NOTES ON FUNDAMENTAL REGIONS OF COVERING TRANSFORMATION GROUPS

HISAO MIZUMOTO

1. Let $\mathfrak{G}_2$ be a transformation group the basis of which is a system of transformations

$$T_1(z)=z+1,\quad T_2(z)=z+i$$

of the finite z-plane $Z=\{z\,|\,|z|<\infty\}$. We shall call a bounded closed domain F on Z a *fundamental region of the transformation group* $\mathfrak{G}_2$ if F satisfies the conditions:

(i) For any point $z\in Z$ there exists a point $z'\in F$ equivalent to z under $\mathfrak{G}_2$, i.e. for any $z\in Z$ there exists a transformation $\chi\in\mathfrak{G}_2$ such that $z'=\chi(z)\in F$;

(ii) Two distinct points z, z' equivalent each other under $\mathfrak{G}_2$ do not simultaneously belong to $(F)^\circ$, [1] i.e. $\chi(z)\neq z'$ for any $z, z'\in(F)^\circ$ $(z\neq z')$ and for any $\chi\in\mathfrak{G}_2$.

Let R be the Riemann surface constructed from Z by identifying the points equivalent under $\mathfrak{G}_2$, and be denoted by $R\equiv Z(\text{mod } \mathfrak{G}_2)$. Then R is a closed Riemann surface of genus 1 (torus) and Z is a covering surface of R with the covering transformation group $\mathfrak{G}_2$. R is also constructed from a fundamental region F of $\mathfrak{G}_2$ by identifying the points equivalent of ∂F under $\mathfrak{G}_2$, where the conformal metric induced from F is taken as that of R. R is uniquely determined by Z and $\mathfrak{G}_2$.

One of the simplest fundamental regions of $\mathfrak{G}_2$ is given by a square

$$F_0=\{z\,|\,0\leqq x\leqq 1,\ 0\leqq y\leqq 1\}\quad (z=x+iy).$$

Let

$$\tilde{\alpha}_1^0=\{z\,|\,y=0,\ 0\leqq x\leqq 1\},\quad \tilde{\alpha}_2^0=\{z\,|\,x=0,\ 0\leqq y\leqq 1\}.$$

Then R is obtained from F_0 by identifying the points equivalent of $\tilde{\alpha}_1^0$, $\tilde{\alpha}_2^0$, $T_2(\tilde{\alpha}_1^0)$ and $T_1(\tilde{\alpha}_2^0)$ under $\mathfrak{G}_2$, and the system of the images α_1^0, α_2^0 on R of $\tilde{\alpha}_1^0$, $\tilde{\alpha}_2^0$ becomes a canonical homology basis of R. Further α_1^0, α_2^0 form a system of generators of the group $\mathfrak{G}_2$ (cf. [3]):

$$\alpha_1^0=T_1,\quad \alpha_2^0=T_2.$$

2. Let K be a bounded set arbitrarily given on Z and consisting of a finite number of continua or isolated points $K_1,\dots,K_n$ which satisfies the conditions:

1) The interior of a set E is denoted by $(E)^\circ$.

(i) The complementary set of K is a domain;

(ii) Two distinct points z, z' equivalent each other under $\mathfrak{G}_2$ do not simultaneously belong to K, i.e. $\chi(z)\neq z'$ for any $z, z'\in K$ $(z\neq z')$ and for any $\chi\in\mathfrak{G}_2$;

(iii) No lattice point (point whose real and imaginary parts are both integers) belongs to K.

Then we have the following theorem which is the main consequence in the present paper.

Theorem 1. *There exist a fundamental region F of $\mathfrak{G}_2$ and a homeomorphic map f of F_0 onto F which have the following properties*:

(a) *Four points* 0, 1, $1+i$, i *are fixed points of* f;

(b) $f\circ T_1(z)=T_1\circ f(z)$ *for any* $z\in\bar{\alpha}_2^0$,
$f\circ T_2(z)=T_2\circ f(z)$ *for any* $z\in\bar{\alpha}_1^0$;

(c) $K\subset(F)^\circ$.

3. For the proof of the theorem 1, we shall prepare some lemmas.

Let Q_ν $(\nu=1,\dots,n)$ be the quadrangles contained in $(F_0)^\circ$ the sides of which are parallel to the coordinate axes. Put

$$Q_\nu=\{z\,|\,a_\nu\leqq x\leqq b_\nu,\ c_\nu\leqq y\leqq d_\nu\}$$
$$(0<a_\nu<b_\nu<1,\ 0<c_\nu<d_\nu<1;\ \nu=1,\dots,n).$$

Suppose further that

$$(1)\qquad l_\mu^x\cap l_\nu^x=\phi,\ l_\mu^y\cap l_\nu^y=\phi\quad(\mu\neq\nu),$$

where

$$l_\nu^x=\{x\,|\,a_\nu\leqq x\leqq b_\nu\},\ l_\nu^y=\{y\,|\,c_\nu\leqq y\leqq d_\nu\}\quad(\nu=1,\dots,n).$$

Then we have a lemma.

Lemma 1. *For an arbitrarily given system of integers $m_{1\nu}$, $m_{2\nu}$ $(\nu=1,\dots,n)$, there exist a fundamental region F of $\mathfrak{G}_2$ and a homeomorphic map f of F_0 onto F which have the following properties*:

(a) *Four points* 0, 1, $1+i$, i *are fixed points of* f;

(b) $f\circ T_1(z)=T_1\circ f(z)$ *for any* $z\in\bar{\alpha}_2^0$,
$f\circ T_2(z)=T_2\circ f(z)$ *for any* $z\in\bar{\alpha}_1^0$;

(c) $T_1^{m_{1\nu}}\circ T_2^{m_{2\nu}}(Q_\nu)\subset(F)^\circ$ $(\nu=1,\dots,n)$,
$f(Q_\nu)=T_1^{m_{1\nu}}\circ T_2^{m_{2\nu}}(Q_\nu)$ $(\nu=1,\dots,n)$.

