IMBEDDINGS OF DOLD MANIFOLDS

TAKUO FUKUDA.

Let $P(m, n)$ be the manitold defined by A. Dold [1], that is the manifold of dimension $m+2n$ obtained from $S^m \times PC(n)$ by identifying (x, z) with $(-x, \bar{z})$, where S^m is the usual m-sphere and $PC(n)$ is the usual complex projective n-space, and $x \in S^m$ is the antipodal point of $x \in S^m$ and $\bar{z} \in PC(n)$ is the complex conjugate point of $z \in PC(n)$. The purpose of this paper is to obtain an imbedding and non-imbedding theorem for these Dold manifolds in euclidean space.

In this paper we say that a homeomorphism of $P(m, n)$ into a euclidean k-space is an imbedding if it is differentiable and regular.

This paper consists of two parts. In the first part we have an imbedding theorem.

Theorem 1. *Let H be the normal line bundle of the cannonical imbedding of $P(m, n)$ imto $P(m+1, n)$. If $P(m, n)$ can be imbedded in a euclidean k-space with normal bundle N so that $N \otimes H$ has a nonzero cross-section, then $P(m+1, n)$ can be topologically imbedded in a $(k+1)$-space.*

By A. Haefliger [3], if $2k > 3(m+2n+1)$ then this can be approximated by a differentiable imbedding. Finding the obstructions to the existence of a non-zero cross-section of $N \otimes H$ we have

Theorem 2. *Suppose $n>0$ $n \equiv 0$ mod 2 $m \equiv 1$ mod 4 and $2n+m \neq 2^k+1$. Then $P(m, n)$ can be imbedded in* $(2m+4n-2)$-space.

In the second part we make an application of Thom's Non-Imbedding Theorem [7]. We calculate the values of the inverse automorphism Q of the total Steenrod square Sq on $H^*(P(m, n); Z_2)$ and we have

Theorem 3. *Suppose $n=2^i$ and $m=2^k-2^{i+1}+2^j>0$, where $j<i$. Then $P(m, n)$ can not be imbedded in $(2m+4n-2)$-space.*

Massey and Peterson showed that a manifold of dimensions $n \neq 2^i$ can be imbedded in $(2n-1)$-space [5], so the result is best for $2n+m \neq 2^h$. Recently J. J. Ucci got the best result for the same codimension in the cases of the types $(0, 2^i)$, $(2^j, 2^i)$; $0 \leqq j \leqq i$, $(2^j-1, 1)$; $j \geqq 1$, and $(m, 2^i)$; $2^{1+i} \leqq m$ and $\binom{2m+2^{i+1}-1}{m} \equiv 1$ mod 2, [8]. A part of theorem III is common to J. J. Ucci's second type.

Notation: Let $X/\sim$ denote the quotient space of X by the relation $\sim$, for example $S^m \times P\bar{C}(n)/(x, z) \sim (-x, \bar{z})$ denotes $P(m, n)$. $A \subset B$ means that A is topologically imbeddable in B.

Part I

1. Proof of theorem I.

The proof of this theorem is the same as the one in [2] for $n=0$, and also analogous to it for the other cases. Let V be a tubular neighborhood of $P(m, n)$ in $P(m+1, n)$. Since H is then the normal line bundle of the cannonical imbedding of $P(m, n)$ into $P(m+1, n)$, V can be naturally imbedded in H. Let I denote the trivial line bundle over $P(m, n)$. If L, M are two vector bundles over $P(m, n)$, let $Hom(L, M)$ be the bundle whose fiber at p is $Hom(L_p, M_p)$, where L_p and M_p are fibers at p respectively. Then $Hom(H, I) \approx H$ and so $N \otimes H \approx N \otimes Hom(H, I) \approx Hom(H, N)$. A non-zero cross-section of $N \otimes H$ is also a non-zero cross-section of $Hom(H, N)$, that is, an imbedding of H as a subbundle of N. So if there exists a non-zero cross-section of $N \otimes H$, we have an imbedding of H into N, and by the natural imbedding of V into H we have $V \subset H \subset N \subset R^k$.

