A FIXPOINT THEOREM ON Sp(n)/U(n)

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Introduction. The Riemann sphere M can be mapped onto the extended complex plane by the stereographic projection, under which the Southern hemisphere of M is sent onto the unit disk M_0, while the Northern one onto $\{z \mid \frac{1}{z} \varepsilon M_0\}$. The latter can be mapped conformally onto the former. These hemispheres are separated by the unit circle, their common boundary. On the other hand, from the group-theoretical point of view, M_0 can be seen to be a Hermitian symmetric manifold on which the multiplicative group G_0 of matrices

$$\left\{\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \Big| \ \alpha, \beta \varepsilon C \text{ and } \alpha\bar{\alpha} - \beta\bar{\beta}=1\right\}$$

acts transitively and M_0, moreover, can be written as $G_0/U(1)$ if $U(1)$ (= $\{\exp\sqrt{-1}\,\theta \mid \theta \varepsilon R\}$ is identified with a subgroup of G_0 by a representation;

$$(1) \qquad u \longrightarrow \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix} \qquad (u \varepsilon U(1)).$$

G_0 is conjugate to $S_p(1, R)$ and the connected group of isometries of M_0 is isomorphic canonically to $G_0/\{\pm 1_2\}$. The Riemann sphere M also can be seen to be a Hermitian symmetric manifold, which is written as $S_p(1)/U(1)$. This symmetric manifold is said to be dual to the preceding one. The Harish-Chandra realization is, as is easily seen, nothing but the restriction on the Southern hemisphere of the stereographic projection. The equator (= the unit circle) has its own group-theoretical meaning. Actually each transvection of $S_p(1)/U(1)$ has exactly two points fixed. The set of these fixed points for all the transvections is precisely the equator. From the viewpoint of analysis the equator is the Bergmann-Silow boundary of the Southern hemisphere.

The aim of the present paper is to generalize these facts to the case of several complex variables.

Let M_0 be a generalized unit disk

$$(2) \qquad \{Z \mid {}^tZ=Z \text{ and } 1_n-\bar{Z}Z > 0\}.$$

Then M_0 can be considered to be a Hermitian symmetric manifold with the symplectic metric

$$ds^2 = Tr(d\bar{Z}(1_n - Z\bar{Z})^{-1}dZ(1_n - \bar{Z}Z)^{-1})$$

as an invariant metric. The compact dual of M_0 is $S_p(n)/U(n)$ where we use the identification (1) of U(n) with a subgronp of $S_p(n)$. The connected group of isometries of M_0 is

$$\frac{1}{2}\begin{pmatrix} 1_n & -\sqrt{-1}\,1_n \\ -\sqrt{-1}1_n & 1_n \end{pmatrix} S_p(n, R) \begin{pmatrix} 1_n & \sqrt{-1}\,1_n \\ \sqrt{-1}\,1_n & 1_n \end{pmatrix}$$

which we denote by G_0. M_0 can be considered to be imbedded in M in the standard way. Then each element of G_0 can de extended to a holomorphic transformation on M in a unique fashion. Then our results are summarized in the following

Theorem a) M is *divided into* $(n+1)(n+2)/2\,G_0$-*orbits, among which there exist* $n+1$(= (*the rank of* M)+1) *open* G_0-*orbits and exactly one compact* G_0-*orbit. The compact one is the Bergmann-Silov boundary of* M_0* (*which is isomophic to* $U(n)/O(n)$).

b) *The number of the fixed ponts of any non-singular transvection is* 2^n.

c) *The set of the fixed points of all the non-singular transvections coincides with the Bergmann-Silov boundary of* M_0.

The set of non-degenerate hermitian forms with a given non-degenerate imaginary part can be considered as a submanifold of $S_p(n)/U(n)$ (see section 15.) That submanifold has $n+1$ connected components corresponding to the $n+1$ signatures of the hermitian forms. This gives rise to the division $S_p(n)/U(n)$ by open G_0-orbits.

1. Notations. For a complex matrix a, ${}^t a$ is the transpose of a, $\bar{a}$ the complex-conjugate, a^* the transposed complex-conjugate and $Tr\,(a)$ the trace of a, 1_n is the n-by-n unit matrix. The diagonal matrix with the i-th diagonal element $\alpha_i (1 \leqq i \leqq n)$ will be denoted by $[\alpha_1, \cdots\cdots \alpha_n]$. Let a and b be n-by-n complex matrices. Then we denote by (a,b) (resp. $(a\ b)'$) the $2n$-by-$2n$ matrix

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \qquad \text{(resp.} \quad \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \text{)}.$$

As above, letters a, b, $\cdots$ stand for n-by-n complex matrices throughout this paper. Lie groups will be denoted by some capital Latin letters and

* This fact is due to Mr. M. Takeuchi.

the Lie algbras by the corresponding small German letters.

