ON GALOIS THEORY OF SIMPLE RINGS

Dedicated to Prof. Kenjiro SHODA on his 60th birthday

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Introduction. Since N. Jacobson and N. Nobusawa found a key to the treating of Galois theory of infinite dimension for division rings, a number of important developments have taken place in this direction for division rings and simple rings. They are given mainly by the present authors and N. Nobusawa in the series cited in the references cited at the end of this paper. The main purpose of this paper is to present a kind of Galois theory of infinite dimension for simple rings such that our previous theories and Walter's one are completely unified.

Our study starts with the preliminary section § **1**, which contains several tool lemmas. And, in § **2**, we shall consider $\mathfrak{G}$-locally Galois extensions. As we can prove that R/S is $\mathfrak{G}$-locally Galois if and only if R/S is Galois and locally Galois, all the cases for simple rings studied till now are concentrated to the one for $\mathfrak{G}$-locally Galois extensions. And, in this case we shall give a somewhat intrinsic characterization of the Galois group. Next, in order to annex Walter's theory [18] to our present study, the notion of "hereditarily Galois" ("h-Galois") will be introduced. In fact, if R/S is $\mathfrak{G}$-locally Galois or if R is a division ring and R/S is Galois then R/S is seen to be h-Galois. § **3** is exclusively devoted to the treaty of h-Galois extensions, and we shall see that the Galois correspondence, the extension theorem, the normality theorem and the transitivity theorem are all valid with the desirable style. Further, in § **4**, we shall state several supplementary remarks for $\mathfrak{G}$-locally Galois extensions, which will be of enough interest for themselves.

In what follows, we shall summarize the notations and definitions which will be used very often in the subsequent study. Throughout the present paper, R be always a simple ring (with minimum condition), and S a simple subring of R containing the identity element of R. And we shall use the following conventions:

$R=\sum_1^n De_{ij}$, where e_{ij}'s are matrix units and $D=V_R(\{e_{ij}\text{'s}\})$ a division ring.

C: the center of R.

Z: the center of S.

V: the centralizer $V_R(S)$ of S in R. And in case V is a simple ring, we set $V=\sum Ug_{pq}$, where g_{pq}'s are matrix units and $U=V_V(\{g_{pq}\text{'s}\})$ a division ring.

H: the centralizer of V in R. And in case H is simple, we set $H=\sum Kd_{hk}$, where d_{hk}'s are matrix units and $K=V_H(\{d_{hk}\text{'s}\})$ a division ring.

C_0: the center of V, which coincides with the center of H.

$\mathfrak{G}$: the group of all the S-(ring) automorphisms of R.

And, as to general notations we follw [8].

R/S is said to be *locally finite* if $S[F]$ is left-finite over S for each finite subset F of R. Now, let $T\ni 1$ be a subring of R. If the fixring $J(\mathfrak{H}, R)$ of an automorphism group $\mathfrak{H}$ in R coincides with T, R/T is said to be *w-Galois* and $\mathfrak{H}$ is called *a Galois group* of R/T. In particular, the totality of the T-(ring) automorphisms of R will be named as *the Galois group* of R/T and denoted by $\mathfrak{G}(R/T)$. If both T and $V_R(T)$ are simple rings, T is called a *regular subring* of R. And, in case R/T is w-Galois and T is regular, we say that R/T is *Galois*. On the other hand, if for each finite subset F of R there exists a simple subring N containing $S[F]$ such that $[N:S]_l<\infty$ and N/S is Galois then R/S is said to be *locally Galois*. In general, for any subset E of R, a simple subring N containing $S[E]$ will be called a *shade* of E (or of $S[E]$) in R if N is Galois and finite over S, and R is N-R-irreducible. Here, one may remark that $V_R(N)$ is necessarily a division ring. To be easily seen, R/S is locally Galois if and only if each finite subset of R possesses its shade in R.

Now, let $\mathfrak{H}$ be a non-empty subset of $\mathfrak{G}$. $\mathfrak{H}$ is said to be *locally finite* if the image of each element of R by $\mathfrak{H}$ is finite. And $(\mathfrak{H}, R/S)$ will be said to be *l.f.d.* if $[S[F\mathfrak{H}]:S]_l<\infty$ for each finite subset F of R. If for each finite subset F of R there exists a simple subring S'' containing $S[F]$ such that S'' is Galois and finite over S and $\mathfrak{G}(S''/S)\subseteq\mathfrak{H}|S''$, then we shall say that R/S is *$\mathfrak{H}$-locally Galois*. If R/S is $\mathfrak{H}$-locally Galois then for an arbitrary finite subset F of R there exists a shade S' of F such that $\mathfrak{G}(S'/S)\subseteq\mathfrak{H}|S'$. Such a shade will be called a *$\mathfrak{H}$-shade* of F. Although one will readily see that if R/S is $\mathfrak{H}$-locally Galois then each shade of a given finite subset is a $\mathfrak{H}$-shade, the term "$\mathfrak{H}$-shade" will make clear the range in which the shade is considered. And, one may remark here that R/S is $\mathfrak{G}$-locally Galois if and only if for each finite subset F of R there exists a shade S' of F such that a Galois group of S'/S is contained in $\mathfrak{G}|S'$. Finally, let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$, and $V_{\mathfrak{H}}=V_R(J(\mathfrak{H}, R))$. If $V_{\mathfrak{H}}$ is simple and $\mathfrak{H}\supseteq\tilde{V}_{\mathfrak{H}}$ then $\mathfrak{H}$ is called a *(*)-regular subgroup*. And a (*)-regular subgroup $\mathfrak{H}$ is said to be *regular or $(*_f)$-regular* according as $J(\mathfrak{H}, R)$ is simple or $[V:V_{\mathfrak{H}}]_r<\infty$. Moreover, if $\mathfrak{H}$ is regular and $(*_f)$-regular then $\mathfrak{H}$ is *f-regular*. On the other hand, a regular intermediate ring T of R/S is called an *f-regular subring* if $[V:V_R(T)]_r<\infty$. For instance, to be easily verified, if T is a regular intermediate ring of R/S with $[T[\{e_{ij}\text{'s}\}]:S]_l<\infty$ then T is f-regular. (Cf. [4, Corollary 1].)