Observe the cellular structure of $P(m+1, n)$:

$$P(m+1, n) = S^{m+1} \times PC(n)/(x, z) \sim (-x, \bar{z})$$
$$= {}^+S^{m+1} \times PC(n)/(x_0, z) \sim (-x_0, \bar{z}),$$

where ${}^+S^{m+1}$ is the upper half of S^{m+1} and $x_0, -x_0 \in S^m$. Therefore V is homeomorphic to $[0, 1) \times S^m \times PC(n)/(0, x, z,) \sim (0, -x, \bar{z})$. Let D^{m+1} denote a disk of dimensions $m+1$, and let $\bar{X}$ and $\dot{X}$ dcnote the closure and the boundary of a set X respectively. Then $P(m+1, n) - V \approx D^{m+1} \times PC(n)$ and we have $P(m+1, n) = \bar{V} \cup_{id} D^{m+1} \times PC(n)$, here $\bar{V} \cup_{id} D^{m+1} \times PC(n)$ is a cell complex which is obtained by attaching $D^{m+1} \times PC(n)$ to $\bar{V}$ by the attaching homeomorphism, i. e., $id: \dot{D}^{m+1} \times PC(n) \approx \dot{V}$. Hence

$$P(m+1, N) \subset R^k \times R^1 \subset R^{k+1}, \quad \text{Q. E. D.}$$

2. The obstructions (proof of theorem II).

2.1 Obstruction theory. Massey and Peterson showed that for any compact differentiable manifold M of dimensions $n \neq 2^k$ there exists an imbedding of M in a $(2n-1)$-space [5]. Let N denote the normal bundle of the imbedding of $P(m, n)$ in a $(2m+4n-1)$-space for $m+2n \neq 2^k$. Then $N \otimes H$ is an $(m+2n-1)$- plane bundle over $P(m, n)$. In order to find a non-zero cross-section of $N\otimes H$, we need observe only the primary and the secondary obstructions to the existence of a cross-section of its associated S^{m+2n-2} — bundle.

We see from M. Mahowald [4] the following.

1). The primary obstruction is the Fuler class

$$\chi(N\otimes H)\in H^{m+2n-1}(P(m,n);Z)$$

2). If $\chi=0$, there exist many extentions of the cross-section to $(m+2n-1)$-skelton of $P(m,n)$. The set of the secondary obstruction classes in $H^{m+2n}(P(m,n);Z_2)$ for such extended sections is a coset of the group $(Sq^2+w_2)H^{m+2n-2}(P(m,n);Z)$. Hence if

$$(Sq^2+w_2)H^{m+2n-2}(P(m,n);Z)=H^{m+2n}(P(m,n);Z_2),$$

then we have an extention of the section to $P(m,n)$, where w_i denotes the Stiefel-Whitney class.

2.2. The Euler class.

Proposition 1. $H^{2n+m-1}(P(m,n);Z)=0$ for $m+n=$even,

i. e, $\chi(N\otimes H)=0$ for $m+n=$even.

Proof. We consider the cellular decomposition and the boundary operator of $P(m,n)$ in Dold's [1]. The notations of cells are the same as in [1]. $\partial(C_i,D_j)=(1+(-1)^{i+j})(C_{i-1},D_j)$. Let (C^i,D^j) denote the cochain which assigns 1 to (C_i,D_j) and 0 to all other $(i+2j)$-cells. Then the cochain group $C^{2n+m-1}(P(m,n);Z)$ is generated by (C^{m-1},D^n).

$$\begin{aligned}\delta(C^{m-1},D^n)(C_m,D_n)&=(C^{m-1},D_n)(\partial(C_m,D_n))\\&=(C^{m-1},D^n)(1+(-1)^{m+n}(C_{m-1},D_n))=2\\&\qquad\qquad for\ m+n=even.\end{aligned}$$

So $Z^{2n+m-1}(P(m,n);Z)=0$, and $H^{2n+m-1}(P(m,n);Z)=0$ for $m+n=$even.

2.3 The class $w_2(N\otimes H)$.

Let $T_{m,n}$ denote the tangent bundle of $P(m,n)$, and let $\mathfrak{N}_T$ denoto the normal bundle of the cannonical imbedding of $T_{m,n-1}$ into $T_{m,n}$. Let c and $d\neq c^2$ be the generators of $H^1(P(m,n);Z_2)$ and $H^2(P(m,n);Z_2)$ respectively.

To prove lemma 2 we use the following Dold's result [1] and the well known fact with respect to the Stiefel-whitney classes of the tensor product of bundles.

(a) 1) $w(T_{m,n})=(1+c)^m(1+c+d)^{n+1}$, especiallzy $w(T_{2,1})=1$.