2. We recall here some known facts concerning symmetric manifolds which will be used in the sequel. Let M be a $2n$-by-$2n$ complex matrix. We write

$$M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then in order that $M \varepsilon S_p(n)$, it is necessary and sufficient that

$$\begin{aligned} c &= -\bar{b} \text{ and } d=a \\ aa^* - bb^* &= 1_n \\ a^tb &= b^ta . \end{aligned}$$

Hence if $M \varepsilon S_p(n)$, we can write M in the form $(a\ b)$.

The unitary group $U(n)$ can de imbedded in $S_p(n)$ by a faithful representation r defined by

$$r(u) = \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix},$$

which allows us to identify $U(n)$ with a subgroup $r(U(n))$ of $S_p(n)$. We write G instead of $S_p(n)$ and K instead of $U(n)$. The Lie algebra $\boldsymbol{g}$ is

$$\{(a\ b) \mid a^* = -a \text{ and } {}^tb=b\}.$$

We write $a=(a_{ij})$ and $b=(b_{ij})$. Then the Killing iorm β of g is given by

$$\beta(X, X) = -Tr(XX^*) = -2\sum_{i,j=1}^{n} \{|a_{ij}|^2 + [b_{ij}|^2\}$$

where $X \varepsilon \boldsymbol{g}$ and $X=(a\ b)$. We denote by $\boldsymbol{p}$ a real vector space $\{(0\ b) \mid {}^tb=b\}$. Then $\boldsymbol{p}$ anb $\boldsymbol{k}$ are mutually orthogonal with respect to β and vector space $\boldsymbol{g}$ turns out to be the direct sum of $\boldsymbol{k}$ and $\boldsymbol{p}$. The corresponding non-compact form $\boldsymbol{g}_0$ is

$$\{(a\ b)' \mid a^* = -a \text{ and } {}^tb=b\}.$$

The Cartan decomposition is

$$\boldsymbol{g}_0 = \boldsymbol{k} + \sqrt{-1}\,\boldsymbol{p}$$

and a Cartan subalgebra common to $\boldsymbol{g}$ and $\boldsymbol{k}$ is $\{([\alpha_1, \cdots, \alpha_n]0) \mid \alpha_1, \cdots, \alpha_n \varepsilon \boldsymbol{R}\}$, which will be denoted by $\boldsymbol{h}$. Let $\boldsymbol{h}^*$ be the dual space of $\boldsymbol{h}$. We can define an ordering in h^* by fixing that $\alpha = m_1\alpha_1 + \cdots + m_n\alpha_n$ (ε h^*) is positive if and only if the first non-vanishing coefficient is so. Then the positive roots relative to this ordering are

$$\alpha_i \pm \alpha_j \; (1 \leqq i < j \leqq n), \;\; 2\alpha_i \; (1 \leqq i \leqq n)$$

and the positive non-compact roots are

$$\alpha_i + \alpha_j \; (1 \leqq i < j \leqq n), \;\; 2\alpha_i \; (1 \leqq i \leqq n).$$

The complexification of g, denoted by g_c, is

$$\left\{ \begin{pmatrix} a & b \\ c & -{}^t a \end{pmatrix} \mid {}^t b = b, \; {}^t c = c \right\}.$$

Since G is simply-connected, there exists a simply-connected complex Lie group G_c with the Lie algebra g_c which contains G as a closed subgroup. It can be considered as $S_p(n, \boldsymbol{C})$. The subgroup G_0 of G_c, corresponding to g_0, is conjngate to $S_p(n, \boldsymbol{R})$. To be exact we have $G_0 = \frac{1}{2}(1_n - \sqrt{-1}\, 1_n)$ $S_p(n, \boldsymbol{R})\,(1_n \; \sqrt{-1}\; 1_n)$. The sum of the eigen-spaces corresponding to the positive non-compact roots is

$$\left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid {}^t b = b \right\}.$$

We shall denote by p^+ this abelian subalgebra of g_c. For the negative non-compact roots we have in the same may another abelian subalgebra p^-, where

$$p^- = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \mid {}^t b = b \right\}.$$

Let p^+ and p^- be the abelian subgroups of G_c that correspond to p^+ and p^- respectively. Then we have

$$p^+ = \left\{ \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} \mid {}^t b = b \right\},$$

$$p^- = \left\{ \begin{pmatrix} 1_n & 0 \\ b & 1_n \end{pmatrix} \mid {}^t b = b \right\}.$$

A maximal abelian subalgebra in p is

$$\{(0\; d) \mid d : \text{real diagonal}\}.$$

It is denoted by h_p. We write $d=[\alpha_1, \cdots, \alpha_n]$. Then then the roots of G/K with respect to h_p are

$$\pm\tilde{\alpha}_i\pm\tilde{\alpha}_j\ (1\leqq i<j\leqq n),\quad \pm 2\tilde{\alpha}_i\ (1\leqq i\leqq n).$$

An element of h_p on which some of the roots vanish is called *singular*. An element of exp h_p is called *singular* if it is the image of a singular element of h_p by the exponential map. An element of *exp* $\boldsymbol{p}$ is called a *transvection* on G/K (with respect to the Cartan deeomposition under consideration). Any transvection is the image of some element of *exp* h_p by K. A transvection is called *singular* if it is the image of a singular element of *exp* h_p by K.

