SUPPLEMENTARY REMARKS TO THE PREVIOUS PAPERS

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The present note contains several improvements of the results obtained in [2], [4] and [5], which are closely related with the recent ones cited in [1]. However, our proofs will be given without making use of Inatomi's [1].

As to notations and terminologies used in this note, we follow the previous ones [4] and [5]. Now, we shall prove our first lemma.

Lemma 1. *Let $U \ni 1$ be an algebra over an infinite field Φ of finite rank, and T an intermediate ring of U/Φ. If $U = T[x]$ then $U = T[u]$ with some regular element u.*

Proof. $A = \Phi[x]$ is evidently a commutative subalgebra of U. If we denote by N the radical of A, then $\bar{A} = A/N = \Phi[\bar{x}] = \bar{A}_1 \oplus \cdots \oplus \bar{A}_s$, where $\bar{A}_i$'s are fields over Φ and $\bar{x}$ is the residue class of x modulo N. We set here $\bar{x} = \bar{a}_1 + \cdots + \bar{a}_s (\bar{a}_i \in \bar{A}_i)$. Since Φ is infinite, we can find such an element $k \in \Phi$ that each $\bar{A}_i$-component of $\overline{x-k}$ is non-zero. And then, it will be clear that $u = x - k$ is a regular element and $U = T[u]$.

In virtue of Lemma 1, we can prove the following sharpening of [4, Theorem 1].

Theorem 1. *Let R be a separable simpe algebra over a field Φ $(\subseteq C)$ of finite rank. If a is an arbitrary element of $R \backslash C$ then $R = \Phi[a, r]$ for some regular element r.*

Proof. In any rate, by [4, Theorem 1], $R = \Phi[a, b]$ for some b. Then, if Φ is infinite, our assertion is clear by Lemma 1. Accordingly, in what follows, we may assume that Φ is finite and $n > 1$, whence $R = (C)_n$. To be easily seen, in the proof (i) of [4, Lemma 10] we may replace the nilpotent element $u^* = \sum_2^{n^*} e_{i i-1}$ by the regular element $1 - u^*$. And so, by the proof Case III of [4, Theorem 1] and the proof Case I of [4, Proposition], we obtain eventually $R = \Phi[a, r]$ with some regular element r.

Now, [1, Theorem 2] is only an easy consequence of Theorem 1, and conversely.

Corollary 1 (Inatomi). *Let R be a simple algebra over C of finite* *rank, and B a subring of R containing* 1 *such that* $B\cap C$ *is a field and* C *is separable and finite over* $B\cap C$. *Then, R is generated by some regular element of R over B, if and only if B is not contained in C or R is commutative.*

Lemma 2. *Let* $U\ni 1$ *be a ring, A a subring of U containing* 1 *that is represented as* $\sum_1^m A'c_{ij}$ *with matrix units* c_{ij}*'s* $(m>1)$ *and* $A'=V_A(\{c_{ij}\text{'s}\})$, *and B a subring of U such that* $B\tilde{A}\subseteq B$.

(i) *If* $Bc_{pq}\subseteq B$ *for some* c_{pq} *then* $BA=B$.

(ii) *Let* $2A=0$, *and x an element of* A' *neither* 0 *nor* 1. *If* $B(x+c_{12})\subseteq B$ *then* $BA=B$.

(iii) *If* $B(\sum_2^m c_{ii-1}+c_{1m}+c_{22})\subseteq B$ *then* $BA=B$.

(Needless to say, in case B contains 1, *our conclusion* $BA=B$ *in* (i)-(iii) *may be replaced by* $A\subseteq B$.)

