SUBMANIFOLDS IN A RIEMANNIAN MANIFOLD WITH GENERAL CONNECTIONS

Chorng-Shi HOUH

Professor T. Ōtsuki developed the theory of general connection of a differentiable manifold. The general connections were defined by a cross-section in an appropriate bundle, that is the tensor product bundle of the tangent bundle (of order 1) and the cotangent bundle of order 2 of a differentiable manifold. The classical affine connections and the tensors of type (1, 2) are remarkable as special general connections. He investigated the tensor calculus of spaces with general connections and showed several formulas which are natural generalizations of those in the spaces with classical affine connections. Development of curves in spaces with general connections which satisfy a certain condition is possible by several methods. The Levi-Civita's connection of Riemannian spaces can be generalized in the theory of general connections under some conditions on an n-dimensional differentiable manifold $\mathfrak{X}$[1].

In the present paper, let $\mathfrak{X}$ be a Riemannian manifold with a metric (g_{ij}), $\mathfrak{X}_l$ be an l-dimensional submanifold of $\mathfrak{X}$. The author tries to induce the general connection of $\mathfrak{X}$ to the submanifold $\mathfrak{X}_l$ and develop a theory of submanifolds with general connections.

In §2 we shall consider some injections of $T(\mathfrak{X}_l)$, $\tau^2(\mathfrak{X}_l)$ into $T(\mathfrak{X})$, $\tau^2(\mathfrak{X})$; $T^*(\mathfrak{X})$, $\mathfrak{D}^2(\mathfrak{X})$ into $T^*(\mathfrak{X}_l)$, $\mathfrak{D}^2(\mathfrak{X}_l)$ and some other things for later use.

In §3, the induced connection in $\mathfrak{X}_l$ is given as follows:

$$P_a^b = \theta_i^a P_j^i \theta_b^j$$
$$\Gamma_{bc}^a = \theta_i^a (P_j^i \partial_c \theta_b^j + \Gamma_{jk}^i \theta_b^j \theta_c^k)$$

where (P_j^i, Γ_{jk}^i) is a general connection of $\mathfrak{X}$ and $\theta_a^i = \partial u^i / \partial x^a$, $g_{ab} = g_{ij}\theta_a^i\theta_b^j$, $(g^{ab}) = (g_{ab})^{-1}$, $\theta_i^a = g^{ab} g_{ij} \theta_b^j$; (u^i), (x^a) denote local coordinates of $\mathfrak{X}$ and $\mathfrak{X}_l$ at the same point. To obtain some results we must restrict our attention to some submanifolds so called *adapted*. The submanifolds considered in §§4-7 are supposed to be adapted.

In §4 we shall induce the normal general connection of $\mathfrak{X}$ to $\mathfrak{X}_l$ and consider an especial normal general connection and its induced connection. A method of development of curve in $\mathfrak{X}_l$ will be obtained.

In §5, we shall consider the regular general connection which is analogous to the classical affine connection such that several results in classical theory of

1) In the present paper, we deal with only manifolds, submanifolds, functions and transformations with suitable differentiabilities for our purpose.

submanifolds can be generalized.

In §6 we show that the induced connection of $\mathfrak{X}_t$ derived from a metric regular general connection of $\mathfrak{X}$ is also metric. Some concepts in classical metric subspace are generalized. We also investigate the connection $\overline{\Gamma}$ which is related closely with submanifolds.

In §7 we investigate the relation between curvature tensors of $\mathfrak{X}$ and $\mathfrak{X}_t$ with respect to the connection $\overline{\Gamma}$ and its induced connection. For this purpose, we should introduce some other general connections and establish some lemmas.

The author wishes to express here his sincere gratitude to Professor T. Ōstuki who has given him various suggestions and frequent chances of discussions. He also wishes to express his hearty thanks to Prof. H. Hombu and Prof. S. Hokari for their kind guidance and encouragement.

§1. **Preliminary.** ([5]—[12])[2]

Let $\mathfrak{L}_n^2$ be the group of all generalized infinitesimal isotropies of order 2 at the origin of the n-dimensional coordinate space R^n, whose element is written as a set of real numbers (a_j^i, a_{jh}^i) such that $|a_j^i| \neq 0$ and whose multiplication is given by the following formulas:

For any $\alpha, \beta \in \mathfrak{L}_n^2$,

$$(1.1) \qquad a_i^j(\alpha\beta) = a_k^j(\alpha) a_i^k(\beta)$$

$$(1.2) \qquad a_{ih}^j(\alpha\beta) = a_k^j(\alpha) a_{ih}^k(\beta) + a_{kl}^j(\alpha) a_i^k(\beta) a_h^l(\beta).$$

Let $\mathfrak{X}$ be any n-dimensional differentiable manifold. With any coordinate neighborhood (U, u^i), where the local coordinates u^j are defined on the neighborhood U in $\mathfrak{X}$, we associate $n^2 \times n$ fields of vectors denoted by ∂u_i, $\partial^2 u_{ih}$. Let ∂v_i, $\partial^2 v_{ih}$ be the vector fields associated with another coordinate neighborhood (V, v^j). When $U \cap V \neq \phi$, we assume that they are related mutually on $U \cap V$ as

$$(1.3) \qquad \partial u_i = \frac{\partial v^j}{\partial u^i} \partial v_j$$

$$(1.4) \qquad \partial^2 v_{ih} = \frac{\partial^2 v^j}{\partial u^h \partial u^i} \partial v_j + \frac{\partial v^j}{\partial u^i} \frac{\partial v^k}{\partial u^h} \partial^2 v_{jk}.$$

Thus we obtain at each point x of $\mathfrak{X}$ an $(n+n^2)$-dimensional vector space spanned by these $n+n^2$ vectors ∂u_i, $\partial^2 u_{ih}$ being independent of coordinate neighborhood containing the point x, which is denoted by $\tau_x^2(\mathfrak{X})$. The union

$$\tau^2(\mathfrak{X}) = \bigcup_{x \in \mathfrak{X}} \tau_x^2(\mathfrak{X})$$

may be considered naturally as the total space of a vector bundle $\{\tau^2(\mathfrak{X}), \mathfrak{X}, \tau\}$

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2) The number in square brackets shows the number of the reference at the end of the present paper.

with the natural projection τ, whose structure group is $\mathfrak{L}_n^2$ (in fact, it is $L_n^2=\{\alpha \mid a_{ih}^j(\alpha)=a_{hi}^j(\alpha),\ \alpha\in\mathfrak{L}_n^2\}$) and the coordinate transformation $g_{VU}: U\cap V\to\mathfrak{L}_n^2$ is given by

$$a_i^j(g_{VU})=\frac{\partial v^j}{\partial u^i},\quad a_{ih}^j(g_{VU})=\frac{\partial^2 v^j}{\partial u^h\partial u^i}.$$

We call any element of $\tau_x^2(\mathfrak{X})$ *a tangent vector of order* 2 of $\mathfrak{X}$ at x. For the sake of simplicity, we denote the vector bundle over $\mathfrak{X}$ by the same notation $\tau^2(\mathfrak{X})$ and call it *the tangent bundle of order 2 of* $\mathfrak{X}$.

With any coordinate neighborhood $(U,\ u^i)$ at each point $x\in U$, we associate an $(n+n^2)$-dimensional vector space which is spanned by $du^i\otimes du^k$ and *the differentials* d^2u^j *of order* 2 which are assumed to be linearly independent among them and of $du^i\otimes du^h$. We relate the two vector spaces corresponding to $(U,\ u^i)$ and $(V,\ v^i)$ at $x\in U\cap V$ with each other by

$$\text{(1.5)}\qquad \begin{cases} dv^j=\dfrac{\partial v^j}{\partial u^i}du^i,\\ d^2v^j=\dfrac{\partial v^j}{\partial u^i}d^2u^i+\dfrac{\partial^2 v^j}{\partial u^h\partial u^i}du^i\otimes du^h.\end{cases}$$

Thus we obtain the cotangent vector space of order 2 of $\mathfrak{X}$ at x denoted by $\mathcal{D}_x^2(\mathfrak{X})$ which is dual to $\tau_x^2(\mathfrak{X})$ and contains the tensor product $T_x^*(\mathfrak{X})\otimes T_x^*(\mathfrak{X})$ of the cotangent space of $\mathfrak{X}$ at x. The base $\{d^2u^i,\ du^i\otimes du^h\}$ is dual to the base $\{\partial u_i,\ \partial^2\partial u_{ih}\}$ of $\tau_x^2(\mathfrak{X})$. The union

$$\mathcal{D}^2(\mathfrak{X})=\bigcup_{x\in\mathfrak{X}}\mathcal{D}_x^2(\mathfrak{X})$$

is the total space of *the cotangent bundle of order* 2 of $\mathfrak{X}$ which we denote by the same notation. $\mathcal{D}^2(\mathfrak{X})$ contains the tensor product bundle $T^*(\mathfrak{X})\otimes T^*(\mathfrak{X})$ of the cotangent bundle $T^*(\mathfrak{X})$ as a subbundle.

We call any cross-section Γ of the vector bundle $T(\mathfrak{X})\otimes\mathcal{D}^2(\mathfrak{X})$ over $\mathfrak{X}$ a *general connection* of $\mathfrak{X}$ by definition. In a coordinate neighborhood $(U,\ u^i)$ let Γ be written as

$$\text{(1.6)}\qquad \Gamma=\partial u_i\otimes(P_j^i d^2u^j+\Gamma_{jk}^i du^j\otimes du^k).$$

In another coordinate neighborhood $(V,\ v^i)$, if $U\cap V\neq\phi$, let Γ be writtten as

$$\Gamma=\partial v_i\otimes(P_j^i(v)d^2v^j+\Gamma_{jk}^i(v)dv^j\otimes dv^k),$$

then we get immediately

$$\text{(1.7)}\qquad \begin{cases} P_i^j(v)=\dfrac{\partial v^j}{\partial u^k}P_h^k\dfrac{\partial u^h}{\partial v^i},\\ \Gamma_{ih}^j(v)=\dfrac{\partial v^j}{\partial u^k}\left(P_l^k\dfrac{\partial^2 u^l}{\partial v^i\partial v^h}+\Gamma_{lm}^k\dfrac{\partial u^l}{\partial v^i}\dfrac{\partial u^m}{\partial v^h}\right).\end{cases}$$

We denote the general connection Γ sometimes by $(P_j^i,\ \Gamma_{jk}^i)$.

Now we define a homomorphism $\mu=\mu_\Gamma: \tau^2(\mathfrak{X})\to T(\mathfrak{X})$ by

(1. 8) $\mu(\partial u_j)=P^i_j\partial u_i,\quad \mu(\partial u_{jk})=\Gamma^i_{jk}\partial u_i,$

and a homomorphism $\varphi=\varphi_\Gamma : \mathcal{D}^2(\mathfrak{X})\to T^*(\mathfrak{X})^{\otimes 2}$ by

(1. 9) $\varphi(d^2u^i)=-\Lambda^i_{jk}\,du^j\wedge du^k,\quad \varphi(du^i\otimes du^h)=P^i_j\,du^j\otimes du^h$

where we put

(1. 10) $$\Lambda^i_{jk}=\Gamma^i_{jk}-\frac{\partial P^i_j}{\partial u^k}.$$

Furthermore we put generally

(1. 11) $\varphi(du^i)=du^i,$

$\varphi(du^{i_1}\otimes\cdots\otimes du^{i_q}\otimes du^h)=P^{i_1}_{j_1}\cdots P^{i_q}_{j_q}du^{j_1}\otimes\cdots\otimes du^{j_q}\otimes du^h$

and

(1. 12) $\varphi(du^{i_1}\otimes\cdots du^{i_{s-1}}\otimes d^2u^{i_s}\otimes du^{i_{s+1}}\otimes\cdots\otimes du^{i_p})$

$=-P^{i_1}_{j_1}\cdots P^{i_{s-1}}_{j_{s-1}}\Lambda^{i_s}_{ij}P^{i_{s+1}}_{j_{s+1}}\cdots P^{i_p}_{j_p}du^{i_1}\otimes\cdots$

$du^{i_{s-1}}\otimes du^i\otimes du^{i_{s+1}}\cdots\otimes du^{i_p}\otimes du^j$

(1. 13) $\varphi\,|\,\tau^2(\mathfrak{X})=\mu,$

then we can also define a homomorphism φ:

(1. 14) $\varphi=\varphi_\Gamma:\ T(\mathfrak{X})^{\bar{\otimes}(p.q+1)}\to T(\mathfrak{X})^{\otimes(p.q+1)}\ (p,\,q=0,1,2,\cdots)$[3)]

The covariant differential operator $D=D_\Gamma$ of the general connection Γ is defined by

$$D=D_\Gamma=\varphi\cdot d:\ \Psi(T(\mathfrak{X})^{\otimes(p.q)})\to\Psi(T(\mathfrak{X})^{(p.q+1)})$$

where $\Psi(T(\mathfrak{X})^{\otimes(p.q)})$ means the vector space consisting of all cross-sections of $T(\mathfrak{X})^{\otimes(p.q)}$ over the algebra $A(\mathfrak{X})$ of all scalar fields on $\mathfrak{X}$. In fact, if $V\in\Psi(T(\mathfrak{X})^{\otimes(p.q)})$

$$V=V^{i_1\cdots i_p}_{j_1\cdots j_q}\partial u_{i_1}\otimes\cdots\otimes\partial u_{i_p}\otimes du^{j_1}\otimes\cdots\otimes du^{j_q},$$

then

$$DV^{i_1\cdots i_p}_{j_1\cdots j_q}=V^{i_1\cdots i_p}_{j_1\cdots j_q},_h\,du^h,$$

(1. 15) $$V^{i_1\cdots i_p}_{j_1\cdots j_q},_h=P^{i_1}_{k_1}\cdots P^{i_p}_{k_p}\frac{\partial V^{k_1\cdots k_p}_{h_1\cdots h_q}}{\partial u^h}P^{h_1}_{j_1}\cdots P^{h_q}_{j_q}$$

$$+\sum_{s=1}^{q}P^{i_1}_{k_1}\cdots P^{i_{s-1}}_{k_{s-1}}\cdots\Gamma^{i_s}_{k_s h}P^{i_{s+1}}_{k_{s+1}}\cdots P^{i_p}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_q}\ P^{h_1}_{j_1}\cdots P^{h_q}_{j_q}$$

$$-\sum_{t=1}^{q}P^{i_1}_{k_1}\cdots P^{i_p}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_q}\quad P^{h_1}_{j_1}\cdots P^{h_{t-1}}_{j_{t-1}}\Lambda^{h_t}_{j_t h}P^{h_{t+1}}_{j_{t+1}}\cdots P^{h_q}_{j_q}.$$

The *torsion tensor* of Γ is a tensor of type (1. 2), with local components

(1. 16) $T^j_{ih}=\Gamma^j_{ih}-\Gamma^j_{hi},$

and the local components R^j_{ihk} of the *curvature tensor* Γ are given by

3) On the symbols $\dot{\otimes}$, $T(\mathfrak{X})^{\bar{\otimes}(p.q+1)}$, and the operation d, see [9] §2.

$$(1.17)\qquad R^j_{ihk}=\left[P^j_l\left(\frac{\partial\Gamma^l_{mk}}{\partial u^h}-\frac{\partial\Gamma^l_{mh}}{\partial u^k}\right)+(\Gamma^j_{lh}\Gamma^l_{mk}-\Gamma^j_{lk}\Gamma^l_{mh})\right]P^m_i$$
$$-\delta^j_{m,h}A^m_{ik}+\delta^j_{m,k}A^m_{ih}.$$

Now we give some definitions for the following sections.

Definition 1.1. *A general connection Γ is said to be regular when P^j_i is an isomorphism of $T(\mathfrak{X})$.*

When Γ is a regular general connection, then $(\delta^i_j,\ '\Gamma^i_{jh})$, where

$$(1.18)\qquad '\Gamma^i_{jh}=Q^i_l\Gamma^l_{jh},\qquad Q=P^{-1},$$

is a classical affine connection. It is called the *contravariant part of Γ.* Putting

$$(1.19)\qquad ''\Gamma^i_{jh}=\Gamma^i_{lh}Q^l_j+P^i_l\frac{\partial Q^l_j}{\partial u^h},$$

$(\delta^i_j,\ ''\Gamma^i_{jh})$ is also a classical affine connection. It is called the *covariant part of Γ.*

Let Γ be a regular general connection. For a tensor $V^{i_1\cdots i_p}_{j_1\cdots j_q}$ the covariant differentiation given by

$$(1.20)\qquad V^{i_1\cdots i_p}_{j_1\cdots j_q|h}=Q^{i_1}_{k_1}\cdots Q^{i_p}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_q,h}Q^{h_1}_{j_1}\cdots Q^{h_q}_{j_q}$$

is called the *basic covariant differentiation of Γ.* In fact, $V^{i_1\cdots i_p}_{j_1\cdots j_q|h}$ is given by

$$(1.21)\qquad V^{i_1\cdots i_p}_{j_1\cdots j_q|h}=\frac{\partial V^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial u^h}+\sum_{s=1}^{p}{'\Gamma}^{i_s}_{kh}V^{i_1\cdots i_{s-1}ki_{s+1}\cdots i_p}_{j_1j_2\cdots\cdots j_q}$$
$$-\sum_{t=1}^{q}V^{i_1i_2\cdots\cdots i_p}_{j_1\cdots j_{t-1}kj_{t+1}\cdots j_q}{''\Gamma}^k_{j_th}.$$

With respect to the basic covariant differentiation, we have the following formula:

$$(1.22)\qquad \delta^j_iV^{i_1\cdots i_p\,i}_{j_1\cdots j_qj|h}=(V^{i_1\cdots i_p\,i}_{j_1\cdots j_q\,j}\,\delta^j_i)_{|h}+V^{i_1\cdots i_p i}_{j_1\cdots j_q j}\,\delta^j_{i|h}.$$

Definition 1.2. *A tensor P of type (1. 1) is called normal when P as a homomorphism of the tangent bundle $T(\mathfrak{X})$ of $\mathfrak{X}$ is an isomorphism on each $P(T_x(\mathfrak{X}))=P_x(\mathfrak{X})$, $x\in\mathfrak{X}$. and* dim $P_x(\mathfrak{X})=$ m *is a constant. A general connection (P^i_j,Γ^i_{jk}) is called normal if the tensor $P=P^i_j\,\partial u_i\otimes du^j$ is normal.*

Definition 1.3. *We say that a general connection Γ satisfies the metric condition for a symmetric covariant tensor $G=g_{ij}\,du^i\otimes du^j$ if*

$$DG=g_{ij,h}du^h=0.$$

Let (P^i_j) be given, we knew there are general connections $\Gamma=(P^i_j,\Gamma^i_{jh})$ which satisfy the metric condition ([11], Theorem 2). We call such a Γ *a metric general connection.*

Definition 1.4. *Let* A^n *be an n-dimensional affine space. If there is* *a general connection of type* $(F^{\lambda}_{\mu}, 0)$ *of* A^n, with respect to affine coordinates *then* A^n *is called a pseudo-affine space of dimension n.*

For a curve C: $u^i(t)$ in $\mathfrak{X}$, if there exists a curve $\bar{C}$: $x^{\lambda}=x^{\lambda}(t)$ in a pseudo-affine space A^n and a field of frame $\{X^i_{\gamma}\}$ of $T(\mathfrak{X})$ along C such that

$$(1.23)\qquad \begin{cases} \dfrac{dx^{\lambda}}{dt}=Y^{\lambda}_i\dfrac{du^i}{dt} \\ P^i_j\dfrac{DX^{\lambda}_i}{dt}=P^i_j(P^j_k\dfrac{dX^k_{\lambda}}{dt}+\Gamma^j_{kh}X^k_{\lambda}\dfrac{du^h}{dt})=0, \end{cases}$$

where $\{Y^{\lambda}_i\}$ is the dual base of $\{X^i_{\lambda}\}$, then $\bar{C}$ is called a *development of C.*

Let C be a curve in a space $\mathfrak{X}$ with a normal general connection such that $\dim P(T_x(\mathfrak{X}))=m$. Then C has a development which depends on $n(n-m)$ arbitrary functions of the parameter of C. ([10], Theorem 3.1.)