3. Now let us consider a more general background and let G be a compact connected semisimple Lie group. Let G/K be a Hermitian symmetric manifold. Then K is of maximal rank and the center of K is a 1-torus. We denote by p the canonical projection of G onto G/K and by $\mathfrak{o}$ the origin $p(e)$. Any element k of K induces a linear transformation $\tilde{k}$ of the tangent vector space $(G/K)_{\mathfrak{o}}$ at $\mathfrak{o}$ of G/K (the linear isotropy representation.). The set of $\tilde{k}$ constitutes a group $\tilde{K}$ called linear isotropy group. Suppose this representation is faithful. The center of K is isomorphic to the torus

$$\{\ exp\sqrt{-1}\,\theta\ |\ 0\leqq\theta<2\pi\ \}.$$

We denote by J the element of $\tilde{K}$ that corresponds to $exp\sqrt{-1}\frac{\pi}{2}(=\sqrt{-1}.)$ It is clear that $J^2=-I$, where I is the identity. J, therefore, defines a complex structure on $(G/K)_{\mathfrak{o}}$ invariant under $\tilde{K}$. This complex structure is extended to an invariant almost complex structure on G/K, which turns out to be integrable. The other complex structure of G/K is defined by J^{-1}, i.e. $-J$.

4. Let us go back to the manifold $S_p(n)/U(n)$. The tangent vector space $(S_p(n)/U(n))_{\mathfrak{o}}$ is identifiable with $\boldsymbol{p}$ by the differential of the projection p. Then for $u\ \varepsilon\ U(n)$ $\tilde{u}$ is identifiable with the restriction on $\boldsymbol{p}$ of $ad(u)$. We denote by $\tilde{r}$ the linear isotropy representation. The kernel of $\tilde{r}$ is $\{(1_n 0),\ -(1_n 0)\}$. Hence we write G for $S_p(n)/\{\pm(1_n\ 0)\}$ and K for $U(n)/\{\pm(1_n\ 0)\}$. Let p' be the canonical projection of $U(n)$ onto K. We write

$$\eta=(\frac{1}{2}(1+\sqrt{-1})1_n\,0\,).$$

Then $\tilde{\gamma}$ is a complex structure on $(G/K)_0$, where $\tilde{\gamma}=\tilde{r}\,p'(\gamma)$. By the map that sends $(0\ b)$ to b, p is identifiable with the vector space over $\boldsymbol{R}$ of symmetric complex matrices. A symmetric complex matrix can be considered as a vector of $C^{\frac{n(n+1)}{2}}$. Using these identifications $\tilde{\gamma}$ may be considered as a complex structure on $C^{\frac{n(n+1)}{2}}$, regarded as a vector space over $\boldsymbol{R}$. In fact this complex structure coincides with the usual multiplication by $\sqrt{-1}$. By this complex structure an invariant complex structure is defined on G/K. The other one is what corresponds to the multiplication by $-\sqrt{-1}$ in $C^{\frac{n(n+1)}{2}}$.

The same argument holds for the dual symmetric manifold of G/K.

5. $S_p(1, C)$ is nothing but the group of unimodular 2-by-2 matrices. $S_p(1)$ coincides with $SU(1)$. We recall that the quaternion algebra can be considered as a subfield of the algebra of 2-by-2 matrices (by the regular representation.) Then $S_p(1)$ is the multiplicative group of quaternions of norm 1. We shall use $a, b, c, d, \ldots$ for complex numbers in this section. Then

$$S_p(1)\,\varepsilon\, g \rightleftarrows g=\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \text{ with } a\bar{a}+b\bar{b}=1.$$

We write

$$r'\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=\text{the linear transformation: } w=\frac{az+b}{cz+d}.$$ Then r' is a representation $S_p(1, C)$ into the group of holomorphic transformations on the extended complex plane $C+\{\infty\}$ and a conformal transformation.

By the othogonal projection the line-element on the sphere becomes a line-element ds on the unit disk and

$$ds^2=\frac{4dz\,d\bar{z}}{(1-|z|^2)^2}.$$

This line-element makes the unit disk into the (Hermitian) hyperbolic plane. The group of isometries is generated by

$$G_0=\{r'((ab)')\,|\,a\bar{a}-b\bar{b}=1\}$$

and $s_0: w=\bar{z}$. The former is the connected component of the identity. The isotropy group at the origin of G_0 is the rotation group $\{\exp\sqrt{-1}\,\theta\,|\,\theta\,\varepsilon\,\boldsymbol{R}\}$. We denote it by K. Then the unit disk can be considered as a Hermitian symmetric manifold G_0/K. We can get a (global) Cartan decoposition