2) $w(\mathfrak{N}_T)=(1+c+d)$.
3) $T_{m,n}\oplus H=g^*(T_{m+1,n})$, where g is the cannonical imbedding of $P(m, n)$ *into* $P(m+1, n)$.
4) $T_{m,n-1}\oplus\mathfrak{N}_T=f^*(T_{m,n})$, where f is the cannonical imbedding of $P(m, n-1)$ into $P(m, n)$.

(b) Let ζ be an r-vector bundle over X with $w(\zeta)=\prod_{i=1}^{r}(1+\alpha_i c)$ and η be a q-vector bundle over X with $w(\eta)=\prod_{j=1}^{q}(1+\beta_j c)$, where $c\in H^1(X; Z_2)$. Then $w(\zeta\otimes\eta)=\prod_{i,j}(1+(\alpha_i+\beta_j)c)$.

Lemma 2.
1) $w(T_{2,1}\otimes H)=(1+c)^4$.
2) $w(\mathfrak{N}_T\otimes H)=(1+c+d)$.

Proof. 1) $w(T_{2,1})=1$ and $w(H)=1+c$, hence by (b) we have immadiately $w(T_{2,1}\otimes H)=(1+c)^4$.

2) Consider the restriction of $\mathfrak{N}_T\otimes H$ to real projective space PR^m of dimensions $m(\subset P(m, n).)$. Let $\mathfrak{N}_T|_{PR^m}$ and $H|_{PR^m}$ denote the restriction of $\mathfrak{N}_T$ and H to PR^m respectively. Then by (a) we have $w(\mathfrak{N}_T|_{PR^m})=1+c$, and $w(H|_{PR^m})=1+c$. By (b) $w(\mathfrak{N}_T|_{PR^m}\otimes H|_{PR^m}\otimes H|_{PR^m})=1+c$. Hence $w(\mathfrak{N}_T\otimes H)$ is equal to whether $1+c$ or $1+c+d$. If we suppose $w(\mathfrak{N}_T\otimes H)=1+c$, then by (b) we have $w((\mathfrak{N}_T\otimes H)\otimes H)=1+c$. but $\mathfrak{N}_T\otimes H\otimes H=\mathfrak{N}_T$, and so $w(\mathfrak{N}_T\otimes H\otimes H)=w(\mathfrak{N}_T)=1+c+d$. This is a contradiction. Hence we have $w(\mathfrak{N}_T\otimes H)=1+c+d$.

Lemma 3. $w_i(T_{m,n}\otimes H)=[(1+c+d)^{n-1}]_i$ *for* $i=1, 2$, *where* $[\ \ ]_i$ *denotes the i-dimensional component.*

Proof. If we leave out of cosideration of the classes of dimensions$\geqq$ 3,

$$\begin{aligned}w(T_{m,n}\otimes H)&=w(T_{m,n-1}\otimes H)w(\mathfrak{N}_T\otimes H)\\&=w(T_{m,1}\otimes H)(w(\mathfrak{N}_T\otimes H))^{n-1}\\&=w((T_{m-1,1}\oplus H)\otimes H)(w(\mathfrak{N}_T\otimes H))^{n-1}\\&=w(T_{m-1,1}\otimes H)(w(H\otimes H))(w\mathfrak{N}_T\otimes H))^{n-1}\\&=w(T_{2,1}\otimes H)(w(H\otimes H))^{m-2}(w(\mathfrak{N}_T\otimes H))^{n-1}\\&=(1+c)^4(1+c+d)^{n-1}\\&=(1+c+d)^{n-1}.\end{aligned}$$

Proposition 4. *Suppose m and n are even. Then* $N\otimes H$ *is an orientable bundle and*

$w_2(N\otimes H)=c^2+d$ *for* $n\equiv 2 \bmod 4$

$w_2(N\otimes H)=d$ *for* $n\equiv 0 \bmod 4$.

Proof. $(T_{m,n}\oplus N)\otimes H=I^{4n+2m-1}\otimes H$. so $w(T_{m,n}\otimes H)w(N\otimes H)=w(I^{4n+2m-1}\otimes H)=(1+c)^{4n+2m-1}$. If m and n are even, then $(1+c)^{4n+2m-1}=(1+c+c^2+\cdots\cdots)$. And

$w(T_{m,n}\otimes H)=1+c+d+\cdots\cdots$ for m is even and $n\equiv 2$ mod 4,

$w(T_{m,n}\otimes H)=1+c+d+c^2+\cdots\cdots$ for m is even and $n\equiv 0$ mod 4.