Proof. We take a real-valued function $y=Y(x)$ continuous on the interval $0\leqq x\leqq 1$ which satisfies the condition

$$Y(x)=\begin{cases}0 & (x=0,1),\\ m_{2\nu} & (a_\nu\leqq x\leqq b_\nu;\ \nu=1,\dots,n),\end{cases}$$

and we consider a mapping function f_1 defined by

$$f_1 \equiv f_1(x, y) = u_1(x, y) + iv_1(x, y)$$

with

$$\begin{cases} u_1(x, y) = x, \\ v_1(x, y) = y + Y(x). \end{cases}$$

Then f_1 maps the fundamental region F_0 homeomorphically onto a closed domain F_1 and satisfies

$$f_1(Q_\nu) = T_2^{m_{2\nu}}(Q_\nu) \quad (\nu = 1, \ldots, n), \tag{2}$$

$$T_2^{m_{2\nu}}(Q_\nu) \subset (F_1)^\circ \quad (\nu = 1, \ldots, n). \tag{3}$$

It is shown that F_1 is a fundamental region of $\mathfrak{G}_2$. In fact, for an arbitrary point $z = x + iy \in F_0$, an integer m_2 satisfying

$$Y(x) \leqq y + m_2 < Y(x) + 1$$

is uniquely determined, and the point $z' = x + i(y + m_2)$ is a unique point of F_1 such that $z' \equiv z (\text{mod } \mathfrak{G}_2)$ except for the case $z' \in \partial F_1$ where there exist more than one points z' such that $z' \equiv z (\text{mod } \mathfrak{G}_2)$. Conversely, for an arbitrary point $z' = x + iy' \in F_1$, the point $z = x + i(y - [y'])$ is a unique point of F_0 such that $z \equiv z' (\text{mod } \mathfrak{G}_2)$ except for the case $z \in \partial F_0$.[2] Since F_1 is a homeomorphic image of F_0 and F_1 corresponds one-to-one to F_0 by the correspondence of equivalent points under $\mathfrak{G}_2$ with the exception of the case of points of ∂F_0 or ∂F_1, F_1 satisfies the conditions (i) and (ii) of **1.**

Next we take a real-valued periodic function $x = X(y)$ continuous on the interval $-\infty < y < +\infty$ which satisfies the conditions:

$$X(y) = \begin{cases} 0 & (y = 0, 1), \\ m_{1\nu} & (c_\nu \leqq y \leqq d_\nu;\ \nu = 1, \ldots, n); \end{cases}$$

$$X(y+1) = X(y),$$

and we consider a mapping function f_2 defined by

$$f_2 \equiv f_2(x, y) = u_2(x, y) + iv_2(x, y)$$

with

$$\begin{cases} u_2(x, y) = x + X(y), \\ v_2(x, y) = y. \end{cases}$$

Then f_2 is a homeomorphic map of the z-plane Z onto itself and each lattice point is a fixed point of f_2. Obviously

$$f_2(T_2^{m_{2\nu}}(Q_\nu)) = T_1^{m_{1\nu}}(T_2^{m_{2\nu}}(Q_\nu)) \quad (\nu = 1, \ldots, n). \tag{4}$$

Let F be the closed domain determined as the image of F_1 under f_2. Then, by (3) and (4)

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2) $[y']$ is the Gauss symbol which means the greatest integer not over y'.

(5) $$T_1^{m_{1\nu}} \circ T_2^{m_{2\nu}}(Q_\nu) \subset (F)^\circ \quad (\nu = 1, \dots, n).$$

It is shown that F is a fundamental region of $\mathfrak{G}_2$. In fact, for an arbitrary point $z = x + iy \in F_1$, a real number x' and an integer m_1 are uniquely determined by the condition

(6) $$x + m_1 = X(y) + x' \quad (0 \leqq x' < 1).$$

Further an integer m_2 is uniquely determined by the condition

$$Y(x') \leqq y + m_2 < Y(x') + 1.$$

Then

(7) $$z' = x' + i(y + m_2) \in F_1,$$

and by the periodicity of $X(y)$ and (6)

$$z'' \equiv f_2(z') = X(y + m_2) + x' + i(y + m_2)$$
$$= x + m_1 + i(y + m_2).$$

Thus $z'' \equiv z (\mathrm{mod}\ \mathfrak{G}_2)$ and, by (7), $z'' \in F$ for $F = f_2(F_1)$. It is immediately verified that any point $z'' \in F$ with $z'' \equiv z (\mathrm{mod}\ \mathfrak{G}_2)$ is obtained from z by the above process if $z'' \in (F)^\circ$. Then we see that for an arbitrary point $z \in F_1$ there exists a unique point $z'' \in F$ such that $z'' \equiv z (\mathrm{mod}\ \mathfrak{G}_2)$ except for the case $z'' \in \partial F$ where there exist more than one points z'' such that $z'' \equiv z (\mathrm{mod}\ \mathfrak{G}_2)$.