(3) $$w=\exp\sqrt{-1}\,\theta\,\frac{z+\lambda th\,\alpha}{\lambda th\,\alpha p+1}\quad (\,|\lambda|=1 \text{ and } \theta,\ \alpha\in \boldsymbol{R})$$

which might be said to be canonical. The transvections relative to the above decomposition are

$$w=\frac{z\lambda th\,\alpha}{\lambda \mathrm{th}\,\alpha z+1}\quad (|\lambda| = 1 \text{ and } \alpha\in \boldsymbol{R}).$$

By the stereographic projection the line-element on the sphere becomes a line-element ds on the complex plane and

$$ds^2 = \frac{4dz\,d\bar{z}}{(1+|z|^2)^2}.$$

This line-element makes the extended complex plane into the (Hermitian) elliptic plane. Then the group of isometries is generated by $r'(S_p(1))$ and s_0. The former is the connected component of the identity. We denote it by G. The isotropy group at the origin of G is again K. The extended complex plane can be considered as a Hermitian symmetric manifold G/K, or $S_p(1)/U(1)$. The decomposition of G corresponding to (3) is

$$w=\exp\sqrt{-1}\,\theta'\,\frac{z+\lambda' t_g\alpha'}{-\bar{\lambda}' t_g\alpha' z+1}\quad (|\lambda'|=1 \text{ and } \theta',\ \in \boldsymbol{R}).$$

The transvections are

$$w=\frac{z+\lambda' t_g\alpha'}{-\lambda' t_g\alpha' z+1}\qquad (|\lambda'|=1 \text{ and } \alpha'\in\boldsymbol{R}).$$

By assigning to each $z_0=|z_0|\exp w$ of the unit disk a transvection with $\lambda=\exp w$ and $\alpha=Arc\ th\ |z_0|$, we can get a one-to-one map of the unit disk onto the set of transvections We can moreover assign to a transvection of hyperbolic geometry a transvection of elliptic geometry, i. e., the one with $\lambda'=\lambda$ and $\alpha'=Arc\ t_g\ th\ \alpha$. This transvection also sends the origin to z_0.

6. We use the well-known

Lemma 1. *Let b be an n-by-n symmetric complex matrix. Then there exists a unitary matrix such that $ub\,{}^tu$ is diagonal.* (See [1] for the proof.)

Lemma 2. *Let b be an n-by-n symmetric complex matrix. Then there exists a unitary matix u such that*

$(u\ 0)\exp(0\ b)(u^*0)=([\cos\alpha_1,\cdots,\ \cos\alpha_n]\ \bar{}\sin\alpha_1,\ \cdots,\ \sin\alpha_n])$.

Proof. Using the preceding lemma we choose a unitary matrix u

such that

$$u b^{t} u = [\alpha_1, \cdots, \alpha_n]$$

where we may suppose $\alpha_1, \cdots, \alpha_n$ are real and positive. For such a matrix u we have

$$u b \bar{b} u^* = u b^{t} u \overline{u b^{t} u} = [\alpha_1 \bar{\alpha}_1, \cdots \alpha_n \bar{\alpha}_n] .$$

We write θ for $\exp(0\ b)$ and R for $(u\ 0)$. Then we have

$$\theta = \sum_{i=0}^{\infty} \frac{(0\ b)^i}{i!} = \sum_{i=0}^{\infty} (-1)^i \frac{((b\ \bar{b})^i 0)}{(2i)!} + \sum_{i=0}^{\infty} (-1)^i \frac{((b\ \bar{b})^i 0)}{(2i+1)!} (0\ b).$$

Hence we can get

$$ad(R)\theta = \sum_{i=0}^{\infty} (-1)^i \frac{([\alpha_1^{2i}, \ldots \alpha_n^{2i}] 0)}{(2i)!}$$
$$+ \sum_{i=0}^{\infty} (-1)^i \frac{([\alpha_1^{2i}, \ldots \alpha_n^{2i}]\ 0)}{(2i+1)!} (0 [\alpha_1, \ldots, \alpha_n]).$$
$$= ([\cos \alpha_1, \ldots, \cos \alpha_n] [\sin \alpha_1, \ldots, \sin \alpha_n]).$$

This completes the proof.

For G_0/K an analogy holds good. Namely we have the

Lemma 3. *Let b be an n-by-n symmetric matric matrix. Then there exists a unitary matrix u such that*

$$(u\ 0)(0\ b)'(u^* 0) = [ch\ \alpha_1, \cdots, ch\ \alpha_n][sh\ \alpha_1, \cdots, sh\ \alpha_n])$$

where $\alpha_1, \cdots, \alpha_n$ are non-negative real numbers.