Proof. (i) If $p\neq q$ then $B\supseteq(1+c_{qp})Bc_{pq}(1+c_{qp})^{-1}=B(1+c_{qp})c_{pq}(1-c_{qp})=B(c_{pq}+c_{qq}-c_{pp}-c_{qp})$ and $B\supseteq Bc_{pq}$ imply $B\supseteq B(c_{qq}-c_{pp}-c_{qp})$, whence it follows $B\supseteq Bc_{pq}(c_{qq}-c_{pp}-c_{qp})=B(c_{pq}-c_{pp})$. Again by our assumption, we obtain $B\supseteq Bc_{pp}$. Now, for each $i\neq p$ and $a'\in A'$, $B\supseteq(1+a'c_{ip})Bc_{pp}(1+a'c_{ip})^{-1}=B(1+a'c_{ip})c_{pp}(1-a'c_{ip})=B(c_{pp}+a'c_{ip})$ and similarly $B\supseteq(1+a'c_{pi})Bc_{pp}(1+a'c_{pi})^{-1}=B(c_{pp}+a'c_{pi})$. Accordingly, we have $B\supseteq Ba'c_{ip}$ and $Ba'c_{pi}$. From those, it will be easy to see that $B\supseteq BA$.

(ii) We set $a_1=x+c_{12}$ and $a^*=c_{22}+c_{11}+c_{21}$. Then, $B\supseteq(1+c_{21})Ba_1(1+c_{21})^{-1}=B(a_1+a^*)$ implies $B\supseteq Ba^*$. And further, $B\supseteq B(a^*-a^{*2})=Bc_{21}$. Now, $BA=B$ is a direct consequence of (i).

(iii) Setting $a_2=\sum_2^m c_{ii-1}+c_{1m}+c_{22}$, $B\supseteq(1+c_{12})Ba_2(1+c_{12})^{-1}=B(a_2+c_{11})$ implies $B\supseteq Bc_{11}$. And so, again by (i), we obtain $BA=B$.

Next, we expose our second theorem. It contains [1, Lemma 4] and [2, Theorem 3]. And its proof is simpler than that of Inatomi's we think.

Theorem 2. *Let U be a ring containing* 1, $A\ni 1$ *a simple subring of U different from* $(GF(2))_2$. *If B is a two-sided simple subring of U and invariant relative to all inner automorphisms effected by regular elements of* $A:B\tilde{A}=B$, *then either* $BA=B$ or $A\subseteq V_U(B)$.

Proof. In fact, for the case where A is a division ring, our proof proceeds just like in that of [3, Lemma 3. 5]. And so, the details may be left to readers. While, in case A is not a division ring, we set $A=\sum_1^m A'c_{ij}$ with matrix units c_{ij}'s $(m>1)$ and a division ring $A'=V_A(\{c_{ij}\text{'s}\})$. One may remark here that $V_U(B)$ is also invariant relative to $\tilde{A}$, and that the same argument as in the proof of [3, Lemma 3. 5] proves that for each biregular element a of A (i. e. a and $1-a$ are regular) there holds either $Ba\subseteq B$ or $a\in V_U(B)$. Now, we shall complete our proof by distinguishing three cases:

Case I: *A is not of characteristic* 2. Evidently, $a_0 = 2 + c_{12}$ is biregular. And so, as we noted above, either $Ba_0 \subseteq B$ or $a_0 \in V_U(B)$, that is, either $Bc_{12} \subseteq B$ or $c_{12} \in V_U(B)$. Recalling here that $V_U(B)\widetilde{A} = V_U(B)$, our assertion is clear by Lemma 2 (i).

Case II: *A is of characteristic* 2 *and* $A' \neq GF(2)$. As $a_1 = x + c_{12}$ ($x \neq 0$, $1 \in A'$) is evidently biregular, it follows either $Ba_1 \subseteq B$ or $a_1 \in V_U(B)$. And then, Lemma 1 (ii) yields at once our assertion.

Case III: *A is of characteristic* 2 *and* $m > 2$. In this case, to be easily verified, $a_2 = \sum_2^m c_{ii-1} + c_{1m} + c_{22}$ is biregular. And so, this time, our assertion is a consequence of Lemma 1 (iii).

Combining Theorem 2 with [5, Corollary 3. 9], one will see at once the next corollary that contains completely [5, Theorem 4. 5].

Corollary 2. *Let a simple ring R be locally finite and h-Galois over a simple subring S,* $[R : H]_l \leq \aleph_0$, *and T an f-regular intermediate ring of R/S. Then, T is* $\mathfrak{G}$*-normal if and only if T/S is Galois and either* $T \subseteq H$ *or* $V \subseteq T$, *provided V is different from* $(GF(2))_2$.