Conversely, for an arbitrary point $z'' = x'' + iy'' \in F$, an integer m_2 is uniquely determined by the condition

$$Y(x'' - [x'']) \leqq y'' + m_2 < Y(x'' - [x'']) + 1,$$

and the point $z = x'' - [x''] + i(y'' + m_2)$ is one and only one point of F_1 such that $z \equiv z'' (\mathrm{mod}\ \mathfrak{G}_2)$ except for the case $z \in \partial F_1$. Thus F satisfies the conditions (i) and (ii) of **1.**

Now the mapping function $f = f_2 \circ f_1$ maps F_0 homeomorphically onto F and obviously satisfies the condition (a). Since

$$f \circ T_1(iy) = f_2 \circ f_1(1 + iy) = 1 + X(y) + iy = T_1 \circ f(iy) \quad (0 \leqq y \leqq 1),$$

$$f \circ T_2(x) = f_2 \circ f_1(x + i) = x + X(1 + Y(x)) + i(1 + Y(x))$$
$$= T_2 \circ f(x) \quad (0 \leqq x \leqq 1),$$

the condition (b) is also satisfied. Further by (2) and (4)

$$f(Q_\nu) = f_2 \circ f_1(Q_\nu) = f_2(T_2^{m_{2\nu}}(Q_\nu)) = T_1^{m_{1\nu}} \circ T_2^{m_{2\nu}}(Q_\nu) \quad (\nu = 1, \dots, n)$$

and thus the condition (c) is satisfied.

4. Let D be a Jordan region[3] arbitrarily given on Z which satisfies the

3) In our note a closed domain surrounded by a Jordan curve is called a Jordan region.

conditions:

(i) Two distinct points z, z' equivalent each other under $\mathfrak{G}_2$ do not simultaneously belong to D;

(ii) No lattice point belongs to D.

Then we have a lemma.

Lemma 2. *There exist a fundamental region F and a homeomorphic map f of F_0 onto F which have the following properties:*

(a) *Four points* 0, 1, $1+i$, i *are fixed points of* f;

(b) $f\circ T_1(z) = T_1\circ f(z)$ *for any* $z \in \tilde{\alpha}_2^0$,

$f\circ T_2(z) = T_2\circ f(z)$ *for any* $z \in \tilde{\alpha}_1^0$;

(c) *There exists a Jordan region D' equivalent to D under $\mathfrak{G}_2$ ($D'\equiv D$ (mod $\mathfrak{G}_2$)) such that*

$$D' \subset (F)^\circ.$$

Proof. Let $D^* \subset R$ be the image of D under the projection map ω of the covering surface Z onto $R\equiv Z(\text{mod } \mathfrak{G}_2)$. Then, by (i) the restriction of ω to D is a homeomorphic map of D onto D^* and by (ii) the image p_0 of the lattice point on Z by ω is an exterior point of D^*.

It is shown that there exists a homeomorphic map φ of R onto itself which maps D^* into the simply-connected domain $R-(\alpha_1^0 \cup \alpha_2^0)$ and fixes p_0, and such that α_1^0, α_2^0 are homotopic to $\alpha_1 = \varphi^{-1}(\alpha_1^0)$, $\alpha_2 = \varphi^{-1}(\alpha_2^0)$, respectively. In fact, it is well known that there exists a conformal (and a fortiori homeomorphic) map φ_1 of $R-D^*$ onto a full covering surface W of the outside $\{w \mid |w|>1\}$ of the unit disk with finitely many sheets (cf. [1]). φ_1 can be easily continued to the homeomorphic map φ_1^* of R onto the full covering surface R_1 of the extended plane with finitely many sheets. Further for an arbitrarily small $\varepsilon>0$ we can construct a homeomorphic map φ_ε of R_1 onto itself which maps the many-sheeted disk $D_1^* = R_1 - W$ over the disk $\{w \mid |w| \leq 1\}$ onto one $D_1^*(\varepsilon)$ over the disk $\{w \mid |w| \leq \varepsilon\}$ and which is identical on the subregion R_1' of R_1 over $\{w \mid |w| \geq 1+\delta\}$, where δ is a fixed positive number such that there is no branch point of R_1 over $\{w \mid 1 \leq |w| \leq 1+2\delta\}$ and $\varphi_1^*(p_0) \in R_1'$. Let φ_2 be a homeomorphic map of R_1 onto R such that $\varphi_2\circ\varphi_\varepsilon\circ\varphi_1^*(p_0) = p_0$, and let D_ε^* be the image of $D_1^*(\varepsilon)$ under φ_2. Then, $\varphi^* = \varphi_2\circ\varphi_\varepsilon\circ\varphi_1^*$ gives a homeomorphic map of R onto itself which maps D^* onto D_ε^* and fixes p_0. Let $\alpha_1^* = \varphi^{*-1}(\alpha_1^0)$ and $\alpha_2^* = \varphi^{*-1}(\alpha_2^0)$, and $\tilde{\alpha}_1^*$, $\tilde{\alpha}_2^*$ be the branches of the images of α_1^*, α_2^* under ω^{-1} which start both from 0. We denote the end-points of $\tilde{\alpha}_1^*$, $\tilde{\alpha}_2^*$ by $m_{11} + im_{21}$, $m_{12}+im_{22}$, respectively, where m_{11}, m_{12}, m_{21} and m_{22} are integers such that

$$\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = 1.$$

Here we should note that the integers m_{11}, m_{12}, m_{21} and m_{22} are independent of the selection of ε in view of the structure of φ_ε.