We can also define a development of C by conditions different from (1.23).

§ 2 Submanifold $\mathfrak{X}_l$ of $\mathfrak{X}$.

Let $\mathfrak{X}$ be a Riemannian manifold with a metric tensor G. In local coordinates (U, u^i),

$$G=g_{ij}du^i\otimes du^j$$

where (g_{ij}) is symmetric, positive definite.

Let $(\iota, \mathfrak{X}_l)$ be a regular submanifold of $\mathfrak{X}$. Let l be the dimension of $\mathfrak{X}_l$. For a point $x\in\mathfrak{X}_l$, the local coordinates (U, u^i) of the point $\iota(x)\in\mathfrak{X}$ and the local coordinates (U_0, x^a) of the point $x\in\mathfrak{X}_l$ have the following relations:

$$u^i\cdot\iota=u^i(x^1, \cdots, x^l)$$

where the matrix $(\partial u^i/\partial x^a)$ is of rank l. In the followings, we always denote the local coordinates of a point by (u^i) if we consider it as a point of $\mathfrak{X}$ and by (x^a) if we consider it as a point of the submanifold $\mathfrak{X}_l$, where $i=1, \cdots, n$ and $a=1, \cdots l$.

Let the natural base of the tangent space of $\mathfrak{X}_l$ at x by $\{\partial x^a\}$. Let the same of the tangent space of $\mathfrak{X}$ at x be $\{\partial u_i\}$. Then ([6], chap. 2)

$$\iota_*\partial x_a=\frac{\partial u^i}{\partial x^a}\partial u_i,\qquad \iota^*du^i=\frac{\partial u^i}{\partial x^a}dx^a,$$

where $\{du^i\}$, $\{dx^a\}$ are the duals of $\{\partial u^i\}$ and $\{\partial x_a\}$. We often omit the notations ι, ι_* and ι^* if it does not lead any confusion. For example, we write

$$u^i=u^i(x^1, \cdots, x^l)$$

instead of $u^i\cdot\iota=u^i(x^1, \cdots, x^l)$.

Putting

$$\frac{\partial u^i}{\partial x^a}=\theta^i_a$$

We have

(2. 1) $\iota_* \partial x_a = \theta_a^i \partial u_i$

(2. 2) $\iota^* du^i = \theta_a^i dx^a$.

We put also

$$g_{ab} = g_{ij}\theta_a^i \theta_b^j;$$

then $g_{ab}\, dx^a \otimes dx^b$ is a positive definite symmetric form on $\mathfrak{X}_l$. Let (g^{ab}) be the inverse matrix of (g_{ab}) and

(2. 3) $\theta_i^a = g^{ab} g_{ij} \theta_b^j$.

Then we have

(2. 4) $\theta_a^j \theta_j^b = \delta_a^b$.

We consider other coordinates (V, v^i) and (V_0, y^a) at x, the corresponding θ_a^i, θ_i^a are written by $\theta_a^i(v)$, $\theta_a^i(x)$ etc., then we get

(2. 5)
$$\begin{cases} \theta_a^i(y) = \theta_b^i(x) \dfrac{\partial x^b}{\partial y^a} & \theta_i^a(y) = \theta_i^b(x) \dfrac{\partial y^a}{\partial x^b} \\ \theta_a^i(v) = \dfrac{\partial v^i}{\partial u^j} \theta_a^j(u) & \theta_i^a(v) = \dfrac{\partial u^j}{\partial v^i} \theta_j^a(u). \end{cases}$$

For any $x \in \mathfrak{X}_l$, we define a linear map $\iota^\# \; T_x^*(\mathfrak{X}_l) \to T_x^*(\mathfrak{X})$:

(2. 6) $\iota^\# \, dx^a = \theta_i^a du^i$;

it is clear that the definifion is independent of (U, u^i). We also define a linear map $\iota_\# \; T_x(\mathfrak{X}) \to T_x(\mathfrak{X}_l)$:

(2. 7) $\iota_\# \partial \, u_i = \theta_i^a \partial x_a$,

which is also independent of the local coordinates (U_0, x^a).

We defined $\tau_x^2(\mathfrak{X})$, the tangent vectors of order 2 at x, in §1. Now we define a linear map $\iota_* : \tau_x^2(\mathfrak{X}_l) \to \tau_x^2(\mathfrak{X})$:

(2. 8)
$$\begin{cases} \iota_* \partial^2 x_{ab} = (\partial_b \theta_a^i) \partial u_i + \theta_a^i \theta_b^j \partial u_{ij} \\ \iota_* / T_x(\mathfrak{X}_l) : \text{the same as (2. 1)}. \end{cases}$$

If we consider another coordinate neighborhood (V, v^i) at x, the coordinates of x can be written as

$$v^i = v^i(u^k(x^l)).$$

The formulas (1. 3), (1. 4) imply the following:

$$\iota_* \partial^2 x_{ab} = \frac{\partial^2 v^j}{\partial v^a \, \partial v^b} \partial v_j + \frac{\partial v^i}{\partial x^a} \frac{\partial v^j}{\partial x^b} \partial^2 v_{ij},$$

which shows that the definition of $\iota_* \, \partial^2 x_{ab}$ is independent of the local coordinates (U, u^i).

On the other hand, let x be contained in two coordinate neighborhoods (U_0, x^a) and (V_0, y^a) of $\mathfrak{X}_l$. Then we have

$$\partial y_a = \frac{\partial x^b}{\partial y^a}\partial x_b$$

$$\partial^2 y_{ab} = \frac{\partial x^c}{\partial y^a}\frac{\partial x^d}{\partial y^b}\partial^2 x_{cd} + \frac{\partial^2 x^c}{\partial y^a \partial y^b}\partial x_c$$

and hence

$$\begin{aligned}\iota_*\partial^2 y_{ab} &= \frac{\partial x^c}{\partial y^a}\frac{\partial x^d}{\partial y^b}\iota_*\partial^2 x_{cd} + \frac{\partial^2 x^c}{\partial y^a \partial y^b}\iota_*\partial x_c \\ &= \frac{\partial^2 u^i}{\partial y^a \partial y^b}\partial u_i + \frac{\partial u^i \partial u^j}{\partial y^a \partial y^b}\partial^2 u_{ij} \\ &= \partial_b \theta^i_a(y)\partial u_i + \theta^i_a(y)\theta^j_b(y)\partial^2 u_{ij}.\end{aligned}$$

Next, we define a linear map $\iota^* : \mathcal{D}^2_x(\mathfrak{X}) \to \mathcal{D}^2_x(\mathfrak{X}_\iota)$:

$$(2.9)\qquad \begin{cases} \iota^* d^2 u^i = \theta^i_a d^2 x^a + \partial_b \theta^i_a \, dx^a \otimes dx^b \\ \iota^*/T^*_x(\mathfrak{X}_\iota) : \text{ the same as (2. 2).}\end{cases}$$

The formula (1. 5) implies:

$$\begin{aligned}\iota^* d^2 v^j &= \frac{\partial v^j}{\partial u^i}\iota^* d^2 u^i + \frac{\partial^2 v^j}{\partial u^i \partial u^h}\iota^* du^i \otimes \iota^* du^h \\ &= \frac{\partial v^j}{\partial x^a} d^2 x^a + \frac{\partial^2 v^j}{\partial x^a \partial x^b} dx^a \otimes dx^b,\end{aligned}$$

which shows that the defintion of ι^*: $\mathcal{D}^2_x(\mathfrak{X}) \to \mathcal{D}^2_x(\mathfrak{X}_\iota)$ is independent of the local coordinate neighborhood (U, u^i). If we transfom the coordinate neighborhood (U_0, x^a) of $\mathfrak{X}_\iota$ into another coordinate neighborhood (V_0, y^a) of $\mathfrak{X}_\iota$, we have easily:

$$\begin{aligned}\iota^* d^2 u^i &= \frac{\partial u^i}{\partial y^a} d^2 y^a + \frac{\partial^2 u^i}{\partial y^a \partial y^b} dy^a \otimes dy^b \\ &= \theta^i_a(y) d^2 y^a + \partial_b \theta^i_a(y) dy^a \otimes dy^b.\end{aligned}$$

Now we want to define a linear map of $\mathcal{D}^2_x(\mathfrak{X}_\iota) \to \mathcal{D}^2_x(\mathfrak{X})$. It is unfortunate that we can not define such a map which is independent of the local coordinates (u^i). We can only define a linear map $\iota^\sharp_U$: $\mathcal{D}^2_x(\mathfrak{X}_\iota) \to \mathcal{D}^2_x(\mathfrak{X})$ *by*:

$$(2.10)\qquad \begin{cases} \iota^\sharp_U d^2 u^a = \theta^a_i d^2 u^i + \theta^b_h \partial_b \theta^a_i du^i \otimes du^h \\ \iota^\sharp_U / T^*_x(\mathfrak{X}_\iota) = \iota^\sharp \text{ of (2. 6).}\end{cases}$$

We substiute (1. 5) into the following:

$$\iota^\sharp_V \; d^2 x^a = \theta^a_i(v) d^2 v^i + \theta^b_j(v)\partial_b \theta^a_i(v) dv^i \otimes dv^j.$$

Comparing this with (2. 10), we have

$$(2.11)\qquad \begin{cases} \iota^\sharp_V d^2 x^a = \iota^\sharp_U d^2 x^a + \theta^a_h(u)\dfrac{\partial u^h}{\partial v^t}\dfrac{\partial^2 v^t}{\partial u^j \partial u^t}(\delta^t_q - A^t_q) du^j \otimes du^q, \\ A^t_q = \theta^t_a \theta^a_q.\end{cases}$$

It is clear that $\iota^* \iota^\sharp_V d^2 x^a = \iota^* \iota^\sharp_U d^2 x^a$. Since the kernel of ι^*: $\mathcal{D}^2(\mathfrak{X}) \to \mathcal{D}^2(\mathfrak{X}_\iota)$

includes $N=\{N^i_j du^j \otimes du^k,\ N^i_k du^j \otimes du^k\}$ where $N^i_k=\delta^i_k - A^i_k$, we may consider $\iota^{\#}\mathcal{D}^2(\mathfrak{X}^l) \subset \overline{\mathcal{D}}^2(\mathfrak{X})$ where $\overline{\mathcal{D}}^2(\mathfrak{X})=\mathcal{D}^2(\mathfrak{X})\,|\,\mathfrak{X}_l/N$[4)]

From the definition of (2. 8) and (2. 10), omitting the notation ι, we may write

$$\tau^2(\mathfrak{X}_l) \subset \tau^2(\mathfrak{X}),$$
$$\mathcal{D}^2(\mathfrak{X}_l) \subset \overline{\mathcal{D}}^2(\mathfrak{X}).$$

Now we consider a tensor field on a coordinate neighborhood (U, u^i) of $\mathfrak{X}$ with the components:

$$T^{ij\cdots}_{kl\cdots}.$$

Let

$$T^{ab\cdots}_{cd\cdots}=\theta^a_i\theta^b_j\cdots\theta^k_c\theta^l_d\cdots T^{ij\cdots}_{kl\cdots}.$$

Then $T^{ab\cdots}_{cd\cdots}$ are components of a tensor field on a neighborhood of $\mathfrak{X}_l$. We call $(T^{ab\cdots}_{cd\cdots})$ the tensor field on $\mathfrak{X}_l$ induced from the tensor field $(T^{ij\cdots}_{kl\cdots})$ on $\mathfrak{X}$.

On the other hand, if there is a tensor field on a neighborhood of $\mathfrak{X}_l$ with components

$$T^{ab\cdots}_{cd\cdots}$$

we have by (2. 1) and (2. 6), omitting the notation ι,

$$\begin{aligned} &T^{ab\cdots}_{cd\cdots}\ \partial x_a \otimes \partial x_b \otimes \cdots \otimes dx^c \otimes dx^d \otimes \cdots \\ &= T^{ab\cdots}_{cd\cdots}\theta^i_a\theta^j_b\cdots\theta^c_k\theta^d_l\cdots\partial u_i \otimes \partial u_j \otimes \cdots \otimes du^k \otimes du^l \cdots. \end{aligned}$$

In this case we say that the tensor field $(T^{ab\cdots}_{cd\cdots})$ on (U_0, x^a) of $\mathfrak{X}^l$ *is represented* on (U, u^i) of $\mathfrak{X}$ by

$$\theta^i_a\theta^j_b\cdots\theta^c_k\theta^d_l\cdots T^{ab\cdots}_{cd\cdots}.$$

§ 3 Induced general connection.

We consider a submanifold $\mathfrak{X}_l$ in $\mathfrak{X}$.

Suppose we are given a general connection

$$\Gamma=\partial u_i \otimes (P^i_j d^2u^j + \Gamma^i_{jh}\ du^j \otimes du^h)$$

of $\mathfrak{X}$. The induced tensor field on $\mathfrak{X}_l$ of the tensor field (P^i_j) on $\mathfrak{X}$ has components as

$$P^a_b=\theta^a_i P^i_j \theta^j_b.$$

Definition 3.1. *For a given general connection* $\Gamma=(P^i_j,\ \Gamma^i_{jk})$ of $\mathfrak{X}$,

$$\gamma\equiv\iota(P)\equiv\iota_{\#}\partial u_i \otimes (P^i_j\iota^* d^2u^j + \Gamma^i_{jk}\iota^* du^j \otimes \iota^* du^k)$$

is called the induced connection of $\mathfrak{X}_l$ *derived from* Γ.

In fact, by definitions in § 2

4) $\mathcal{D}^2(\mathfrak{X})/\mathfrak{X}_l$ means the portion of $\mathcal{D}^2(\mathfrak{X})$ on $\mathfrak{X}_l$. That is, if we consider $\mathfrak{X}_l \subset \mathfrak{X}$ and ι is the identify map of $\mathfrak{X}_l \to \mathfrak{X}$, then $\mathcal{D}^2(\mathfrak{X})/\mathfrak{X}_l$ is the induced bundle $\iota^{-1}(\mathcal{D}^2(\mathfrak{X}))$

$$\iota_\sharp \partial u_i \otimes (P^i_j \iota^* d^2 u^j + \Gamma^i_{jk} \iota^* du^j \otimes \iota^* du^k)$$
$$= \theta^a_i \partial x_a \otimes [P^i_j(\theta^j_b d^2 x^b + \partial_c \theta^j_b dx^b \otimes dx^c) + \Gamma^i_{jk} \theta^j_b \theta^k_c dx^b \otimes dx^c]$$
$$= \partial x_a \otimes [\theta^a_i P^i_j \theta^j_b d^2 x^b + \theta^a_i P^i_j \partial_c \theta^j_b dx^b \otimes dx^c + \theta^a_i \Gamma^i_{jk} \theta^j_b \theta^k_c dx^b \otimes dx^c]$$
$$= \partial x_a \otimes [P^a_b d^2 x^b + \theta^a_i (P^i_j \partial_c \theta^j_b + \Gamma^i_{jk} \theta^j_b \theta^k_c) dx^b \otimes dx^c].$$

If we put

$$\Gamma^a_{bc} = \theta^a_i (P^i_j \partial_c \theta^j_b + \Gamma^i_{jk} \theta^j_b \theta^k_c) \tag{3.1}$$

then

$$\iota(\Gamma) = \partial x_a \otimes [P^a_b d^2 x^b + \Gamma^a_{bc} dx^b \otimes dx^c], \tag{3.2}$$

hence γ is a cross-section on $T(\mathfrak{X}_\iota) \otimes \mathcal{D}^2(\mathfrak{X}_\iota)$. For any other local coordinate neighborhood (V_0, y^a) of $\mathfrak{X}_\iota$, if $U_0 \cap V_0 \neq \phi$, it is easy to verify by (2.5) and (3.1):

$$\Gamma^a_{bc}(y) = \frac{\partial y^a}{\partial x^f} \frac{\partial x^d}{\partial y^b} \frac{\partial x^e}{\partial y^c} \Gamma^f_{de}(x) + P^f_e(x) \frac{\partial y^a}{\partial x^f} \frac{\partial^2 x^e}{\partial y^b \partial y^c}.$$

Corresponding to $\Lambda^i_{jh} = \Gamma^i_{jh} - \partial_h P^i_j$ in the general connection theory of $\mathfrak{X}$, let us put

$$\Lambda^a_{bc} = \Gamma^a_{bc} - \frac{\partial P^a_b}{\partial x^c} \tag{3.3}$$
$$= \theta^a_i \Lambda^i_{jk} \theta^j_b \theta^k_c - \theta^j_b P^i_j \partial_c \theta^a_i.$$

If $U_0 \cap V_0 \neq \phi$, then $\Lambda^a_{bc}(x)$ and $\Lambda^a_{bc}(y)$ are related with

$$\Lambda^a_{bc}(y) = \left\{ -P^f_d(x) \frac{\partial^2 y^a}{\partial x^f \partial x^e} + \frac{\partial y^a}{\partial x^f} \Lambda^f_{de}(x) \right\} \frac{\partial x^d}{\partial y^b} \frac{\partial x^e}{\partial y^c}. \tag{3.4}$$

Now we shall define the covariant differentiation of a mixed tensor with respect to the given general connection and its induced connection.