Now, let $\mathfrak{H}$ be a group of automorphisms of R, $\mathfrak{H}_0$ the group of inner automorphisms contained in $\mathfrak{H}$, and $I(\mathfrak{H})$ the subring of R generated by all the regular elements inducing inner automorphisms belonging to $\mathfrak{H}_0$. If $I(\mathfrak{H})$ is a simple ring and $\mathfrak{H}\supseteq\widetilde{I(\mathfrak{H})}$, $\mathfrak{H}$ will be called an *N-regular group.* For an N-regular group $\mathfrak{H}$ with $[I(\mathfrak{H}): C]<\infty$ and $(\mathfrak{H}:\mathfrak{H}_0)<\infty$, it is known that $I(\mathfrak{H})=V_R(J(\mathfrak{H}, R))$, and this fact plays an important rôle in Galois theory of finite degree. However, as one will see in the following example, this is not true for the case of infinite degree. Because of this reason, we have defined above a regular group $\mathfrak{H}$ by making use of $V\mathfrak{H}$ instead of $I(\mathfrak{H})$.

Example. Let C be the algebraic closure of the rational number field P. As is well-known, C is Galois and algebraic (and so locally finite) over P and $\mathfrak{G}(C/P)$ contains an automorphism σ of infinite order. Now, cosider the 2×2 complete matrix ring $R=(C)_2=(P)_2\times_P C$ over C. To be easily seen, R is Galois and locally finite over $S=P$. As $[V_R(S): V_R(R)]=[R: C]=4$, $\mathfrak{G}=\mathfrak{G}(R/S)$ is locally compact by Theorem 2.6. And $\mathfrak{G}(C/P)$ may be naturally regarded as a subgroup of $\mathfrak{G}$. Let $a=1+e_{21}(\in(P)_2)$. Then, noting that $a^{-1}=1-e_{21}$, we readily see that $a^k=1+ke_{21}$ for each integer k. Now, let $\mathfrak{H}$ be the subgroup $[\sigma\tilde{a}]$ of $\mathfrak{G}$ generated by $\sigma\tilde{a}$. Noting here that $\sigma\tilde{a}=\tilde{a}\sigma$ and σ is of infinite order, one will easily see that $\mathfrak{H}$ is an outer group of infinite order. Moreover, if τ is an arbitrary element of the topological closure $\bar{\mathfrak{H}}$ of $\mathfrak{H}$ in $\mathfrak{G}$ then for each intermediate rings S_1, S_2 of $R/(P)_2$ with $[S_i: P]<\infty$ there exist some integers k_1, k_2 such that $\tau|S_i=(\sigma\tilde{a})^{k_i}|S_i(i=1,2)$. And then, $\tilde{a}^{k_1}|(P)_2=(\sigma\tilde{a})^{k_1}|(P)_2=\tau|(P)_2=(\sigma\tilde{a})^{k_2}|(P)_2=\tilde{a}^{k_2}|(P)_2$, whence $\tilde{a}^{k_1-k_2}|(P)_2=1$, that is, $1+(k_1-k_2)e_{21}=a^{k_1-k_2}\in P$. Hence, we have $k_1=k_2$, which means $\tau=(\sigma\tilde{a})^{k_1}\in\mathfrak{H}$. Thus, we have seen that $\mathfrak{H}$ is a closed outer (and so N-regular) group. Finally, as $H=V_R(V_R(S))=C$, $S_0=(P)_2$ contains a linearly independent basis of R over H and a system of matrix units of R as well. We set here $\mathfrak{G}_0=\mathfrak{G}(R/(P)_2)$. If $\tau=(\sigma\tilde{a})^k$ is an element of $\mathfrak{G}_0\cap\mathfrak{H}$ then $\tilde{a}^k|(P)_2=\tau|(P)_2=1$ will yields at once $k=0$, which proves that $\mathfrak{G}_0\cap\mathfrak{H}=1$. Accordingly, it follows that $(\mathfrak{H}: (\mathfrak{G}_0\cap\mathfrak{H})\cdot\widetilde{I(\mathfrak{H})})=(\mathfrak{H}: 1)=\infty$. Hence, $\mathfrak{H}$ can not be regular by Theorem 2.9.

1. Preliminary lemmas. The fact stated in the next lemma has been used in our previous study. We shall present here a notably simple proof.

Lemma 1.1. *Let T be a subring of R. If there exists a directed set $\{R_\alpha$'s$\}$ consisting of simple subrings R_α's of R such that $T=\cup_\alpha R_\alpha$ then T is a simple ring.*

Proof. Let $R_0=\sum D_0 f_{ij}\in\{R_\alpha$'s$\}$ be of maximal capacity ($\leqq$ the capacity $[R|R]$ of R). Then, for each $R_\alpha\supseteq R_0$ we have $R_\alpha=\sum D_\alpha f_{ij}$ with the division ring $D_\alpha = V_{R_\alpha}(\{f_{ij}\text{'s}\})$. As evidently $T = \bigcup_{R_\alpha \supseteq R_0} R_\alpha = \sum (\bigcup D_\alpha) f_{ij}$ and $\bigcup D_\alpha$ is a division ring, T is a simple ring.

Lemma 1.2. *If R/S is locally Galois, and V a division ring, then R is S-R-irreducible.*

Proof. For any non-zero element $x \in R$, there exists a shade N of $\{x\}$. Since N is S-N-irreducible by [8, Lemma 1. 1], $SxN = N \ni 1$, whence $SxR = R$.

The following three lemmas will play important rôles in our subsequent consideration.

Lemma 1.3. *Let S be a regular subring of R, $\mathfrak{H}$ a set of S-(ring) automorphisms of R, and S' an intermediate ring of R/S such that R is S'-R-irreducible.*

(i) *For each $\sigma \in \mathfrak{H}$, $(\sigma | S')R_r$ is S'_r-R_r-irreducible and $x_r \to (\sigma | S')x_r$ is an R_r-isomorphism of R_r onto $(\sigma | S')R_r$.*

(ii) *For any subset $\mathfrak{H}'$ of $\mathfrak{H}$, $\mathfrak{H}' | S'$ is linearly independent over R_r if and only if so it is over V_r.*

(iii) *The R_r-module $(\mathfrak{H} | S')R_r$ possesses a subset of $\mathfrak{H} | S'$ as a linearly independent R_r-right basis, and $\mathfrak{H}' | S' (\subseteq \mathfrak{H} | S')$ is a linearly independent R_r-right basis of $(\mathfrak{H} | S')R_r$ if and only if it is a linearly independent V_r-right basis of $(\mathfrak{H} | S')V_r$.*

(iv) *Let α be an S-(ring) isomorphism of S' into R. Then αR_r is S'_r-R_r-isomorphic to $(\sigma | S')R_r (\sigma \in \mathfrak{G})$ if and only if $\alpha = \sigma \tilde{v} | S'$ for some regular element $v \in V$.*

Proof. (i)-(iii) will be shown in the same way as in [3, Lemma 2].