We define an affine transformation g of Z onto itself by

$$g(z) = u(x, y) + iv(x, y) \quad (z = x + iy),$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $\tilde{\alpha}_1'$, $\tilde{\alpha}_2'$ be the images of $\tilde{\alpha}_1^0$, $\tilde{\alpha}_2^0$ under g^{-1}, respectively, and α_1', α_2' be the images of $\tilde{\alpha}_1'$, $\tilde{\alpha}_2'$ under the projection map ω. Here we may assume that α_1', α_2' have no common point with D_ε^*. In fact, we have only to deform α_1', α_2' infinitesimally and homotopically and further to take ε small enough. Then, α_1^0, α_2^0 are obviously homotopic to $\alpha_1 = \varphi^{*-1}(\alpha_1')$, $\alpha_2 = \varphi^{*-1}(\alpha_2')$, respectively, and the map $\varphi = \omega \circ g \circ \omega^{-1} \circ \varphi^*$ gives a required one.

Let $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ be the branches of the images of $\alpha_1 = \varphi^{-1}(\alpha_1^0)$, $\alpha_2 = \varphi^{-1}(\alpha_2^0)$ under the inverse map ω^{-1} which follow from 0 to 1, i, respectively. Then, the closed domain F surrounded by $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $T_2(\tilde{\alpha}_1)$ and $T_1(\tilde{\alpha}_2)$ is a fundamental region of $\mathfrak{G}_2$, and $(F)^\circ$ contains a connected component D' of the image of D^* by ω^{-1}, for D^* does not intersect α_1, α_2. Since $D' \equiv D(\text{mod } \mathfrak{G}_2)$, the condition (c) is satisfied. Further the map $f = \omega^{-1} \circ \varphi^{-1} \circ \omega$ gives a homeomorphic map of F_0 onto F by taking a suitable branch of ω^{-1} and f satisfies obviously the conditions (a) and (b).

5. Lemma 3. *There exist a fundamental region F of $\mathfrak{G}_2$ and a homeomorphic map f of F_0 onto F which have the following properties:*

(a) *Four points* 0, 1, $1+i$, i *are fixed points of* f;

(b) $f \circ T_1(z) = T_1 \circ f(z)$ *for any* $z \in \tilde{\alpha}_2^0$,

$f \circ T_2(z) = T_2 \circ f(z)$ *for any* $z \in \tilde{\alpha}_1^0$;

(c) *For* K_ν $(\nu = 1, \ldots, n)$ *defined in* **2**, *there exist continua or isolated points* K_ν' *equivalent to* K_ν *under* $\mathfrak{G}_2$ $(K_\nu' \equiv K_\nu(\text{mod } \mathfrak{G}_2))$ *such that*

$$K' = \bigcup_{\nu=1}^{n} K_\nu' \subset (F)^\circ.$$

Proof. Let K^*, $K_\nu^* \subset R$ $(\nu = 1, \ldots, n)$ be the images of K, K_ν $(\nu = 1, \ldots, n)$ under the projection map ω of Z onto $R \equiv Z(\text{mod } \mathfrak{G}_2)$, respectively. Then by (ii) of **2**, the restriction of ω to K is a homeomorphic map of K onto K^*, and by (iii) of **2**, the image p_0 of the lattice points on Z under ω is an exterior point of K^*. Now, if we connect $K_\nu^*(\nu = 1, \ldots, n-1)$ to $K_{\nu+1}^*$ by a cross-cut L_ν^* on $R - K^* - \{p_0\}$, then $C^* = K^* \cup \bigcup_{\nu=1}^{n-1} L_\nu^*$ is a continuum, p_0 is an exterior point of C^* and $R - C^*$ is connected. Then we can take a Jordan region D^* on R such that $C^* \subset D^*$ and p_0 is an exterior point of D^*. In fact, there exists a conformal (and a fortiori homeomorphic) map φ of $R-C^*$ onto a finitely many sheeted unit disk W on the w-plane. For a sufficiently small $\varepsilon>0$, the set of the points of W having the projection $\{w \mid |w|=1-\varepsilon\}$ forms a Jordan curve Γ homotopic to ∂W. Then, for the Jordan region D^* on R surrounded by $\varphi^{-1}(\Gamma)$, $C^*\subset D^*$. Further, we can take ε such that p_0 is an exterior point of D^*.

If we take a connected component D of the image on Z of D^* under ω^{-1}, then the restriction of ω to D is a homeomorphic map of D onto D^*. Thus D is a Jordan region on Z and satisfies the conditions (i) and (ii) of **4.** Then, if we take the fundamental region F and the homeomorphic map f which satisfy for the present D the relations stated in the lemma 2, F and f satisfy the conditions in the present lemma. For, D contains a connected component K_ν' of the image of K_ν^* under ω^{-1} for each ν $(\nu=1,\dots,n)$ which is equivalent to K_ν under $\mathfrak{G}_2$.

6. Now we shall prove the theorem **1.**

We denote F and f satisfying the relations stated in the lemma 3 by F_1 and f_1, respectively. For K_ν' $(\nu=1,\dots,n)$ in the lemma 3, there exists a system of integers $m_{1\nu}$, $m_{2\nu}$ $(\nu=1,\dots,n)$ such that

$$(8) \qquad K_\nu = T_1^{m_{1\nu}}\circ T_2^{m_{2\nu}}(K_\nu') \qquad (\nu=1,\dots,n).$$