Let μ_Γ be the linear map defined in § 1 (1.8),

$$\iota_\sharp \mu_\Gamma(\partial u_i) = \iota_\sharp(P^j_i \partial u_j) = P^j_i \theta^a_j \partial x_a,$$
$$\iota_\sharp \mu_\Gamma(\partial^2 u_{jk}) = \iota_\sharp(\Gamma^i_{jk} \partial u_i) = \Gamma^i_{jk} \theta^a_i \partial x_a.$$

Then we obtain $\iota_\sharp \mu_\Gamma / \iota_* \tau^2_x(\mathfrak{X}_\iota)$:

$$\iota_\sharp \mu_\Gamma \iota_*(\partial x_a) = \iota_\sharp \mu_\Gamma(\theta^i_a \partial u_i) = \theta^i_a P^j_i \theta^b_j \partial x_b = P^b_a \partial x_b,$$
$$\iota_\sharp \mu_\Gamma \iota_*(\partial_2 x_{ab}) = \iota_\sharp \mu_\Gamma(\partial_b \theta^i_a \partial u_i + \theta^i_a \theta^j_b \partial^2 u_{ij})$$
$$= \partial_b \theta^i_a P^j_i \theta^c_j \partial x_c + \theta^i_a \theta^j_b \Gamma^k_{ij} \theta^c_k \partial x_c$$
$$= \theta^c_k (P^k_i \partial_b \theta^i_a + \theta^i_a \theta^j_b \Gamma^k_{ij}) \partial x_c$$
$$= \Gamma^c_{ab} \partial x_c.$$

Hence we have

$$\mu_\gamma = \iota_\sharp \mu_\Gamma \iota_*,$$

where

$$(3.5)\qquad \begin{cases} \mu_\gamma(\partial x_a) = P_a^b \partial x_b, \\ \mu_\gamma(\partial^2 x_{ab}) = \Gamma_{ab}^c \partial x_c. \end{cases}$$

Similarly, we consider the linear map $\iota^*\varphi_\Gamma$ (§ 1, (1. 11) and (1. 14)) at each point of $\mathfrak{X}_\iota$:

$$\iota^*\varphi_\Gamma(du^i) = \iota^*(du^i) = \theta_a^i dx^a,$$
$$\iota^*\varphi_\Gamma(du^i \otimes du^j) = \iota^*(P_k^i du^k \otimes du^j)$$
$$= P_k^i \theta_a^k \theta_b^j dx^a \otimes dx^b.$$
$$\iota^*\varphi_\Gamma(d^2u^i) = \iota^*[-(\Gamma_{jk}^i - \partial_k P_j^i)]du^j \otimes du^k$$
$$= -(\Gamma_{jk}^i - \partial_k P_j^i)\theta_a^j \theta_b^k dx^a \otimes dx^b.$$

Then we have $\iota^*\varphi_\Gamma / \mathcal{D}_x^2(\mathfrak{X}_\iota)$ as follows:

$$\iota^*\varphi_\Gamma(\iota^\sharp dx^a) = \iota^*\varphi_\Gamma(\theta_i^a du^i) = \theta_i^a \theta_b^i dx^b = dx^a,$$
$$\iota^*\varphi_\Gamma(\iota^\sharp dx^a \otimes dx^b) = \theta_i^a \theta_j^b \iota^*\varphi_\Gamma(du^i \otimes du^j)$$
$$= \theta_i^a \theta_j^b \; P_k^i \theta_c^k \theta_d^j dx^c \otimes dx^d$$
$$= P_c^a dx^c \otimes dx^b,$$
$$\iota^*\varphi_\Gamma \iota^\sharp_U(d^2x^a) = \iota^*\varphi_\Gamma(\theta_i^a d^2u^i + \theta_h^b \partial_b \theta_i^a du^i \otimes du^h)$$
$$= -\theta_i^a(\Gamma_{jk}^i - \partial_k P_j^i)\theta_b^j \theta_c^k dx^b \otimes dx^c + P_k^i \theta_h^b \partial_b \theta_i^a \theta_d^k \theta_c^h dx^d \otimes dx^c$$
$$= [-\theta_i^a \Gamma_{jk}^i \theta_b^j \theta_c^k + \theta_i^a \partial_k P_j^i \theta_b^j \theta_c^k + P_k^i \partial_c \theta_i^a \theta_b^k] dx^b \otimes dx^c$$
$$= [-\theta_i^a(\Gamma_{jk}^i \theta_b^j \theta_c^k + P_j^i \partial_c \theta_b^j) + \theta_i^a P_j^i \partial_c \theta_b^j + \theta_i^a \partial_c P_j^i \theta_b^j$$
$$+ P_k^i \partial_c \theta_i^a \theta_b^k] dx^b \otimes dx^c$$
$$= [-\Gamma_{bc}^a + \partial_c(\theta_i^a P_j^i \theta_b^j)] dx^b \otimes dx^c$$
$$= (-\Gamma_{bc}^a + \partial_c P_b^a) dx^b \otimes dx^c$$
$$= -A_{bc}^a dx^b \otimes dx^c.$$

To compute $\iota^*\varphi_\Gamma \iota^\sharp_V(d^2x^a)$, we only have to operate the map $\iota^*\varphi_\Gamma$ to the last term of the right hand side of (2. 11),

$$\iota^*\varphi_\Gamma\left[\theta_h^a \frac{\partial u^h}{\partial v^i} \frac{\partial^2 v^i}{\partial u^p\,\partial u^t}(\delta_q^t - A_q^t)\, du^p \otimes du^q\right]$$
$$= \iota^*\left[\theta_h^a \frac{\partial u^h}{\partial v^i} \frac{\partial^2 v^i}{\partial u^p\,\partial u^t}(\delta_q^t - A_q^t)\, P_s^p du^s \otimes du^q\right]$$
$$= \theta_h^a \frac{\partial u^h}{\partial v^i} \frac{\partial^2 v^i}{\partial u^p\,\partial u^t}(\delta_q^t - A_q^t)\, \theta_c^q P_s^p \theta_b^s dx^b \otimes dx^c$$
$$= 0$$

since $A_q^t \theta_c^q = \theta_c^t$. Hence we have

$$\iota^*\varphi_\Gamma \iota^\sharp_U d^2x^a = \iota^*\varphi_\Gamma \iota^\sharp_V d^2x^a,$$

that is $\iota^*\varphi_\Gamma \iota^\sharp_U$ is independent of the coordinate neighborhood (U, u^i). Therefore, we have

$$\varphi'_\gamma \equiv \iota^* \varphi_\Gamma \cdot \iota^{\#}_U = \iota'^* \varphi_\Gamma \iota^{\#}_V,$$

where

$$(3.6)\quad \begin{cases} \varphi'_\gamma(dx^a) = dx^a, \\ \varphi'_\gamma(dx^a \otimes dx^b) = P^a_c dx^c \otimes dx^b, \\ \varphi'_\gamma(d^2x^a) = -\Lambda^a_{bc} dx^b \otimes dx^c. \end{cases}$$

We generalize (3. 5) and (3. 6) to define a linear map by the followings:

$$\varphi_\gamma(\partial x_{a_1} \otimes \cdots \otimes \partial x_{a_m}) = P^{b_1}_{a_1} \cdots P^{b_m}_{a_m} \partial x_{b_1} \otimes \cdots \otimes \partial x_{b_m},$$

$$\varphi_\gamma(dx^{a_1} \otimes \cdots \otimes dx^{a_m} \otimes dx^{a_{m+1}}) = P^{a_1}_{b_1} \cdots P^{a_m}_{b_m} dx^{b_1} \otimes \cdots \otimes dx^{b_m} \otimes dx^{a_{m+1}},$$

$$\varphi_\gamma(\partial^2 x_{ab}) = \Gamma^c_{ab}\, \partial x_c,$$

$$\varphi_\gamma(d^2x^a) = -\Lambda^a_{bc} dx^b \otimes dx^c.$$

This map φ_γ leads us to define the covariant differentiation D_γ of the induced general connection γ of $\mathfrak{X}_l$:

$$D \equiv D_\gamma = \varphi_\gamma \cdot d:\ \Phi(T(\mathfrak{X}_l)^{\otimes(p,q)}) \to \Phi(T(\mathfrak{X}_l)^{\otimes(p,q+1)}).$$

Now we try to define the coovariant differentiation of so called mixed tensor on the submanifold $\mathfrak{X}_l$. The tensor

$$V = V^{a_1 a_2 \cdots \iota_1 \iota_2 \cdots}_{b_1 b_2 \cdots j_1 j_2 \cdots} \partial x_{a_1} \otimes \partial x_{a_2} \otimes \cdots \otimes \partial u_{\ 1} \otimes \partial u_{\iota_2} \otimes \cdots$$
$$\otimes dx^{b_1} \otimes dx^{b_2} \otimes \cdots \otimes du^{j_1} \otimes du^{j_2} \otimes \cdots$$

is called a mixed tensor on $\mathfrak{X}_l$. We define dV by the following:

$$dV = \partial_b V^{a_1 \cdots \iota_1 \cdots}_{b_1 \cdots j_1 \cdots} \partial x_{a_1} \otimes \cdots \otimes \partial u_{\iota_1} \otimes \cdots \otimes du^{j_1} \otimes \cdots \otimes dx^b$$
$$+ \sum_k V^{a_1 \cdots \iota_1 \cdots}_{b_1 \cdots j_1 \cdots} \partial x_{a_1} \otimes \cdots \otimes \partial^2 x_{a_k b} \otimes \cdots \otimes \partial u_{\iota_1} \otimes \cdots \otimes dx^{b_1} \otimes \cdots$$
$$\otimes du^{j_1} \otimes \cdots \otimes dx^b$$
$$+ \sum_l V^{a_1 \cdots \iota_1 \cdots}_{b_1 \cdots j_1 \cdots} \partial x_{a_1} \otimes \cdots \otimes \partial u_{\iota_1} \otimes \cdots \otimes \partial^2 u_{\iota_l \iota} \otimes \cdots$$
$$\otimes dx^{b_1} \otimes \cdots \otimes du^{j_1} \otimes \cdots \otimes du^{\iota}$$
$$+ \sum_h V^{a_1 \cdots \iota_1 \cdots}_{b_1 \cdots j_1 \cdots} \partial x_{a_1} \otimes \cdots \otimes \partial u_{\iota_1} \otimes \cdots \otimes dx^{b_1} \otimes \cdots \otimes d^2 x^{b_h}$$
$$\dot{\otimes} (\cdots) \otimes du^{j_1} \otimes \cdots$$
$$+ \sum_m V^{a_1 \cdots \iota_1 \cdots}_{b_1 \cdots j_1 \cdots} \partial x_{a_1} \otimes \cdots \otimes \partial u_{\ 1} \otimes \cdots \otimes dx^{b_1} \otimes \cdots \otimes du^{j_1} \otimes \cdots$$
$$d^2 u^{j_m} \dot{\otimes} (\cdots).$$

If we denote the portion $T(\mathfrak{X})^{\otimes q}/\mathfrak{X}_l$, $T^*(\mathfrak{X})^{\otimes r}/\mathfrak{X}_l$ briefly by $T(\mathfrak{X})^{\otimes q}$. $T^*(\mathfrak{X})^{\otimes r}$, then d maps a vector field of the vector bundle $T(\mathfrak{X}_l)^{\otimes p} \otimes T(\mathfrak{X})^{\otimes q} \otimes T^*(\mathfrak{X}_l)^{\otimes r} \otimes T^*(\mathfrak{X})^{\otimes s}$ into one of the following vector bundle:

$$T(\mathfrak{X}_l)^{\otimes p} \otimes T(\mathfrak{X})^{\otimes q} \otimes T^*(\mathfrak{X}_l)^{\otimes r} \otimes T^*(\mathfrak{X})^{\otimes s} \otimes T^*(\mathfrak{X}_l)$$
$$+ \sum_{s=1}^{q} T(\mathfrak{X}_l)^{\otimes (s-1)} \otimes \tau^2(\mathfrak{X}_l) \otimes T(\mathfrak{X}_l)^{\otimes (p-s)} \otimes T(\mathfrak{X})^{\otimes q}$$
$$\otimes T^*(\mathfrak{X}_l)^{\otimes r} \otimes T^*(\mathfrak{X})^{\otimes s} \otimes T^*(\mathfrak{X}_l)$$

$$+\sum_{s=1}^{q} T(\mathfrak{X}_l)^{\otimes p}\otimes T(\mathfrak{X})^{\otimes(s-1)}\otimes\tau^2(\mathfrak{X})\otimes T(\mathfrak{X})^{\otimes(q-s)}\otimes T^*(\mathfrak{X}_l)^{\otimes r}$$
$$\otimes T^*(\mathfrak{X})^{\otimes(s+1)}$$
$$+\sum_{t=1}^{r} T(\mathfrak{X}_l)^{\otimes p}\otimes T(\mathfrak{X})^{\otimes q}T^*(\mathfrak{X}_l)^{\otimes(t-1)}\otimes\mathcal{D}^2(\mathfrak{X}_l)\dot{\otimes} T(\mathfrak{X}_l)^{\otimes(r-t)}\otimes T^*(\mathfrak{X})^{\otimes s}$$
$$+\sum_{t=1}^{s} T(\mathfrak{X}_l)^{\otimes p}\otimes T(\mathfrak{X})^{\otimes q}\otimes T^*(\mathfrak{X}_l)^{\otimes r}\otimes T^*(\mathfrak{X})^{\otimes(t-1)}\otimes\mathcal{D}^2(\mathfrak{X})\dot{\otimes} T^*(\mathfrak{X})^{\otimes(s-t)}.$$

Now we define a map φ on the image of d which is a generalization of (3. 6) and (1. 14) as follows

$$\varphi(\partial x_{a_1}\otimes\cdots\otimes\partial u_{i_1}\otimes\cdots\otimes dx^{b_1}\otimes\cdots\otimes du^{j_1}\otimes\cdots\otimes dx^{b})$$
$$=P_{a_1}^{c_1}\cdots P_{i_1}^{h_1}\cdots P_{d_1}^{b_1}\cdots P_{k_1}^{j_1}\cdots\partial x_{c_1}\otimes\cdots\otimes\partial u_{h_1}\otimes\cdots\otimes dx^{d_1}\otimes\cdots\otimes du^{k_1}$$
$$\otimes\cdots\otimes dx^{b},$$

$$\varphi(\partial x_{a_1}\otimes\cdots\otimes \partial^2 x_{a_k b}\otimes\cdots\otimes \partial u_{i_1}\otimes\cdots\otimes dx^{b_1}\otimes\cdots\otimes du^{j_1}\otimes\cdots\otimes dx^{b})$$
$$=P_{a_1}^{c_1}\cdots \Gamma_{a_k b}^{c_k}\cdots P_{i_1}^{h_1}\cdots P_{d_1}^{b_1}\cdots P_{k_1}^{j_1}\cdots\partial x_{c_1}\otimes\cdots\otimes\partial u_{h_1}\otimes\cdots\otimes dx^{d_1}\otimes\cdots$$
$$\otimes du^{k_1}\otimes\cdots\otimes dx^{b},$$

$$\varphi(\partial u_{a_1}\otimes\cdots\otimes\partial u_{1}\otimes\cdots\otimes\partial^2 u_{i_l i}\otimes\cdots\otimes dx^{b_1}\otimes\cdots\otimes du^{j_1}\otimes\cdots\otimes du^{i})$$
$$=P_{a_1}^{c_1}\cdots P_{i_1}^{h_1}\cdots\Gamma_{i_l i}^{h_l}\cdots P_{d_1}^{b_1}\cdots P_{k_1}^{j_1}\cdots\partial x_{c_1}\otimes\cdots\otimes \partial u_{h_1}\otimes\cdots$$
$$\otimes dx^{d_1}\otimes\cdots\otimes du^{k_1}\otimes\cdots\otimes du^{i},$$

$$\varphi(\partial x_{a_1}\otimes\cdots\otimes\partial u_{i_1}\otimes\cdots\otimes dx^{b_1}\otimes\cdots\otimes d^2x^{b_h}\dot{\otimes}(\cdots)\otimes du^{j_1}\otimes\cdots)$$
$$=-P_{a_1}^{c_1}\cdots P_{i_1}^{h_1}\cdots P_{d_1}^{b_1}\cdots A_{d_h b}^{b_h}\cdots P_{k_1}^{j_1}\cdots\partial x_{c_1}\otimes\cdots\otimes\partial u_{h_1}\otimes\cdots\otimes dx^{d_1}\otimes\cdots$$
$$\otimes du^{k_1}\otimes\cdots\otimes dx^{b},$$

$$\varphi(\partial x_{a_1}\otimes\cdots\otimes\partial u_{i_1}\otimes\cdots\otimes dx^{b_1}\otimes\cdots\otimes du^{j_1}\otimes\cdots\otimes d^2u^{j_m}\dot{\otimes}(\cdots))$$
$$=-P_{a_1}^{c_1}\cdots P_{i_1}^{h_1}\cdots P_{d_1}^{b_1}\cdots P_{k_1}^{j_1}\cdots A_{k_m k}^{j_m}\cdots\partial x_{c_1}\otimes\cdots\otimes\partial u_{h_1}\otimes\cdots$$
$$\otimes dx^{d_1}\otimes\cdots\otimes du^{k_1}\otimes\cdots\otimes du^{k}.$$

If we consider the last term dx^b of the above right hand side as an element $B_i^b\, du^i$ of $T^*(\mathfrak{X})$, we have

$$\varphi:\ d(T(\mathfrak{X}_l)^{\otimes p}\otimes T(\mathfrak{X})^{\otimes q}\otimes T^*(\mathfrak{X}_l)^{\otimes r}\otimes T^*(\mathfrak{X})^{\otimes s})$$
$$\to T(\mathfrak{X}_l)^{\otimes p}\otimes T(\mathfrak{X})^{\otimes q}\otimes T^*(\mathfrak{X}_l)^{\otimes r}\otimes T^*(\mathfrak{X})^{\otimes s}\otimes T^*(\mathfrak{X}).$$

Now we define the covariant differentiation of a mixed tensor by the following:

$$DV=\varphi dV$$

In fact, let

$$DV=\partial x_{a_1}\otimes\cdots\otimes dx^{b_1}\otimes\cdots\otimes\partial u_{i_1}\otimes\cdots\otimes du^{j_1}\otimes\cdots\otimes DV_{b_1 b_2\cdots j_1 j_2\cdots}^{a_1 a_2\cdots i_1 i_2\cdots}$$

and

$$DV^{a_1\cdots i_1\cdots}_{b_1\cdots j_1\cdots}=D_bV^{a_1\cdots i_1\cdots}_{b_1\cdots j_1\cdots}\,dx^b.$$

Then

$$\begin{aligned} D_bV^{a_1\cdots i_1\cdots}_{b_1\cdots j_1\cdots}= & P^{a_1}_{c_1}\cdots P^{i_1}_{h_1}\cdots\partial_b V^{c_1\cdots h_1\cdots}_{d_1\cdots k_1\cdots}P^{d_1}_{b_1}\cdots P^{k_1}_{j_1}\cdots \\ & +\sum_l P^{a_1}_{c_1}\cdots \Gamma^{a_l}_{bc}\cdots P^{i_1}_{h_1}\cdots V^{c_1\cdots c\cdots h_1\cdots}_{d_1\cdots\cdots k_1\cdots}P^{d_1}_{b_1}\cdots P^{k_1}_{j_1}\cdots \\ & +\sum_l P^{a_1}_{c_1}\cdots P^{i_1}_{h_1}\cdots \Gamma^{i_l}_{h_j}\theta^j_b\cdots V^{c_1\cdots h_1\cdots h\cdots}_{d_1\cdots k_1\cdots\cdots\cdots}P^{d_1}_{b_1}\cdots P^{k_1}_{j_1}\cdots \\ & -\sum_l P^{a_1}_{c_1}\cdots P^{i_1}_{h_1}\cdots V^{c_1\cdots\cdots h_1\cdots}_{d_1\cdots d\cdots k_1\cdots}P^{d_1}_{b_1}\cdots \Lambda^{d}_{d_l b}\cdots P^{k_1}_{j_1}\cdots \\ & -\sum_l P^{a_1}_{c_1}\cdots P^{i_1}_{h_1}\cdots V^{c_1\cdots\cdots\cdots h_1\cdots\cdots}_{d_1\cdots k_1\cdots k\cdots}P^{d_1}_{b_1}\cdots P^{k_1}_{j_1}\cdots \Lambda^{k}_{j_l i}\theta^i_b\cdots. \end{aligned}$$

For a tensor field with components $(V^{i_1\cdots}_{j_1\cdots})$ of $\mathfrak{X}$ defined on $\mathfrak{X}_l$, the covariant differentiation of $(V^{i_1\cdots}_{j_1\cdots})$ defined by (3. 7) is the same as that defined by the general connection Γ of $\mathfrak{X}$. For a tensor with components $(V^{a_1\cdots}_{b_1\cdots})$ on $\mathfrak{X}_l$, the covariant differentiation of $(V^{a_1\cdots}_{b_1\cdots})$ is the same as that defined by induced connection γ of $\mathfrak{X}_l$ derived from Γ of $\mathfrak{X}$.

