^*/S, R) \times_{H^* \cap R'} \mathfrak{G}(R'/S, R)$ (or $\mathfrak{G}(H^*/S, R) \times \mathfrak{G}(R'/S, R)$) the set of such maps:

$$\{\sigma \times_{H^* \cap R'} \tau \,;\, \sigma \in \mathfrak{G}(H^*/S, R),\ \tau \in \mathfrak{G}(R'/S, R),\ \sigma | H^* \cap R' = \tau | H^* \cap R'\}.$$

With these notations, we have the following lemma which is useful in the sequel.

Lemma 19. *Let R be q-Galois and locally finite over S. Let H^* be a subring of H containing S, and $R' \in \mathfrak{R}$. Then*

(i) $\mathfrak{G}(H^*/S, R) = \mathfrak{G}(H^*/S, H)$.

(ii) $\mathfrak{G}([H^*, R']/S, R) \subset \mathfrak{G}(H^*/S, R) \times \mathfrak{G}(R'/S, R)$.

(iii) Let $\sigma \times \tau \in \mathfrak{G}(H^*/S, R) \times \mathfrak{G}(R'/S, R)$. *Then, the following hold:*

(a) *If, for every subring $H^{*\prime}$ of H^* containing S which is finite over S and for every $S' \in \mathfrak{R}_{l.f}$ such that $S' \subset R'$, $(\sigma \times \tau) | [H^{*\prime}, S']$ is a ring isomorphism, then $\sigma \times \tau$ is also a ring isomorphism;*

(b) *if $\sigma \times \tau$ is a ring isomorphism then $\sigma \times \tau \in \mathfrak{G}([H^*, R']/S, R)$.*

(iv) *If* $\mathfrak{G}([H^*, R']/S, R) = \mathfrak{G}(H^*/S, R) \times \mathfrak{G}(R'/S, R)$, *then the following hold:*

(a) $\mathfrak{G}([H^*, R']/S, R) | R' = \mathfrak{G}(R'/S, R)$;

(b) $\mathfrak{G}([H^*, R']/H^*, R) | R' = \mathfrak{G}(R'/H^* \cap R', R)$;

(c) $\mathfrak{G}([H^*, R']/R', R) | H^* = \mathfrak{G}(H^*/H^* \cap R', R)$;

(d) *if* $\mathfrak{G}(R'/S, R) | H^* \cap R' = \mathfrak{G}(H^* \cap R'/S, R)$ *then* $\mathfrak{G}([H^*, R']/S, R) | H^* = \mathfrak{G}(H^*/S, R)$, (*if* $[R' : S] < \infty$ *then* $\mathfrak{G}(R'/S, R) | H^* \cap R' = \mathfrak{G}(H^* \cap R'/S, R)$ *by Remark 9*).

Proof. (i) is a direct consequence of Lemma 18 (iii). (ii): If $\rho \in \mathfrak{G}([H^*, R']/S, R)$ then $H^*\rho \subset H$ by Lemma 18 (iii). Hence $V_R([H^*\rho, R'\rho]) = V_H(R'\rho)$ and this is simple; thus $R'\rho \in \mathfrak{R}$. This implies that $\rho | R' \in \mathfrak{G}(R'/S, R)$. Moreover, we have $\rho | H^* \in \mathfrak{G}(H^*/S, R)$. Therefore we obtain $\rho=\rho|H^*\times\rho|R'\in\mathfrak{G}(H^*/S, R)\times\mathfrak{G}(R'/S, R)$. (iii): (a): This follows immediately from Lemma 18 (i) and the fact that H^* and R' are both locally finite over S. (b): Since $H^*\sigma\subset H$ and $R'\tau\in\mathfrak{R}$, we have again $[H^*\sigma, R'\tau]\in\mathfrak{R}$ by Lemma 18 (iv); whence $\sigma\times\tau\in\mathfrak{G}([H^*, R']/S, R)$. (iv): (a): If $\tau\in\mathfrak{G}(R'/S, R)$ then $(H^*\cap R')\tau\subset H$ by Lemma 18 (iii). Since H is outer Galois over S and $H\supset H^*\supset H^*\cap R'\supset S$, it follows that $\tau|H^*\cap R'\in\mathfrak{G}(H^*\cap R'/S, H)=\mathfrak{G}(S,H)|H^*\cap R'=\mathfrak{G}(H^*/S, H)|H^*\cap R'=\mathfrak{G}(H^*/S,R)|H^*\cap R'$, that is, $\tau|H^*\cap R'=\sigma|H^*\cap R'$ for some $\sigma\in\mathfrak{G}(H^*/S, R)$. Hence $\sigma\times\tau\in\mathfrak{G}(H^*/S, R)\times\mathfrak{G}(R'/S, R)=\mathfrak{G}([H^*, R']/S, R)$; this implies that $\mathfrak{G}([H^*, R']/S, R)|R'\supset\mathfrak{G}(R'/S, R)$. On the other hand, we have $\mathfrak{G}([H^*, R']/S, R)|R'\subset\mathfrak{G}(R'/S, R)$ by (ii). Consequently, we obtain $\mathfrak{G}([H^*, R']/S, R)|R'=\mathfrak{G}(R'/S, R)$. By similar methods, we can prove the remainder of (iv).

Lemma 20. *Let R be q-Galois and locally finite over S. Then*
(i) *If H_1, H_2 are subrings of H containing S such that $[H_i : H_1\cap H_2]<\infty$ $(i=1, 2)$ and one of H_i's is Galois over $H_1\cap H_2$, then $\mathfrak{G}(H_1/S, R)\times\mathfrak{G}(H_2/S, R)=\mathfrak{G}([H_1, H_2]/S, R)$.*
(ii) *If H^* is a subring of H containing S which is finite over S and if $S'\in\mathfrak{R}_{l.f}$ and $H^*\cap S'=H\cap S'$, then $\mathfrak{G}(H^*/S, R)\times\mathfrak{G}(S'/S, R)=\mathfrak{G}([H^*, S']/S, R)$.*

Proof. (i): By Lemma 19 (i, ii), it suffices to prove that $\mathfrak{G}(H_1/S, H)\times\mathfrak{G}(H_2/S, H)\subset\mathfrak{G}([H_1, H_2]/S, H)$. Let H_1 be Galois over $H_1\cap H_2$, and $\sigma\times\tau\in\mathfrak{G}(H_1/S, H)\times\mathfrak{G}(H_2/S, H)$. We shall distinguish two cases:

Case 1. $\tau=1$. Since $\sigma|H_1\cap H_2=\tau|H_1\cap H_2=1$, σ is contained in $\mathfrak{G}(H_1\cap H_2, H_1)$. By Lemma 17 (i), we have $\mathfrak{G}(H_1\cap H_2, H_1)=\mathfrak{G}(H_2, H)|H_1$. Hence $\sigma=\sigma'|H_1$ for some $\sigma'\in\mathfrak{G}(H_2, H)$. It follows that $\sigma\times\tau=\sigma\times 1=\sigma'|[H_1, H_2]\in\mathfrak{G}([H_1, H_2]/S, H)$.