We apply the lemma 1 for the system of integers $m_{1\nu}$, $m_{2\nu}$ $(\nu=1,\dots,n)$ satisfying (8) and for an arbitrarily fixed system of Q_ν $(\nu=1,\dots,n)$ satisfying the conditions of **3**, and we denote F and f satisfying the relations stated in the lemma 1 for this system by F_2 and f_2, respectively. Let K_ν'' $(\nu=1,\dots,n)$ be the images of K_ν' under the inverse map f_1^{-1}:

$$(9) \qquad K_\nu''=f_1^{-1}(K_\nu') \qquad (\nu=1,\dots,n).$$

There exists a homeomorphic map g of F_0 onto itself which satisfies the conditions

$$(\mathrm{a}') \qquad g(z)=z \qquad \text{for any } z\in\partial F_0;$$

$$(\mathrm{b}') \qquad K_\nu'''\equiv g(K_\nu'')\subset(Q_\nu)^\circ \qquad (\nu=1,\dots,n).$$

In fact, it is well known that there exists a conformal (and a fortiori homeomorphic) map φ of $(F_0)^\circ-\bigcup_{\nu=1}^{n}K_\nu''$ onto a circular domain D on the w-plane surrounded by $n+1$ circles Γ_ν $(\nu=0,\dots,n)$, where Γ_0 and Γ_ν $(\nu=1,\dots,n)$ correspond to ∂F_0 and $\partial K_\nu''(\nu=1,\dots,n)$, respectively (cf. [2]). Let $\Gamma_\nu^*(\nu=1,\dots,n)$ be circles on D locally surrounding $\Gamma_\nu(\nu=1,\dots,n)$, respectively. Then $C_\nu=\varphi^{-1}(\Gamma_\nu^*)$ $(\nu=1,\dots,n)$ constitute Jordan curves on $(F_0)^\circ$ locally surrounding $K_\nu''(\nu=1,\dots,n)$, respectively. Let φ_1 be a homeomorphic map of the domain surrounded by ∂F_0 and $C_\nu(\nu=1,\dots,n)$ onto a circular domain D_1 on the w-plane surrounded by $n+1$ circles $\Gamma_\nu^1=\{w \mid |w-w_\nu^1|=r_\nu^1\}$ $(\nu=0,\dots,n)$, where Γ_0^1 and $\Gamma_\nu^1(\nu=1,\dots,n)$ correspond to ∂F_0 and $C_\nu(\nu=1,\dots,n)$ respectively. Here we may assume that D_1 is bounded, Γ_0^1 is the outer circle of D_1 and $\Gamma_0^1=\{w \mid |w|=1\}$. φ_1 can be easily continued to a homeomorphic map $\varphi_1{}^*$ of F_0 onto the unit disk $G_1=\{w \mid |w|\leqq 1\}$. Further for an arbitrarily small $\varepsilon>0$ we can construct a homeomorphic map φ_ε of G_1 onto itself which maps the interior of Γ_ν^1 onto the interior of $\Gamma_\nu^\varepsilon=\{w \mid |w-w_\nu^1|=\varepsilon\}$ for each $\nu=1,\dots,n$ and such that $\varphi_\varepsilon(z)=z$ for any $z\in\partial G_1$. Let φ_2 be a homeomorphic map of $(F_0)^\circ-\bigcup_{\nu=1}^n Q_\nu$ onto a bounded circular domain D_2 on the w-plane surrounded by $n+1$ circles $\Gamma_0^2=\{w \mid |w|=1\}$ and $\Gamma_\nu^2=\{w \mid |w-w_\nu^2|=r_\nu^2\}$ $(\nu=1,\dots,n)$, where Γ_0^2 and Γ_ν^2 $(\nu=1,\dots,n)$ correspond to ∂F_0 and $\partial Q_\nu(\nu=1,\dots,n)$ respectively, and Γ_0^2 forms the outer circle of D_2. Let φ_ν^* be the continuation of φ_2 which maps F_0 homeomorphically onto the unit disk $G_2=\{w \mid |w|\leqq 1\}$. There exists a homeomorphic map φ_3 of G_1 onto G_2 which maps w_ν^1 to w_ν^2 $(\nu=1,\dots,n)$ respectively and such that $w=\varphi_\varepsilon\circ\varphi_1{}^*(z)$ corresponds to $w'=\varphi_2{}^*(z)$ for each $z\in\partial F_0$. Then, for a sufficiently small $\varepsilon>0$, $\varphi_3(\Gamma_\nu^\varepsilon)\subset\{w \mid |w-w_\nu^2|<r_\nu^2\}(\nu=1,\dots,n)$. Thus the homeomorphic map $g=\varphi_2{}^{*-1}\circ\varphi_3\circ\varphi_\varepsilon\circ\varphi_1{}^*$ constitutes the required one.

The function $h=f_1\circ g^{-1}$ is a homeomorphic map of F_0 onto F_1 which has the properties:

(a$''$) Four points 0, 1, $1+i$, i are fixed points of h;

(b$''$) $$h\circ T_1(z) = T_1\circ h(z) \qquad \text{for any } z\in\tilde{\alpha}_2^0,$$
$$h\circ T_2(z) = T_2\circ h(z) \qquad \text{for any } z\in\tilde{\alpha}_1^0;$$

(c$''$) $$h(K_\nu{}''') = K_\nu{}' \qquad (\nu=1,\dots,n).$$

(a$''$) and (b$''$) follow from (a) and (b) of the lemma 3 and (a$'$), and (c$''$) does from (9) and (b$'$).

By the property (b$''$), we can uniquely extend the map h to the homeomorphic map $\tilde{h}$ of Z onto itself by the conditions:

(α) $$\tilde{h}(z)=h(z) \qquad \text{for any } z\in F_0;$$

(β) $$\tilde{h}\circ T_1(z)=T_1\circ\tilde{h}(z),\ \tilde{h}\circ T_2(z)=T_2\circ\tilde{h}(z) \qquad \text{for any } z\in Z.$$

The map $\tilde{h}$ is a doubly quasi periodic function with the primitive periods 1, i and each lattice point is a fixed point of $\tilde{h}$.