Next, we are going to consider a special general connection of $\mathfrak{X}$ and its induced connection of $\mathfrak{X}_l$. Let A^i_j be any 1-1 tensor of $\mathfrak{X}$. Then

$$A\Gamma=(A^i_jP^j_k,\quad A^i_j\Gamma^j_{hk})$$

and

$$\Gamma A=(P^i_jA^j_k,\quad \Gamma^i_{jk}A^j_h+P^i_j\partial_kA^j_h)$$

are also general connections of $\mathfrak{X}$, and hence

$$\widetilde{\Gamma}=A\Gamma A=(A^i_kP^k_hA^h_j,\quad A^i_k\Gamma^k_{lh}A^l_j+A^i_kP^k_l\frac{\partial A^l_j}{\partial u^h})$$

is a general connection of $\mathfrak{X}$ ([13], §1). Let us put

$$\widetilde{P}^i_j=A^i_kP^k_hA^h_j,$$

$$\widetilde{\Gamma}^i_{jh}=A^i_k\Gamma^k_{lh}A^l_j+A^i_kP^k_l\partial_hA^l_j.$$

Then

$$\widetilde{\Lambda}^i_{jh}=A^i_l\Lambda^l_{kh}A^k_j-\partial_hA^i_lP^l_kA^k_j.$$

Especially, we consider the case when $A^i_j=\theta^i_a\theta^a_j$ on $\mathfrak{X}_l$. In this case, we have

$$\theta^a_iA^i_j=\theta^a_j,\quad \theta^j_aA^i_j=\theta^i_a$$

and

$$\theta^a_iA^i_kP^k_hA^h_j\theta^j_b=\theta^a_kP^k_h\theta^h_b=P^a_b$$

$$\begin{aligned} & \theta^a_i(\widetilde{P}^i_j\partial_c\theta^j_b+\widetilde{\Gamma}^i_{jh}\theta^j_a\theta^h_c) \\ & \quad =\theta^a_i[A^i_kP^k_hA^h_j\partial_c\theta^j_b+A^i_k\Gamma^k_{lh}A^l_j\theta^j_b\theta^h_c+A^i_kP^k_l\partial_hA^l_j\theta^j_b\theta^h_c] \\ & \quad =\theta^a_kP^k_l\partial_c(A^l_j\theta^j_b)+\theta^a_k\Gamma^k_{lh}\theta^l_b\theta^h_c \end{aligned}$$

$$=\Gamma^a_{bc}.$$

Hence we have the following theorem:

Theorem 3. 1. *If* $A^i_j=\theta^i_a\theta^a_j$ *on* $\mathfrak{X}_l$ *then the induced connection of* $\mathfrak{X}_l$ *derived from the general connection* $A\Gamma A$ *of* $\mathfrak{X}$ *is the same as that derived from the general connection* Γ *of* $\mathfrak{X}$.

We denote the covariant differentiation with respect to the general connection $\widetilde{\Gamma}$ by $\widetilde{D}$. Then it is easy to verify:

$$\widetilde{D}_j V^{i_1 i_2 \cdots}_{k_1 k_2 \cdots} = A^{i_1}_{s_1} A^{i_2}_{s_2}\cdots D_j(A^{s_1}_{l_1}\cdots V^{l_1 l_2 \cdots}_{m_1 m_2 \cdots} A^{m_1}_{v_1}\cdots) A^{v_1}_{k_1} A^{v_2}_{k_2}\cdots. \tag{3. 8}$$

Now we consider a tensor field with components $(V^{a_1 a_2 \cdots}_{b_1 b_2 \cdots})$, which is represented in a neighborhood of $\mathfrak{X}$ by:

$$'V^{l_1 l_2 \cdots}_{m_1 m_2 \cdots} = \theta^{l_1}_{a_1}\theta^{l_2}_{a_2}\cdots V^{a_1 a_2 \cdots}_{b_1 b_2 \cdots}\theta^{b_1}_{m_1}\theta^{b_2}_{m_2}\cdots.$$

Since

$$A^{s_1}_{l_1}A^{s_2}_{l_2}\cdots {}'V^{l_1 l_2 \cdots}_{m_1 m_2 \cdots} A^{m_1}_{v_1}A^{m_2}_{v_2}\cdots = {}'V^{s_1 s_2 \cdots}_{v_1 v_2 \cdots},$$

(3. 8) implies

$$\widetilde{D}' V^{i_1 i_2 \cdots}_{j_1 j_2 \cdots} = A^{i_1}_{s_1}A^{i_2}_{s_2}\cdots D('V^{s_1 s_2 \cdots}_{v_1 v_2 \cdots}) A^{v_1}_{j_1}A^{v_2}_{j_2}\cdots. \tag{3. 9}$$

Since $A^{i_1}_{j_1}=\theta^{i_1}_a\theta^a_{j_1}$ on $\mathfrak{X}_l$, we have

$$\widetilde{D}'_b V^{i_1 i_2 \cdots}_{j_1 j_2 \cdots} = \theta^{i_1}_{a_1}\theta^{i_2}_{a_2}\cdots(D_b V^{a_1 a_2 \cdots}_{b_1 b_2 \cdots})\theta^{b_1}_{j_1}\theta^{b_2}_{j_2}\cdots.$$

Theorem 3. 2. *Let* $V^{a_1 a_2 \cdots}_{b_1 b_2}$ *be components of a tensor field of* $\mathfrak{X}_l$. *If we represent them in a neighborhood* (U, u^i) *of* $\mathfrak{X}$ *by*;

$$'V^{i_1 i_2 \cdots}_{j_1 j_2 \cdots} = \theta^{i_1}_{a_1}\theta^{i_2}_{a_2}\cdots V^{a_1 a_2 \cdots}_{b_1 b_2 \cdots}\theta^{b_1}_{j_1}\theta^{b_2}_{j_2}\cdots,$$

Then,

$$\overline{D}'_b V^{i_1 i_2 \cdots}_{j_1 j_2 \cdots} = \theta^{i_1}_{a_1}\theta^{i_2}_{a_2}\cdots(D_b V^{a_1 a_2 \cdots}_{b_1 b_2 \cdots})\theta^{b_1}_{j_1}\theta^{b_2}_{j_2}\cdots\cdots$$

where $\overline{D}$ *denotes the covariant differentiation with respect to the general connection* $\overline{\Gamma}=A\Gamma A$ *and* $A^i_j=\theta^i_a\theta^a_j$ *on* $\mathfrak{X}_l$.

In fact,

$$\begin{aligned}
\overline{D}'_j V^h_k &= A^h_s D_j' V^s_t A^t_k \\
&= A^h_s(P^s_l\partial_j{}'V^l_m P^m_t + \Gamma^s_{lj}{}'V^l_m P^m_t\theta^j_j - \Lambda^m_{tj}{}'V^l_m P^s_l\theta^j_j)A^t_k \\
&= \theta^h_a(\theta^a_s P^s_l\partial_j{}'V^l_m P^m_t\theta^t_b + \theta^a_s\Gamma^s_{lj}{}'V^l_m P^m_t\theta^t_b\vartheta^j_j - \theta^t_b\Lambda^m_{tj}\theta^j_j{}'V^l_m P^s_l\theta^a_s)\theta^b_k \\
&= \theta^h_a(\theta^a_s P^s_l\partial_j{}'V^l_m P^m_t\theta^t_b + \theta^a_s\Gamma^s_{lj}{}'V^l_m P^m_t\theta^t_b\vartheta^j_j - \theta^t_b\Lambda^m_{tj}\theta^j_j{}'V^l_m P^s_l\theta^a_s)\theta^b_k \\
&= \theta^h_a(\theta^a_s P^s_l\theta^l_c\partial_j V^c_d\vartheta^d_m P^m_t\theta^t_b + \theta^a_s P^s_l\partial_j\vartheta^l_c V^c_d\vartheta^d_m P^m_t\theta^t_b \\
&\quad + \theta^a_s P^s_l\theta^l_c V^c_d\partial_j\vartheta^d_m\cdot P^m_t\theta^t_b + \theta^a_s\Gamma^s_{lj}{}'V^l_m P^m_t\theta^t_b\theta^j_j \\
&\quad - \theta^t_b\Lambda^m_{tj}\theta^j_j{}'V^l_m P^s_l\theta^a_s)\theta^b_k \\
&= \theta^h_a[P^a_c\partial_j V^c_d P^d_b + \theta^a_s(P^s_l\partial_j\vartheta^l_c + \Gamma^s_{lj}\theta^l_c\theta^j_j)V^c_d P^d_b
\end{aligned}$$

$$+ P_c^a V_a^c \theta_b^t (P_t^m \partial_J g_m^d - \theta_m^d A_{tj}^m \theta_J^j)] \theta_k^b$$

From (3. 1) (3. 3) we have

$$\bar{D}_J{}'V_k^h = \theta_a^h [P_c^a \partial_J V_d^c P_b^d + \Gamma_{cJ}^a V_d^c P_b^d - P_c^a V_a^c A_{bJ}^d] \theta_k^b = \theta_a^h D_J V_b^a \theta_k^b.$$

Finally we shall generalize a theorem on parallel vector field in the classical subspace theory to our general connection theory.

Let (V^a) be a vector tangent to a submanifold $\mathfrak{X}_l$ and be represented in a neighborhood of $\mathfrak{X}$ by

$${}'V^i = \theta_a^i V^a.$$

We consider the covariant differentiation of the vector ${}'V^i$ along a curve in $\mathfrak{X}_l$ with respect to the general connection Γ of $\mathfrak{X}$;

$$\frac{D'V^i}{ds} = P_j^i \frac{d'V^j}{ds} + \Gamma_{jk}^i {}'V^j \frac{dx^k}{ds}$$

$$= (P_j^i \partial_b {}'V^j + \Gamma_{jk}^i \ {}'V^j \theta_b^k) \frac{dx^b}{ds}$$

$$= \left[P_j^i \partial_b \theta_a^j \ V^a + P_j^i \theta_a^j \partial_b V^a + \Gamma_{jk}^i {}'V^j \theta_b^k \right] \frac{dx^b}{ds},$$

$$\theta_i^c \frac{D'V^i}{ds} = \left[\theta_i^c P_j^i (\partial_b \theta_a^j) V^a + P_a^c \partial_b V^a + \theta_i^c \Gamma_{jk}^i \theta_a^j \theta_b^k V^a \right] \frac{dx^b}{ds}$$

$$= \left[\theta_i^c (P_j^i \partial_b \theta_a^j + \Gamma_{jk}^i \theta_a^j \theta_b^k) V^a + P_a^c \partial_b V^a \right] \frac{dx^b}{ds}$$

$$= \left[P_a^c \partial_b V^a + \Gamma_{ab}^c V^a \right] \frac{dx^b}{ds}$$

$$= D_b V^c \frac{dx^b}{ds},$$

where dx^b/ds is the tangent vector of the given curve in $\mathfrak{X}_l$.

Hence we have

$$\theta_i^c \frac{D(\theta_a^i V^a)}{ds} = \frac{DV^c}{ds}$$

along the given curve.

Theorem 3. 3. *Let $\mathfrak{X}_l$ be a submanifold $\mathfrak{X}$. If a vector field tangent to $\mathfrak{X}_l$ is a parallel vector field along a curve of $\mathfrak{X}_l$ with respect to a given connection Γ of $\mathfrak{X}$, then it is also parallel along the same curve with respect to the induced connection of $\mathfrak{X}_l$ derived from the given Γ of $\mathfrak{X}$.*

§ 4. Induced normal connection.

Let the given general connection (P_j^i, Γ_{jk}^i) of $\mathfrak{X}$ be normal. Then P_j^i is a normal tensor (Definition 1. 2.). We denote the image $P(T_x(\mathfrak{X}))$ by $P_x(\mathfrak{X})$ or P_x and the kernel of $P: T_x(\mathfrak{X}_l) \to P_x$ by $N_x(\mathfrak{X})$ or N_x. Now we have

$$T_x(\mathfrak{X}) = P_x(\mathfrak{X}) + N_x(\mathfrak{X}).$$

We assume in this section that the normal tensor P is *orthogonally related with* the given Riemann metric G, that is, $P_x(\mathfrak{X})$ and $N_x(\mathfrak{X})$ are mutually orthogonal with respect to G.

We are going to consider some submanifolds satisfying a condition given by the following definition.

Definition 4. 1. *A submanifold $\mathfrak{X}_l$ of $\mathfrak{X}$ is called adapted to the general connection* $(P^i_j,\ \Gamma^i_{jh})$ *of $\mathfrak{X}$ if each tangent space $T_x(\mathfrak{X}_l)$, $x \in \mathfrak{X}_l$, and its orthogonal complementary space in $T_x(\mathfrak{X})$ are invariant by the homomor phism P.*

Briefly, we say sometimes such a submanifold $\mathfrak{X}_l$ to *be adapted in* $\mathfrak{X}$.

Let $\mathfrak{X}_l$ be adapted in $\mathfrak{X}$. Then the tangent space at each point $x \in \mathfrak{X}_l$ can be written by

$$(4.1) \qquad T_x(\mathfrak{X}_l) = P_x(\mathfrak{X}_l) + N_x(\mathfrak{X}_l)$$

where $P_x(\mathfrak{X}_l)$ is invariant by P[5] and $N_x(\mathfrak{X}_l)$ is a subspace of N_x. Let $\{\theta^i_a\}$ $(a = 1, \cdots, l)$ be a basis of $T_x(\mathfrak{X}_l)$. There exists a matrix (P^a_b) such that

$$(4.2) \qquad P^i_j \theta^j_a = P^b_a \theta^i_b,$$

where (P^a_b) is not necessarily regular.

We can verify that the dimension p_x of $P(T_x(\mathfrak{X}_l))$ is constant. In fact, at a fixed point $x_0 \in \mathfrak{X}_l$, a basis of $P(T_x(\mathfrak{X}_l))$ can be represented by an (n, l) matrix with respect to a basis of $T_x(\mathfrak{X}_l)$. The rank of this matrix is $p(x_0)$, that is the dimension of $P(T_x(\mathfrak{X}_l)$. In a neighborhood of x_0, each point has such a matrix of rank not less than $p(x_0)$, that is

$$\varliminf_{x \to x_0} p(x) \geqq p(x_0),$$

hence $p(x)$ is a lower semi-continuous function. Similarly the dimension $n(x)$ of $N_x(\mathfrak{X}_l)$ is a lower semi-continuous function. Since $p(x) + n(x) =$ constant, we have that $p(x)$ is a continuous function and hence it is a constant. Let the dimension of $P_x(\mathfrak{X}_l)$ be l_1. The dimension of $N_x(\mathfrak{X}_l)$ is $l_2 = l - l_1$. We consider a basis $\{V^a_{a_0} / a_0 = 1, \cdots, l_1\}$ of $P_x(\mathfrak{X}_l)$ with respect to a natural frame in $\mathfrak{X}_l$, and a basis $\{V^a_{a_1} / a_1 = l_1 + 1, \cdots, l\}$ of $N_x(\mathfrak{X}_l)$ with respect to the same frame. If we represent the basis of $T_x(\mathfrak{X}_l)$ by the given corresponding neighborhood and the natrual frame in $\mathfrak{X}$, we shall have $\{\theta^i_a V^a_{a_0}\}$ and $\{\theta^i_a V^a_{a_1}\}$ as its components. By (4. 2), we have $P^b_a = \theta^b_i P^i_j \theta^j_a$. Since $\theta^i_a V^a_{a_0} \in P_x(\mathfrak{X}_l) \subset P_x$ and $\mathfrak{X}_l$ is adapted in $\mathfrak{X}$, we have

$$P^a_b V^b_{a_0} = \theta^a_i P^i_j (\theta^j_b V^b_{a_0}) = \theta^a_i (\theta^i_b V^b_{b_0} W^{b_0}_{a_0}) = V^a_{b_0} W^{b_0}_{a_0},$$

where $(W^{a_0}_{b_0})$ is a regular (l_1, l_1) matrix. Since $\theta^i_a V^a_{a_1} \in N_x(\mathfrak{X}_l) \subset N_x$, we have

5) "$T_x(\mathfrak{X}_l)$ is invariant by P" means $P(T_x(\mathfrak{X}_l)) \subset T_x(\mathfrak{X}_l)$.

$P^i_j\theta^j_a V^a_{a_1}=0$, that is $\theta^i_b P^b_a V^a_{a_1}=0$, in other words,

$$P^b_a V^a_{a_1}=0.$$

The P^a_b given in (4. 2) is the induced tensor field on $\mathfrak{X}_l$ from the tensor field P^i_j on $\mathfrak{X}$. We have an induced connection (P^a_b, Γ^a_{bc}) of $\mathfrak{X}_l$ derived from Γ. Hence we have the following:

Theorem 4. 1. *Let $\Gamma=(P^i_j, \Gamma^i_{jk})$ be a normal general connection of $\mathfrak{X}$, $\mathfrak{X}_l$ be adapted in $\mathfrak{X}$. Then the induced tensor P^a_b on $\mathfrak{X}_l$ from P^i_j is also normal and hence the induced general connection : $\gamma=(P^a_b, \Gamma^a_{bc})$ of $\mathfrak{X}_l$ derived from $\Gamma=(P^i_j, \Gamma^i_{jh})$ of $\mathfrak{X}$ is normal.*

Now we consider the case $T_x(\mathfrak{X})= P_x(\mathfrak{X})$. At this time, (P^a_b) is a regular matrix.