(iv) Assume that αR_r is S'_r-R_r-isomorphic to $(\sigma | S')R_r$. If $\sigma | S' \leftrightarrow \alpha v_r$ under the isomorphism, then one will easily see that $v \in V$. Moreover, $\alpha v_r R_r = \alpha R_r$ yields at once $vR = R$. Hence, v is a regular element of R. Now, it will be easy to see that $\alpha = \sigma \tilde{v} | S'$. And the converse part is an easy consequence of (i).

Corollary 1.1. *Let S be a regular subring* of *R, and S' an intermediate ring of R/S such that R is S'-R-irreducible. If an S-(ring) isomorphism α of S' into R is contained in $\sum (\sigma_\iota | S')R_r (\sigma_\iota \in \mathfrak{G})$ then $\alpha = \sigma_j \tilde{v} | S'$ $(\in \mathfrak{G} | S')$ for some regular element $v \in V$ and some $\sigma_j \in \{\sigma_\iota\text{'s}\}$.*

Proof. By Lemma 1.3, each $(\sigma_\iota | S')R_r$ is S'_r-R_r-irreducible and R_r-isomorphic to R_r. And so, the non-zero S'_r-R_r-submodule αR_r of $\sum (\sigma_\iota | S')R_r$ being R_r-homomorphic to R_r, we see that there exists some $\sigma_j \in \{\sigma_\iota\text{'s}\}$ such that αR_r is S'_r-R_r-isomorphic to $(\sigma_j | S')R_r$. Hence, by Lemma 1.3 (iv), $\alpha = \sigma_j \tilde{v} | S'$ for some $\tilde{v} \in \widetilde{V}$.

Lemma 1.4. *Let S be a regular subring of R, and S' an intermediate ring of R/S such that R is S'-R-irreducible.*

(i) *For each non-zero* $v \in V$, $(v_l | S')R_r$ *is* S'_r-R_r-*irreducible and* $x_r \to (v_l | S')x_r$ *is an* R_r-*isomorphism of* R_r *onto* $(v_l | S')R_r$.

(ii) *For any subset* W *of* V, $W_l | S'$ *is linearly independent over* R_r *if and only if* W *is linearly right-independent over* $V' = V_R(S')$. *Particularly, in case* W *is consisting of regular elements,* $\widetilde{W} | S'$ *is linearly right-independent over* R_r *if and only if* W *is linearly right-independent over* V'.

(iii) *If* W *is a linearly independent* V'-*right basis of* V *consisting of regular elements then* $\widetilde{W} | S'$ *is a linearly independent* R_r-*basis of* $(\widetilde{V} | S')R_r$.

Proof. (i) The proof is the same with that of Lemma 1.3 (i).

(ii) Assume that $W_l | S'$ is linearly dependent over R_r and $\sum_1^m (v_{il} | S') x_{ir} = 0$ $(v_i \in W,\ x_i \neq 0 \in R)$ is a non-trivial relation of the shortest length. Then, by (i), without loss of generality, we may set $x_1 = -1$. And so, we may assume that $v_{1l} | S' = \sum_2^m (v_{il} | S') x_{ir}$. Now, for arbitrary $y \in S'$ there holds

$$0 = y_r(v_{1l} | S') - (v_{1l} | S')y_r = \textstyle\sum_2^m (v_{il} | S')\ (yx_i - x_i y)_r.$$

Since our relation is of the shortest length, it follows that $x_i \in V' (i = 2, \cdots, m)$. Consequently, we have $v_1 = \sum_2^m v_i x_i (x_i \in V')$. The converse will be almost trivial. To prove the last assertion, it will suffice to remark that in case W is consisting of regular elements $\widetilde{W} | S'$ is linearly independent over R_r if and only if so is $W_l | S'$.

(iii) Let v be an arbitrary regular element of V. Then, $v = \sum w_i v_i$ with some $w_i \in W$ and $v_i \in V'$. We obtain therefore $\tilde{v} | S' = v_l v_r^{-1} | S' = \sum v_{il} w_{il} v_r^{-1} | S' = \sum w_{il} v_{il} v_r^{-1} | S' = \sum \widetilde{w}_i (w_i v_i v^{-1})_r | S'$, whence it follows $(\widetilde{W} | S')R_r = (\widetilde{V} | S')R_r$. Hence, our assertion is clear by (ii).

Lemma 1.5. *Let* S *be a regular subring of* R, *and* S' *an intermediate ring of* R/S *such that* $[S' : S]_l < \infty$ *and* R *is* S'-R-*irreducible.*

(i) $[V : V']_r \leq [S' : S]_l$, *where* $V' = V_R(S')$.

(ii) $[(\widetilde{V\sigma} | S')R_r : R_r]_r = [V : V']_r$ *for each* $\sigma \in \mathfrak{G}$.

(iii) *If* $\mathfrak{G}'$ *is a subgroup of* $\mathfrak{G}$ *containing* $\widetilde{V}$ *then the* S'_r-R_r-*module* $(\mathfrak{G}' | S')R_r$ *is completely reducible, its homogeneous component is of the form* $(\widetilde{V}\sigma | S')R_r (\sigma \in \mathfrak{G}')$, *and the number of homogeneous components is finite.*

Proof. (i) $\infty > [S' : S]_l = [\mathrm{Hom}_{S_l}(S', R) : R_r]_r \geq [(\widetilde{V} | S')R_r : R_r]_r$ by Lemma 1.3 (i). Since V possesses a linearly independent V'-right basis consisting of regular elements, $\infty > [(\widetilde{V} | S')R_r : R_r]_r = [V : V']_r$ by Lemma 1.4 (iii).

(ii) By (i), we can find a linearly independent finite V'-right basis $\{w_1, \cdots, w_t\}$ of V consisting of regular elements. Then, in virtue of Lemma

1.4 (iii), there holds $(\widetilde{V}\sigma|S')R_r=(\widetilde{V}|S')R_r\sigma=(\sum_1^t\oplus(\widetilde{w}_i|S')R_r)\sigma=\sum_1^t\oplus(\widetilde{w}_i\sigma|S')R_r$.

(iii) Our former assertions are direct consequences of Lemma 1.3 (i) and Lemma 1.3 (iv). And the last one is easy by $[\mathrm{Hom}_{S_l}(S', R): R_r]_r=[S': S]_l<\infty$.