Now we shall verify that the mapping function

$$f = \tilde{h}\circ f_2$$

and the homeomorphic image F of F_0 by f constitute the required pair in the theorem. It is evident that f is a homeomorphic map and have the property (a). Since by (b) in the lemma 1 and (β)

$$f\circ T_1(z) = \tilde{h}\circ f_2\circ T_1(z) = T_1\circ f(z) \qquad \text{for any } z\in\tilde{\alpha}_2^0,$$
$$f\circ T_2(z) = \tilde{h}\circ f_2\circ T_2(z) = T_2\circ f(z) \qquad \text{for any } z\in\tilde{\alpha}_1^0,$$

the property (b) is satisfied. Since by (β)

$$\tilde{h}\circ T_1^{m_{1\nu}}\circ T_2^{m_{2\nu}}(z) = T_1^{m_{1\nu}}\circ T_2^{m_{2\nu}}\circ \tilde{h}(z) \quad (\nu=1,\dots,n),$$

we have

$$\begin{aligned} f(K_\nu''') &= \tilde{h}\circ f_2(K_\nu''') = \tilde{h}(T_1^{m_{1\nu}}\circ T_2^{m_{2\nu}}(K_\nu''')) \\ &= T_1^{m_{1\nu}}\circ T_2^{m_{2\nu}}\circ \tilde{h}(K_\nu''') = T_1^{m_{1\nu}}\circ T_2^{m_{2\nu}}(K_\nu') \\ &= K_\nu \qquad (\nu=1,\dots,n). \end{aligned}$$

Further, since $K_\nu'''\subset(F_0)^\circ$ $(\nu=1,\dots,n)$, it must hold $K_\nu\subset(F)^\circ$. Thus, the property (c) is satisfied. Finally, F is a fundamental region of $\mathfrak{G}_2$. In fact, F is the image of the fundamental region F_2 under $\tilde{h}$, which satisfies (β).

7. Remark 1. Let $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ be the images of $\tilde{\alpha}_1^0$, $\tilde{\alpha}_2^0$ under f in the theorem 1, respectively. We can select the fundamental region F and the map f in the theorem such that $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are sufficiently smooth (e. g. analytic) curves. For that purpose, we take a neighborhood of $\tilde{\alpha}_1\cup\tilde{\alpha}_2$ which does not intersect K, reselect $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ in the neighborhood such that they are smooth, and further reselect F and f in accordance with them.

Remark 2. Let α_1, α_2 be the images of $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ under the projection map ω of Z onto $R\equiv Z(\text{mod } \mathfrak{G}_2)$. Then the system of the cycles α_1, α_2 forms a canonical homology basis of R. It is notable that the cycles α_1, α_2 are obtained by a homotopic deformation of α_1^0, α_2^0 with the fixed point p_0 which is the intersection point of α_1^0 and α_2^0.

Remark 3. By the property (b) of the theorem 1, we can extend the map f of the theorem to the homeomorphic map $\bar{f}$ of Z onto itself by the conditions:

(α) $\quad \bar{f}(z) = f(z) \quad$ for any $z\in F_0$;

(β) $\quad \bar{f}\circ T_1(z) = T_1\circ\bar{f}(z),\ \bar{f}\circ T_2(z) = T_2\circ\bar{f}(z) \quad$ for any $z\in Z$.

8. Let m_{11}, m_{12}, m_{21} and m_{22} be an arbitrary system of integers satisfying

$$\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} = 1.$$

Then the closed parallelogram F_0' which has the vertices

$$0,\ z_1 = m_{11} + im_{21},\ z_2 = m_{11} + m_{12} + i(m_{21} + m_{22}),\ z_3 = m_{12} + im_{22}$$

forms a fundamental region of $\mathfrak{G}_2$. Let $\tilde{\alpha}_1'$, $\tilde{\alpha}_2'$ be the sides of F_0' from 0 to $m_{11}+im_{21}$, $m_{12}+im_{22}$, respectively, and α_1', α_2' be the images of $\tilde{\alpha}_1'$, $\tilde{\alpha}_2'$ under the projection map ω of Z onto $R\equiv Z(\text{mod } \mathfrak{G}_2)$. Then the system of cycles α_1', α_2' forms a canonical homology basis of R.

We have the following corollary.

Corollary 1. *For the set* K *given in* **2**, *there exist a fundamental region* F *of* $\mathfrak{G}_2$ *and a homeomorphic map* f *of* F_0' *onto* F *which have the properties*:

(a) *Four points* $0,\ z_1,\ z_2,\ z_3$ *are fixed points of* f;

(b) $f\circ T_1^{m_{11}}\circ T_2^{m_{21}}(z) = T_1^{m_{11}}\circ T_2^{m_{21}}\circ f(z)$ *for any* $z\in\tilde{\alpha}_2'$,

$f\circ T_1^{m_{12}}\circ T_2^{m_{22}}(z) = T_1^{m_{12}}\circ T_2^{m_{22}}\circ f(z)$ *for any* $z\in\tilde{\alpha}_1'$;

(c) $K\subset (F)^{\circ}$.