Case 2. $\tau\neq 1$. Let $\mathfrak{G}(H_1\cap H_2, H_1)=\{\sigma_1, \cdots, \sigma_n\}$ $(n=[H_1 : H_1\cap H_2]$ by outer Galois theory). Then, by Case 1, we have $\{\sigma_i\times 1\}\subset\mathfrak{G}([H_1, H_2]/S, H)$. Since H is outer Galois over S, we can find some $\tau^*\in\mathfrak{G}(S, H)$ such that $\tau=\tau^*|H_2$ Then, we have

(1) $\{(\sigma_i\times 1)\tau^*\}\subset\mathfrak{G}([H_1, H_2]/S, H)$ and $(\sigma_i\times 1)\tau^*|H_2=\tau$

for every i. Furthermore, from $(\sigma_i\times 1)\tau^*|H_1\cap H_2=\tau|H_1\cap H_2=\sigma|H_1\cap H_2$, we have

(2) $\{(\sigma_i\times 1)\tau^*|H_1\}\subset\{\sigma'\in\mathfrak{G}(H_1/S, H)\,;\ \sigma'|H_1\cap H_2=\sigma|H_1\cap H_2\}$,

and the right hand side of (2) contains σ. By outer Galois theory, we have $\sigma=\sigma^*|H_1$ for some $\sigma^*\in\mathfrak{G}(S, H)$ and $\sigma'=\sigma''|H_1$ for some $\sigma''\in\mathfrak{G}(S, H)$. Furthermore, from $\sigma''|H_1\cap H_2=\sigma'|H_1\cap H_2=\sigma|H_1\cap H_2=\sigma^*|H_1\cap H_2$, we have $\sigma''=\rho\sigma^*$ for some $\rho\in\mathfrak{G}(H_1\cap H_2, H)$. Noting here $\sigma''|H_1=\sigma'$, we obtain that the right hand side of $(2)=\mathfrak{G}(H_1\cap H_2, H)\sigma^*|H_1$ and $\sharp\mathfrak{G}(H_1\cap$

$H_2, H)\sigma^*|H_1 = \sharp\mathfrak{G}(H_1\cap H_2, H)|H_1 = \sharp\mathfrak{G}(H_1/H_1\cap H_2, H) = \sharp\mathfrak{G}(H_1\cap H_2, H_1) = \sharp\{(\sigma_i\times 1)\sigma^*|H_1\}$. From this and (2), the left hand side of (2) is equal to the right hand side of (2). Hence $\sigma=(\sigma_{i'}\times 1)\tau^*|H_1$ for some $\sigma_{i'}\in\{\sigma_i\}$. Therefore, it follows from (1) that $\sigma\times\tau=(\sigma_{i'}\times 1)\tau^*\in\mathfrak{G}([H_1, H_2]/S, H)$.
(ii): By Lemma 19 (i, ii), it suffices to prove that $\mathfrak{G}(H^*/S, H)\times\mathfrak{G}(S'/S, R)\subset\mathfrak{G}([H^*, S']/S, R)$. Let $\sigma\times\tau\in\mathfrak{G}(H^*/S, H)\times\mathfrak{G}(S'/S, R)$. We shall distinguish two cases:

Case 1. $\tau=1$. Since $\sigma|H^*\cap S'=\tau|H^*\cap S'=1$ and $H^*\cap S'=H\cap S'$, σ is contained in $\mathfrak{G}(H^*/H^*\cap S', H)=\mathfrak{G}(H^*/H\cap S', H)$. By Lemma 17-(ii, d), we have $\mathfrak{G}(H^*/H\cap S', H)=\mathfrak{G}([H^*, S']/S', R)|H^*$. Hence $\sigma=\sigma'|H^*$ for some $\sigma'\in\mathfrak{G}([H^*, S']/S', R)$. It follows that $\sigma\times\tau=\sigma\times 1=\sigma'\in\mathfrak{G}([H^*, S']/S', R)$.

Case 2. $\tau\neq 1$. Let $\mathfrak{G}(H^*/H^*\cap S', H)=\{\sigma_1, \cdots, \sigma_n\}$ $(n=[H^*: H^*\cap S'])$. We choose a $T^*\in\mathfrak{R}^0_{l.f}/[S', H^*\sigma_1, \cdots, H^*\sigma_n]$. Then $\tau=\tau^*|S'$ for some $\tau^*\in\mathfrak{G}(T^*/S, R)$ (Theorem 2), and then we have

(1) $\{(\sigma_i\times 1)\tau^*\}\subset\mathfrak{G}([H^*, S']/S, R)$, $(\sigma_i\times 1)\tau^*|S'=\tau$

for every i. Furthermore, from $(\sigma_i\times 1)\tau^*|H^*\cap S'=\tau|H^*\cap S'=\sigma|H^*\cap S'$, we have

(2) $\{(\sigma_i\times 1)\tau^*|H^*\}\subset\{\sigma'\in\mathfrak{G}(H^*/S, H); \sigma'|H^*\cap S'=\sigma|H^*\cap S'\}$,

and the right hand side of (2) contains σ. Then, by the same methods in the proof of (i, Case 2), we obtain

(3) The left hand side of (2) = the right hand side of (2).

From (1)—(3), it follows that $\sigma\times\tau\in\mathfrak{G}([H^*, S']/S, R)$.

Theorem 11. *Let R be q-Galois and locally finite over S. Let H^* be a subring of H containing S, and $R'\in\mathfrak{R}$. Then*
(i) *If $H^*\cap R'=H\cap R'$ then $\mathfrak{G}(H^*/S, R)\times\mathfrak{G}(R'/S, R)=\mathfrak{G}([H^*, R']/S, R)$.*
(ii) *In general, if one of the subrings H^*, $H\cap R'$ is Galois over $H^*\cap R'$ then $\mathfrak{G}(H^*/S, R)\times\mathfrak{G}(R'/S, R)=\mathfrak{G}([H^*, R']/S, R)$.*