Lemma 1.6. *Let R be locally finite over a regular subring S. Then, $\mathfrak{G}$ is locally finite if and only if $\# V<\infty$ or $V=C$, or what is the same, if and only if $\mathfrak{G}$ is almost outer.*

Proof. At first, the only if part is a direct consequence of [16, Theorem 1].[1] Now, we shall prove the if part. Let r be an arbitrary element of R. Then, there exists an intermediate ring S' of $R/S[r]$ such that $[S': S]_l<\infty$ and R is S'-R-irreducible. In virtue of Lemma 1.3 (iii), we have $\mathfrak{G}|S'\subseteq\sum_1^s\oplus(\sigma_i|S')V_r$ for some $\sigma_1\cdots, \sigma_s\in\mathfrak{G}$. If $\# V<\infty$, then $\mathfrak{G}|S'$ is evidently finite. On the other hand, if $V=C$ then each $\sigma|S'\in\mathfrak{G}|S'$ coincides with some $\sigma_i|S'$ by Corollary 1.1. And so, in either cases, $\mathfrak{G}|S'$ is finite. We have proved therefore that $\mathfrak{G}$ is locally finite. Finally, if $\mathfrak{G}$ is almost outer: $\#\widetilde{V}<\infty$ i.e. $(V^*: C^*)<\infty$, then $\# V<\infty$ or $V=C$ by [16, Lemma 1].[2]

Now, on the group $\mathfrak{G}$ we may place the finite topology. Here a basis for the neighborhoods of $\sigma\in\mathfrak{G}$ consists of the sets $U(\sigma, F)=\{\tau\in\mathfrak{G}; \tau|F=\sigma|F\}$, where F runs over all the (non-empty) finite subsets of R (or subrings of R finitely generated over S). Now, it is clear that each $U(\sigma, F)$ is open and coincides with $\mathfrak{G}(F)\sigma$. Moreover, noting that $U(1, F)=\mathfrak{G}(F)$ is a subgroup of $\mathfrak{G}$, we readily see that $U(\sigma, F)$ is closed as well. Accordingly, for any subset T of R it follows that $\mathfrak{G}(T)$ is closed. Next, we shall prove that $\mathfrak{G}$ is a topological group. In fact, for an arbitrary finite subset F of R and arbitrary $\sigma\in\mathfrak{G}$, there holds $\sigma^{-1}\mathfrak{G}(F)\sigma=\mathfrak{G}(F\sigma)$, that is, $\mathfrak{G}(F)\sigma=\sigma\mathfrak{G}(F\sigma)$. And so, we obtain $(\mathfrak{G}(F\sigma^{-1})\sigma)^{-1}=\sigma^{-1}\mathfrak{G}(F\sigma^{-1})=\mathfrak{G}(F)\sigma^{-1}=U(\sigma^{-1}, F)$, and $\mathfrak{G}(F)\sigma\cdot\mathfrak{G}(F\sigma)\tau=\mathfrak{G}(F)\sigma\tau=U(\sigma\tau, F)$ for each $\sigma, \tau\in\mathfrak{G}$. Hence, $\mathfrak{G}$ is a topological group. Consequently, if there exists a compact neighborhood $U(1, F)$ then $\mathfrak{G}$ is locally compact.

Suppose now that $\mathfrak{G}$ is compact. Then, for any $r\in R$ we have $\mathfrak{G}=\bigcup_\lambda\mathfrak{G}(\{r\})\sigma_\lambda$, where $\{\sigma_\lambda\text{'s}\}$ is a complete representative system of $\mathfrak{G}$ modulo $\mathfrak{G}(\{r\})$. Hence, it must be a finite open covering: $\mathfrak{G}=\bigcup_1^m\mathfrak{G}(\{r\})\sigma_\lambda$. Accordingly, $\{r\}\mathfrak{G}=\{r\sigma_1, \cdots, r\sigma_m\}$. In particular, if S is a regular subring then $\# V<\infty$ or $V=C$ by the proof of Lemma 1.6. Conversely, if $\mathfrak{G}$ is locally finite and R/S is locally finite, then $S[F\mathfrak{G}]$ is $\mathfrak{G}$-normal and left-finite over S for each finite subset F of R. And so, wo have $R=\bigcup_\lambda N_\lambda$, where

1) We should like to note here two typographical errors in the statements of [16, Theorems 1 and 1']. $\#(\bar{R})$ should replace $\#(R)$.

2) The second theorem stated in [1, Remarks, p. 482] is essentially contained in [16, Lemma 1]. For, if $(R^*: S^*)<\infty$ (where S is not necessarily simple) then it is clear that S is a π-subring of R.

N_λ runs over all the $\mathfrak{G}$-normal intermediate rings of R/S with $[N_\lambda: S]_l < \infty$. We set here $\mathfrak{G}_\lambda = \mathfrak{G} | N_\lambda$, which is evidently a finite group of S-(ring) automorphisms of N_λ. And then, the topological group $\mathfrak{G}$ can be regarded as the inverse limit of the system $\{\mathfrak{G}_\lambda\}$. Hence, as is well-known, $\mathfrak{G}$ is compact.

Moreover, we can prove the next that contains [18, Theorem 4].

Lemma 1.7. *Let R be locally finite over a regular subring S. Then, $\mathfrak{G}$ is locally compact if and only if $[V: C] < \infty$.*

Proof. Suppose at first that $\mathfrak{G}$ is locally compact. Then, there exists a compact neighborhood $U(1, F)$. Here, without loss of generality, we may assume that $S' = S[F]$ is a regular subring and R is S'-R-irreducible. Then, as is remarked above, it follows that $\# V_R(S') < \infty$ or $V_R(S') = C$. And so, in any rate, we have $[V_R(S'): C] < \infty$. On the orher hand, $[V: V_R(S')]_r \leqq [S': S]_l < \infty$ by Lemma 1.5. It follows therefore $[V: C] = [V: V_R(S')]_r \cdot [V_R(S'): C] < \infty$. Now, we shall prove the converse part. As $\infty > [V: C] = [R: H]$, we can find a finite subset F such that $S' = S[F]$ is simple and $V_R(S') = C$. Then, R/S' is locally finite. And so, as is noted above, $\mathfrak{G}(S')$ is compact, which proves the local compactness of $\mathfrak{G}$.