Proof. We define an affine transformation g of Z onto itself by

$$g(z) = u(x,\ y) + iv(x,\ y) \qquad (z = x + iy),$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

g maps F_0 homeomorphically onto F_0', satisfies

$$g(0) = 0,\ g(1) = z_1,\ g(1+i) = z_2,\ g(i) = z_3;$$

$$g(\tilde{\alpha}_1^0) = \tilde{\alpha}_1',\ g(\tilde{\alpha}_2^0) = \tilde{\alpha}_2',$$

and further has a doubly quasi periodicity

$$g\circ T_1(z) = T_1^{m_{11}}\circ T_2^{m_{21}}\circ g(z),$$

$$g\circ T_2(z) = T_1^{m_{12}}\circ T_2^{m_{22}}\circ g(z).$$

Let K' be the homeomorphic image of K under the inverse map g^{-1}: $K' = g^{-1}(K)$. We denote F and f satisfying the relations stated in the theorem 1 by taking K' in place of K by F_1 and f_1, respectively. Then, it is immediately verified that the fundamental region $F=g(F_1)$ and the map $f=g\circ f_1\circ g^{-1}$ have the properties of the present corollary.

9. Throughout the present section we denote the set consisting of a finite number of continua or isolated points by the same symbol K as **2** which satisfies only the conditions (i) and (ii) of **2**. Then we have the following corollary:

Corollary 2. *Let* z_0 *be an arbitrarily fixed point of* Z *equivalent to no point of* K *under* $\mathfrak{G}_2$. *Then there exist a fundamental region* F *and a homeomorphic map* f *of* F_0' *onto* F *which have the properties*:

(a) $f(0) = z_0,\ f(z_1) = z_0 + z_1,\ f(z_2) = z_0 + z_2,\ f(z_3) = z_0 + z_3$;

(b) $f\circ T_1^{m_{11}}\circ T_2^{m_{21}}(z) = T_1^{m_{11}}\circ T_2^{m_{21}}\circ f(z)$ *for any* $z\in\tilde{\alpha}_2'$,

$f\circ T_1^{m_{12}}\circ T_2^{m_{22}}(z) = T_1^{m_{12}}\circ T_2^{m_{22}}\circ f(z)$ *for any* $z\in\tilde{\alpha}_1'$;

(c) $K\subset (F)^{\circ}$.

Proof. Let K' be the image of K under the parallel translation

$$g(z) = z - z_0.$$

Then the set K' satisfies the conditions (i), (ii) and (iii) for K of **2.** We denote F and f satisfying the relation stated in the corollary 1 by taking K' in place of K by F_1 and f_1. Then the fundamental region $F = g^{-1}(F_1)$ and the map $f = g^{-1} \circ f_1 \circ g$ have the properties of the present corollary.

10. Let $\mathfrak{G}_1$ be the transformation group the basis of which is the transformation

$$T(z) = tz \qquad (t = e^{2\pi})$$

of the finite z-plane $Z = \{z \mid |z| < \infty\}$. We define a *fundamental region* F *of the transformation group* $\mathfrak{G}_1$ like in 2 by taking $\mathfrak{G}_1$ in place of $\mathfrak{G}_2$.

Let R be the Riemann surface constructed from Z by identifying the points equivalent under $\mathfrak{G}_1$, and be denoted by $R \equiv Z(\text{mod } \mathfrak{G}_1)$. Then R is a closed Riemann surface of genus 1 (torus) and Z is a covering surface of R with the covering transformation group $\mathfrak{G}_1$. R is also constructed from a fundamental region F of $\mathfrak{G}_1$ by identifying the points equivalent of ∂F under $\mathfrak{G}_1$ where the conformal metric induced from F is taken as that of R. R is uniquely determined by Z and $\mathfrak{G}_1$.

One of the simplest fundamental regions of $\mathfrak{G}_1$ is given by an annulus

$$F_0 = \{z \mid 1 \leqq |z| \leqq t\}.$$

Let

$$\tilde{\alpha}_1^0 = \{z \mid 1 \leqq x \leqq t,\ y = 0\},\ \tilde{\alpha}_2^0 = \{z \mid |z| = 1\},\ (z = x + iy).$$

Then R is obtained from F_0 by identifying the points equivalent of $\tilde{\alpha}_2^0$ and $T(\tilde{\alpha}_2^0)$ under $\mathfrak{G}_1$, and the system of the images α_1^0, α_2^0 on R of $\tilde{\alpha}_1^0$, $\tilde{\alpha}_2^0$ becomes a canonical homology basis of R.

11. Let K be a bounded set arbitrarily given on Z and consisting of a finite number of continua or isolated points $K_1, \dots, K_n$ which satisfies the conditions:

(i) The complementary set of K is a domain;

(ii) Two distinct points z, z' equivalent each other under $\mathfrak{G}_1$ do not simultaneously belong to K;

(iii) The points $z = t^m$ $(m = 0, \pm 1, \dots)$ do not belong to K.

In the present case we obtain the following theorem similar to the theorem 1.

Theorem 2. *There exist a fundamental region* F *of* $\mathfrak{G}_1$ *and a homeomorphic map* f *of* F_0 *onto* F *which have the properties*:

(a) *Two points* 1, t *are fixed points of* f;

(b) $f \circ T(z) = T \circ f(z)$ *for any* $z \in \tilde{\alpha}_2^0$;

(c) $K \subset (F)^\circ - f(\tilde{\alpha}_1^0)$.

Proof. Let K' be some branch of the image of the set K under the many-valued function

$$g(z) = \frac{1}{2\pi} \log z.$$

Then the restriction of $g(z)$ to K is a homeomorphic map of K onto K', and K' satisfies the conditions (i), (ii) and (iii) for the set K of **2.** We denote F and f satisfying the relations stated in the theorem 1 for the present K' in place of K of 2 by F_1 and f_1. Then $F=g^{-1}(F_1)$ forms a fundamental region of $\mathfrak{G}_1$ and the map $f=g^{-1}\circ f_1 \circ g$ is a homeomorphic map of F_0 onto F on taking the branch of g which maps the fundamental region F_0 onto the square $\{z \mid 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1\}$ $(z = x+iy)$. f and F have the properties of the present theorem.