Proof. (i): Let $\sigma\times\tau\in\mathfrak{G}(H^*/S, R)\times\mathfrak{G}(R'/S, R)$. Let $H^{*\prime}$ be a subring of H^* containing S which is finite over S, and $S'\in\mathfrak{R}_{l.f}$ such that $S'\subset R'$. Then $[H^{*\prime}, H\cap S']$ is finite over S and this is contained in H^*. Since $\sigma|[H^{*\prime}, H\cap S']\in\mathfrak{G}([H^{*\prime}, H\cap S']/S, R)$ and $\tau|S'\in\mathfrak{G}(S'/S, R)$ (Lemma 18 (ii)), it follows from Lemma 20 (ii) that $\sigma|[H^{*\prime}, H\cap S']\times\tau|S'\in\mathfrak{G}([H^{*\prime}, H\cap S', S']/S, R)$. Hence $(\sigma\times\tau)|[H^{*\prime}, S',]=(\sigma|[H^{*\prime}, H\cap S']\times\tau|S')|[H^{*\prime}, S']$, which is a ring isomorphism. Therefore by Lemma 19 (iii), $\sigma\times\tau$ is contained in $\mathfrak{G}([H^*, R']/S, R)$; thus $\mathfrak{G}(H^*/S, R)\times\mathfrak{G}(R'/S, R)\subset\mathfrak{G}([H^*, R']/S, R)$. Combining this with Lemma 19 (ii), we obtain (i).
(ii): First, we shall prove

(1) $\mathfrak{G}(H^*/S, R)\times\mathfrak{G}(H\cap R'/S, R)=\mathfrak{G}([H^*, H\cap R']/S, R)$.

Without loss of generality, we may suppose that H^* is Galois over $H^*\cap R'$. Let $\sigma\times\tau\in\mathfrak{G}(H^*/S, R)\times\mathfrak{G}(H\cap R'/S, R)$. Let $H^{*\prime}$ be a subring of H^* containing S which is finite over S, and H'' a subring of $H\cap R'$ containing S which is finite over S. Then, by outer Galois theory, there exists a subring N of H^* containing $[H^{*\prime}, H^*\cap R']$ which is Galois and finite over $H^*\cap R'$. Clearly $\sigma|N\in\mathfrak{G}(N/S, R)$ and $\tau|[H^*\cap R', H'']\in\mathfrak{G}([H^*\cap R', H'']/S, R)$ (Lemma 18 (iii)). Noting that $N\cap[H^*\cap R', H'']=H^*\cap R'$, by Lemma 20 (i), we have $\sigma|N\times\tau|[H^*\cap R', H'']\in\mathfrak{G}([N, H^*\cap R', H'']/S, R)$. Hence $(\sigma\times\tau)|[H^{*\prime}, H'']$ is a ring isomorphism. Therefore by Lemma 19 (iii), $\sigma\times\tau$ is contained in $\mathfrak{G}([H^*, H\cap R']/S, R)$; thus $\mathfrak{G}(H^*/S, R)\times\mathfrak{G}(H\cap R', R)\subset\mathfrak{G}([H^*, H\cap R']/S, R)$. From this and Lemma 19 (ii), we obtain (1). Now, we shall prove (ii). From (i), we have

(2) $$\mathfrak{G}([H^*, H\cap R']/S, R)\times_{H\cap R'}\mathfrak{G}(R'/S, R)=\mathfrak{G}([H^*, R']/S, R).$$

Let $\sigma\times\tau\in\mathfrak{G}(H^*/S, R)\times\mathfrak{G}(R'/S, R)$. Since $(H\cap R')\tau\subset H$ (Lemma 18 (iii)), $\tau|H\cap R'$ is contained in $\mathfrak{G}(H\cap R'/S, R)$. Hence $\sigma\times_{H^*\cap R'}\tau|H\cap R'\in\mathfrak{G}([H^*, H\cap R']/S, R)$ by (1). From this and (2), it follows that $\sigma\times_{H^*\cap R'}\tau=(\sigma\times_{H^*\cap R'}\tau|H\cap R')\times_{H\cap R'}\tau\in\mathfrak{G}([H^*, R']/S, R)$; thus $\mathfrak{G}(H^*/S, R)\times\mathfrak{G}(R'/S,R)\subset\mathfrak{G}([H^*, R']/S, R)$. Combining this with Lemma 19 (ii), we obtain (ii).

Corollary 12. *Let R be q-Galois and locally finite over S. Let H^* be a subring of H containing S, and $R'\in\mathfrak{R}$. Then*

(i) $\mathfrak{G}([H^*, H\cap R']/S, R)\times\mathfrak{G}(R'/S, R)=\mathfrak{G}([H^*, R']/S, R)$.

(ii) *If $H^*\cap R'=H\cap R'$ then for every subring $H^{*\prime}$ of H^* containing $H^*\cap R'$, $\mathfrak{G}(H^*/S, R)\times\mathfrak{G}([H^{*\prime}, R']/S, R)=\mathfrak{G}([H^*, R']/S, R)$.*

Proof. (i) is a direct consequence of Theorem 11 (i). (ii): By Lemma 18 (v), $H\cap[H^{*\prime}, R']=H^{*\prime}$ and this is equal to $H^*\cap[H^{*\prime}, R']$. Hence, our assertion follows from Theorem 11 (i).

In [3], the following theorems 12, 13 have been shown under the assumption that R is h-q-Galois and left locally finite over S.

Theorem 12. *Let R be q-Galois and locally finite over S. If $S_1, S_2\in\mathfrak{R}$ such that $S_1\supset S_2$ and S_i's are f-regular, then $\mathfrak{G}(S_1/S, R)|S_2=\mathfrak{G}(S_2/S, R)$.*

Proof. Set $H_i=H\cap S_i$ $(i=1, 2)$. Then, by Corollary 9, we can find some $S_1', S_2'\in\mathfrak{R}_{l.f}$ such that $S_1'\supset S_2'$ and $[H_i, S_i']=S_i$ $(i=1, 2)$. Then, we have

(1) $H_2\cap S_2'=(H\cap S_2)\cap S_2'=H\cap S_2'$,

(2) $H_1\cap S_2'=(H\cap S_1)\cap S_2'=H\cap S_2'=H_2\cap S_2'$,

(3) $H_1\supset H_1\cap S_1'\supset H_1\cap S_2'=H\cap S_2'$,

(4) $H_1\cap S_1'=(H\cap S_1)\cap S_1'=H\cap S_1'$.