Next, we shall prove the following improvement of [4, Theorem 4] that contains [1, Theorem 6] as well.

Theorem 3. *Let a simple ring R be Galois and finite over a simple subring S, and T a* $\widetilde{V}$*-normal simple intermediate ring of R/S. Then,* $n(T/S) = 1$ *if and only if* $S \not\subseteq V_T(T)$ *or T is commutative.*

Proof. It will suffice to prove the if part only. For the case where $[S : Z] = \infty$, we have seen in [3, Corollary 2. 1] that our assertion is true even for arbitrary intermediate ring T. And so, in what follows, we may restrict our attention to the case $[S : Z] < \infty$ (whence $[R : C] < \infty$ by [6, Lemma]), and we distinguish two cases:

Case I: *C is finite.* Since (R and so) T is finite, $T/S \cap C$ is a separable simple algebra of finite rank. If $S \not\subseteq V_T(T)$, by Theorem 1, $T = S[t]$ with some regular t. On the other hand, if $T = V_T(T)$, there is nothing to prove.

Case II: *C is infinite.* By Theorem 2, we obtain $V \subseteq T$ or $T \subseteq H$. Since, in case $S \supsetneqq Z$ our assertion is clear by [4, Theorem 3], in what follows, we shall assume that $S = Z$. Now, $S = Z \subseteq C_0 = V_V(V)$ implies $V_R(C_0) = V = V_R(V_R(V))$, whence it follows H is commutative: $H = C_0$. If T is commutative, then $T \subseteq H$ in any rate. Accordingly, $n(T/S) = 1$ by [4, Theorem 2]. On the other hand, if $S \not\subseteq V_T(T)$ then $T \subseteq H$ yields the contradiction $S \subseteq T = V_T(T)$. It follows therefore $V \subseteq T$, whence $V_T(T) \subseteq V_R(T) \subseteq H = C_0$. Hence, R, T and S satisfy the assumptions in [4, Proposition]. And so, if s is an arbitrary element of $S \backslash V_T(T)$, then there exists some t such that $T=Z[s, t]=S[t]$.

We shall present here another proof to [1, Theorem 5].

Corollary 3. *Let a simple ring R be Galois and finite over a simple subring S. If T is a $\widetilde{V}$-normal simple intermediate ring of R/S, then $T=S[t, t\tilde{r}]$ with some t and r.*

Proof. By Theorem 3, it will suffice to prove our assertion for the case $S\subseteq V_T(T)$ (whence $[R:C]<\infty$ by [6, Lemma]). If C is finite, the finite ring T is a separable algebra over S. And then, our assertion is clear by [7, Theorem 2]. On the other hand, if C is infinite, $T\subseteq H$ or $T\supseteq V(=V_R(S)\supseteq V_R(V_R(T))=T$, whence $T=V$) by Theorem 2. Hence, our assertion is a consequence of [4, Theorem 2] and [7, Theorem 1].

Finally, we shall prove a partial extension of [4, Theorem 6]. To this end, the next lemma will be neeed.

Lemma 3. *Let a simple ring R be Galois and finite over a simple subring S, and T a regular intermediate ring of R/S. If $S\not\subseteq V_T(T)$ then $n(T/S)\leqq \mathrm{Max}\,\{0,\ n(Z[V_T(T)]/Z)-[S:Z]\}+2$.*