Hence we have the proposition.

2.4 $(\mathbf{Sq}^2+\mathbf{w}_2)\mathbf{H}^{2n+m-2}(\mathbf{P}(\mathbf{m},\mathbf{n});\mathbf{Z})$

Lemma 5. *Suppose $m+n$ is even. Then $H^{2n+m-2}(P(m,n);Z)\cong Z+Z_2$, which is generated by the cohomology classes of cocycles (C^m, D^{n-1}) and (C^{m-2}, D^n).*

Proof. Cochain group $C^{2n+m-2}(P(m,n);Z)=\{(C^m,D^{n-1})\}+\{(C^{m-2},D^n)\}$.

$\delta(C^m, D^{n-1})(C_{m-1}, D^n)=0$, so $\delta(C^m, D^{n-1})=0$.

$\delta(C^{m-2}, D^n)(C_{m-1}, D_n)=(C^{m-2}, D^n)(\delta(C_{m-1}, D^n))$

$=(C^{m-2}, D^n)((1+(-1)^{m+n-1})(C_{m-2}, D_n))=0$, so

$\delta(C^{m-2}, D^n)=0$: Hence $Z^{2n+m-2}(P(m, n);Z)=\{(C^m, D^{n-1})\}+\{(C^{m-2}, D_n)\}$.

On the other hand,

$\delta(C^{m-1}, D^{n-1})(C_m, D_{n-1})=(C^{m-1}, D^{n-1})((1+(-1)^{m+n-1})(C_{m-1}, D_{n-1}))=0$, so $\delta(C^{m-1}, D^{n-1})=0$.

$\delta(C^{m-3}, D^n)(C_{m-2}, D_n)=(C^{m-3}, D^n)((1+(-1)^{m+n-2})(C_{m-3}, D_n)=2$, so $\delta(C^{m-3}, D^n)=2(C^{m-2}, D^n)$. Hence if $m+n$ is even, then $H^{2n+m-2}(P(m, n);Z)=\{(C^m, D^{n-1})\}+\{(C^{m-2}, D^n)\}/\{2(C^{m-2}, D^n)\}\cong Z+Z_2$.

Here $\{x\}$ denotes the free group generated by x.

Let $[x]$ denote the cohomology class of x. Then

$$Sq^2[(C^{m-2}, D^n)]=\binom{m-2}{2}c^m d^n+\binom{m-2}{1}\binom{n}{1}c^m d^n.$$

$$Sp^2[(C^m, D^{n-1})]=\binom{n-1}{1}c^m d^n.$$

Hence we have thethe following table.

m, u *mod* 4	$w_2. H^{2n+m-2}(P(m, n); Z)$	$S_q^2(C^{m-2}, D^n)$	$S_q^2(C^m, D^{n-1})$
$n\equiv 0$ and $m\equiv 0$.	$H^{2n+m}(P(m, n);Z_2)$	$c^m d^n$	$c^m d^n$
$n\equiv 2$ and $m\equiv 0$.	0	0	$c^m d^n$
$n\equiv 0$ and $m\equiv 2$.	$H^{2n+m}(P(m, n);Z_2)$	0	$c^m d^n$
$n\equiv 2$ and $m\equiv 2$.	0	$c^m d^n$	$c^m d^n$

Hence if $n\equiv 0$ *mod* 2 and $m\equiv$ *mod* 4, then $(Sq^2+w^2)H^{2n+m-2}(P(m,n);Z)H^{2n+m}(P(m,n);Z_2$, that is, then $N\otimes H$ has a non-zero cross-section. By theorem I we have theorem II.

Part II

Let Sq denote the total Steenrod square. It is known that Sq is an automorphism of the *mod* 2 cohomology ring $H^*(X;Z_2)$ of a space X such that $H^q(X:Z_2)=0$ for all sufficiently large q. We will denote the inverse automorphism by Q and its component of degree i by Q^i. $Q^0=1$, and Q^i can be defined inductively by the relation $\sum_{i+j=R} Q^iSq^j=0$. The following properties of Q^i are immediate.

Lemma 6.