7. Let $q_1, \ldots, q_n$ be quaternions of norm 1. Then the set of diagonal by matrices $[q_1, \cdots, q_n]$ constitutes a multiplicative group, which we denote $\boldsymbol{H}$. We write

$$q_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\bar{\beta}_1 & \bar{\alpha}_1 \end{pmatrix}, \quad \cdots, \quad q_n = \begin{pmatrix} \alpha_n & \beta_n \\ -\bar{\beta}_n & \bar{\alpha}_n \end{pmatrix}.$$

If we define

$$\hat{r}(q_1, \cdots, q_n) = ([\alpha_1, \cdots, \alpha_n][\beta_1, \cdots, \beta_n]),$$

then $\hat{r}$ is a representation of $\boldsymbol{H}$ into $S_p(n)$. Remember that p is the canonical projection of $S_p(n)$ onto $S_p(n)/U(n)$, Then $(p \circ \hat{r})^{-1}(o)$ is $\{(\exp \sqrt{-1}\,\theta_1, \cdots, \exp \sqrt{-1}\,\theta_n) \mid \theta_1, \cdots, \theta_n \in \boldsymbol{R}\}$. Hence $p \circ \hat{r}$ induces an imbedding i of S into G/K ($G=S_p(n)$, $K=U(n)$), where S denotes the product of n-copies of the extended complex plane, a complex manifold. We shall show that i is holomorphic. For that it suffices to prove that the differential at any point of i is $\boldsymbol{C}$-linear. For notational simplicity we identify S with a subset of G/K by i. Then our purpose is to show that the tangent vector space S_0 of S at the origin o is a complex vector subspace of $(G/K)_0$. We have seen (section 4) that $(G/K)_0$ is identifiable with $\{(0\ b)\mid {}^tb=b\}$ and the complex structure is given by the usual multiplication by $\sqrt{-1}$. On the other hand, as is easily seen S_0 is

$$\{(0[\beta_1, \cdots, \beta_n])\mid \beta_1, \cdots, \beta_n \varepsilon \boldsymbol{C}\}.$$

The complex structure is the tensor product of n copies of the complex structure of the complex plane, i. e., this is also the usual multiplication by $\sqrt{-1}$. Hence S_0 is a complex vector subspace of $(G/K)_0$.

8. Let us consider the so-called Harish-Chandra realization of a portion of G/K. Any element Z of $\boldsymbol{p}^-$ has the form

$$\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.$$

Hence $\exp Z=$

$$\begin{pmatrix} 1_n & 0 \\ c' & 1_n \end{pmatrix}.$$

G/K is naturally identical with the complex homogeneous manifold G_c/K_cP^+. $\exp Z$ belongs to the coset

$$\left\{\begin{pmatrix} a & ab \\ c'a & c'ab+{}^ta^{-1} \end{pmatrix} \mid \alpha \varepsilon GL(n, C),\ {}^tb=b\right\}.$$

On the other hand, for any $(a\ b)\ \varepsilon\ S_p(n)$ we can choose a unitary matrix u such that $a=u[\cos\alpha_1, \cdots, \cos\alpha_n]u^*$ and $b=u[\sin\alpha_1, \cdots, \sin\alpha_n]{}^tu$. Let us suppose that a is inversible, e. g., $0\leqq\alpha_1, \cdots, \alpha_n<\frac{\pi}{2}$. If we write Z for $-\bar{b}a^{-1}$, then we have

$$1_n-Z\bar{Z}=\bar{u}[1-t_g\alpha_1, \cdots, 1-t_g\,\alpha_n]u^t.$$

Hence we see that

$$1_n-Z\bar{Z}>0 \leftrightarrow 0\leqq\alpha_1, \cdots, \alpha_n<\frac{\pi}{4}\ .$$

Consequently the set of $x=p(a\ b)\ \varepsilon\ G/K$ with $0\leqq\alpha_1,\cdots,\alpha_n<\frac{\pi}{4}$ constitutes a portion in G/K which is complex analytically homeomorphic to G_0/K. The realization is the map which sends x of the portion to $h(x)=-\bar{b}a^{-1}$.

We note that we can fix this portion by another method. Let us first consider the natural imbedding of D into S where D is the product of n isomorphic copies of the unit disk. D can be considered as a complex submanifold of M_0 in exactly the same way as S in section 7. We denote the above imbedding of D into S by j and we recall that any transvection of M_0 is written $ad(k)g$ with $k\ \varepsilon\ K$ by the use of a transvection of the form stated in Lemma 3. We define

$$j^*(ad\ (k)gK)=ad(k)j(g)K.$$

It is easily seen that j^* is well-defined. j^* is a holomorphic imbedding of M_0 into M such that $j^*\ (M_0)=h(M_0)$.

9. The algebra $\mathfrak{A}$ of continuous function f on $\overline{M}_0$ that are holomophic in M_0 becomes a Banach algebra if we define the norm $|f|$ by *max* $|f|$. A maximal ideal $\mathfrak{H}$ of $\mathfrak{A}$ is the set of functions $\varepsilon\ \mathfrak{A}$ which vanish at a definit point x. $\mathfrak{A}/\mathfrak{H}$ is isomorphic to $\boldsymbol{C}$ in the natural way. The value corresponding to the residue class $f-\mathfrak{H}$ is written $f(\mathfrak{H})$. Then $f(\mathfrak{H})$ is a functional defined on the space X of maximal ideals. If we introduce the usual topology into X, then the map which sends $\mathfrak{H}$ to x is a homeomorphism of X onto $\overline{M}_0$. In analysis it is known that X has a uniquely defined (ring) boundary namely a minimal closed set on which every $f\ \varepsilon\ \mathfrak{A}$ achieves its maximum. The image by the above homeomorphism of the (ring) boundary is called *Bergmann-Silov boundary* of M_0.