Then if we apply Theorem 11 (i) and Corollary 12 (ii) we will have
(1′) $\mathfrak{G}(H_2/S, R)\times_{H_2\cap S_2'}\mathfrak{G}(S_2'/S, R)=\mathfrak{G}([H_2, S_2']/S, R)$ (from (1)),
(2′) $\mathfrak{G}(H_1/S, R)\times_{H_2\cap S_2'}\mathfrak{G}(S_2'/S, R)=\mathfrak{G}([H_1, S_2']/S, R)$ (from (2)),
(3′) $\mathfrak{G}(H_1/S, R)\times_{H_1\cap S_1'}\mathfrak{G}([H_1\cap S_1', S_2']/S, R)=\mathfrak{G}([H_1, S_2']/S, R)$ (from (3)),
(4′) $\mathfrak{G}(H_1/S, R)\times_{H_1\cap S_1'}\mathfrak{G}(S_1'/S, R)=\mathfrak{G}([H_1, S_1']/S, R)$ (from (4)).
Moreover, by Lemma 19 (i), outer Galois theory and Theorem 2, we have
(5) $\mathfrak{G}(H_2/S, R)=\mathfrak{G}(H_1/S, R)|H_2$,
(6) $\mathfrak{G}([H_1\cap S_1', S_2']/S, R)=\mathfrak{G}(S_1'/S, R)|[H_1\cap S_1', S_2']$.
Hence, it follows from (1′)—(6) that
(7) $\mathfrak{G}([H_2, S_2']/S, R)=\mathfrak{G}([H_1, S_2']/S, R)|[H_2, S_2']$,
(8) $\mathfrak{G}([H_1, S_2']/S, R)=\mathfrak{G}([H_1, S_1']/S, R)|[H_1, S_2']$.
Combining (7) with (8), we can conclude that

$\mathfrak{G}([H_2, S_2']/S, R)=\mathfrak{G}([H_1, S_1']/S, R)|[H_2, S_2']$,

i. e., $\mathfrak{G}(S_2/S, R)=\mathfrak{G}(S_1/S, R)|S_2$.

Theorem 13. *Let R be q-Galois and locally finite over S. Then, for every $S'\in\mathfrak{R}$ such that S' is f-regular, R is q-Galois and locally finite over S'.*

Proof. By Corollary 9, S' is finite over $H\cap S'$. Hence, by Theorem 8, it suffices to prove that if $S'\subset H$ then R satisfies the condition (V_l) of Theorem 5 over S'. Let $S'\subset H$. Then, (V_l : iv) follows from Corollary 8. Since R is $S\cdot V$-R-irreducible, R is $S'\cdot V$-R-irreducible; this implies (V_l : iii). Let $T\in\mathfrak{R}^0/S'$ be such that $[T:S']_l<\infty$. Then T is f-regular (Corollary 9). We set $H'=H\cap T$. Then $H'\mathfrak{G}(T/S', R)\subset H'\mathfrak{G}(T/S, R)\subset H$ (Lemma 18 (iii)); this implies (V_l : ii). By outer Galois theory, we have $J(\mathfrak{G}(H'/S', R), H')=S'$. Moreover, we have $\mathfrak{G}(H'/S', R)\subset\mathfrak{G}(H'/S, R)=\mathfrak{G}(T/S, R)|H'$ (Theorem 12) and $J(\langle V\rangle|T, T)=H'$. Therefore, it follows that $J(\mathfrak{G}(T/S', R), T)=S'$; this implies ($V_l$: i).

Remark 12. From Theorems 3, 9, 13 and Corollary 9, R is q-Galois and left locally finite over S if and only if R is h-q-Galois and locally finite over S, and the results of [3, Theorem 9 and Corollary 7] follow immediately.

Corollary 13. *Let R be q-Galois and locally finite over S. If S_1, $S_2\in\mathfrak{R}$ such that $S_1\supset S_2$ and S_i's are f-regular, then $J(\mathfrak{G}(S_1/S_2, R), S_1)=S_2$.*

Proof. Since R is q-Galois and locally finite over S_2 (Theorem 13), for each element a of S_1 there exists some $S'\in\mathfrak{R}$ such that $S_1\supset S'\supset[S_2, a]$ and $[S':S_2]<\infty$ (Corollary 9, Lemma 18 (i)). Then, by Theorem 12, we have $\mathfrak{G}(S_1/S_2, R)|S'=\mathfrak{G}(S'/S_2, R)$. Moreover, we have $J(\mathfrak{G}(S'/S_2, R), S')=S_2$ (Corollary 3 (ii)). Hence, if $a\notin S_2$ then we can find some $\sigma\in\mathfrak{G}(S_1/S_2, R)$

such that $a\sigma \neq a$; this proves our assertion.

From Lemma 19 (iv), Theorem 11 and Theorem 12, we have the following

Corollary 14. *Let R be q-Galois and locally finite over S. Let H^* be a subring of H containing S, and $S' \in \mathfrak{R}$ such that S' is f-regular. If one of subrings H^*, $H \cap S'$ is Galois over $H^* \cap S'$, then*

$$\mathfrak{G}(H^*/S, R) \times \mathfrak{G}(S'/S, R) = \mathfrak{G}([H^*, S']/S, R),$$
$$\mathfrak{G}([H^*, S']/S, R) | H^* = \mathfrak{G}(H^*/S, R),$$
$$\mathfrak{G}([H^*, S']/S, R) | S' = \mathfrak{G}(S'/S, R).$$

Theorem 14. *Let R be q-Galois and locally finite over S. Let $S' \in \mathfrak{R}_{l.f}$. Let $\{\sigma_1, \cdots, \sigma_m\}$ be a subset of $\mathfrak{G}(S'/S, R)$ such that $\{\sigma_i\} | H \cap S' = \mathfrak{G}(S'/S, R) | H \cap S'$ and $\sigma_i | H \cap S' \neq \sigma_{i'} | H \cap S'$ for every $i \neq i'$, and let $\{v_1, \cdots, v_n\}$ be a linearly independent $V_R(S')$-right basis of V which consists of regular elements. Let $S^* \in \mathfrak{R}_{l.f} / [S', \{v_j\}]$. Then, the following hold:*
(i) $\mathfrak{G}(S'/H \cap S', R) = \langle V \rangle | S'$; $\mathrm{Hom}_{(H \cap S')_l}(S', R) = \sum_{\oplus_j} (\langle v_j \rangle | S') R_r$.
(ii) $\mathfrak{G}(S'/S, R) = \bigcup_i \sigma_i \langle V \rangle$; *there exists a subset* $\{\sigma_1^*, \cdots, \sigma_m^*\} \subset \mathfrak{G}(S^*/S, R)$ *such that* $\sigma_i^* | S' = \sigma_i$ $(1 \leqq i \leqq m)$; *and for such subset* $\{\sigma_i^*\}$, $\mathrm{Hom}_{S_l}(S', R) = \sum_{\oplus i, j} \sigma_i \langle v_j \sigma_i^* \rangle R_r$.