Proof. In case $[S:Z]=\infty$, our assertion is clear by [3, Corollary 2.1]. And so, we may, and shall, restrict our proof to the case $[S:Z]<\infty$ (whence $[T:V_T(T)]<\infty$ by [6, Lemma]). Then, by Theorem 1, $T=V_T(T)[S, u]$ for some u. Now, let $S=\sum_{i=1}^{k}\oplus Zd_i$ and $Z[V_T(T)]=Z[a_1, \cdots, a_h]$ where $h=n(Z[V_T(T)]/Z)$. We set here $v=\sum_{i=1}^{s}a_id_i$, where $s=\mathrm{Min}\,\{k, h\}$. Then, $T'=S[u, v, \{a_i\,;\, s<i\leqq h\}]$ is a simple subring of T by [4, Lemma 11], for $T'[V_T(T)]=T$. Noting that $S[V]=S\times_Z V$, we see that $\{d_i\text{'s}\}$ is linearly independent over V. And so, for any element $x\in V_R(T')$ $(\subseteq V)$. $0=xv-vx=\sum_{i=1}^{s}(xa_i-a_ix)d_i$ yields at once $xa_i=a_ix$ $(i=1, \cdots, s)$. It follows therefore that $V_R(T')=V_R(S[u, a_1, \cdots, a_h])=V_R(T)$, that is, T' is a regular subring of R. Accordingly, R is Galois and finite over T'. Recalling here that V is $\mathfrak{G}$-normal, $0=v-v\sigma=\sum_{i=1}^{s}(a_i-a_i\sigma)d_i$ with every $\sigma\in\mathfrak{G}(R/T')$ implies $a_i=a_i\sigma$ $(i=1, \cdots, s)$, whence it follows $a_i\in T'$ $(i=1, \cdots, h)$. We have proved therefore that $T=T'[V_T(T)]=T'$. Our lemma is now a direct consequence of $T'=S[u, v, \{a_i\,;\, s<i\leqq h\}]$.

Theorem 4. *Let a simple ring R be Galois and finite over a simple subring S. If $S\supsetneqq Z$ then, for any regular intermediate ring T of R/S, $n(T/S)\leqq n_0=\mathrm{Max}\ n(W/Z)$, where W runs over all the intermediate rings of V/Z.*

Proof. Firstly $S\supsetneqq Z$ yields evidently $S\not\subseteq V_T(T)$. If $n_0=1$, V is commutative and then $n(T/S)=1$ by [4, Theorem 2]. And so in what follows, we may assume that $n_0>1$. If $n(Z[V_T(T)]/Z)-[S:Z]\leqq 0$ then $n(T/S)\leqq 2$

by Lemma 3. While, if $n(Z[V_T(T)]/Z)-[S:Z]>0$ then $n(T/S)\leqq n(Z[V_T(T)]/Z)-[S:Z]+2\leqq n(Z[V_T(T)]/Z)\leqq n_0$ again by Lemma 3.

Another consequence of Lemma 3 is the next

Corollary 4. *Let a simple ring R be Galois and finite over a simple subring S. If $[S:Z]\geqq m_0=\text{Max}\ n(U/Z)$, where U runs over all the commutative intermediate rings of V/Z, then $n(T/S)\leqq 2$ for any regular intermediate ring T of R/S.*

Proof. In case $S\not\subseteq V_T(T)$, our assertion is clear by Lemma 3. On the other hand, if $S\subseteq V_T(T)$ then $T\subseteq V$, and $1=[S:Z]\geqq m_0$ means $m_0=1$. Since $T=V_T(T)[u,v]$ for some u, v by Theorem 1, it will be easy to see that $n(T/S)\leqq n(V_T(T)[u]/Z)+1=1+1=2$.

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Page 160, line 11. For "(iii) *If* $B(\sum_2^m c_{ii-1}+c_{1m}+c_{22})\subseteq B$ *then* $BA=B$" read "(iii) *Let* $2A=0$ *and* $m>2$. *If* $B(\sum_2^m c_{ii-1}+c_{1m}+c_{1m-1})\subseteq B$ *then* $BA=B$".

Page 160, lines 24—25. For "(iii) Setting$\cdots$, we obtain $BA=B$" read "We set $a_2=\sum_2^m c_{ii-1}+c_{1m}+c_{1m-1}$ and $a^{**}=c_{22}+c_{11}+c_{12}$. Then, $B\supseteq(1+c_{12})Ba_2(1+c_{12})^{-1}=B(a_2+a^{**})$ implies $B\supseteq Ba^{**}$, and hence $B\supseteq B(a^{**}+(a^{**})^2)=Bc_{12}$. Now, $BA=B$ is a consequence of (i)".

Page 161, line 10. For "$a_2=\sum_2^m c_{ii-1}+c_{1m}+c_{22}$" read "$a_2=\sum_2^m c_{ii-1}+c_{1m}+c_{1m-1}$".