10. $S_p(n)$ is a subgroup of $U(2n)$ and $U(n)$, contained in $S_p(n)$, is a subgroup of $U(n)\times U(n)$. These inclusions induce that of the manifolds: $M\subset G_{n,n}(\boldsymbol{C})$. This inclusion can be explained in another way. Suppose $x\ \varepsilon\ M$ and $x=gK$ with $g\ \varepsilon\ G\ (G=S_p(n),\ K=U(n))$. We write $g=(a\ b)$. Then the n column vectors of the matrix

$$\begin{pmatrix} a \\ -\bar{b} \end{pmatrix} \tag{4}$$

span an n-plane in $\boldsymbol{C}^{2n}$. This n-plane does not depend on the choice of g. Hence we can get a map of M into $G_{n,n}(\boldsymbol{C})$. In fact this map is a holomorphic imbedding as an easy check shows it. We write

$$E_n=\begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}.$$

Let us consider the hermitian matrix

$$\begin{pmatrix} a \\ -\bar{b} \end{pmatrix}^* E_n \begin{pmatrix} a \\ -\bar{b} \end{pmatrix}.$$

The signature of this matrix depends only on x. We can therefore divide M_0 into $(n+1)(n+2)/2(=1+\cdots+(n+1))$ subsets by the use of signatures. Since G_0 is a subgroup of $U(n, n)$, we see $g^*E_n g=E_n$ for $g \varepsilon G_0$. Hence G_0 leaves these subsets invariant Let us go back to S. $x \varepsilon S$ is represented by a transvection in Lemma 2. Consequently the Hermitian matrix is written as

$$[cos^2\alpha_1-sin^2\alpha_1, \cdots, cos^2\alpha_n-sin^2\alpha_1](0\leqq\alpha_i\leqq\frac{\pi}{2};\ i=1, \cdots, n).$$

Hence the signature of x is

$$(\text{the number of } i \text{ with } \alpha_i<\frac{\pi}{4},\ \text{the number of } i \text{ with } \alpha_i>\frac{\pi}{4}).$$

It follows that S is divided into 3^n subsets. The K-orbits of these subsets is $(n+1)(n+2)/2$ in number by the Weyl group being taken into consideration. Each of these K-invariant subsets must be contained in one of the preceding G_0-invariant subsets. There exist the same number of them. Hence we can conclude that both of the divisions must coincide. It is immediate that G_0 acts on each of these subsets transitively. Thus we can find that these $(n+1)(n+2)/2$ subsets are G_0-orbits. Only the one of the lowest dimension is compact and K acts on it transitively as is easily seen. It is on the boundary of M_0. Hence the only compact G_0-orbit is the Bergmann-Silov boundary of M_0 [3].

11. We confine ourselves to the domain of n-planes (4) with det $a\neq0$. Then this domain is in 1-1 correspondence with n-by-n matrix Z by putting $Z=-\bar{b}a^{-1}$. The matrix $(c\ d)$ sends Z to

(5) $$-\bar{b}+\bar{c}Z/c+dZ$$

if det $(c+dZ)\neq0$. The preceding Bergmann-Silov boundary is contained in this domain. It is the K-orbit of $Z=1_n$. Suppose $(c\ d)=(u\ 0)$. Then (5) turns out to be uZu^{-1}. Hence the isotropy group at 1_n of K is the group of such a matrix $(u\ O)$ with $\bar{u}1_n u^{-1}=1_n$, namely u: real. Hence it is isomorphic to $O(n)$. Consequently the Bergmann-Silov boundary is isomorphic to the reducible symmetric manifold $U(n)/O(n)$. Its dimension is half the dimension of the whole manifold.

12. We shall here in this section discuss how the fixed points of transvections are distributed and how many of them each transvection leaves invariant. Let θ be a transvection of G/K. The fixed points of θ are the left cosets gK which satisfy

$$g^{-1}\theta\, g \in K. \tag{6}$$

Then it is seen that kx is a fixed point of $ad\,(k)\,\theta$ if and only if x is a fixed point of θ where $k \in K$. Consequently we can confine our attention to a transvection θ of the form $([cos\ \alpha_1, \cdots, cos\ \alpha_n]\,[sin\alpha_1, \cdots, sin\ \alpha_n])$ without any loss of generality.