Let $\{\theta^i_A\}$ be a basis of $N_x(\mathfrak{X})$, and

(4. 3) $\qquad g_{AB}=g_{ij}\theta^i_A\theta^j_B.$

Then (g_{AB}) is regular, whose inverse we denote by (g^{AB}). If we put

(4. 4) $\qquad \theta^A_i=g_{ij}\,g^{BA}\theta^j_B,$

since P is orthogonally related with G, we have

(4. 5) $\qquad \theta^i_A\theta^B_i=\delta^B_A, \quad \theta^i_A\theta^a_i=0, \quad \theta^i_a\theta^A_i=0.$

The inverse of the matrix (θ^i_a, θ^i_A) is

$$\begin{pmatrix}\theta^a_i\\ \theta^A_i\end{pmatrix}$$

that is

(4. 5) $\qquad \theta^i_a\theta^a_i+\theta^i_A\theta^A_i=\delta^j_i.$

we shall compute the mixed tensor $D_b\theta^i_a$ defined by (3. 7). In this case

$$D_b\theta^i_a= P^c_a\partial_b\theta^j_c P^i_j+\Gamma^i_{jk}\theta^j_c P^c_a\theta^k_b-\Lambda^c_{ab}\theta^j_c P^i_j$$
$$= P^c_a(\partial_b\theta^j_c P^i_j+\Gamma^i_{jk}\theta^j_c\theta^k_b)-\Lambda^c_{ab}\theta^j_c P^i_j.$$

Hence we have by (3. 1)

$$\theta^c_i D_b\theta^i_a= P^d_a\theta^c_i(\partial_b\theta^j_d P^i_j+\Gamma^i_{jk}\theta^j_d\theta^k_b)-\theta^c_i\Lambda^d_{ab}\theta^j_d P^i_j$$
$$= P^d_a\Gamma^c_{db}-\Lambda^d_{ab}P^c_d$$
$$=D_b\delta^a_c.$$

Theorem 4. 2. *Let $\Gamma=(P^i_j, \Gamma^i_{jh})$ be a given normal general connection of $\mathfrak{X}$ and $\mathfrak{X}_l$ be a submanifold of $\mathfrak{X}$ such that each tangent space $T_x(\mathfrak{X}_l)$ coincides with $P_x(\mathfrak{X})$. With respect to Γ of $\mathfrak{X}$ and its induced general connection γ of $\mathfrak{X}_l$, $D_b\theta^i_a$ is orthogonal to $\mathfrak{X}$ if and only if*

$$D\delta^a_b=0.$$

At each point $X\in\mathfrak{X}_l$, we consider a coordinate neighborhood (U, u^i) of $\mathfrak{X}$ which satisfies the following conditions:

(a) In U, $\mathfrak{X}_l$ can be represented by the following equations:

$$u^i = u^i(x^1, \cdots, x^l, c^{l+1}, \cdots\cdots c^n)$$

where $(x^1, \cdots, x^l)$ are variables and $c^{l+1}, \cdots, c^n$ are constants.

(b) U is covered by a system of l dimensional submanifolds denoted by

$$u^i = u^i(x^a, x^B)$$

where (x^B) $(B=l+1, \cdots, n)$ can be considered as a system of parameters which represent the system of submanifolds.

(c) Any two vectors $(\partial u^i/\partial x^a)$ and $(\partial u^i/\partial x^B)$ $(a=1, \cdots l;\ B=l+1, \cdots, n)$ are orthogonal with respect to the given metric G.

We call such a coordinate neighborhood "*a coordinate neighborhood associated to* $\mathfrak{X}_l$". It is clear that (x^a, x^A) can also be considered as coordinates in U.

Let the indices α, β, $\cdots$ run through 1, 2, $\cdots$, n; the indices a, b, $\cdots$ run through 1, $\cdots$, l and the indices A, B, $\cdots$ run through $l+1$, $\cdots$, n. With respect to any associated coordinates, we put

$$(4.7) \qquad \theta^i_\alpha = \frac{\partial u^i}{\partial x^\alpha}$$

and

$$(4.8) \qquad \theta^\alpha_i = g^{\alpha\beta} g_{ij} \theta^j_\beta,$$

where

$$g_{\alpha\beta} = g_{ij}\theta^i_\alpha \theta^j_\beta, \quad (g^{\alpha\beta}) = (g_{\alpha\beta})^{-1};$$

clearly the matrix $(g_{\alpha\beta})$ is reducible:

$$\begin{pmatrix} g_{ab} & 0 \\ 0 & g_{AB} \end{pmatrix}.$$

We are going to consider the general connection $A\Gamma A$ defined in §3. In the associated coordinates (x^α) the tensor field A^i_j may be written by:

$$A^\alpha_\beta = \theta^\alpha_i A^i_j \theta^j_\beta.$$

The general connection Γ can be written in the coordinates (x^α) as

$$P^\alpha_\beta = \theta^\alpha_i P^i_j \theta^j_\beta$$

$$\Gamma^\alpha_{\beta\gamma} = \theta^\alpha_i (P^i_l \partial_\gamma \theta^l_\beta + \Gamma^i_{jh} \theta^j_\beta \theta^h_\gamma).$$

Especially, let A^i_j be the following tensor:

$$(4.9) \qquad A^i_j = \theta^i_a \theta^a_j,$$

that is, if we consider the tensor A^i_j as an endomorphism A of $T_x(\mathfrak{X})$ in itself, A is a projection of the tangent space $T_x(\mathfrak{X})$ onto the tangent space $T_x(\mathfrak{X}_l)$ for any $x \in U \cap \mathfrak{X}_l$. If A is defined by (4.9), we denote the general connection $A\Gamma A$ by $\overline{\Gamma}$, in other words

$$\bar{P}^i_j = A^i_l P^l_h A^h_j$$

(4.10) $$\begin{cases} \bar{\Gamma}^i_{jk} = A^i_l \Gamma^l_{hk} A^h_j + A^i_l P^l_h \partial_k A^h_j \\ A^i_j = \theta^i_a \theta^a_j. \end{cases}$$

Since A is a projection, we have $A^2 = A$. By the definition (4.9), the following relations are easily verified:

(4.11) $$\bar{P}^i_j A^j_k = A^i_j \bar{P}^j_k = \bar{P}^i_k,$$

(4.12) $$\begin{cases} A^i_j \theta^j_a = \theta^i_a & A^i_j \theta^j_B = 0 \\ \theta^a_i A^i_j = \theta^a_j & \theta^B_i A^i_j = 0. \end{cases}$$

Hence we have

$$\bar{P}^i_j \theta^j_a = \theta^i_b \theta^b_l P^l_h \theta^h_a = \theta^i_b P^b_a$$

$$\bar{P}^i_j \theta^j_B = A^i_l P^l_h A^h_j \theta^j_B = 0.$$

Thus it follows immediately:

Theorem 4.3: *Let $\Gamma = (P^i_j, \Gamma^i_{jk})$ be a normal general connection of $\mathfrak{X}$. If $\mathfrak{X}_l$ be adapted in $\mathfrak{X}$, then $(\bar{P}^i_j, \bar{\Gamma}^i_{jk})$ is also a normal general connection of $\mathfrak{X}$.*

Now we try to write the general connection $(\bar{P}^i_j, \bar{\Gamma}^i_{jk})$ in the associated coordinates (x^α).

$$\bar{P}^a_b = \theta^a_i \bar{P}^i_j \theta^j_b = \theta^a_i P^i_j \theta^j_b = P^a_b,$$

$$\bar{P}^B_b = \theta^B_i \bar{P}^i_j \theta^j_b = \theta^B_i A^i_l P^l_k A^k_j \theta^j_b = 0,$$

$$\bar{P}^a_B = \bar{P}^B_c = 0,$$

that is,

(4.13) $$(\bar{P}^\alpha_\beta) = \begin{pmatrix} P^a_b & 0 \\ 0 & 0 \end{pmatrix}.$$

We can also easily see:

(4.14) $$(A^\alpha_\beta) = \begin{pmatrix} \delta^a_c & 0 \\ 0 & 0 \end{pmatrix}.$$

Concerning the $\bar{\Gamma}^\alpha_{\beta\gamma}$, we have by (4.10)

$$\begin{aligned} \bar{\Gamma}^\alpha_{\beta\gamma} &= \theta^\alpha_i (\bar{P}^i_j \partial_\gamma \theta^j_\beta + \bar{\Gamma}^i_{jk} \theta^j_\beta \theta^k_\gamma) \\ &= \theta^\alpha_i [A^i_l P^l_h A^h_j \partial_\gamma \theta^j_\beta + A^i_l (\Gamma^l_{hk} A^h_j + P^l_h \partial_k A^h_j) \theta^j_\beta \theta^k_\gamma] \\ &= \theta^\alpha_i A^i_l [P^l_h A^h_j \partial_\gamma \theta^j_\beta + \Gamma^l_{hk} A^h_j \theta^j_\beta \theta^k_\gamma + P^l_h \partial_k A^h_j \theta^j_\beta \theta^k_\gamma]. \end{aligned}$$

Hence we have

$$\bar{\Gamma}^B_{\beta\gamma} = 0,$$

$$\begin{aligned}\bar{\Gamma}^{\alpha}_{B\gamma} &= \theta^{\alpha}_{i} A^{i}_{l} [P^{l}_{h} A^{h}_{j} \partial_{\gamma} \vartheta^{j}_{B} + P^{l}_{h} \partial_{k} A^{h}_{j} \theta^{j}_{B} \theta^{k}_{\gamma}] \\ &= \theta^{\alpha}_{i} A^{i}_{l} P^{l}_{h} [A^{h}_{j} \partial_{\gamma} \vartheta^{j}_{B} + \partial_{\gamma} A^{h}_{j} \theta^{j}_{B}] \\ &= \theta^{\alpha}_{i} A^{i}_{l} P^{l}_{h} \partial_{\gamma} (A^{h}_{j} \theta^{j}_{B}) = 0.\end{aligned}$$

In the same way as Theorem 3. 1., the adove $\bar{\Gamma}^{\alpha}_{\beta\gamma}$ implies

$$\bar{\Gamma}^{a}_{bc} = \Gamma^{a}_{bc}.$$

$\bar{\Gamma}^{\alpha}_{\beta\gamma}$ can be written now by:

$$(4.15)\qquad \begin{cases} \bar{\Gamma}^{B}_{\beta\gamma} = 0, \qquad \bar{\Gamma}^{\alpha}_{B\gamma} = 0, \\ \bar{\Gamma}^{a}_{bc} = \Gamma^{a}_{bc}. \end{cases}$$

Next, we are going to give a method of develpoment of curves in $\mathfrak{X}_l$ by use of the associated coordinates (U, x^{α}) of $\mathfrak{X}$.

Let A^n be a pseudo-affine space of dimension n (§ 1 Definition 1. 4) and $C: x^{\alpha} = x^{\alpha}(t)$ be a curve in $\mathfrak{X}$. If there exists a curve $\bar{C}: v^{\lambda} = v^{\lambda}(t)$ in A^n and a frame field $\{X^{\alpha}_{\lambda}\}$ on $T(\mathfrak{X})$ along C such that

$$(4.16)\qquad \begin{cases} \dfrac{dv^{\lambda}}{dt} = Y^{\lambda}_{\alpha} \dfrac{dx^{\alpha}}{dt} \\ P^{\alpha}_{\beta} \dfrac{DX^{\beta}_{\lambda}}{dt} = P^{\alpha}_{\beta} (P^{\beta}_{\gamma} \dfrac{dX^{\gamma}_{\lambda}}{dt} + \Gamma^{\beta}_{\gamma\delta} X^{\gamma}_{\lambda} \dfrac{dx^{\delta}}{dt}) = 0, \end{cases}$$

where $\{Y^{\lambda}_{\alpha}\}$ is the dual basis of $\{X^{\alpha}_{\lambda}\}$, and the given general connection $(F^{\lambda}_{\mu}, 0)$ of A^n satisfies

$$F^{\lambda}_{\mu} = Y^{\lambda}_{\alpha} P^{\alpha}_{\beta} X^{\beta}_{\mu}$$

along C, and is free otherwhere except the curve $\bar{C}$, then $\bar{C}$ is called the development of C with respect to the general connection Γ.

Now let C be a curve in the submanifold $\mathfrak{X}_l$. We shall find a development of C with respect to the general connection $\bar{\Gamma}$. Local coordinates used in the following are the adapted ones. Since C is contained in $\mathfrak{X}_l$, we have

$$\frac{dx^{B}(t)}{dt} = 0.$$

By (4. 13), (4. 15), the formulas (4. 16) are written as

$$(4.17)\qquad \begin{cases} \dfrac{dv^{\lambda}}{dt} = Y^{\lambda}_{a} \dfrac{dx^{a}}{dt} \\ P^{a}_{b} (P^{b}_{c} \dfrac{dX^{c}_{\lambda}}{dt} + \Gamma^{b}_{cd} X^{c}_{\lambda} \dfrac{dx^{d}}{dt}) = 0. \end{cases}$$

The matrix (P^{a}_{b}) is regular, hence the second formula is

$$\frac{dX^{a}_{\lambda}}{dt} + Q^{a}_{b} \Gamma^{b}_{cd} X^{c}_{\lambda} \frac{dx^{d}}{dt} = 0.$$

Let us put

$$Q^a_b \Gamma^b_{cd} \frac{dx^d}{dt} = K^a_c.$$

Then it can be written in

(4. 18) $\frac{dX^a_\lambda}{dt} + K^a_c X^c_\lambda = 0.$

The λ's run through from 1 to n. Let us denote

$$\bar{\lambda} = 1, \cdots, l\,; \qquad \bar{\bar{\lambda}} = l+1, \cdots, n.$$

In (4. 18) let us put

(4. 19) $X^a_{\bar{\bar{\lambda}}} = 0$

and solve the equations

$$\frac{dX^a_{\bar{\lambda}}}{dt} + K^a_c X^c_{\bar{\lambda}} = 0$$

about $(X^a_{\bar{\lambda}})$. If we give such an initial condition as:

$$X^a_{\bar{\lambda}}(t_0) = c^a_{\bar{\lambda}} \qquad |c^a_{\bar{\lambda}}| \neq 0$$

where $(c^a_{\bar{\lambda}})$ is a constant matrix, then we can get a system of linearly independent solutions. Thus we have a system of solutions of (4. 17), that is

$$(X^a_{\bar{\lambda}}, 0) \qquad |X^a_{\bar{\lambda}}(t)| \neq 0.$$

Now we attach any $(n-l)$ vectors

$$(X^B_{\bar{\lambda}},\ X^B_{\bar{\bar{\lambda}}})$$

to the above solution such that the following matrix

(4. 20) $\begin{pmatrix} X^a_{\bar{\lambda}} & 0 \\ X^B_{\bar{\lambda}} & X^B_{\bar{\bar{\lambda}}} \end{pmatrix}$

to be regular. Let the dual frame of (4. 20) be

(4. 21) $\begin{pmatrix} Y^{\bar{\lambda}}_a & 0 \\ Y^{\bar{\lambda}}_B & Y^{\bar{\bar{\lambda}}}_B \end{pmatrix} \qquad (Y^{\bar{\lambda}}_a) = (X^a_{\bar{\lambda}})^{-1}.$

Now the first formulas of (4. 17) can be written as

$$\frac{dv^{\bar{\bar{\lambda}}}}{dt} = 0,$$

that is $v^{\bar{\bar{\lambda}}} = c^{\bar{\bar{\lambda}}}$ (constant), in other words, the development $\bar{C}$ of C is contained in a submanifold A^l of A^n, where A^l is defined by $v^{\bar{\bar{\lambda}}} = c^{\bar{\bar{\lambda}}}$.

We have proved the following theorem:

Theorem 4.4 *Let $\mathfrak{X}_1$ be adapted in $\mathfrak{X}$ as in Theorem* 4.2. *Let Γ be a given normal general connection of $\mathfrak{X}$. $\bar{\Gamma}$ is a general normal connection given in* (4. 10). *In an associated coordinate neighborhood any curve C in*

$\mathfrak{X}_l$ has at least one development in a submanifold A^l of a pseudo-affine space A^n, which can be determined except $(n-l)$ frame vectors which are not tangent to $\mathfrak{X}_l$. In this case, A^n has a general connection $(F^\lambda_\mu, 0)$ which satisfies

$$F^{\bar\lambda}_{\bar\mu}=Y^{\bar\lambda}_a P^a_b X^b_{\bar\mu}, \qquad F^{\bar\lambda}_{\bar{\bar\mu}}=0, \qquad F^{\bar{\bar\lambda}}_{\mu}=0$$

along $\bar C$ and F^λ_μ is free outside $\bar C$.

From Theorem 3. 1. and Theorem 4. 4. we have

Corollary. *The development $\bar C$ of C in $\mathfrak{X}_l$ of Theorem 4. 4. coincides with the development of C in the pseudo-affine space A^l with respect to the induced connection γ of $\mathfrak{X}_l$ derived from the normal connection Γ of $\mathfrak{X}$.*

Remark[6] Under the conditions $F^{\bar\lambda}_{\bar\mu}=Y^{\bar\lambda}_a P^a_b X^b_{\bar\mu}$, $F^{\bar\lambda}_{\bar{\bar\mu}}=0$, $F^{\bar{\bar\lambda}}_{\mu}=0$, the development $\bar C$ in A^l in A^n of any geodesic C will be also a geodesic. Because, under the given conditions

$$F^{\bar\lambda}_{\bar\mu} Y^{\bar\mu}_a = Y^{\bar\lambda}_c P^c_a$$

along C, and taking account of (4. 17) we have

$$\begin{aligned}
\frac{D}{dt}\left(\frac{dv^{\bar\lambda}}{dt}\right) &= F^{\bar\lambda}_{\bar\mu}\frac{d^2v^{\bar\mu}}{dt} = F^{\bar\lambda}_{\bar\mu}\frac{d}{dt}\left(Y^{\bar\mu}_a\frac{dx^a}{dt}\right)\\
&= F^{\bar\lambda}_{\bar\mu}\left(Y^{\bar\mu}_a\frac{d^2x^a}{dt}+\frac{dY^{\bar\mu}_a}{dt}\frac{dx^a}{dt}\right)\\
&= Y^{\bar\lambda}_c P^c_b\left(\frac{d^2x^b}{dt^2}+X^b_{\bar\mu}\frac{dY^{\bar\mu}_a}{dt}\frac{dx^a}{dt}\right)\\
&= Y^{\bar\lambda}_c\left[\frac{D}{dt}\left(\frac{dx^c}{dt}\right)-\Gamma^c_{ab}\frac{dx^b}{dt}\frac{dx^a}{dt}-P^c_b\frac{dX^b_{\bar\mu}}{dt}Y^{\bar\mu}_a\frac{dx^a}{dt}\right]\\
&= Y^{\bar\lambda}_c\left[\frac{D}{dt}\left(\frac{dx^c}{dt}\right)-\Gamma^c_{ba}\frac{dx^b}{dt}\frac{dx^a}{dt}+\Gamma^c_{bd}X^b_{\bar\mu}\frac{dx^d}{dt}Y^{\bar\mu}_a\frac{dx^a}{dt}\right]\\
&= Y^{\bar\lambda}_c\frac{D}{dt}\left(\frac{dx^c}{dt}\right) = Y^{\bar\lambda}_c\psi P^c_b\frac{dx^b}{dt}\\
&= \psi F^{\bar\lambda}_{\bar\mu}\frac{dv^{\bar\mu}}{dt}.
\end{aligned}$$

§ 5 Induced regular connection

Let the given general connection of $\mathfrak{X}$ be regular (§ 1 Definition 1. 1.), $\mathfrak{X}_l$ be an adapted submanifold in $\mathfrak{X}$, $\{\theta^i_a\}$, $\theta^i_a=\partial u^i/\partial x^a$ be a basis of the tagnent space $T_x(\mathfrak{X}_l)$ at each point $x\in\mathfrak{X}_l$ and $\{\theta^i_A\}$ be $n-l$ independent tangent vectors of $T_x(\mathfrak{X})$ which are orthogonal to $T_x(\mathfrak{X}_l)$. Then they satisfy:

6) See [10] § 3.

(5. 1) $g_{ij}\theta^i_a\theta^j_A=0.$

We put $g_{ab}=g_{ij}\theta^i_a\theta^j_b$, $\theta^a_i=g^{ab}g_{ij}\theta^j_b$ as in § 2, and

(5. 2) $g_{AB}=g^{ij}\theta^i_A\theta^j_B$,

(5. 3) $\theta^A_i=g^{AB}g_{ij}\theta^j_B$, $(g_{AB})^{-1}=(g^{AB})$.