Proof. We choose a $T^* \in \mathfrak{R}^0_{l.f}/S'$, and set $H^* = H \cap T^*$. Then $H^* \cap S' = (H \cap T^*) \cap S' = H \cap S'$. Hence, by Theorem 11, we have $\mathfrak{G}(H^*/H^* \cap S', R) \times \mathfrak{G}(S'/H^* \cap S', R) = \mathfrak{G}([H^*, S']/H^* \cap S', R)$. From $\mathfrak{G}([H^*, S']/H^* \cap S', R) = \mathfrak{G}(T^*/H^* \cap S', R) | [H^*, S']$, we obtain $\mathfrak{G}(H^*/H^* \cap S', R) \times {}_{H^* \cap S'} 1 | S' = \{\rho_1 | [H^*, S'], \cdots, \rho_s | [H^*, S']\}$ for some $\{\rho_k\} \subset \mathfrak{G}(T^*/H^* \cap S', R)$ ($\rho_k | [H^*, S'] \neq \rho_{k'} | [H^*, S']$ for every $k \neq k'$). Then one will see
(1) $\{\rho_k | H^*\} = \mathfrak{G}(H^*/H^* \cap S', R)$ and $\rho_k | H^* \neq \rho_{k'} | H^*$ for every $k \neq k'$.
(2) $\rho_k | S' = 1$ for every k.
Now, we shall prove (i). Since $[T^* : H^*] = [V : V_R(T^*)]$ (Theorem 9) and $[H^* : H^* \cap S'] = s$ (from (1)), it follows from Lemma 3 (iii, iv, v) that $\mathfrak{G}(T^*/H^* \cap S', R) = \bigcup_k \rho_k \langle V \rangle$. Hence, noting that $\mathfrak{G}(S'/H^* \cap S', R) = \mathfrak{G}(T^*/H^* \cap S', R) | S'$, we have $\mathfrak{G}(S'/H^* \cap S', R) = (\bigcup_k \rho_k \langle V \rangle) | S' = \langle V \rangle | S'$ (from (2)). Therefore $\mathrm{Hom}_{(H \cap S')_l}(S', R) = \mathrm{Hom}_{(H^* \cap S')_l}(S', R) = \mathfrak{G}(S'/H^* \cap S', R) R_r = (\langle V \rangle | S') R_r = \sum_{j=1}^n (\langle v_j \rangle | S') R_r$, and this is a direct sum, since $n = [V : V_R(S')] = [S' : H \cap S'] = [S' : H^* \cap S'] = [\mathrm{Hom}_{(H^* \cap S')_l}(S', R) : R_r]_r$ (Theorem 9). (ii): We choose a $T' \in \mathfrak{R}^0_{l.f}/[T^*, T^*\rho_1, \cdots, T^*\rho_s]$. Then, from $\mathfrak{G}(S'/S, R) = \mathfrak{G}(T'/S, R) | S'$, we can find some $\{\sigma_1', \cdots, \sigma_m'\} \subset \mathfrak{G}(T'/S, R)$ such that
(3) $\sigma_i' | S' = \sigma_i$ for every i.
Then, combining (1) with the fact that $\{\sigma_i' | H^* \cap S'\} = \{\sigma_i | H^* \cap S'\} =$

$\mathfrak{G}(H\cap S'/S, R)=\mathfrak{G}(H^*\cap S'/S, R)$ (from (3) and Remark 9), we have
(4) $\{\rho_k\sigma_i' | H^*\} = \mathfrak{G}(H^*/S, R)$ (where $\{\rho_k\sigma_i'\}\subset\mathfrak{G}(T^*/S, R)$)
by outer Galois theory. Since $[T^*: H^*]=[V: V_R(T^*)]$ (Theorem 9) and $[H^*: S]=sm$ (from (4)), it follows from Lemma 3 (iii, iv, v) that $\mathfrak{G}(T^*/S, R)=\cup_{k,i}\rho_k\sigma_i'\langle V\rangle$. Therefore, from $\mathfrak{G}(S'/S, R)=\mathfrak{G}(T^*/S, R)|S'$, (2) and (3), we obtain $\mathfrak{G}(S'/S\ R)=(\cup_{k,i}\rho_k\sigma_i'\langle V\rangle)|S'=\cup_{k,i}(\rho_k\sigma_i'|S')\langle V\rangle=\cup_i(\sigma_i'|S')\langle V\rangle=\cup_i\sigma_i\langle V\rangle$. Now, from $\mathfrak{G}(S^*/S, R)|S'=\mathfrak{G}(S'/S, R)$, there exists a subset $\{\sigma_1^*, \cdots, \sigma_m^*\}\subset\mathfrak{G}(S^*/S, R)$ such that $\sigma_i^*|S'=\sigma_i$ $(1\leqq i\leqq m)$, where $S^*\in\mathfrak{R}_{l.f}/[S', \{v_j\}]$. Then, noting that $\mathfrak{G}(S_1/S, R)|S_2=\mathfrak{G}(S_2/S, R)$ for every $S_1, S_2\in\mathfrak{R}_{l.f}$ such that $S_1\supset S_2$, we have $\sum_i\sigma_i\langle V\rangle R_r=\sum_{i=1,}^{m}{}_{j=1}^{n}\sigma_i\langle v_j\sigma_i^*\rangle\cdot R_r$ by making use of the same method as in the proof of Lemma 3 (v). Since $\mathrm{Hom}_{S_l}(S', R)=\mathfrak{G}(S'/S, R)R_r=\sum_i\sigma_i\langle V\rangle R_r$ and $[\mathrm{Hom}_{S_l}(S', R): R_r]_r=[S': S]=mn$ (Theorem 9 and Remark 9), we obtain $\mathrm{Hom}_{S_l}(S', R)=\sum_{\oplus i=1,}^{m}{}_{j=1}^{n}\sigma_i\langle v_j\sigma_i^*\rangle R_r$.

Theorem 15. *Let R be q-Galois and locally finite over S. Let $R'\in\mathfrak{R}$ be such that R' is Galois over S. Then $H\cap R'$ is outer Galois over S, and the following hold:*
(i) *If $S'\in\mathfrak{R}_{l.f}$ such that $S'\subset R'$, then $\mathfrak{G}(S'/S, R)=\mathfrak{G}(R'/S, R)|S'=\mathfrak{G}(S, R')\langle V\rangle|S'$.*
(ii) *$\mathfrak{G}(S, R')\langle V\rangle$ is dense in $\mathfrak{G}(R'/S, R)$ in the finite topology.*
(iii) *If $R'\supset V$ then R' is h-Galois over S, $\mathfrak{G}(S, R')$ is dense in $\mathfrak{G}(R'/S, R)$, and $\mathfrak{G}(R'/S, R)=\mathfrak{G}(R'/S, R')$.*