Let $g=(a\ b)$. Then $g^{-1}=(a^*\ -{}^tb)$. We write $x=gK$. suppose x is a fixed point of $\theta=([cos\ \alpha_1, \cdots, cos\ \alpha_n]\,[sin\ \alpha_1, \cdots, sin\ \alpha_n])$. Then (6) holds. On the other hand we have

$$\begin{aligned} g^{-1}\theta g &= (a^*[cos\ \alpha_1, \cdots, cos\ \alpha_n]+{}^tb[sin\ \alpha_1, \cdots, sin\ \alpha_n] \\ & \quad a^*[sin\ \alpha_1, \cdots, sin\ \alpha_n]-{}^tb[cos\ \alpha_1, \cdots, cos\ \alpha_n])\ g \\ &=(a^*[cas\ \alpha_1, \cdots, cos\ \alpha_n]a+{}^tb[sin\ \alpha_1, \cdots, sin\ \alpha_n]a \\ & \quad -a^*[sin\ \alpha_1, \cdots, sin\ \alpha_n]\bar{b}+{}^tb[cos\ \alpha_1, \cdots, cos\ \alpha_n]b \\ & \quad a^*[sin\ \alpha_1, \cdots, sin\ \alpha_n]\bar{a}-{}^tb[cos\ \alpha_1, \cdots, cos\ \alpha_n]\bar{a} \\ & \quad +a^*[cos\ \alpha_1, \cdots, cos\ \alpha_n]b+{}^tb[sin\ \alpha_1, \cdots, sin\ \alpha_u]b). \end{aligned}$$

Hence we obtain

$$\begin{aligned} & a^*[sin\ \alpha_1, \cdots, sin\ \alpha_n]\bar{a}+{}^tb[sin\ \alpha_1, \cdots, sin\ \alpha_n]b \\ &={}^tb[cos\ \alpha_1, \cdots, cos\ \alpha_n]\bar{a}-a^*[cos\ \alpha_1, \cdots, cos\ \alpha_n]b. \end{aligned}$$

We note that the left hand side of the above equality is symmetric, while the right hand side anti-symmetric. It follows from this fact that

$$\begin{aligned} & a^*[cos\ \alpha_1, \cdots, cos\ \alpha_n]b={}^tb[cos\ \alpha_1, \cdots, cos\ \alpha_n]\bar{a}. \\ & a^*[sin\ \alpha_1, \cdots, sin\ \alpha_n]\bar{a}=-{}^tb[sin\ \alpha_1, \cdots, sin\ \bar{\beta}_n]b. \end{aligned} \tag{7}$$

There exists a unitary matrix u such that

$$\begin{aligned} a &= u[cos\ \beta_1, \cdots, cos\ \beta_n]u^* \\ b &= u\,[exp\sqrt{-1}\varphi_1, sin\ \beta_1, \cdots, exp\sqrt{-1}\varphi_n\ sin\ \beta_n]\,{}^tu. \end{aligned}$$

We note that we can change φ arbitrarily. (In that case u of course changes depending on it.) From (7) it follows that

$$u[\cos\beta_1, \cdots, \cos\beta_n]u^*[\cos\alpha_1, \cdots, \cos\alpha_n]u$$
$$[\exp-\sqrt{-1}\varphi_1 \sin\beta_1, \cdots, \exp-\sqrt{-1}\varphi_n \sin\beta_n]^t u$$
$$=u[\exp-\sqrt{-1}\varphi_1 \sin\beta_1, \cdots, \exp-\sqrt{-1}\varphi_n \sin\beta_n]^t u$$
$$[\cos\alpha_1, \cdots, \cos\alpha_n]\bar{u}[\cos\beta_1, \cdots, \cos\beta_n]^t u.$$

If we write $w=u^*[\cos\alpha_1, \cdots, \cos\alpha_n]u$, than we have

$$[\cos\beta_1, \cdots, \cos\beta_n]w[\exp\sqrt{-1}\varphi_1 \sin\beta_1, \cdots. \exp\sqrt{-1}\varphi_n \sin\beta_n]$$
$$=[\exp\sqrt{-1}\varphi_1 \sin\beta_1, \cdots, \exp\sqrt{-1}\varphi_n \sin\beta_n]\bar{w}[\cos\beta_1, \cdots, \cos\beta_n].$$

Hence we can obtain

$$\exp\sqrt{-1}(\varphi_i-\varphi_j)w_{ji}t_g\beta_i=t_g\beta_j\bar{w}_{ji},$$

where $w=(w_{ij})$. Let $i\neq j$. Then $\varphi_i-\varphi_j$ are considered as arbitrary. Actually they are variable independent of w_{ij}. Consequently we have

$$w_{ij}t_g\beta_j=\bar{w}_{ij}t_g\beta_i=0. \tag{8}$$

For the sake of brevity we suppose

$$t_g\beta_1, \cdots, t_g\beta_k\neq 0\,;\, t_g\beta_{k+1}=\cdots=t_g\beta_n=0.$$

Then it follows from (8) that $w_{ij}=0$ for each of these cases:

1) $1\leqq i\leqq k$, j: arbitrary
2) i: arbitray, $1\leqq j\leqq k$.

$(i\neq j)$

Consequently we can write w in the form

$$\begin{pmatrix} d & 0 \\ 0 & w' \end{pmatrix} \tag{9}$$

where d denotes a diagonal k-by-k real matrix and w' an $(n-k)$-by-$(n-k)$ hermitian matrix. The matrix (9) can be diagonalized by a unitary matrix of the form

$$\begin{pmatrix} 1_k & 0 \\ 0 & u' \end{pmatrix}$$

Hence if we denote this matrix by $\tilde{u}$, $(u\tilde{u})^*[\cos\alpha_1, \cdots, \cos\alpha_n]$ $(u\tilde{u})$ is diagonal.