Lemma 1.8. *Let R be outer Galois and algebraic over S, $\mathfrak{H}$ a Galois group of R/S, and S' an intermediate ring of R/S. If $[S': S]_l < \infty$ then $S' = S[t']$ for some t' and $\#(\mathfrak{H}|S') = \#\{t'\}\mathfrak{H} = [S': S]$. And conversely, if $\#(\mathfrak{H}|S') < \infty$ then $[S': S] < \infty$.*

Proof. At first, one may remark that R/S is locally finite by [3, Theorem 2], whence $\mathfrak{G}$ is l. f. d. If $[S': S]_l < \infty$ then $S' = S[t']$ by [3, Corollary 1][3]. Further, if T is a $\mathfrak{H}$-normal shade of S' then $\mathfrak{G}(T/S) = \mathfrak{H} | T$, that is a finite group. Accordingly, $\mathfrak{H} | S' = \{\sigma_1 | S', \cdots, \sigma_s | S'\}$. Hence, we have $\mathrm{Hom}_{S_l}(S', T) = (\mathfrak{H}|S')T_r = \sum_1^s (\sigma_i | S')T_r = \sum_1^s \oplus (\sigma_i | S')T_r$ by Lemma 1.2 and Lemma 1.5 (iii). We obtain therefore $s = [\mathrm{Hom}_{S_l}(S', T): T_r]_r = [S': S]_l = [S': S]$. Now, we shall prove the converse part. For an arbitrary element t of S' there holds $\#\{t\}\mathfrak{H} = \#(\mathfrak{H}|S[t]) \leqq \#(\mathfrak{H}|S') < \infty$. And so, we can choose an element $t_0 \in S'$ such that $\#\{t_0\}\mathfrak{H}$ is maximal. Then, $S' = S[t_0]$. For, if not, there exists a subring S'' of S' such that $S'' \supsetneqq S[t_0]$ and $[S'': S]_l < \infty$. As $S'' = S[t'']$ for some t'' by [3, Corollary 1], by the assertion cited previously, it follows that $\#\{t''\}\mathfrak{H} = [S'': S] > [S[t_0]: S] = \#\{t_0\}\mathfrak{H}$, which is a contradiction.

If for each k elements $x_1, \cdots, x_k$ of R we have $[S[x_1, \cdots, x_k]: S]_l < \infty$ then R/S is said to be *k-(left) algebraic*. Corresponding to [3, Theorem 2], we shall prove the following two lemmas.

3) As is noted in [9], the proof of [8, Theorem 3.1] is not complete. Nevertheless [9, Theorem 2] secures that [8, Theorem 3.1] is certainly true, whence we see that [3, Corollary 1] (whose proof needed [8, Theorem 3.1]) holds good.

Lemma 1.9. *Let R/S be Galois and 2-algebraic. If $[V: C]<\infty$ then R/S is locally finite.*

Proof. Since our assertion for the case $[S: Z]<\infty$ is contained in [8, Theorem 5. 1], we shall restrict our subsequent consideration to the case $[S: Z]=\infty$. Since $S[V]=S\times_Z V$, we have $[S[V]: V]_l=\infty$, whence $[H[V]: V]_l=\infty$. And so, noting that $H[V]=H\times_{C_0}V$, we obtain $[H: C_0]=\infty$. Thus, R is inner Galois and finite over the simple ring H that is infinite over its center C_0. Accordingly, we have $R=H[a]$ by [8, Theorem 2. 1]. Hence, for each finite subset F of R there exists a finite subset E of H such that $S[E][a]\supseteq F$. Since $S[E]=S[b]$ for some b by Lemma 1. 8, $S[F]$ is contained in $S[a, b]$ that is left-finite over S.

Lemma 1.10. *Let R/S be Galois and 3-algebraic. If $[V: C_0]<\infty$ then R/S is locally finite.*

Proof. Since V is inner Galois and finite over C_0, by [17, Theorem 1], $V=C_0[a_2, a_3]$ for some a_2, a_3. If $S\subseteq C$, then one will readily see that $C_0=C$, whence $[V: C]<\infty$. Accordingly, our assertion is clear by Lemma 1. 9. Thus, in what follows, we may restrict our attention to the case where $S\not\subseteq C$. At first, we shall prove the existence of an intermediate ring S' of R/S with the following properties:

(a) $[S': S]_l<\infty$.

(b) R is S'-R-irreducible.

(c) $V_R(S')\subseteq C_0$.

(d) $H'=V_R(V_R(S'))$ is simple and of capacity n($=$ the capacity of R).

(e) $S\not\subseteq C'=V_{H'}(H')$.

To this end, we shall distinguish here two cases:

Case I: *S contains merely diagonal elements.* V contains evidently $\{e_{11}, \cdots, e_{nn}\}$, whence the capacity of V is equal to n. And so, without loss of generality, we may assume that e_{ij}'s are all contained in V, whence $S\subseteq D$. As S is not contained in $C=V_D(D)$, there exist some $x\in S$ and $y\in D$ such that $xy\neq yx$. Set $a_1=\sum_2^n e_{ii-1}+ye_{1n}$ or $a_1=y$ according as $n>1$ or $n=1$. Then, $S'=S[a_1, a_2, a_3]$ is our desired one. In fact, by making use of the same method as in the proof of [8, Lemma 2. 3], we see that $S'\supseteq\{e_{ij}\text{'s}\}$. And so, R is S'-R-irreducible and $H'=V_R(V_R(S'))$ is a simple ring of capacity n. Further, $V_R(S')\subseteq V_R(S[a_2, a_3])=V_V(C_0[a_2, a_3])=C_0$. And finally, (e) is evident by the fact $x\in S$ and $y=a_1^n\in S'\subseteq H'$.

Case II: S *contains a non-diagonal element* $b=\sum e_{ij}c_{ij}(c_{ij}\in D)$. Here, without loss of generality, we may assume that $c_{1n}\neq 0$ (cf. [3, p.62—63]). If $r=(\sum_1^{n-1}e_{ii}+e_{nn}c_{1n})(1-e_{n1}c_{nn}c_{1n}^{-1})\cdots(1-e_{21}c_{2n}c_{1n}^{-1})$, then $r^{-1}=(1+e_{21}c_{2n}c_{1n}^{-1})\cdots(1+e_{n1}c_{nn}c_{1n}^{-1})(\sum_1^{n-1}e_{ii}+e_{nn}c_{1n}^{-1})$ and $rbr^{-1}=\sum e_{ij}d_{ij}$ with $d_{1n}=1$ and $d_{in}=0$ ($i\geqq 2$). Hence, $b=\sum(r^{-1}e_{ij}r)(r^{-1}d_{ij}r)$ is contained in S, which means that we may assume from the beginning that $c_{1n}=1$ and $c_{in}=0$ ($i\geqq 2$).

We set here $a_1=\sum_2^n e_{ii-1}$ and $S'=S[a_1, a_2, a_3]$. Then, R is S'-R-irreducible by [3, Lemma 6 (ii)] and $V_R(S')\subseteq V_R(S[a_2, a_3])=C_0$. Next, $a_1^{n-1}=e_{n1}$ implies $be_{n1}=e_{11}\in S'$. And so, $e_{k1}=a_1^{k-1}e_{11}\in S'$ $(k=2, \cdots, n)$. Now, to be easily verified, $V_R(S')\subseteq V_R(\{e_{11}, e_{21}, \cdots, e_{n1}\})=D$. Hence, $H'=V_R(V_R(S'))\supseteq V_R(D)\supseteq\{e_{ij}\text{'s}\}$, from which (d) will be clear. And, (e) is a direct consequence of the fact $b\in S$ and $\{e_{ij}\text{'s}\}\subseteq H'$.