Proof. (i): By Lemma 18 (iii), we have $(H\cap R')\mathfrak{G}(S, R')\subset(H\cap R')\cdot\mathfrak{G}(R'/S, R)\subset H$, and so
$$(H\cap R')\mathfrak{G}(S, R')\subset H\cap R'.$$
Hence $H\cap R'$ is outer Galois over S. Because of $J(\mathfrak{G}(S, R')|H\cap R', H\cap R')=S$, $\mathfrak{G}(S, R')|H\cap R'$ is dense in $\mathfrak{G}(S, H\cap R')$ by outer Galois theory. Let $S'\in\mathfrak{R}_{l.f}$ be such that $S'\subset R'$. Then $\mathfrak{G}(S'/S, R)|H\cap S'=\mathfrak{G}(H\cap S'/S, R)=\mathfrak{G}(H\cap S'/S, H)=\mathfrak{G}(H\cap S'/S, H\cap R')=(\mathfrak{G}(S, R')|H\cap R')|H\cap S'=\mathfrak{G}(S, R')|H\cap S'$. Hence we can find some $\{\sigma_1, \cdots, \sigma_m\}\subset\mathfrak{G}(S, R')$ such that $\{\sigma_i\}|H\cap S'=\mathfrak{G}(S'/S, R)|H\cap S'$ and $\sigma_i|H\cap S'\neq\sigma_{i'}|H\cap S'$ for every $i\neq i'$. Since $\cup_i\sigma_i\langle V\rangle\subset\mathfrak{G}(S, R')\langle V\rangle\subset\mathfrak{G}(R'/S, R)$, it follows from Lemma 18 (ii) and Theorem 14 (ii) that $\mathfrak{G}(S'/S, R)=\cup_i(\sigma_i|S')\langle V\rangle=(\cup_i\sigma_i\langle V\rangle)|S'\subset\mathfrak{G}(S, R')\cdot\langle V\rangle|S'\subset\mathfrak{G}(R'/S, R)|S'\subset\mathfrak{G}(S'/S, R)$. From this, our assertion (i) follows immediately. (ii): By Lemma 18 (i), we can choose a $U\in\mathfrak{R}_{l.f}$ such that $U\subset R'$ and $[U, F]\in\mathfrak{R}_{l.f}$ for every finite subset F of R'. Then, for an arbitrary finite subset F of R', we have $\mathfrak{G}(R'/S, R)|[U, F]=\mathfrak{G}(S, R')\langle V\rangle|[U, F]$ by (i); whence $\mathfrak{G}(R'/S, R)|F=\mathfrak{G}(S, R')\langle V\rangle|F$. This proves our assertion (ii). (iii): Since R is $S\cdot V\text{-}R$-irreducible, R' is $S\cdot V\text{-}R'$-irreducible by Lemma 1 (iii, b). Hence, R' is h-Galois over S. The remainder of (iii)

follows immediately from (ii) and the fact that $\mathfrak{G}(S, R')\langle V\rangle=\mathfrak{G}(S, R')$.

Remark 13. Let R be q-Galois and locally finite over S. Let $R'\in\mathfrak{R}$. If $R'\supset V$ and $\mathfrak{G}(S'/S, R)=\mathfrak{G}(S'/S, R')$ for every $S'\in\mathfrak{R}_{l,f}$ such that $S'\subset R'$, then R' is q-Galois and locally finite over S. In fact, since R is $S\cdot V\text{-}R$-irreducible, R' is $S\cdot V\text{-}R'$-irreducible by Lemma 1 (iii, b), and moreover, noting that for a subring S' of R' containing S, S' is a regular subring of R' if and only if $S'\in\mathfrak{R}$, one will see that R' satisfies the condition (IV_l) of Theorem 4 over S.

Now, we shall prove the following theorem which contains the result of [3, Theorem 5].

Theorem 16. *Let R be q-Galois and locally finite over S. Let H^* be a subring of H containing S, and $R'\in\mathfrak{R}$. Then the following hold:*

(i): *If one of subrings H^*, $H\cap R'$ is Galois over $H^*\cap R'$ then*

(a) *the contraction map f_1:*

$$\mathfrak{G}([H^*, R']/R', R) \longrightarrow \mathfrak{G}(H^*/H^*\cap R', R) \ (i.e., f_1: \sigma \longrightarrow \sigma|H^*)$$

is onto, one-to-one, and $J(\mathfrak{G}([H^*, R']/R', R), [H^*, R'])=R'$, and

(b) *the contraction map f_2:*

$$\mathfrak{G}([H^*, R']/H^*, R) \longrightarrow \mathfrak{G}(R'/H^*\cap R', R)$$

is onto, one-to-one, and if $\mathfrak{G}(S'/H^*\cap R', R)=\mathfrak{G}(R'/H^*\cap R', R)|S'$ for every $S'\in\mathfrak{R}/H^*\cap R'$ such that $S'\subset R'$ and $[S': H^*\cap R']<\infty$ then $J(\mathfrak{G}([H^*, R']/H^*, R), [H^*, R'])=H^*$.

(ii): (a) *If H^* is Galois over $H^*\cap R'$ then $[H^*, R']$ is outer Galois over R', and*

$$\mathfrak{G}(R', [H^*, R'])\cong\mathfrak{G}(H^*\cap R', H^*)$$

(algebraically and topologically) under the map f_1.

(b) *If R' is Galois over $H^*\cap R'$ and $R'\supset V$ then $H\cap R'$ is outer Galois over $H^*\cap R'$ and $[H^*, R']$ is h-Galois over H^*, and*

$$\mathfrak{G}(H^*, [H^*, R'])\cong\mathfrak{G}(H^*\cap R', R')$$

(algebraically and topologically), under the map $f_2|\mathfrak{G}(H^*, [H^*, R'])$.

(iii): (a) *If H^* and R' are both Galois over S and $R'\supset V$ then $[H^*, R']$ is h-Galois over S, and*

$$\mathfrak{G}(S, [H^*, R'])=\mathfrak{G}(S, H^*)\times\mathfrak{G}(S, R')$$

(b) *If H^* and R' are both Galois over $H^*\cap R'$ and $R'\supset V$ then $[H^*, R']$ is h-Galois over $H^*\cap R'$, and*

$$\mathfrak{G}(H^*\cap R', [H^*, R'])=\mathfrak{G}(H^*\cap R', H^*)\times\mathfrak{G}(H^*\cap R', R')$$
$$\cong\mathfrak{G}(H^*\cap R', H^*)\otimes\mathfrak{G}(H^*\cap R', R')$$

(algebraically and topologically), where $\otimes$ means a direct product of groups.

Proof. (i): Since R is q-Galois and locally finite over $H^*\cap R'$, by Theorem 11, we have

$$\mathfrak{G}(H^*/H^*\cap R', R)\times_{H^*\cap R'}\mathfrak{G}(R'/H^*\cap R', R)=\mathfrak{G}([H^*, R']/H^*\cap R', R).$$

Hence, we obtain

(1) $\mathfrak{G}(H^*/H^*\cap R', R)\times_{H^*\cap R'}1|R'=\mathfrak{G}([H^*, R']/R', R)$,

(2) $1|H^*\times_{H^*\cap R'}\mathfrak{G}(R'/H^*\cap R', R)=\mathfrak{G}([H^*, R']/H^*, R)$.