Then we have

(5. 4) $\theta^i_B\theta^A_i=\delta^A_B$, $\theta^i_a\theta^B_i=0$, $\theta^a_i\theta^i_A=0.$

Since $\{\theta^i_a\}$ and $\{\theta^i_A\}$ are invariant by P and P is regular, then if we denote

$$P^\alpha_\beta=\theta^\alpha_i P^i_j\theta^j_\beta \quad \alpha,\beta\cdots=1,2,\cdots,n.$$

we have

(5. 5) $$\begin{aligned} P^A_a&=\theta^A_i P^i_j\theta^j_a=\theta^A_i\theta^i_b W^b_a=0,\\ P^a_A&=\theta^a_i P^i_j\theta^j_A=\theta^a_i\theta^i_B W^B_A=0. \end{aligned}$$

Hence the matrix (P^α_β) can be written:

$$\begin{pmatrix}\theta^a_i\\ \theta^A_i\end{pmatrix}(P^i_j)(\theta^j_b,\theta^j_B)=\begin{pmatrix}P^a_b & 0\\ 0 & P^A_B\end{pmatrix}$$

where (P^a_b), (P^A_B) are regular matrices.

Theorem 5. 1. *Let $\Gamma=(P^i_j,\Gamma^i_{jk})$ be a given regular general connection of $\mathfrak{X}$, $\mathfrak{X}_l$ be an adapted submanifold in $\mathfrak{X}$. Then the induced general connection γ of $\mathfrak{X}_l$ from Γ is regular.*

From the definition of θ^a_i and θ^A_i, it is clear that

$$\theta^a_i\theta^i_b=\delta^a_b,\quad \theta^A_i\theta^i_B=\delta^A_B,$$

(5. 6) $\theta^a_i\theta^j_a+\theta^A_i\theta^j_A=\delta^j_i.$

By (5. 5) (5. 6) we get:

$$\begin{aligned} P^a_b\theta^b_k&=\theta^a_i P^i_j\theta^j_b\theta^b_k\\ &=\theta^a_i P^i_j(\delta^j_k-\theta^j_B\theta^B_k)\\ &=\theta^a_i P^i_k. \end{aligned}$$

Similarly we have $\theta^k_b P^b_a=P^k_j\theta^j_a$, hence

(5. 7) $\theta^a_i P^i_k=P^a_b\theta^b_k$, $\theta^i_a P^a_b=P^i_k\theta^k_b$.

Denoting the inverse matrix of (P^i_j) and (P^a_b) by

$$(Q^i_j)=(P^i_j)^{-1},\quad (Q^a_b)=(P^a_b)^{-1},$$

we obtain from (5. 7)

(5. 8) $\theta^a_k Q^k_i=\theta^b_i Q^a_b$, $\theta^k_b Q^b_a=Q^k_j\theta^j_a$.

Let us put

(5. 9) $$\begin{aligned} &Q^i_k\Gamma^k_{jh}={}'\Gamma^i_{jh}, && \Lambda^i_{kh}Q^k_j={}''\Gamma^i_{jh}.\\ &Q^a_f\Gamma^f_{bc}={}'\Gamma^a_{bc}, && \Lambda^a_{fc}Q^f_b={}''\Gamma^a_{bc}. \end{aligned}$$

Now the basic covariant differentiation (§ 1, (1. 20)) can be generalized on the mixed tensor field.

Definetion 5. 1. *For a mixed tensor field with components* $V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots}$ *on* $\mathfrak{X}_l$, *the operation*

$$\overline{D}_b V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots}=Q^{a_1}_{c_1}Q^{a_2}_{c_2}\cdots Q^{i_1}_{h_1}Q^{i_2}_{h_2}\cdots D_b V^{c_1c_2\cdots h_1h_2\cdots}_{d_1d_2\cdots k_1k_2\cdots}Q^{d_1}_{b_1}Q^{d_2}_{b_2}\cdots Q^{k_1}_{j_1}Q^{k_2}_{j_2}\cdots$$

is called basic covariant differentiation with respect to the connection Γ *and its induced connection* γ.

From (3. 7), we get easily:

$$\begin{aligned}\overline{D}_b V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots} &=\partial_b V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots}+\sum_l {}'\Gamma^{a_l}_{cb} V^{a_1\cdots c\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots}\\ &-\sum_l {}''\Gamma^{c}_{b_lb} V^{a_1a_2\cdots i_1i_2\cdots}_{b_1\cdots c\cdots j_1j_2\cdots}+\sum_l {}'\Gamma^{i_l}_{hk} V^{a_1a_2\cdots i_1\cdots h\cdots}_{b_1b_2\cdots j_1j_2\cdots}\theta^k_b\\ &-\sum_l {}''\Gamma^{h}_{j_lk} V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1\cdots h\cdots}\theta^k_b.\end{aligned}\tag{5.10}$$

Now we consider the relation between basic covariant differentiations and contractions. We have

$$\begin{aligned}\delta^{j_1}_{i_1}\overline{D}_b V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots} &=\partial_b V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots i_1j_2\cdots}+\sum_l {}'\Gamma^{a_l}_{cb} V^{a_1\cdots c\cdots i_1i_2\cdots}_{b_1b_2\cdots i_1j_2\cdots}\\ &-\sum_l {}''\Gamma^{c}_{b_lb} V^{a_1a_2\cdots i_1i_2\cdots}_{b_1\cdots c\cdots i_1j_2\cdots}+\sum_l {}'\Gamma^{i_l}_{hk} V^{a_1a_2\cdots i_1\cdots h\cdots}_{b_1b_2\cdots i_1j_2\cdots\cdots}\theta^k_b\\ &-\sum_l {}''\Gamma^{h}_{j_lk} V^{a_1a_2\cdots i_1i_2\cdots\cdots}_{b_1\cdots\cdots i_1j_2\cdots h\cdots}\theta^k_b\\ &=\overline{D}_b(V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots i_1j_2\cdots})+{}'\Gamma^{i_1}_{hk} V^{a_1a_2\cdots h\ i_2\cdots}_{b_1b_2\cdots i_1\ j_2\cdots}\theta^k_b\\ &-{}''\Gamma^{h}_{i_1k} V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots h\ j_2\cdots}\theta^k_b\\ &=\overline{D}_b(V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots i_1j_2\cdots})+V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots}({}'\Gamma^{j_1}_{i_1k}-{}''\Gamma^{j_1}_{i_1k})\theta^k_b\\ &=\overline{D}_b(V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots i_1j_2\cdots})+V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots}\overline{D}_b\delta^{j_1}_{i_1},\end{aligned}$$

that is

$$\delta^{j_1}_{i_1}\overline{D}_b V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots}=\overline{D}_b(V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots}\delta^{j_1}_{i_1})+V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots}\overline{D}_b\delta^{j_1}_{i_1}.\tag{5.11}$$

Similary we get

$$\delta^{b_1}_{a_1}\overline{D}_b V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots}=\overline{D}_b(V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots}\delta^{b_1}_{a_1})+V^{a_1a_2\cdots i_1i_2\cdots}_{b_1b_2\cdots j_1j_2\cdots}\overline{D}_b\delta^{b_1}_{a_1}.\tag{5.12}$$

Hence we can assert

Theorem 5. 2. *If* $\overline{D}\delta^i_j=0$ *with respect to the general regular connection* Γ *of* $\mathfrak{X}$ *and if* $\overline{D}\delta^a_b=0$[7] *with respect to the induced connection* γ *of* $\mathfrak{X}_l$, *then the basic covariant differentiation of a tensor and the contraction of the tensor are commutative.*

We shall compute the basic covariant differentiation of the mixed tensor θ^i_a,

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7) In fact, $\overline{D}\delta^i_j=0$ implies $\overline{D}\delta^a_b=,0$ (see §6, Theorem 6. 1.)

which is given by

(5. 13) $$\overline{D}_b\theta^i_a=\partial_b\theta^i_a+{}'\Gamma^i_{jk}\theta^j_a\theta^k_b-{}''\Gamma^c_{ab}\theta^i_c.$$

From (5. 7) and (3. 1) we get

$$\Gamma^a_{bc}=P^a_f\theta^f_j\partial_c\theta^j_b+\theta^a_i\Gamma^i_{jh}\theta^j_b\theta^h_c$$

$$Q^a_f\Gamma^f_{bc}=\theta^a_j\partial_c\theta^j_b+Q^a_f\theta^f_i\Gamma^i_{jh}\theta^j_b\theta^h_c$$

$$=\theta^a_j\partial_c\theta^j_b+\theta^a_kQ^k_i\Gamma^i_{jh}\theta^j_b\theta^h_c.$$

(5. 14) $$'\Gamma^a_{bc}=\theta^a_j\partial_c\theta^j_b+\theta^a_k{}'\Gamma^k_{jh}\theta^j_b\theta^h_c.$$

From (5. 13) and (5. 14) we get

$$\theta^c_i\overline{D}_b\theta^i_a={}'\Gamma^c_{ab}-{}''\Gamma^c_{ab}=\overline{D}_b\delta^c_a.$$

If we put $D_b\theta^i_a=H^i_{ba}$, then

(5. 15) $$H^i_{ab}=\overline{D}_b\delta^c_a\theta^i_c+\Omega^i_{ab},$$

where

$$\Omega^i_{ab}=\sum_A\omega^A_{ab}\theta^i_A,$$

$$\omega^A_{ab}=\theta^A_iH^i_{ab}.$$

Ω^i_{ab} is a vector orthogonal to $T_x(\mathfrak{X}_l)$. ω^A_{ab} can be calculated as follows:

$$\omega^A_{ab}=\theta^A_i(\partial_b\theta^i_a+{}'\Gamma^i_{jk}\theta^j_a\theta^k_b-{}''\Gamma^c_{ab}\theta^i_c)$$

$$=\theta^A_i\partial_b\theta^i_a+\theta^A_i{}'\Gamma^i_{jk}\theta^j_a\theta^k_b.$$

Lastly, we generalize some theorems in the classical theory of subspace to the submanifold with a regular general connection.

Let C be a curve in $\mathfrak{X}_l$ which is a geodesic[8] of $\mathfrak{X}$. The tangent vector at each point of C may be denoted dy ξ^a if we consider it as a vector of $\mathfrak{X}_l$ and by ξ^i if we consider it as a vector of $\mathfrak{X}$, that is, $\xi^i=\theta^i_a\xi^a$. Let the affine parameter of the geodesic C be s. Then

$$\frac{\overline{D}\xi^i}{ds}=0.$$

Since $\xi^i=\theta^i_a\xi^a$, we have by (5. 12) and (5. 15)

$$\frac{\overline{D}\xi^i}{ds}=\frac{\overline{D}(\theta^i_a\xi^a)}{ds}$$

(5. 16) $$=\theta^i_a\frac{\overline{D}\xi^a}{ds}+\xi^a\Big(\theta^i_b\frac{\overline{D}\delta^b_a}{ds}+\Omega^i_{ac}\xi^c\Big)-\theta^i_b\xi^c\frac{\overline{D}\delta^b_c}{ds}$$

$$=\theta^i_a\frac{\overline{D}\xi^a}{ds}+\xi^a\Omega^i_{ac}\xi^c.$$

Hence $\frac{\overline{D}\xi^i}{ds}=0$ implies $\frac{\overline{D}\xi^a}{ds}=0$ and $\Omega^i_{ac}\xi^a\xi^c=0$.

8) See [9], §4, Definition 4. 1.

Theorem 5.3. *Let* $\Gamma=(P^i_j,\ \Gamma^i_{jh})$ *be a given regular general connection of* $\mathfrak{X}$, $\mathfrak{X}_l$ *be an adapted submanifold in* $\mathfrak{X}$. *If C is a curve on* $\mathfrak{X}_l$ *and is a geodesic of* $\mathfrak{X}$ *with respect to* Γ, *then C is also a geodesic of* $\mathfrak{X}_l$ *with respect to the induced connection* γ *of* $\mathfrak{X}_l$ *derived from* Γ.

Let the given general connection Γ of $\mathfrak{X}$ be regular, and $\mathfrak{X}_l$ be an adapted sudmanifold in $\mathfrak{X}$. If any geodesic on $\mathfrak{X}_l$ with respect to the induced connection γ of $\mathfrak{X}_l$ is also a geodesic of $\mathfrak{X}$ with respect to Γ, then $\mathfrak{X}_l$ is called *a geodcsic submanifold* of $\mathfrak{X}$. Now we have evidently

Theorem 5.4. *The necessary and sufficient condition of* $\mathfrak{X}_l$ *to be geodesic in* $\mathfrak{X}$ *is*

(5.17) $\quad \Omega^i_{(ab)}=0.$

Similarly, let $\eta^a(s)$ be a vector field tangent to $\mathfrak{X}_l$ along a curve C in $\mathfrak{X}_l$. It is represented by $\eta^i=\theta^i_a\eta^a$ in coordinates of $\mathfrak{X}$. We have

$$\frac{\overline{D}(\theta^i_a\eta^a)}{ds}=\theta^i_a\frac{\overline{D}\eta^a}{ds}+\eta^c\Omega^i_{ca}\xi^a. \tag{5.18}$$

Hence, if η^a is any vector field which is parallel along C with respect to the induced connection of $\mathfrak{X}_l$, then $\eta^i=\theta^i_a\eta^a$ is parallel along C with respect to the general connection Γ of $\mathfrak{X}$ if and only if $\Omega^i_{ab}=0$.

Let $\mathfrak{X}_l$ be adapted in $\mathfrak{X}$ and $\mathfrak{X}$ be given a regular general connection Γ. We displace parallelly a vector tangent to $\mathfrak{X}_l$ along any curve in $\mathfrak{X}_l$ with respect to Γ. If the displaced vector is always tangent to $\mathfrak{X}_l$, then $\mathfrak{X}_l$ is called a *flat submanifold*. Now we get the following theorem:

Theorem 5.5. *Let* $\mathfrak{X}$ *be given a regular general connection and* $\mathfrak{X}_l$ *be adapted in* $\mathfrak{X}$. $\mathfrak{X}_l$ *is a flat submanifold of* $\mathfrak{X}$ *if and only if* $\Omega^i_{ac}=0$.

§ 6. Induced metric connection.

In this section, we suppose $\mathfrak{X}$ has a regular general connection and the submanifold $\mathfrak{X}_l$ is adapted in $\mathfrak{X}$.

We have a relation between $''\Gamma^a_{bc}$ and $''\Gamma^i_{jh}$:

$$\begin{aligned}
''\Gamma^a_{bc} &= \Delta^a_{fc}Q^f_b=\Gamma^a_{fc}\,Q^f_b-\frac{\partial P^a_f}{\partial x^c}Q^f_b\\
&=\theta^a_i(\Gamma^i_{jh}\vartheta^j_f\vartheta^h_c+P^i_j\partial_c\vartheta^j_f)Q^f_b-\frac{\partial P^a_f}{\partial x^c}Q^f_b\\
&=\theta^a_i\Gamma^i_{jh}Q^j_k\theta^k_b\theta^h_c+\theta^a_iP^i_j\partial_c\vartheta^j_fQ^f_b-\frac{\partial(\theta^a_iP^i_jQ^j_f)}{\partial x^c}Q^f_b\\
&=\theta^a_i\Gamma^i_{jh}Q^j_k\theta^k_b\vartheta^h_c-\theta^a_i\frac{\partial P^i_j}{\partial u^h}\theta^h_cQ^j_k\vartheta^k_b-\frac{\partial\theta^a_i}{\partial x^c}P^i_j\theta^j_fQ^f_b\\
&=\theta^a_i\Delta^i_{jh}Q^j_k\vartheta^k_b\vartheta^h_c-\frac{\partial\theta^a_i}{\partial x^c}P^i_jQ^j_k\theta^k_b
\end{aligned}$$

$$=\theta_i^{a\prime\prime}\Gamma_{kh}^i\theta_b^k\theta_c^h+\theta_i^a\frac{\partial\theta_b^i}{\partial x^c},$$

that is

$$''\Gamma_{bc}^a=\theta_i^{a\prime\prime}\Gamma_{hk}^i\theta_b^h\theta_c^k+\theta_i^a\partial_c\vartheta_b^i.$$

From (5. 14) we have

$$'\Gamma_{bc}^a-''\Gamma_{bc}^a=\theta_i^a('\Gamma_{jh}^i-''\Gamma_{jh}^i)\theta_b^j\theta_c^h.$$

In other words,

$$\overline{D}_c\delta_b^a=\theta_i^a\overline{D}_h\delta_j^i\theta_b^j\theta_c^h.$$

Theorem 6. 1. *Let* $\Gamma=(P_j^i,\Gamma_{jk}^i)$ *be a regular general connection of* $\mathfrak{X}$ *and* $\mathfrak{X}_l$ *be adapted in* $\mathfrak{X}$. *Then* $\overline{D}_k\delta_j^i=0$ *implies* $\overline{D}_c\delta_b^a=0$.