Suppose now θ is regular in the sense of section 2. Then $(u\bar{u})^*[\cos\alpha_1, \cdots, \cos\alpha_n]u\bar{u}) = [\cos\alpha_{\sigma(1)}, \cdots, \cos\alpha_{\sigma(n)}]$ for a permutation of 1, ⋯, n. Consequently $u\bar{u} = \bar{d}u_\sigma$ where $\bar{d}$ is a diagonal unitary matrix and where $\sigma \longrightarrow u_\sigma$ is the usual unitary representation of the permutation group. It follows that $u = \bar{d}u_\sigma\bar{u}^*$. Hence $g = \bar{d}u_\sigma\bar{u}^*([\cos\beta_1, \cdots, \cos\beta_t, 1, \cdots, 1]\ [\sin\beta_1, \cdots, \sin\beta_k, 0, \cdots, 0]\bar{u}u^*\ \bar{d}^*$. By an easy computation we can find that g has the form

$$([\cos\beta'_1, \cdots, \cos\beta'_n]\ [\lambda'_1\sin\beta_1', \cdots, \lambda_n'\sin\beta_n])$$

where $\beta'_1, \cdots, \beta'_n \varepsilon \boldsymbol{R}$ and $|\lambda_1'| = \cdots = |\lambda'_n|$.

After all we can state our conclusion as follows:
All the fixed points of the transvection θ exist in S.

13. Consider the case of the elliptic plane and let $\theta = (\cos\alpha\ \lambda\ \sin\alpha)$. In order that the coset containing $g = (\cos\beta\ \bar{\lambda}\sin\beta)$ be a fixed point of θ, it is necessary and sufficient that $g^{-1}\ \theta\ g\ \varepsilon\ U(1)$ and therefore that

$$(\lambda)^2\cos^2\beta + (\bar{\lambda})^2\sin^2\beta = 0.$$

Hence we have

$$\bar{\lambda}t_g\beta = \pm\sqrt{-1}\lambda.$$

Consequently we can obtain the

Proposition. *A transvection* $(\cos\alpha\ \ \lambda\ \sin\alpha)$ *of the elliptie plane has as its fixed points the two cosets containing*

$$\frac{1}{\sqrt{2}}(1\ \ \sqrt{-1}\,\lambda) \text{ and } \frac{1}{\sqrt{2}}(1\ \ -\sqrt{-1}\,\lambda)$$

respectively.

14. Recall that S denotes the product manifold of n copies of the Riemann sphere. Let θ be a transvection $([\cos\alpha_1, \cdots, \cos\alpha_n]\ [\sin\alpha_1, \cdots, \sin\alpha_n])$. According to the conclusion of section 12, θ has all its fixed points in S. On the other hand each copy remains invariant under θ. θ acts on the i-th copy in the same way as the 2-by-2 matrix $(\cos\alpha_i\ \ \sin\alpha_i)$. Hence the fixed points of θ in each copy are the cosets containing $\frac{1}{2}(1\ \ \pm\sqrt{-1})$ (section 13). We therefore come the following conclusion.

Theorem. *Let θ be an arbitrary transvection on $S_p(n)/U(n)$. We write it in the form $u([\cos \alpha_1, \cdots, \cos \alpha_n] \,|\, \sin \alpha_1, \cdots, \sin \alpha_n])$, where u is a unitary matrix. Then the fixed points of θ are these 2^n points :*

$$\frac{1}{\sqrt{2}} u^*(1_n \quad [\pm\sqrt{-1}, \cdots, \pm\sqrt{-1}])U(n). \tag{10}$$

If u goes through $U(n)$, the locus of the points (10) is precisely the Bergmann-Silov boundary stated in Sections **10** and **11** which can be seen easily.

15. Let A be a non-degenerate alternate bilinear form over $V \times V$ where V denotes a real vector space of dimension $2n$. Consider a hermitian form h with A as *imaginary part*, by introducing a complex structure J into V. The set of h is in 1-1 correspondence with the set of J satisfying

$$A(J(x), J(y)) = A(x, y)$$

where $x, y \varepsilon V$. The eigenspace of J belonging to the eigenvalue $\sqrt{-1}$ can be considered as a point of the Grassmann manifold $G_{n,n}(C)$, the manifold of n-planes in $C \otimes E$. We denote that point by $f(h)$. Let N be the set of non-degenerate hermitian forms with the imaginary part A. Then the closure of $f(N)$ is a complex snbmanifold wh'ch is complex analytically homeomorphic to $S_p(n)/U(n)$. Corresponding to the signatures of hermitian forms $i(N)$ can be divided into $(n+1)$ connected components. This gives a division of $S_p(n)/U(n)$ equivalent to what has been discussed in the present paper. The details will appear elsewhere. See also [2].

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