1) $Q^k(x\cdot y)=\sum_{j=0}^{k} Q^i(x)\cdot Q^{k-i}(y)$. (*Cartan formula.*)

2) $Q^{2n+1}=Q^{2n}Sq^1$.

3) $Q^{2n+1}Sq^{2k+1}=0$.

4) *Suppose* $x\in H^1(X;Z_2)$. *Then* $Q^kx=0$ *for* $k\neq 2^h-1$ and $Q^kx=x^{2^h}$ for $k=2^h-1$.

In this part we use R. Thom's Theorem: If Y is a compact space such that, for some $i, r>0$

$$Q^i: H^r(Y;Z_2)\longrightarrow H^{r+i}(Y;Z_2)$$

is not zero, then Y is not imbeddable in S^{r+2i}[7].

We remember $H^*(P(m,n);Z_2)$ and Sq^i operating on $H^*(P(m,n);Z_2)$ given by [1].

I. Cohomology ring $H^*(P(m,n);Z^2)$ is generated by $c\in H^1(P(m,n);Z_2)$ and $d\in H^2(P(m,n);Z_2)$ with $c^{m+1}=0$ and $d^{n+1}=0$.

II. $Sq^1d=cd$.

Then theorem III follows from

Theorem IV.

a) $Q^{2^m-1}d=\sum_{j=0}^{m-1} c^{2^m-2^{l+1}+1}d^{2^l}$.

b) $Q^{2^{m+l}+2^{l+1}-2}d=\sum_{j=1}^{m-1} c^{2^{m+l}-2^{l+j+1}+2^{l+1}}d^{2^{l+j}}$.

In this part all calculations will be done *mod* 2.

3. Proof of theorem IV.

Lemma 7.

1) $Q^{2k}d=(Q^{k-1}d)^2$.

2) $Q^{2^m-2}d=d^{2^{m-1}}$.

Proof. 1). $Q^{2k}d=\sum_{i+j=2k}Q^iSq^jd=Q^{2j-1}Sq^1d+Q^{2k-2}Sq^2d$

$$=Q^{2k-2}d^2 \qquad \text{by lemma 6. 3)}$$

$$=\sum_{i+j=2k-2}Q^id\,Q^jd,$$

the terms vanish except the middle term,

$$=Q^{k-1}dQ^{k-1}d=(Q^{k-1}d)^2.$$

2). By 1)

$$Q^{2^m-2}d=(Q^{2^{m-1}-2}d)^2=\cdots\cdots\cdots=(Q^0d)^{2^{m-1}}=d^{2^{m-1}}.$$

Lemma 8.

$$Q^{2k+1}d=\begin{cases}0 & \textit{for } 2k+1\neq 2^m-1\\ \sum_{i=0}^{m-1}c^{2m-2^{i+1}+1}d^{2^i} & \textit{for } 2k+1=2^m-1.\end{cases}$$

Proof. This is proved by induction. It is clear that $Q^1d=cd$ for $2k+1=1$. Suppose $2k+1=2^m-1$ for an integer $m(>0)$ and that the lemma is true for all odd numbers $<2k+1$. We have by lemma 6. 1

$$Q^{2^m-1}d=\sum_{i+j=2m-1}Q^iSq^jd=Q^{2^m-2}cd+Q^{2^m-3}d^2$$

$$=\sum_{i+j=2m-2}Q^icQ^jd+\sum_{i+j=2m-3}Q^idQ^jd.$$

Because the second sum vanishes, we have by lemma 6. 4, lemma 7. 2 and the hypothesis of induction

$$Q^{2^m-1}d=Q^{2^h-1}cQ^{2^m-2^h-1}d.$$

$$=\sum_{2^i+2j=2^m}Q^{2^i-1}cQ^{2^j-1}d+cQ^{2^m-2}d$$

$$=Q^{2^{m-1}-1}cQ^{2^{m-1}-1}d+Qc^{2^{m-2}}d$$

$$=c^{2^{m-1}}\sum_{j=0}c^{2^{m-1}-2^{i+1}+1}d^{2^i}+cd^{2^{m-1}}$$

$$=\sum_{i=0}^{m-1}c^{2^m-2^{i+1}+1}d^{2^i}.$$

Hence we have proved the case $2k+1=2^m-1$.