Therefore both f_1 and f_2 are onto, one-to-one. First, we shall prove the last part of (i, b). By Lemma 18 (i), we can choose a $U\in\mathfrak{R}/H^*\cap R'$ such that $U\subset R'$, $[U:H^*\cap R']<\infty$ and $[U, F]\in\mathfrak{R}$ for every finite subset F of R'. Let $a\in[H^*, R']\setminus H^*$ (the complement of H^* in $[H^*, R']$). Then $a\in[F_1, F_2]$ for some finite subset F_1 of H^* and for some finite subset F_2 of R'. Set $H^{*\prime}=[H^*\cap R', F_1]$ and $S'=[U, F_2]$. Then $H^{*\prime}$, $[H^{*\prime}, S']\in\mathfrak{R}/H^*\cap R'$, and they are finite over $H^*\cap R'$. Hence $J(\mathfrak{G}([H^{*\prime}, S']/H^{*\prime}, R), [H^{*\prime}, S'])=H^{*\prime}$ (Corollary 13). Noting that $a\in[H^{*\prime}, S']\setminus H^{*\prime}$, we can find some $\tau\in\mathfrak{G}([H^{*\prime}, S']/H^{*\prime}, R)$ such that $a\tau\neq a$. Clearly $\tau=1|H^{*\prime}\times_{H^*\cap R'}\tau|S'\in 1|H^{*\prime}\times_{H^*\cap R'}\mathfrak{G}(S'/H^*\cap R', R)$. From $\mathfrak{G}(S'/H^*\cap R', R)=\mathfrak{G}(R'/H^*\cap R', R)|S'$ (assumption), we have $\tau|S'=\tau'|S'$ for some $\tau'\in\mathfrak{G}(R'/H^*\cap R', R)$. Then, by (2) we obtain $1|H^*\times_{H^*\cap R'}\tau'\in\mathfrak{G}([H^*, R']/H^*, R)$, and $a(1|H^*\times_{H^*\cap R'}\tau')=a(1|H^{*\prime}\times_{H^*\cap R'}\tau'|S')=a(1|H^{*\prime}\times_{H^*\cap R'}\tau|S')=a\tau\neq a$. Therefore, it follows that $J(\mathfrak{G}([H^*, R']/H^*, R), [H^*, R'])=H^*$. (ii, b): By Theorem 15, $H\cap R'$ is Galois over $H^*\cap R'$. Hence

$$f_2:\quad \mathfrak{G}([H^*, R']/H^*, R)\longrightarrow\mathfrak{G}(R'/H^*\cap R', R)$$

is onto, one-to-one. By Theorem 15 (iii), we have $\mathfrak{G}(R'/H^*\cap R', R)=\mathfrak{G}(R'/H^*\cap R', R')$. From this and (2), we obtain $\mathfrak{G}([H^*, R']/H^*, R)=\mathfrak{G}([H^*, R']/H^*, [H^*, R'])$. Hence, for an isomorphism $\tau\in\mathfrak{G}([H^*, R']/H^*, R)$, τ is an automorphism if and only if $\tau|R'$ is an automorphism. Therefore $\mathfrak{G}(H^*, [H^*, R'])\cong\mathfrak{G}(H^*\cap R', R')$ (algebraically and topologically) under the map $f_2|\mathfrak{G}(H^*, [H^*, R'])$. Now, from Theorem 15 (iii) and the last part of (i, b), we have $J(\mathfrak{G}([H^*, R']/H^*, R), [H^*, R'])=H^*$. Let us recall the proof of the last part of (i, b). Then, by Theorem 15 (iii), we can find some $\tau''\in\mathfrak{G}(H^*\cap R', R')$ such that $\tau''|S'=\tau'|S'=\tau|S'$. Clearly $1|H^*\times_{H^*\cap R'}\tau''\in\mathfrak{G}(H^*, [H^*, R'])$. From this we obtain $J(\mathfrak{G}(H^*, [H^*, R']), [H^*, R'])=H^*$, so that $[H^*, R']$ is Galois over H^*. By Theorem 15 (iii), $[H^*, R']$ is h-Galois over H^*. The proofs of the last part of (i, a) and (ii, a) follow in much the same manner as the proofs of the last part of (i, b) and (ii. b) respectively from the fact: If $H^{*\prime}$ is a subring of H^* containing $H^*\cap R'$ which is finite over $H^*\cap R'$ then $\mathfrak{G}(H^{*\prime}/H^*\cap R', R)=\mathfrak{G}(H^*/H^*\cap R', R)|H^{*\prime}$; if H^* is Galois over $H^*\cap R'$ then $H^*\mathfrak{G}(H^*/H^*\cap R', R)\subset H^*$ and so $H^*\sigma=H^*$ for every $\sigma\in\mathfrak{G}(H^*/H^*\cap R', R)$. (iii, a): Since H^* is Galois over S, H^* is Galois over $H^*\cap R'$ by outer Galois theory. Hence, by Theorem 11, we have

$$\mathfrak{G}([H^*, R']/S, R)=\mathfrak{G}(H^*/S, R)\times\mathfrak{G}(R'/S, R).$$

From $\mathfrak{G}(H^*/S, R)=\mathfrak{G}(H^*/S, H)=\mathfrak{G}(S, H^*)$ and $\mathfrak{G}(R'/S, R)=\mathfrak{G}(R'/S, R')$ (Theorem 15 (iii)), it follows that

$\mathfrak{G}([H^*, R']/S, [H^*, R'])=\mathfrak{G}(S, H^*)\times\mathfrak{G}(R'/S, R')$.

Hence, for an isomorphism $\rho=\sigma\times\tau\in\mathfrak{G}([H^*, R']/S, [H^*, R'])$ ($\sigma\in\mathfrak{G}(S, H^*)$, $\tau\in\mathfrak{G}(R'/S, R')$), ρ is an automorphism if and only if τ is an automorphism. Therefore $\mathfrak{G}(S, [H^*, R'])=\mathfrak{G}(S, H^*)\times\mathfrak{G}(S, R')$. By outer Galois theory, H^* is Galois over $H^*\cap R'$. Hence, by (ii, a), $[H^*, R']$ is Galois over R'. Since $\mathfrak{G}(S, R')|H^*\cap R'\subset\mathfrak{G}(H^*\cap R'/S, H^*)=\mathfrak{G}(S, H^*)|H^*\cap R'$, we have $\mathfrak{G}(S, [H^*, R'])|R'=\mathfrak{G}(S, R')$. Therefore $J(\mathfrak{G}(S, [H^*, R']), [H^*, R'])\subset R'\cap J(\mathfrak{G}(S, [H^*, R'])|R', R')=R'\cap J(\mathfrak{G}(S,R'), R')=S$; this implies that $[H^*, R']$ is Galois over S, so that $[H^*, R']$ is h-Galois over S by Theorem 15. The last assertion (iii, b) follows from (iii, a).