12. Let m_1, m_2, m_1^*, m_2^* be an arbitrary system of integers satisfying

$$\begin{vmatrix} m_1 & m_2 \\ m_1^* & m_2^* \end{vmatrix} = 1.$$

Let $\tilde{\alpha}_1'$ and $\tilde{\alpha}_2'$ be the arcs of logarithmic spirals starting from 1 to $e^{2\pi m_1}$ and $e^{2\pi m_2}$:

$$\tilde{\alpha}_1' = \left\{ z \,\middle|\, m_1 \arg z - m_1^* \log|z| = 0,\ 0 \leqq \frac{1}{2\pi} |\log z| < \sqrt{m_1^2 + m_1^{*2}} \right\},$$

$$\tilde{\alpha}_2' = \left\{ z \,\middle|\, m_2 \arg z - m_2^* \log|z| = 0,\ 0 \leqq \frac{1}{2\pi} |\log z| < \sqrt{m_2^2 + m_2^{*2}} \right\},[4]$$

respectively. Then the closed domain F_0' surrounded by $\tilde{\alpha}_1'$, $\tilde{\alpha}_2'$, $T^{m_2}(\tilde{\alpha}_1')$ and $T^{m_1}(\tilde{\alpha}_2')$ forms a fundamental region of $\mathfrak{G}_1$. If we denote the images of $\tilde{\alpha}_1'$, $\tilde{\alpha}_2'$ under the projection map ω of Z onto $R \equiv Z(\text{mod } \mathfrak{G}_1)$ by α_1', α_2', respectively, then the system α_1', α_2' forms a canonical homology basis of R.

In the present case we obtain the following corollary similar to the corollary 1.

Corollary 3. *For the set K given in* **11**, *there exist a fundamental region F of $\mathfrak{G}_1$ and a homeomorphic map f of F_0' onto F which have the properties:*

(a) *Four points* 1, $e^{2\pi m_1}$, $e^{2\pi m_2}$, $e^{2\pi(m_1+m_2)}$ *are fixed points of f;*

(b) $f \circ T^{m_1}(z) = T^{m_1} \circ f(z)$ *for any* $z \in \tilde{\alpha}_2'$,

$f \circ T^{m_2}(z) = T^{m_2} \circ f(z)$ *for any* $z \in \tilde{\alpha}_1'$;

4) It may happen that $\tilde{\alpha}_1'$ or $\tilde{\alpha}_2'$ is a circle or a segment for the special case where m_1, m_2, m_1^* or m_2^* is zero.

(c) $K \subset (F)^{\circ} - f(\alpha_1') - f(\alpha_2')$.

The proof is similar to the method of the theorem 2 if we make use of the many valued function

$$g(z) = \frac{1}{2\pi} \log z$$

and the corollary 1. Hence we omit the detail.

13. Let $\mathfrak{G}_1'$ be the transformation group the basis of which is the transformation

$$T(z) = z + 1$$

of the finite z-plane $Z = \{z \mid |z| < \infty\}$. We shall call a closed domain F on Z a *fundamental region of the transformation group* $\mathfrak{G}_1'$ if F satisfies the conditions (i) and (ii) of **1** for $\mathfrak{G}_1'$ in place of $\mathfrak{G}_2$.

In the present case, $R \equiv Z(\text{mod } \mathfrak{G}_1')$ is homeomorphic to the punctured finite plane $\{z \mid 0 < |z| < \infty\}$. One of the simplest fundamental regions of $\mathfrak{G}_1'$ is given by the parallel strip

$$F_0 = \{z \mid 0 \leqq x \leqq 1\} \qquad (z = x + iy).$$

Let

$$\tilde{\beta}_0 = \{z \mid 0 \leqq x \leqq 1,\ y = 0\},$$
$$\tilde{\beta}_0^* = \{z \mid x = 0\}.$$

Then R is obtained from F_0 by identifying the points equivalent of $\tilde{\beta}_0^*$ and $T(\tilde{\beta}_0^*)$ under $\mathfrak{G}_1'$. If we denote the images on R of $\tilde{\beta}_0$, $\tilde{\beta}_0^*$ by β_0, β_0^*, respectively, then β_0, β_0^* constitute dividing and relative cycles of R, respectively.

Let K be a set arbitrarily given on Z (not necessarily bounded) and consisting of a finite number of continua or isolated points $K_1, \dots, K_n$ which satisfies the conditions:

(i) The complementary set of K is a domain;

(ii) Two distinct points z, z' equivalent each other under $\mathfrak{G}_1'$ do not simultaneously belong to K;

(iii) The points $z = 0, \pm 1, \dots$ do not belong to K.

In the present case we obtain the following theorem similar to the theorem 1.

Theorem 3. *There exist a fundamental region F and a homeomorphic map f of F_0 onto F which have the properties*:

(a) *Two points* 0, 1 *are fixed points of* F;

(b) $f \circ T(z) = T \circ f(z)$ *for any* $z \in \tilde{\beta}_0^*$;

(c) $K \subset (F)^{\circ}$.

The proof is similar to and rather easier than that of the theorem 1 and hence we omit it.

We shall exhibit applications of the consequences of the present paper in a forthcoming paper [4].

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DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

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