Suppose $2k+1\neq 2^m-1$ for any integer m and that the lemma is true for all odd numbers $<2k+1$, Then

$$Q^{2k+1}d=Q^{2k}cd=\sum_{2^i+2^j=2k+2} Q^{2^i-1}cQ^{2^j-1}d+cQ^{2k}d.$$

a). If $2k+2=2^i+2^j$ for some $i>j$, then

$$\begin{aligned}Q^{2k+1}d&=cQ^{2^i+2^j-2}d+Q^{2^i-1}cQ^{2^j-1}d+Q^{2^j-1}cQ^{2^i-1}d\\&=cQ^{2^{i-j}-1}d^{2^j}+c^{2^i}(\sum_{h=0}^{j-1}c^{2^j-2^{h+1}+1}d^{2^h})\\&\quad+c^{2^j}(\sum_{h=0}^{j+1}c^{2^i-2^{h+1}+1}d^{2^h}).\end{aligned}$$

We apply lemma 7. 1) for the first term and the hypothesis of the induction for the second and the third terms. We then see that the above equals

$$\begin{aligned}&\sum_{h=j}^{i-i}c^{2^i+2^j-2^{h+1}+1}d^{2^h}+\sum_{h=0}^{j-1}c^{2^i+2^j-2^{h+1}+1}d^{2^h}\\&+\sum_{h=0}^{1-1}c^{2^i+2^j-2^{h+1}+1}d^{2^h}=0\end{aligned}$$

b). If $2k+2\neq 2^i+2^j$ for any $i>j$, then $Q^{2k+1}d=cQ^{2k}d$. In order to show $Q^{2k}d=0$ for such $2k$, it is sufficient to prove that if $Q^{2k}d\neq 0$ then $2k=2^i+2^j-2$ or $2k+1=2^i-1$. Suppose $Q^{2k}d\neq 0$. Then since $Q^{2k}d=(Q^{k-1}d)^2\neq 0$, we see $Q^{k-1}d\neq 0$. If $k-1$ is odd, then by the hypothesis of the induction we have $k-1=2^i-1$, that is, $2k=2^i+2-2$. If $k-1$ is even, then $Q^{k-1}d=\left(Q^{\frac{k-1}{2}-1}d\right)^2\neq 0$. Hence $Q^{\frac{k-1}{2}-1}d\neq 0$. If $\frac{k-1}{2}$ is odd, then by the hypothesis $\frac{k-1}{2}-1=2^i-1$, that is, $2k=2^{i+1}+2^2-2$. If $\frac{k-1}{2}-1$ is even we can repeat the same argument. Thus if $Q^{2k}d\neq 0$, we see then

$$\cfrac{\cfrac{\cfrac{k-1}{2}-1}{2}-1}{\cfrac{\cdot\ \cdot\ \cdot}{2}}-1=2^i-1 \text{ or } 2,$$

that is, $2k=2^{n+i}+2^n-2$ or $2k+1=2^{i+1}-1$. Hence if $Q^{2k}k\neq 0$, then $2k=2^i+2^j-2$ or $2k+1=2^i-1$. Hence $Q^{2k+1}d=0$ for $2k+2\neq 2^i+2^j$.

The lemma is thus proved.

Lemma 8 contains theorem IV a). It remains the proof of IV b).

Lemma 9.

$$Q^{2^m}d=\sum_{i=1}^{m-1}c^{2^m-2^{i+1}-2}d^{2^i}.$$

Proof.

$$Q^{2^m}d \cdots (Q^{2^{m-1}-1}d)^2 \qquad \text{(by lemma 7. 1)}$$

$$= (\sum_{j=0}^{m-2} c^{2^{m-1}-2^{i+1}+1} d^{2^i})^2 \qquad \text{(by lemma 8)}$$

$$= \sum_{j=1}^{m-1} c^{2^m-2^{i+1}+2} d^{2^i}.$$

PROOF OF THEOREM IV b).

$$Q^{2^{m+i}+2^{i+1}-2}d = (Q^{2^{m+i-1}+2^i-2}d)^2 = \cdots\cdots\cdots$$

$$= (Q^{2^{m+1}+2^2-2}d)^{2^{i-1}} \qquad \text{(by lemma 7. 1)}$$

$$= (Q^{2^m}d)^{2^i} = \sum_{j=1}^{m-1} c^{2^{m+i}-2^{i+j+1}+2^{i+j}} d^{2^{i+1}} \qquad \text{(by lemma 9).}$$

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DEPARTMENT of MATHEMATICS,
OKAYAMA UNIVERSITY

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