Now, for $S', \in\mathfrak{R}$, we set $N(S')=[S'\mathfrak{G}(S'/S, R), H, V]$.
Then, we have the following

Lemma 21. *Let R be q-Galois and locally finite over S. Let $S'\in\mathfrak{R}_{l.f}$. Then*

(i) *If $\mathfrak{G}(S'/S, R)|H\cap S'=\{\sigma_1|H\cap S', \cdots, \sigma_n|H\cap S'\}$ for some $\{\sigma_i\}\subset\mathfrak{G}(S'/S, R)$ $(n<\infty)$ then $N(S')=[S'\sigma_1, \cdots, S'\sigma_n, H, V]$.*

(ii) *If $S''\in\mathfrak{R}_{l.f}$* such that *$S''\subset S'$ then $N(S'')\subset N(S')$.*

(iii) *If $S'\in\mathfrak{R}^0_{l.f}$ then $\mathfrak{G}(S''/S, R)=\mathfrak{G}(S''/S, N(S'))$ for every $S''\in\mathfrak{R}_{l.f}$ such that $S''\subset N(S')$.*

Proof. (i): By Theorem 14, we have $\mathfrak{G}(S'/S, R)=\cup_i\sigma_i\langle V\rangle$. From this our assertion (i) follows immediately. The assertion (ii) follows $\mathfrak{G}(S''/S, R)=\mathfrak{G}(S'/S, R)|S''$. (iii): Let $\{\sigma_i\}$ be as in (i). Then $N(S')=[S'\sigma_1, \cdots, S'\sigma_n, H, V]$. Hence $S''\subset[S'\sigma_1, \cdots, S'\sigma_n, F_1, F_2]$ for some finite subset F_1 of H and for some finite subset F_2 of V. We set $S^*=[S'\sigma_1, \cdots, S'\sigma_n, F_1, F_2]$. Then $S^*\in\mathfrak{R}^0_{l.f}$, and so $\mathfrak{G}(S''/S, R)=\mathfrak{G}(S^*/S, R)|S''$. Hence, it suffices to prove that $\mathfrak{G}(S^*/S, R)=\mathfrak{G}(S^*/S, N(S'))$. For an arbitrary $\sigma\in\mathfrak{G}(S^*/S, R)$, we have $S^*\sigma=[S'\sigma_1\sigma, \cdots, S'\sigma_n\sigma, F_1\sigma, F_2\sigma]$. Noting that $\sigma_i\sigma\in\mathfrak{G}(S'/S, R)$ for every i and $F_1\sigma\subset H$ and $F_2\sigma\subset V$, we obtain $S^*\sigma\subset[S'\mathfrak{G}(S'/S, R), H, V]=N(S')$; this implies that $\sigma\in\mathfrak{G}(S^*/S, N(S'))$.

Now, by C_0 we denote the center of V. Then $C_0=H\cap V$, and we shall prove the following

Theorem 17. *Let R be q-Galois and locally finite over S. Let $[V:C_0]\leqq\aleph_0$. Then, for each countable subset $\{x_i\}\subset R$, there exists a $R'\in\mathfrak{R}$ such that $R'\supset[H, \{x_i\}]$, $[R':H]\leqq\aleph_0$ and R' is h-Galois over S.*

Proof. We choose a $T\in\mathfrak{R}^0_{l.f}$, and set $T_i=[T, x_1, \cdots, x_i]$. Then

$$T_1\subset T_2\subset\cdots\subset T_i\subset\cdots,$$

where $T_i\in\mathfrak{R}^0_{l.f}$ for every i. Hence, by Lemma 21 (ii), we have

$$N(T_1)\subset N(T_2)\subset\cdots\subset N(T_i)\subset\cdots$$

Now, we set $N=\cup_i N(T_i)$. If $S'\in\mathfrak{R}_{l,f}$ such that $S'\subset N$ then $S'\subset N(T_i)$ for some i. Hence, by Lemma 21 (iii), we have $\mathfrak{G}(S'/S, R)=\mathfrak{G}(S'/S, N(T_i))=\mathfrak{G}(S'/S, N)$. Therefore, it follows from Remark 13 that

(1) N is q-Galois and locally finite over S.

By Lemma 21 (i), one will see that $[N(T_i):H]\leqq\aleph_0$, so that

(2) $[N:H]\leqq\aleph_0$.

Then, in virtue of (1), (2) and Theorem 12, we have $\mathfrak{G}(S, H)=\mathfrak{G}(S, N)|H$ $(H=V_N^2(S))$ by making use of the same method as in the proof [6, Lemma 3.9]. From this, it follows that N is Galois over S, and so N is h-Galois over S by Theorem 15 (iii).

Corollary 15. *Let R be q-Galois and locally finite over S. If $[V:C_0]<\infty$ then R is locally Galois over S.*

Proof. Let F be a finite subset of R. Then, by Theorem 17, there exists an h-Galois extension R' over S which contains $[V, F]$. Then R' is locally Galois over S by [6, Theorem 2.4]. Hence we can find a subring S' of R' containing $[S, F]$ which is Galois and finite over S. This completes the proof.

If $[R:H]\leqq\aleph_0$ then $[V:C_0]=[H\cdot V:H]\leqq[R:H]\leqq\aleph_0$. Hence, by Theorem 17, we have the following result which is contained in [3].

Corollary 16. *Let R be q-Galois and locally finite over S. If $[R:H]\leqq\aleph_0$ then R is h-Galois over S.*

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References

[1] H. TOMINAGA: Galois theory of simple rings, Math. J. Okayama Univ., 6 (1956), 29—48.

[2] H. TOMINAGA: Galois theory of simple rings II, Math. J. Okayama Univ., 6 (1957), 153—170.

[3] H. TOMINAGA: On q-Galois extensions of simple rings, Nagoya Math. J., 27 (1966), 485—507.

[4] T. NAGAHARA: On Galois conditions and Galois groups of simple rings, Trans. Amer. Math. Soc., 116 (1965), 417—434.

[5] T. NAGAHARA and H. TOMINAGA: On Galois and lacally Galois extensions of simple rings, Math. J. Okayama Univ., 10 (1961), 143—166.

[6] T. NAGAHARA and H. TOMINAGA: On Galois theory of simple rings, Math. J. Okayama

Univ., 11 (1963), 79—117.
[7] T. NAGAHARA and H. TOMINAGA: Some theorems on Galois theory of simple rings, J. Fac. Sci. Hokkaido Univ., Ser. I, 17 (1963), 1—13.
[8] T. NAGAHARA and H. TOMINAGA: On quasi-Galois extensions of division rings, J. Fac. Sci. Hokkaido Univ., Ser. I, 17 (1963), 73—78.
[9] N. NOBUSAWA and H. TOMINAGA: On Galois theory of division rings III, Math. J. Okayama Univ., 10 (1960), 67—73.

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