Now we consider the basic covariant differentiation of the induced metric tensor g_{ba} of $\mathfrak{X}_l$.

$$\begin{aligned}
\overline{D}_cg_{ab}&=\overline{D}_c(g_{ij}\theta_a^i\theta_b^j)\\
&=(\overline{D}_cg_{ij})\theta_a^i\theta_b^j+g_{ij}(\overline{D}_c\vartheta_a^i)\theta_b^j+g_{ij}\theta_a^i\overline{D}_c\theta_b^j-g_{hj}\theta_a^k\theta_b^j\overline{D}_c\delta_k^h-g_{ih}\theta_a^i\theta_b^k\overline{D}_c\delta_k^h\\
&=(\overline{D}_cg_{ij})\theta_a^i\theta_b^j+g_{ij}(\theta_d^i\overline{D}_c\delta_a^d+\Omega_{ac}^i)\theta_b^j+g_{ij}\theta_a^i(\theta_d^j\overline{D}_c\delta_b^d+\Omega_{bc}^j)-g_{hj}\theta_a^k\theta_b^j\overline{D}_c\delta_k^h\\
&\quad-g_{ih}\theta_a^i\theta_b^k\overline{D}_c\delta_k^h.\\
&=(\overline{D}_cg_{ij})\theta_a^i\theta_b^j+g_{ij}\theta_d^i\overline{D}_c\delta_a^d\theta_b^j+g_{ij}\theta_a^i\theta_d^j\overline{D}_c\delta_b^d-g_{hj}\theta_a^k\theta_b^j\overline{D}_c\delta_k^h-g_{ih}\theta_a^i\theta_b^k\overline{D}_c\delta_k^h\\
&=(\overline{D}_cg_{ij})\theta_a^i\theta_b^j+g_{bd}\overline{D}_c\delta_a^d+g_{ad}\overline{D}_c\delta_b^d-g_{hj}\theta_a^k\theta_b^j\overline{D}_c\delta_k^h-g_{ih}\theta_a^i\theta_b^k\overline{D}_c\delta_k^h.
\end{aligned}$$

But since

$$g_{bd}\overline{D}_c\delta_a^d=g_{bd}\theta_i^d\overline{D}_c\delta_k^i\theta_a^k=g_{il}\theta_b^l\overline{D}_c\delta_k^i\theta_a^k$$

$$g_{ad}\overline{D}_c\delta_b^d=g_{ad}\theta_h^d\overline{D}_c\delta_k^h\theta_b^k=g_{ih}\theta_a^i\overline{D}_c\delta_k^h\theta_b^k.$$

we obtain

$$\overline{D}_c(g_{ij}\theta_a^i\theta_b^j)=(\overline{D}_cg_{ij})\theta_a^i\theta_b^j.$$

Theorem 6. 2. *If* $\Gamma=(P_j^i,\Gamma_{jk}^i)$ *is a regular metric general connection of* $\mathfrak{X}$, *and* $\mathfrak{X}_l$ *is adapted in* $\mathfrak{X}$, *then the induced connection* γ *of* $\mathfrak{X}_l$ *from* $\mathfrak{X}$ *is also metric.*

For a reglar metric general connection $(P_j^i,\ \Gamma_{jh}^i)$ of $\mathfrak{X}$, it must be

$$g_{ij,h}=\frac{\partial g_{lk}}{\partial u^h}P_i^lP_j^k-g_{lk}\Lambda_{ih}^lP_j^k-g_{lk}\Lambda_{jh}^kP_i^l=0$$

where

$$\Lambda_{jh}^i=\Gamma_{jh}^i-\frac{\partial P_j^i}{\partial u^h},$$

that is

$$\frac{\partial g_{ik}}{\partial u^h}-g_{st}A^s_{ih}P^t_jQ^i_lQ^j_k-g_{st}A^t_{jh}P^s_iQ^i_lQ^j_k=0,$$

$$\frac{\partial g_{ik}}{\partial u^h}-g_{sk}{}''\Gamma^s_{ih}-g_{it}{}''\Gamma^t_{kh}=0.$$

Hence $''\Gamma$ is a metric connection derived from g_{ij}. It is clear that if $''\Gamma$ is a metric connection derived from g_{ij}, then $(P^l_j,\ \Gamma^l_{jh})$ is metric. Furthermore, if the regular metric connection Γ satisfies the condition:

$$(6.1)\qquad S^j_{ih}=\frac{1}{2}(\Gamma^j_{ih}-\Gamma^j_{hi})=\frac{1}{2}(P^j_{i;h}-P^j_{h;i})$$

where the semi-colon " ; " denotes the covariant differentiation with respect to the Levi-Civita's connection derived from (g_{ij}), then $\overline{\Gamma}=A\Gamma A$ is a normal metric general connection with respect to (g_{ij}) and satisfies the following condition:

$$(6.2)\qquad \overline{S}^j_{kh}A^k_i=\frac{1}{2}A^j_l(\overline{P}^l_{k;h}-\overline{P}^l_{h;k})A^k_i.$$ [9]

Now we shall use a coordinate neighborhood associated to the submanifold $\mathfrak{X}_l$ to write the metrric general connection $\overline{\Gamma}$.

At first, we are going to write (6.2) in associate coordinates (x^α):

$$(6.3)\qquad \overline{S}^\alpha_{\delta\beta}A^\delta_\gamma=\frac{1}{2}A^\alpha_\delta(\overline{P}^\delta_{\varepsilon;\beta}-\overline{P}^\delta_{\beta;\varepsilon})A^\varepsilon_\gamma.$$

From (4.13) and (4.14), (6.3) can be written as

$$\overline{S}^a_{bc}=\frac{1}{2}(\overline{P}^a_{c;b}-\overline{P}^a_{b;c})=\frac{1}{2}(P^a_{c;b}-P^a_{b;c}),$$

$$\overline{S}^B_{c\beta}=0,$$

$$\overline{S}^a_{cB}=\frac{1}{2}\overline{P}^a_{c;B}=\frac{1}{2}P^a_{c;B}.$$

Let us put

$$(6.4)\qquad g_{\alpha\delta}\overline{S}^\delta_{\beta\gamma}=\overline{S}_{\beta\alpha\gamma},\qquad g_{\alpha\gamma}P^\gamma_{\beta;\delta}=P_{\beta\alpha;\delta},$$

then the above three formulas are

$$(6.5)\qquad \overline{S}_{cab}=\frac{1}{2}(P_{ca;b}-P_{ba;c}),$$

$$(6.6)\qquad \overline{S}_{cB\beta}=0,$$

$$(6.7)\qquad \overline{S}_{caB}=\frac{1}{2}P_{ca;B}.$$

Concerning the metric general connectton $\overline{\Gamma}$, (6) of [13] shows that $\overline{\Gamma}_{ikj}=g_{kl}\overline{\Gamma}^l_{ij}$ satisfies the following relation:

9) See [13] Theorem 3.3.

$$\overline{[ij,\, l]}\, \overline{Q}^l_h = \frac{1}{2}(\overline{\Gamma}_{jli} + \overline{\Gamma}_{lij})A^l_h + (\overline{S}_{lki}\overline{P}^k_j + \overline{S}_{lkj}\overline{P}^k_i)\overline{Q}^l_h$$

where

$$\overline{[ij,\, h]} = \frac{1}{2}\left(\frac{\partial \overline{g}_{ih}}{\partial u^j} + \frac{\partial \overline{g}_{hj}}{\partial u^i} - \frac{\partial \overline{g}_{ij}}{\partial u^h}\right), \qquad \overline{g}_{ij} = g_{kh}\overline{P}^k_i\overline{P}^h_j,$$

$$\overline{Q}^l_h = A^l_i Q^i_j A^j_h.$$

We shall write these formulas in terms of the associated coordinates. It holds clearly:

$$(6.8) \qquad (\overline{Q}^\alpha_\beta) = \begin{pmatrix} Q^a_b & 0 \\ 0 & 0 \end{pmatrix} \qquad (Q^a_b) = (P^a_b)^{-1},$$

$$(6.9) \qquad \overline{g}_{ab} = g_{cd}P^c_a P^d_b, \qquad \overline{g}_{Ab} = \overline{g}_{bA} = 0. \qquad \overline{g}_{AB} = 0.$$

$$\overline{[ab,\, c]} = \frac{1}{2}\left(\frac{\partial \overline{g}_{ac}}{\partial x^b} + \frac{\partial \overline{g}_{bc}}{\partial x^a} - \frac{\partial \overline{g}_{ab}}{\partial x^c}\right),$$

$$(6.10) \qquad \overline{[BC,\, a]} = 0, \qquad \overline{[Bb,\, C]} = 0,$$

$$\overline{[Bb,\, c]} = \frac{1}{2}\frac{\partial \overline{g}_{bc}}{\partial x^B}, \qquad \overline{[ab,\, C]} = -\frac{1}{2}\frac{\partial \overline{g}_{ab}}{\partial x^C},$$

$$(6.11) \qquad \overline{[\alpha\beta,\, b]}\, Q^b_d = \frac{1}{2}(\overline{\Gamma}_{\alpha d\beta} + \overline{\Gamma}_{\beta d\alpha}) + (\overline{S}_{b\gamma\alpha}\overline{P}^\gamma_\beta + \overline{S}_{b\gamma\beta}\overline{P}^\gamma_\alpha)Q^b_d.$$

Let α be a and β be b in (6. 11). Then

$$\overline{\Gamma}_{adb} + \overline{\Gamma}_{bda} = 2\overline{[ab,\, c]}\, Q^c_d - 2(\overline{S}_{fea}P^e_b + \overline{S}_{feb}P^e_a)Q^f_d.$$

Since

$$\overline{\Gamma}_{adb} - \overline{\Gamma}_{bda} = 2S_{adb},$$

we have

$$(6.12) \qquad \begin{cases} \overline{\Gamma}_{abd} = \overline{[ab,\, c]}\, Q^c_d + \overline{S}_{adb} - (\overline{S}_{fea}P^e_b + \overline{S}_{feb}P^e_a)Q^f_d \\ \overline{S}_{cab} = \dfrac{1}{2}(P_{ca\,;\,b} - P_{ba\,;\,c}). \end{cases}$$

Next, we have from (4. 14),

$$(6.13) \qquad \overline{\Gamma}_{Ba\gamma} = 0,$$

$$(6.14) \qquad \overline{\Gamma}_{aB\gamma} = 0.$$

Lastly, it remains to find $\overline{\Gamma}_{acB}$. By (6. 6) we have

$$(6.15) \qquad \overline{\Gamma}_{acB} = \overline{\Gamma}_{acB} - \overline{\Gamma}_{Bca} = 2\overline{S}_{acB} = P_{ac\,;\,B}.$$

Theorem 6. 3. *Let* $\mathfrak{X}$ *have a regular metric general connection* $\Gamma = (P^i_j, \Gamma^i_{jh})$ *satisfying* (6. 1). *In a coordinate neighborhood associated to the adapted submanifold* $\mathfrak{X}_l$, *the normal metric connection* $A\Gamma A = \overline{\Gamma}$ *can be* *written by* (6. 12), (6. 13), (6. 14) *and* (6. 15).

From theorem 3. 1. and (4. 15) we get

Corollary. The induced connection of $\mathfrak{X}_l$ *from* $(\overline{P}, \overline{\Gamma})$ *is metric and given by* (6. 12).

§ 7. Curvatures of $\overline{\Gamma}$ and its induced connection.

Let $\Gamma = (P^i_j, \Gamma^i_{jh})$ be a regular general connection of $\mathfrak{X}$, and $\mathfrak{X}_l$ be a submanifold adapted in $\mathfrak{X}$. $\overline{\Gamma}$ is the connection $A\Gamma A$ where $A^i_j = \theta^i_a \theta^a_j$ on $\mathfrak{X}_l$. Since $\mathfrak{X}_l$ is adapted in $\mathfrak{X}$, the matrix (P^a_b):

$$P^a_b = \theta^a_i P^i_j \theta^j_b$$

is regular. Let (Q^a_b) be the inverse matrix of (P^a_b) and put

$$(7.1) \qquad \overline{Q}^i_j = \theta^i_a Q^a_b \theta^b_j,$$

then $(\overline{Q}^i_j)$ has the following properties:

$$(7.2) \qquad \overline{P}^i_j \overline{Q}^j_k = A^i_k, \qquad \overline{Q}^i_j \overline{P}^j_k = A^i_k,$$

$$(7.3) \qquad A^i_j \overline{Q}^j_k = \overline{Q}^i_k, \qquad \overline{Q}^i_j A^j_k = \overline{Q}^i_k.$$

$$(7.4) \qquad \theta^a_i \overline{Q}^i_j = Q^a_b \theta^b_j, \qquad \theta^j_b \overline{Q}^i_j = \theta^i_a Q^a_b.$$

Suppose that $\mathfrak{X}_l$ is covered by coordinate neighborhoods of $\mathfrak{X}$ associated to $\mathfrak{X}_l$. Then Q^i_j is a tensor and we can derive the following general connection:

$$\begin{aligned} {}'\overline{\Gamma} &= \overline{Q}\overline{\Gamma} \\ &= (\overline{Q}^i_j \overline{P}^j_k, \ \overline{Q}^i_l \overline{\Gamma}^l_{jk}) \\ &= (A^i_k, \ {}'\overline{\Gamma}^i_{jk}) \end{aligned}$$

where

$$(7.5) \qquad {}'\overline{\Gamma}^i_{jk} = \overline{Q}^i_l \overline{\Gamma}^l_{jk}.$$

From the definition of $\overline{\Gamma}^i_{jk}$ (see (4. 9)), we have $A^i_l \overline{\Gamma}^l_{jk} = \overline{\Gamma}^i_{jk}$. Hence (7. 5) implies the following relation:

$$(7.6) \qquad \overline{P}^i_l\, {}'\overline{\Gamma}^l_{jk} = \overline{\Gamma}^i_{jk}.$$

${}'\overline{\Lambda}^i_{jk}$ can be written as

$$(7.7) \qquad {}'\overline{\Lambda}^i_{jk} = {}'\overline{\Gamma}^i_{jk} - \partial_k A^i_j.$$

Now we shall give some lemmas concerning the general connection.

Lemma 1. *The induced connection of* $\mathfrak{X}_l$ *derived from the general connection* ${}'\overline{\Gamma}$ *of* $\mathfrak{X}$ *is a classical connection* $(\delta^a_b, {}'\Gamma^a_{bc})$ *where*

$${}'\Gamma^a_{bc} = Q^a_d \Gamma^d_{bc}$$

and Γ^d_{bc} *are the same as* (3. 1).

Proof. It is clear that

$$\theta_i^a A_j^i \theta_b^j = \delta_b^a.$$

From (7. 4) and Theorem 3. 1., we have

$$\begin{aligned}
&\theta_i^a(A_j^i \partial_c \theta_b^j + {}'\overline{\Gamma}_{jk}^i \theta_b^j \theta_c^k) \\
&= \theta_j^a \partial_c \theta_b^j + \theta_i^a \overline{Q}_l^i \overline{\Gamma}_{jk}^l \theta_b^j \theta_c^k \\
&= \theta_j^a \partial_c \theta_b^j + Q_e^a \theta_l^e \overline{\Gamma}_{jk}^l \theta_b^j \theta_c^k \\
&= Q_c^a(P_d^e \theta_j^d \partial_c \theta_b^j + \theta_l^e \overline{\Gamma}_{jk}^l \theta_b^j \theta_c^k) \\
&= Q_e^a \theta_i^e(\overline{P}_j^i \partial_c \theta_b^j + \overline{\Gamma}_{jk}^i \theta_b^j \theta_c^k) \\
&= Q_d^a \Gamma_{bc}^d. \qquad \text{(Q. E. D.)}
\end{aligned}$$

We denote the covariant differentiation with respect to $'\overline{\Gamma}$ by $'\overline{D}$.

Lemma 2. $'\overline{D}\delta_i^j = 0.$

Proof.

$$\begin{aligned}
'\overline{D}_h \delta_i^j &= {}'\overline{\Gamma}_{lh}^j \delta_s^l A_i^s - A_l^j \delta_s^l \overline{\Delta}_{ih}^s \\
&= {}'\overline{\Gamma}_{lh}^j A_i^l - A_l^j({}'\overline{\Gamma}_{ih}^l - \partial_h A_i^l) \\
&= {}'\overline{\Gamma}_{lh}^j A_i^l - {}'\overline{\Gamma}_{ih}^j + A_l^j \partial_h A_i^l.
\end{aligned}$$

$'\overline{\Gamma}_{lh}^j A_i^l$ *can be calculated as follows:*

$$\begin{aligned}
'\overline{\Gamma}_{lh}^j A_i^l &= \overline{Q}_t^j \overline{\Gamma}_{lh}^t A_i^l \\
&= \overline{Q}_t^j (A_s^t \Gamma_{kh}^s A_l^k + A_s^t P_k^s \partial_h A_l^k) A_i^l \\
&= \overline{Q}_t^j (A_s^t \Gamma_{kh}^s A_i^k + A_s^t P_k^s \partial_h A_l^k A_i^l) \\
&= \overline{Q}_t^j (A_s^t \Gamma_{kh}^s A_i^k + A_s^t P_k^s \partial_h A_i^k - A_s^t P_k^s A_l^k \partial_h A_i^l) \\
&= \overline{Q}_t^j (A_s^t \Gamma_{kh}^s A_i^k + A_s^t P_k^s \partial_h A_i^k) - \overline{Q}_t^j \overline{P}_l^t \partial_h A_i^l \\
&= \overline{Q}_t^j \overline{\Gamma}_{ih}^t - A_l^j \partial_h A_i^l \\
&= {}'\overline{\Gamma}_{ih}^j - A_l^j \partial_h A_i^l.
\end{aligned}$$

Hence we get

$$'\overline{D}\delta_i^j = 0. \qquad \text{(Q. E. D.)}$$

Lemma 3. $\overline{P}_l^j {}'\overline{D}_h \overline{P}_i^l = \overline{D}_h \delta_i^j$ *where* $\overline{D}$ *denotes the covariant differentiation with respect to* $\overline{\Gamma}$.

Proof.

$$\begin{aligned}
'\overline{D}_h \overline{P}_i^l &= A_s^l \partial_h \overline{P}_t^s A_i^t + {}'\overline{\Gamma}_{sh}^l \overline{P}_t^s A_i^t - A_s^l \overline{P}_t^s \overline{\Delta}_{ih}^t \\
&= A_s^l \partial_h \overline{P}_t^s A_i^t + {}'\overline{\Gamma}_{sh}^l \overline{P}_i^s - \overline{P}_t^l {}'\overline{\Delta}_{ih}^t \\
&= A_s^l \partial_h \overline{P}_t^s A_i^t + {}'\overline{\Gamma}_{sh}^l \overline{P}_i^s - \overline{P}_t^l {}'\overline{\Gamma}_{ih}^t + \overline{P}_t^l \partial_h A_i^t \\
&= \partial_h \overline{P}_t^l A_i^t - \partial_h A_s^l \overline{P}_i^s + {}'\overline{\Gamma}_{sh}^l \overline{P}_i^s - \overline{\Gamma}_{ih}^l + \overline{P}_t^l \partial_h A_i^t \\
&= \partial_h \overline{P}_i^l - \partial_h A_s^l \overline{P}_i^s + {}'\overline{\Gamma}_{sh}^l \overline{P}_i^s - \overline{\Gamma}_{ih}^l \\
&= -\partial_h A_s^l \overline{P}_i^s - \overline{\Delta}_{ih}^l + {}'\overline{\Gamma}_{sh}^l \overline{P}_i^s.
\end{aligned}$$

On the other hand,

$$\overline{P}_l^j \partial_h A_s^l \overline{P}_i^s = \overline{P}_l^j \partial_h \overline{P}_i^l - \overline{P}_l^j A_s^l \partial_h \overline{P}_i^s = 0,$$

hence from (7. 16) we get

$$\overline{P}_l^j {}'\overline{D}_h \overline{P}_i^l = \overline{P}_i^s \overline{\Gamma}_{sh}^j - \overline{P}_l^j \overline{A}_{ih}^l = \overline{D}_h \delta_i^j. \qquad \text{(Q. E. D.)}$$

Next, we are going to investigate the relation between the curvature tensor $\overline{R}_{jkl}^i$ of the connection $\overline{\Gamma}$ and the curvature tensor R_{bdc}^a of the induced connection of $\mathfrak{X}_l$ from $\overline{\Gamma}$.