Now, let S' be an arbitrary intermediate ring of R/S with the properties (a)-(e). Needless to say, every subring containing S' left-finite over S has also these properties. $H'=V_R(V_R(S'))\supseteq V_R(C_0)\supseteq V$ implies evidently $C'=H'\cap V_R(S')=V_R(S')$. And so, $[V_{H'}(S): C']=[V: V_R(S')]_r\leqq[S': S]_l<\infty$ by Lemma 1.5 (i). Hence, H' is inner Galois and finite over $V_{H'}(V)=H$(whence H is simple). As $(S\subseteq)H\not\subseteq C'$, in virtue of [9, Corollary 2], $H'=H[a_1^*]$ with some a_1^*. Hence, as S' is a subring of H' left-finite over S, we have $S'\subseteq S[F][a_1^*]$ for some finite subset F of H. Recalling here that $S[F]=S[a_2^*]$ with some a_2^* by Lemma 1.8, it follows that $S'\subseteq S[a_1^*, a_2^*]$. Now, for arbitrary $u_1\in R$ the above argument yields $S[a_1^*, a_2^*, u_1]\subseteq S[a_1^{**}, a_2^{**}]$ with some a_1^{**}, a_2^{**}. Continuing the same procedures, for each finite subset $\{u_1, \cdots, u_s\}\subseteq R$ we obtain eventually $S[u_1, \cdots, u_s]\subseteq S[a_1^\S, a_2^\S]$ for some $a_1^\S$, $a_2^\S$. Hence, we have proved that R/S is locally finite.

For the later use, we shall state the following as a corollary.

Corollary 1.2. *Let R/S be Galois and 3-algebraic. If $[V: C_0]<\infty$ then H is simple and there exists an intermediate ring S' of R/S such that R is S'-R-irreducible and, for each subring S^* containing S' such that $[S^*: S]_l<\infty$, $H^*=V_R(V_R(S^*))$ is a simple ring containing V and $[V_{H^*}(S): V_{H^*}(H^*)]<\infty$.*

Proof. At first, R/S is locally finite by Lemma 1.10. And, as is noted at the beginning of the proof of Lemma 1.10, if $S\subseteq C$ then $[V: C]<\infty$. And so, $S'=S[\{e_{ij}\text{'s}\}, \{u_\nu\text{'s}\}]$ is our desired one, where $\{u_\nu\text{'s}\}$ is a linearly independent C-basis of V. On the other hand, in case $S\not\subseteq C$, there exists an intermediate ring S' of R/S with the properties (a)-(e) cited in the proof of Lemma 1.10. And, our assertion will be easy by the proof of Lemma 1.10.

2. $\mathfrak{G}$-locally Galois extensions.

At first, we shall prove the next theorem that contains evidently [11, Theorem 2].

Theorem 2.1. *Let R be locally finite oves a regular subring S, and $\mathfrak{H}R_r$ dense in $Hom_{S_l}(R, R)$ for some subgroup $\mathfrak{H}$ of $\mathfrak{G}$ containing $\tilde{V}$. If S' is an arbitrary intermediate ring of R/S with $[S': S]_l<\infty$ such that R is S'-R-irreducible then $J(\mathfrak{H}(S'), R)=S'$.*

Proof. By Lemma 1.5 (iii) and our assumption, we obtain

$$\mathrm{Hom}_{S_l}(S', R) = (\mathfrak{H}|S')R_r = \textstyle\sum_1^s \oplus (\widetilde{V}\sigma_i|S')R_r.$$

If $J(\mathfrak{H}(S'), R) \neq S'$, for arbitrary $y \in J(\mathfrak{H}(S'), R) \backslash S'$ we set $S'' = S[y]$, which is evidently finite over S. Since $S'' \supsetneqq S'$ and $V_R(S'') = V_R(S')$, we have

$[(\widetilde{V}\sigma_i|S')R_r: R_r]_r = [V: V_R(S')]_r = [V: V_R(S'')]_r = [(\widetilde{V}\sigma_i|S'')R_r: R_r]_r$ by Lemma 1. 5 (ii). Moreover, of course, $\sum_1^s(\widetilde{V}\sigma_i|S'')R_r = \sum_1^s \oplus (\widetilde{V}\sigma_i|S'')R_r$. Now, noting that $[(\mathfrak{H}|S')R_r: R_r]_r = [S': S]_l < [S'': S]_l = [(\mathfrak{H}|S'')R_r: R_r]_r$, by Lemma 1. 5 (iii) we obtain

$$(\mathfrak{H}|S'')R_r = \textstyle\sum_1^s \oplus (\widetilde{V}\sigma_i|S'')R_r \oplus \sum_1^t \oplus (\widetilde{V}\tau_j|S'')R_r$$

for some $\tau_j \in \mathfrak{H}$ $(t \geqq 1)$. Since $\tau_1|S'$ is contained in $\sum_1^s \oplus (\widetilde{V}\sigma_i|S')R_r$, there exists some σ_k such that $\tau_1|S' = \sigma_k \tilde{v}|S'$ for some $\tilde{v} \in \widetilde{V}$ by Corollary 1. 1. And then, $\tau_1 \tilde{v}^{-1} \sigma_k^{-1} \in \mathfrak{H}(S') = \mathfrak{H}(S'')$ implies $\tau_1 = \tau \sigma_k \tilde{v}$ with some $\tau \in \mathfrak{H}(S'')$. Accordingly, we have $\tau_1|S'' = \tau\sigma_k\tilde{v}|S'' = \sigma_k\tilde{v}|S'' \in \sigma_k\widetilde{V}|S'' = \widetilde{V}\sigma_k|S''$, which is a contradiction. We have proved therefore $J(\mathfrak{H}(S'), R) = S'$.

Lemma 2.1. *If R/S is $\mathfrak{H}$-locally Galois then it is Galois.*

Proof. Let a be an arbitrary element of $R \backslash S$. Then there exists a $\mathfrak{H}$-shade N of $\{a\}$. Since there exists some $\sigma \in \mathfrak{G}(N/S) \subseteq \mathfrak{H}|N$ such that $a\sigma \neq a$, we readily see that $J(\mathfrak{H}, R) = S$. Next, it will be easy to see that $V = \bigcup V_N(S)$, where N runs over all the shade of finite subsets of R. Since each $V_N(S)$ is simple, V is simple by Lemma 1. 1. (Cf. [8, Lemma 4. 3].)

Theorem 2.2. *Let $\mathfrak{H}$ be a non-empty subset of $\mathfrak{G}$. If R/S is $\mathfrak{H}$-locally Galois then $\mathfrak{H}R_r$ is dense in $\mathrm{Hom}_{S_l}(R, R)$.*

Proof. Let $M = \sum_1^m Su_i$ be an arbitrary submodule of R left-finite over S. Then, by our assumption, there exists a $\mathfrak{H}$-shade S' of $\{u_i\text{'s}\}$. Then, as is well-known, $\mathrm{Hom}_{S_l}(S', S') = \mathfrak{G}(S'/S)S'_r$. Now let $\{x_1, \cdots, x_s\}$ be a linearly independent S-left basis of S', and α_i's elements of $\mathrm{Hom}_{S_l}(S', S')$ such that

$$x_j \alpha_i = \delta_{ij} \quad (i, j = 1, \cdots, s).$$

Then, $\{\alpha_i\text{'s}\}$ is evidently a linearly independent S'_r-basis of $\mathrm{Hom}_{S_l}(S', S')$. And it can be a linearly independent R_r-basis of $\mathrm{Hom}_{S_l}(S', R)$ as well. Hence, by our assumption, $\mathrm{Hom}_{S_l}(S', R) = \mathfrak{G}(S'/S)R_r = (\mathfrak{H}|S')R_r$. Accordingly, as is well-known, $\mathrm{Hom}_{S_l}(M, R) = \mathrm{Hom}_{S_l}(S', R)|M = (\mathfrak{H}|S')R_r|M = \mathfrak{H}R_r|M$.

Combining Theorem 2. 1 with Theorem 2.2, we obtain

Corollary 2.1. *Let R/S be $\mathfrak{H}$-locally Galois for a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ containing $\widetilde{V}$. If S' is an intermediqte ring of R/S with $[S': S]_l < \infty$ such that R is S'-R-irreducible then $J(\mathfrak{H}(S'), R) = S'$.*

Lemma 2.2. *Let R be Galois and locally finite over S. If $[V: C]<\infty$ then R/S is $\mathfrak{G}$-locally Galois.*

Proof. Let $\{v_1, \cdots, v_t\}$ (v_i's are regular) and $\{r_1, \cdots, r_t\}$ be a linearly independent C-basis of V and an H-left basis of R respectively. We set here $R=\sum R'd_{hk}$ with the simple ring $R'=V_R(\{d_{hk}\text{'s}\})$, where d_{hk}'s are matrix units of H. Noting here that $\mathfrak{G}(H/S)$ is l. f. d. and $\mathfrak{G}^*=\mathfrak{G}|H$ is a Galois group of H/S, one will readily see that H/S is is $\mathfrak{G}^*$-locally Galois. If $T=S[\{d_{hk}\text{'s}\}]$, then $T=\sum T'd_{hk}$ with the division ring $T'=V_T(\{d_{hk}\text{'s}\})$ and $J(\mathfrak{G}(T), R)=J(\tilde{V}, R)\cap J(\mathfrak{G}(T), R)=J(\mathfrak{G}^*(T)|H, H)=T$ by Corollary 2. 1. And so, if $U=\sum V_T(T)d_{hk}$ then $J(\mathfrak{G}(T)\cdot\tilde{U}, R)=V_T(U)=T'$. Moreover, one will readily see that $V_R(T')=\sum Vd_{hk}$. We have proved therefore that R is Galois and locally finite over T'. Next, we shall prove that $\mathfrak{G}(R/T')R_r$ is a dense subring of $\mathrm{Hom}_{T'_l}(R, R)$. Since $V_R(T')$ is finite over C, R is finite and inner Galois over $V_R(V_R(T'))=V_{R'}(V_{R'}(T))=V_{R'}(V_R(T))=R'\cap V_R(V_R(H))=K(=V_H(\{d_{hk}\text{'s}\}))$. And so, noting that K is a division ring, we see that R is $\widetilde{V_R(T')}R_r$-irreducible by finite Galois theory. It follows therefore that R is $\mathfrak{G}(R/T')R_r$-irreducible. Recalling here that $V_{\mathrm{Hom}(R,R)}(\mathfrak{G}(R/T')R_r)=T'_l$, Jacobson's density theorem [2] shows that $\mathfrak{G}(R/T')R_r$ is a dense subring of $\mathrm{Hom}_{T'_l}(R, R)$. Accordingly, by Theorem 2. 1, R is Galois over $S_0=T[\{e_{ij}\text{'s}\}\{r_v\text{'s}\}]$. Now, to see that R/S is $\mathfrak{G}$-locally Galois, it will suffice to prove that there exists a Galois group $\hat{\mathfrak{G}}$ of R/S such that $(\hat{\mathfrak{G}}, R/S)$ is l. f. d. Noting here that R/S_0 is outer Galois and $J(\mathfrak{G}(S_0)|H, H)=S_0\cap H=H_0$ (a simple ring), one will readily see that $\mathfrak{G}(S_0)|H=\mathfrak{G}(H/H_0)$. And so, it will be easily verified that $H'\to\sum_1^t\oplus H'r_v$ and $S'\to S'\cap H$ are mutually converse correspondences between intermediate (simple) rings H' of H/H_0 and intermediate ones S' of R/S_0. In what follows, this fact will be used often without mention. Now, let H' be a $\mathfrak{G}(H/S)$-normal shade of $S_0[v_1, \cdots, v_t]\cap H$, and set $S'=\sum\oplus H'r_v$. As $\mathfrak{G}(H'/S)=\{\sigma_1, \cdots \sigma_u\}|H'$ for some σ_i's in $\mathfrak{G}$, we have $\mathfrak{G}(H/S)=\{\sigma_1\mathfrak{G}', \cdots, \sigma_u\mathfrak{G}'\}|H$, where $\mathfrak{G}'=\mathfrak{G}(S')$. We set here $r_j\sigma_i=\sum h_{ijv}r_v (i=1, \cdots, u;\ j=1, \cdots, t)$, and let N be a $\mathfrak{G}(H/S)$-normal shade of $H'[\{h_{ijv}\text{'s}\}]$. If we set $M=\sum\oplus Nr_v$ then $\hat{\mathfrak{G}}=\{\sigma\in\mathfrak{G};\ M\sigma=M\}$ contains $\{\sigma_1\mathfrak{G}', \cdots, \sigma_u\mathfrak{G}'\}$ as well as $\widetilde{V_M(S)}$, and so we have $J(\hat{\mathfrak{G}}, R)\subseteq J(\{\sigma_1\mathfrak{G}', \cdots, \sigma_u\mathfrak{G}'\}, R)\cap J(\{\tilde{v}_1, \cdots, \tilde{v}_t\}, R)=J(\{\sigma_1\mathfrak{G}', \cdots, \sigma_u\mathfrak{G}'\}|H, H)=S$, that is, $J(\hat{\mathfrak{G}}, R)=S$. Next, let $F=\{a_k=\sum h_{kv}r_v (h_{kv}\in H;\ k=1, \cdots, s)\}$ be an arbitrary finite subset of R, and set $E=\{h_{kv}\sigma;\ k=1, \cdots, s;\ v=1, \cdots, t;\ \sigma\in\hat{\mathfrak{G}}\}$, which is evidently a finite set. Then $S[F\hat{\mathfrak{G}}]\subseteq M[E]$, and $M[E]$ is left-finite over S. Hence, we have proved that $(\hat{\mathfrak{G}}, R/S)$ is l. f. d.