Since $(\delta_b^a, {}'\Gamma_{bc}^a)$ is a classical connection, its curvature tensor ${}'R_{bdc}^a$ is given by

$$(7.8) \qquad {}'R_{bdc}^a = \partial_d {}'\Gamma_{bc}^a - \partial_c {}'\Gamma_{bd}^a + {}'\Gamma_{ed}^a {}'\Gamma_{bc}^e - {}'\Gamma_{ec}^a {}'\Gamma_{bd}^e.$$

From (7. 4) and (7. 5) we have

$$\begin{aligned} {}'\Gamma_{bc}^a &= Q_d^a \Gamma_{bc}^d \\ &= Q_d^a \theta_i^d (\overline{P}_j^i \partial_c \theta_b^j + \overline{\Gamma}_{jk}^i \theta_b^j \theta_c^k) \\ &= \theta_l^a \overline{Q}_i^l (\overline{P}_j^i \partial_c \theta_b^j + \overline{\Gamma}_{jk}^i \theta_b^j \theta_c^k) \\ &= \theta_l^a (A_j^l \partial_c \theta_b^j + {}'\overline{\Gamma}_{jk}^l \theta_b^j \theta_c^k) \end{aligned}$$

that is

$${}'\Gamma_{bc}^a = \theta_j^a \partial_c \theta_b^j + \theta_l^a {}'\overline{\Gamma}_{jk}^l \theta_b^j \theta_c^k.$$

We substitute this relation into (7. 8), then the result is

$$\begin{aligned} {}'R_{bdc}^a &= \partial_d {}'\Gamma_{bc}^a - \partial_c {}'\Gamma_{bd}^a + {}'\Gamma_{ed}^a {}'\Gamma_{bc}^e - {}'\Gamma_{ec}^a {}'\Gamma_{bd}^e \\ &= \theta_i^a (\partial_h {}'\overline{\Gamma}_{jk}^i - \partial_k \overline{\Gamma}_{jh}^i + {}'\overline{\Gamma}_{lh}^i {}'\overline{\Gamma}_{jk}^l - {}'\overline{\Gamma}_{lk}^i {}'\overline{\Gamma}_{jh}^l) \theta_d^h \theta_b^j \theta_c^k \\ &+ \partial_d \theta_i^a \theta_b^l \partial_c A_l^i - \partial_c \theta_i^a \theta_b^l \partial_d A_l^i - \theta_i^a {}'\overline{\Gamma}_{jk}^i \theta_d^k \theta_b^l \partial_c A_l^j + \theta_i^a {}'\overline{\Gamma}_{jk}^i \theta_c^k \theta_b^l \partial_d A_l^j. \end{aligned}$$

In the proof of Lemma 2, we got the following relation :

$${}'\overline{\Gamma}_{ih}^j A_i^l = {}'\overline{\Gamma}_{lh}^j - A_l^j \partial_h A_i^l.$$

Hence the last two terms in the right hand side of ${}'R_{bdc}^a$ can be written as :

$$\begin{aligned} \theta_i^a {}'\overline{\Gamma}_{jk}^i \theta_d^k \theta_b^l \partial_c A_l^j &= \theta_i^a {}'\overline{\Gamma}_{jk}^i \theta_d^k \partial_c \theta_b^j - \theta_i^a {}'\overline{\Gamma}_{jk}^i \theta_d^k A_l^j \partial_c \theta_b^l \\ &= \theta_i^a {}'\overline{\Gamma}_{jk}^i \theta_d^k \partial_c \theta_b^j - \theta_i^a ({}'\overline{\Gamma}_{lk}^i - A_j^i \partial_k A_l^j) \theta_d^k \partial_c \theta_b^l. \\ &= \theta_j^a \partial_d A_l^j \partial_c \theta_b^l \\ &= \partial_d \theta_l^a \partial_c \theta_b^l - A_l^j \partial_d \theta_j^a \partial_c \theta_b^l \\ &= \partial_d \theta_l^a \partial_c \theta_b^l - \partial_d \theta_j^a \partial_c \theta_b^j + \partial_d \theta_j^a . \theta_b^l \partial_c A_l^j \\ &= \partial_d \theta_j^a . \theta_b^l \partial_c A_l^j. \end{aligned}$$

Now we obtain

$$(7.9) \qquad {}'R_{bdc}^a = \theta_i^a (\partial_h {}'\overline{\Gamma}_{jk}^i - \partial_k {}'\overline{\Gamma}_{jh}^i + {}'\overline{\Gamma}_{lh}^i {}'\overline{\Gamma}_{jk}^l - {}'\overline{\Gamma}_{lk}^i {}'\overline{\Gamma}_{jh}^l) \theta_d^h \theta_b^j \theta_c^k.$$

On the other hand, the curvature tensor of the general connection ${}'\overline{\Gamma}$ is given by

$$'R^j_{ihk} = [A^j_l(\partial_h{}'\overline{\Gamma}^l_{mk} - \partial_k{}'\overline{\Gamma}^l_{mh}) + ('\overline{\Gamma}^j_{lh}{}'\overline{\Gamma}^l_{mk} - '\overline{\Gamma}^j_{lk}{}'\overline{\Gamma}^l_{mh})]A^m_i$$
$$- '\overline{D}_h\delta^j_m{}'\overline{A}^m_{ik} + '\overline{D}_k\delta^j_m{}'\overline{A}^m_{ih}.$$

Lemma 2 asserts that $'\overline{D}_c\delta^m_j = 0$, hence we get

$$(7.10) \qquad '\overline{R}^j_{ihk} = [A^j_l(\partial_h{}'\overline{\Gamma}^l_{mk} - \partial_k{}'\overline{\Gamma}^l_{mh}) + ('\overline{\Gamma}^j_{lh}{}'\overline{\Gamma}^l_{mk} - '\overline{\Gamma}^j_{lk}{}'\overline{\Gamma}^l_{mh})]A^m_i.$$

Now (7.9) and (7.10) give the following theorem:

Theorem 7.1. *On the adapted submanifold* $\mathfrak{X}_l$, *the curvature tensor of the general connection* $'\overline{\Gamma}$ *of* $\mathfrak{X}$ *and the curvature tensor* $'R^a_{bdc}$ *of the induced classical connection* $'\Gamma^a_{bc}$ *derived from* $'\overline{\Gamma}$ *of* $\mathfrak{X}$ *have the following relation:*

$$(7.11) \qquad 'R^a_{bdc} = \theta^a_j{}'\overline{R}^j_{ihk}\theta^i_b\theta^h_d\theta^k_c.$$

Next, we investigate the relation between $'R^j_{ihk}$ and the curvature tensor $\overline{R}^j_{ihk}$ of the general connection $\overline{\Gamma}$.

(7.10) can be written in the differential form as follows:

$$(7.12) \qquad '\overline{\Omega}^j_i = (A^j_l d{}'\overline{\Gamma}^l_{mk} \wedge du^k + '\overline{\Gamma}^j_{lh}du^h \wedge '\overline{\Gamma}^l_{mk}\, du^k)A^m_i.$$

The curvature form of $\overline{\Gamma}$ is

$$\overline{\Omega}^j_i = (\overline{P}^j_l d\overline{\Gamma}^l_{mk} \wedge du^k + \overline{\Gamma}^j_{lh}\, du^h \wedge \overline{\Gamma}^l_{mk}du^k)\,\overline{P}^m_i - \overline{D}\delta^j_m \wedge \overline{A}^m_{ik}du^k.$$

From (7.6) and Lemma 3, we have

$$\overline{\Omega}^j_i = \{\overline{P}^j_l d(\overline{P}^l_t{}'\overline{\Gamma}^t_{mk}) \wedge du^k + \overline{P}^j_t{}'\overline{\Gamma}^t_{lh}du^h \wedge \overline{P}^l_s{}'\overline{\Gamma}^s_{mk}du^k\}\,\overline{P}^m_i$$
$$- \overline{P}^j_t{}'\overline{D}\,\overline{P}^t_m \wedge \overline{A}^m_{ik}du^k.$$

In the proof of Lemma 3, we obtained the following relation:

$$-\overline{A}^m_{ik}du^k = '\overline{D}\,\overline{P}^m_i - '\overline{\Gamma}^m_{lk}\overline{P}^l_i du^k + \overline{P}^l_i dA^m_l.$$

We substitute this relation in $\overline{\Omega}^j_i$ and put

$$\overline{M}^j_i = \overline{P}^j_l\overline{P}^l_i.$$

Then we have

$$\overline{\Omega}^j_i = \{\overline{M}^j_l d{}'\overline{\Gamma}^l_{mk} \wedge du^k + \overline{P}^j_l d\overline{P}^l_t \wedge '\overline{\Gamma}^t_{mk}du^k + \overline{P}^j_t{}'\overline{\Gamma}^t_{lh}du^h \wedge \overline{P}^l_s{}'\overline{\Gamma}^s_{mk}du^k$$
$$- \overline{P}^j_t{}'\overline{D}\,\overline{P}^t_l \wedge '\overline{\Gamma}^l_{mk}du^k + \overline{P}^j_t{}'\overline{D}\,\overline{P}^t_l \wedge dA^l_m\}\,\overline{P}^m_i + \overline{P}^j_t{}'\overline{D}\,\overline{P}^t_m \wedge '\overline{D}\,\overline{P}^m_i.$$

The term $\overline{P}^j_t{}'\overline{D}\,\overline{P}^t_l \wedge '\overline{\Gamma}^l_{mk}\, du^k$ can be written as

$$\overline{P}^j_t{}'\overline{D}\,\overline{P}^t_l \wedge '\overline{\Gamma}^l_{mk}du^k = \overline{P}^j_l d\overline{P}^t_s \wedge '\overline{\Gamma}^s_{mk}du^k + \overline{P}^j_t{}'\overline{\Gamma}^t_{sh}du^h \wedge \overline{P}^s_l{}'\overline{\Gamma}^l_{mk}du^k$$
$$- \overline{M}^j_s{}'\overline{\Gamma}^s_{lh}du^h \wedge '\overline{\Gamma}^l_{mk}du^k + \overline{M}^j_s dA^s_l \wedge '\overline{\Gamma}^l_{mk}du^k.$$

Now we obtain

$$\overline{\Omega}^j_i = \{\overline{M}^j_l d{}'\overline{\Gamma}^l_{mk} \wedge du^k + \overline{M}^j_l{}'\overline{\Gamma}^l_{lh}du^h \wedge '\overline{\Gamma}^l_{mk}du^k - \overline{M}^j_s dA^s_l \wedge '\overline{\Gamma}^l_{mk}du^k$$
$$+ \overline{P}^j_t{}'\overline{D}\,\overline{P}^t_l \wedge dA^l_m\}\,\overline{P}^m_i + \overline{P}^j_t{}'\overline{D}\,\overline{P}^t_m \wedge '\overline{D}\,\overline{P}^m_i.$$

On the other hand (7. 12) is

$$'\overline{\Omega}_i^j = (d'\overline{\Gamma}^j_{mk} \wedge du^k - '\overline{\Gamma}^l_{mk} dA^j_l \wedge du^k + '\overline{\Gamma}^j_{lh} du^h \wedge '\overline{\Gamma}^l_{mk} du^k) A^m_i.$$

We get the following theorem:

Theorem 7. 2. *Between the curvature form of* $\overline{\Gamma}$ *and the curvature form of* $'\overline{\Gamma}$, *we have the following relation:*

$$\overline{\Omega}_i^j = \overline{M}^j_l\, '\overline{\Omega}^l_m \overline{P}^m_i + \overline{P}^j_t\, '\overline{D}\,\overline{P}^t_l \wedge dA^l_m \overline{P}^m_i + \overline{P}^j_t\, '\overline{D}\,\overline{P}^t_m \wedge '\overline{D}\,\overline{P}^m_i. \tag{7. 13}$$

Let us put

$$M^a_e = P^a_f P^f_e,$$

and combine Theorem 7. 1. and Theorem 7. 2. Then we obtain

Corollary. Let $\mathfrak{X}_l$ *be adapted in* $\mathfrak{X}$. *Between the curvature tensor of* $\mathfrak{X}$ *with respect to the general connection* $\overline{\Gamma}$ *and the curvature tensor of* $\mathfrak{X}_l$ *with respect to the induced connection* (P^a_b, Γ^a_{bc}) *of* Γ, *the following relation holds:*

$$\begin{aligned} \theta^a_j \overline{R}^j_{ihk} \theta^i_b \vartheta^h_d \theta^k_c = M^a_e\, 'R^e_{fdc} P^f_b + 2P^a_e \theta^e_t\, '\overline{D}_{[d} \overline{P}^t_{|l|} \partial_{c]} A^l_m \theta^m_f P^f_b \\ + 2P^a_e \theta^e_t\, '\overline{D}_{[d} \overline{P}^t_{|m|}\, '\overline{D}_{c]} \overline{P}^m_i \theta^i_b. \end{aligned} \tag{7. 14}$$

Lastly, between R^a_{bdc} and $'R^a_{bdc}$ we have[10]

$$R^a_{bdc} = M^a_e\, 'R^e_{fdc} P^f_b + 2P^a_e\, 'D_{[d} P^e_{|f|}\, 'D_{c]} P^f_b. \tag{7. 15}$$

To simplify the last term of the right hand side of (7. 14), we establish the following lemma:

Lemma 4. (a) $'\overline{D}_c \theta^i_b = 0$, (b) $'\overline{D}_c \theta^b_i = 0$.

Proof. (a) $'\overline{D}_c \theta^i_b = A^i_j \partial_c \theta^j_b + '\Gamma^i_{lk} \theta^l_b \theta^k_c - '\Gamma^d_{bc} \theta^j_d A^i_j$.

From (7. 3), (7. 5) and the definition of $'\Gamma^d_{bc}$, we get

$$\begin{aligned} '\overline{D}_c \theta^i_b &= A^i_j \partial_c \vartheta^j_b + '\overline{\Gamma}^i_{lk} \theta^l_b \theta^k_c - (\theta^d_h \partial_c \theta^h_b + \theta^d_l\, '\overline{\Gamma}^l_{hk} \theta^h_b \theta^k_c) \theta^j_d A^i_j \\ &= A^i_j \partial_c \theta^j_b + '\overline{\Gamma}^i_{lk} \theta^l_b \theta^k_c - A^i_h \partial_c \vartheta^h_b - '\overline{\Gamma}^i_{hk} \theta^h_b \theta^k_c = 0. \end{aligned}$$

$$\begin{aligned} \text{(b)}\quad '\overline{D}_c \theta^b_i &= \partial_c \theta^b_j A^j_i + '\Gamma^b_{ac} \theta^a_j A^j_i - '\overline{A}^k_{il} \theta^b_k \theta^l_c \\ &= \partial_c \theta^b_j A^j_i + \theta^a_i\, '\Gamma^b_{ac} - ('\overline{\Gamma}^k_{il} - \partial_l A^k_i) \theta^b_k \theta^l_c. \\ &= \partial_c \theta^b_i + \theta^a_i (\theta^b_h \partial_c \theta^h_a + \theta^b_h\, '\overline{\Gamma}^h_{jk} \theta^j_a \theta^k_c) - '\overline{\Gamma}^k_{il} \theta^b_k \theta^l_c. \end{aligned}$$

From the relation $A^j_i\, '\overline{\Gamma}^h_{jk} = '\overline{\Gamma}^h_{ik} - A^h_l \partial_k A^l_i$ which is proved in Lemma 2, we have

$$'\overline{D}_c \theta^b_i = \theta^b_h \partial_c A^h_i + ('\overline{\Gamma}^h_{ik} - A^h_l \partial_k A^l_i) \theta^k_c \vartheta^b_h - '\overline{\Gamma}^k_{il} \theta^b_k \theta^l_c = 0. \qquad \text{(Q. E. D.)}$$

By Lemma 2, Lemma 4, we can write the last term of the right side of (7. 14) as

$$P^a_e \theta^e_t\, '\overline{D}_d \overline{P}^t_m\, '\overline{D}_c \overline{P}^m_i \theta^i_b = P^a_e\, '\overline{D}_d (\theta^e_t \overline{P}^t_m)\, '\overline{D}_c (\overline{P}^m_i \theta^i_b)$$

10) See [9] §7 (1. 2).

$$= P^a_e{}'\overline{D}_d(P^e_f\theta^f_m)'\overline{D}_c(P^g_b\theta^m_g) = P^a_e{}'\overline{D}_d P^e_f{}'\overline{D}_c P^f_b.$$

Hence from (7. 14) and (7. 15) we have

$$\theta^a_j\overline{R}^j_{ihk}\theta^i_b\theta^h_d\theta^k_c = R^a_{bdc} + 2P^a_e\theta^e_t{}'\overline{D}_{[d}\overline{P}^t_{|l|}\partial_{c]}A^l_m\theta^m_f P^f_b.$$

The last term of the right hand of this formula can be simplified by Lemma 4, that is

$$'\overline{D}_d\overline{P}^t_l\partial_c A^l_m\theta^m_f = {}'\overline{D}_d(\theta^t_a P^a_b\theta^b_l)\partial_c A^l_m\theta^m_f = \theta^t_a{}'\overline{D}_d P^a_b\theta^b_l\partial_c A^l_m\theta^m_f,$$

but since

$$\theta^b_l\partial_c A^l_m\theta^m_f = 0,$$

we have

$$\theta^a_j\overline{R}^j_{ihk}\theta^i_b\theta^h_d\theta^k_c = R^a_{bdc}.$$

Then we have established the following theorem:

Theorem 7. 3. *If $\mathfrak{X}_l$ is adapted in $\mathfrak{X}$, and $\mathfrak{X}$ has a regular connection $(P^i_j,\ \Gamma^i_{jh})$, then between the curvature tensor $\overline{R}^i_{jhk}$ of $\overline{\Gamma} = A\Gamma A$ and the curvature tensor R^a_{bdc} of $\mathfrak{X}_l$ with respect to the induced connection $(P^a_b,\ \Gamma^a_{bc})$ from Γ of $\mathfrak{X}$, there is the following relation:*

(7. 16) $\qquad \theta^a_j\overline{R}^j_{ihk}\theta^i_b\theta^h_d\theta^k_c = R^a_{bdc}.$

In the above, we used the general connection $'\overline{\Gamma}$ to simplify the curvature tensors. Similarly, we can use a general connection defined as follows

$$''\overline{\Gamma} = \overline{\Gamma}\overline{Q} = (A^i_j,\ ''\overline{\Gamma}^i_{jk})$$

$$''\overline{\Gamma}^i_{jk} = \overline{\Gamma}^i_{lk}\overline{Q}^l_j + \overline{P}^i_l\partial_k\overline{Q}^l_j, \qquad ''\overline{\Lambda}^i_{jk} = \overline{\Lambda}^i_{lk}\overline{Q}^l_j.$$

The induced connection $(P^c_b,\ \Gamma^a_{bc})$ of $\mathfrak{X}_l$ from the regular general connection Γ of $\mathfrak{X}$ is regular. Its covariant part $''\Gamma^a_{bc}$ is given by

$$''\Gamma^a_{bc} = \Gamma^a_{dc}Q^d_b + P^a_d\partial_c Q^d_b.$$

It is easy to show that the classical connection $''\Gamma^a_{bc}$ is the induced connection of $\mathfrak{X}_l$ from $''\overline{\Gamma}$ of $\mathfrak{X}$, that is

(7. 17) $\qquad ''\Gamma^a_{bc} = \theta^a_i(A^i_j\partial_c\theta^j_b + {}''\overline{\Gamma}^i_{jk}\theta^j_b\theta^k_c).$

Corresponding to Lemmas 2, 3, and 4, we can prove:

(7. 18) $\qquad ''\overline{D}_h\delta^j_i = 0,$

(7. 19) $\qquad ''\overline{D}_h\overline{P}^i_l\overline{P}^l_k = \overline{D}_h\delta^i_k,$

(7. 20) $\qquad ''\overline{D}\theta^i_a = 0, \qquad ''\overline{D}\theta^a_i = 0.$

Lastly, from § 5, Lemma 4 of this section and (7. 20), we get

Theorem 7. 4. *Let $\mathfrak{X}_l$ be adapted in $\mathfrak{X}$, and $\mathfrak{X}$ have a regular general connection $(P^i_j,\ \Gamma^i_{jh})$. Then $\mathfrak{X}_l$ is flat with respect to the general connections $'\overline{\Gamma}$, $''\overline{\Gamma}$ of $\mathfrak{X}$ and their induced classical connections of $\mathfrak{X}_l$.*

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TOKYO METROPOLITAN UNIVERSITY.

(Received January 9, 1963)