Lemma 2.3. *Let S be a regular subring of R, and S' a regular in-*

termediate ring of R/S such that S'/S is Galois and $[S': S] < \infty$. If $\mathfrak{G}(S'/S) \subseteq \mathfrak{G}|S'$ and R is S'-R-irreducible then $S' \cap H$ is simple and $[S': S' \cap H]_l = [V: V_R(S')]_r$.

Proof. Let $\mathfrak{G}' = \{\sigma \in \mathfrak{G};\ S'\sigma = S'\}$, $V_0 = \{v \in V;\ S'\tilde{v} = S'\}$. Then, $\widetilde{V}_0$ is a normal subgroup of $\mathfrak{G}'$ and $\mathfrak{G}(S'/S) = \mathfrak{G}'|S'$. Since $\widetilde{V}_0|S'$ contains $\widetilde{V_{S'}(S)}|S'$, $\widetilde{V}_0|S'$ is a regular subgroup of $\mathfrak{G}(S'/S)$ and the simple ring $S_0 = J(\widetilde{V}_0|S', S')$ is contained in $V_{S'}(V_{S'}(S))$. And so, S_0/S is outer Galois and $[S_0: S] = (\mathfrak{G}(S'/S): \widetilde{V}_0|S') = s$. Now, let $\{\sigma_1|S' = 1|S', \cdots, \sigma_s|S'\}$ $(\sigma_i \in \mathfrak{G}')$ be a complete representative system of $\mathfrak{G}(S'/S)/\widetilde{V}_0|S'$. Then, as is noted in the proof of Theorem 2.2, there holds $\mathrm{Hom}_{S_l}(S', R) = (\mathfrak{G}'|S')R_r = \sum_1^s(\widetilde{V}_0\sigma_i|S')R_r$. If $\sigma_i|S' = \sigma_j\tilde{v}|S'$ $(i \neq j)$ for some $\tilde{v} \in \widetilde{V}$, then we see that $S' = S'\sigma_i = S'\sigma_j\tilde{v} = S'\tilde{v}$, whence $v \in V_0$. But this is a contradiction. Hence, by Lemma 1.5 (iii) we obtain

$$(2.1) \qquad \mathrm{Hom}_{S_l}(S', R) = \sum_1^s(\widetilde{V}_0\sigma_i|S')R_r = \sum_1^s \oplus (\widetilde{V}\sigma_i|S')R_r.$$

Accordingly, by Lemma 1.5 (ii) and (iii),

$$[S': S_0]_l \cdot [S_0: S] = [S': S]_l = [\mathrm{Hom}_{S_l}(S', R): R_r]_r = s \cdot [V: V_R(S')]_r.$$

Recalling here that $[S_0: S] = s$, we have $[S': S_0] = [V: V_R(S')]_r$. On the other hand, (2.1) implies $(\widetilde{V}_0|S')R_r = (\widetilde{V}|S')R_r$. Hence, in virtue of Lemma 1.4 (i), (ii) and Lemma 1.5 (ii) we see that V possesses a linearly independent $V_R(S')$-basis consisting of elements belonging to V_0. Recalling here that V_0 contains all the regular elements of $V_R(S')$, we obtain $S_0 = V_{S'}(V_0) = S' \cap V_R([V_0]) = S' \cap V_R(V) = S' \cap H$. We have proved therefore that $S' \cap H$ is simple and $[S': S' \cap H]_l = [V: V_R(S')]_r$.

Lemma 2.4. *If R/S is locally Galois then H is simple and Galois over S.*

Proof. Let F be an arbitrary finite subset of H, and S' a shade of F. As $S \subseteq S' \cap H = S' \cap V_R(V) \subseteq S' \cap V_R(C) = S'$ and $[S': S]_l < \infty$, there exists a finite subset W of V such that $S' \cap H = S' \cap V_R(W) = V_{S'}(W)$. We consider here an arbitrary shade S'' of $S'[W]$, and set $V'' = V_{S''}(S)$, $H'' = V_{S''}(V'')$. Then, by [8, Lemma 1.4] $H'' \cap S' (= V_{S'}(V''))$ is a simple ring containing $H \cap S' = V_{S'}(V)$. On the other hand, $H'' \cap S' = V_{S'}(V'') \subseteq V_{S'}(W) = H \cap S'$. Hence, $H \cap S'$ is a simple subring of H containing F which is Galois over $S\ (= J(\mathfrak{G}(S''/S)|S' \cap \mathfrak{G}(S'/S), S'))$. Whence, our assertion is a direct consequence of Lemma 1.1 and [8, Lemma 4.2].

Corollary 2.2. *Let R/S be $\mathfrak{G}$-locally Galois.*

(i) *If S' is an arbitrary $\mathfrak{G}$-shade then $R' = V_R(V_R(S'))$ is simple and $[R': H]_l < \infty$.*

(ii) R/H *is* $\mathfrak{G}(H)$*-locally Galois.*

Proof. (i) At first, as R/S' is locally Galois, R' is simple by Lemma 2.4. Let F be an arbitrary finite subset of R' that is linearly left-independent over H. We choose here a $\mathfrak{G}$-shade N of $S'[F, \{e_{ij}\text{'s}\}]$, and set $S^* = (N \cap H)[S', \{e_{ij}\text{'s}\}, F]$, which is evidently a simple subring of N. Now, by Lemma 2.3, we see that $N \cap H$ is simple and $\infty > [N: N \cap H]_l = [V: V_R(N)]_r$. Accordingly, noting that $S^* \supseteq S'$ and $R/N \cap H$ is $\mathfrak{G}(N \cap H)$-locally Galois, we obtain $\infty > [N: S^*]_l \cdot [S^*: N \cap H]_l = [V: V_R(S^*)]_r \cdot [V_R(S^*): V_R(N)]_r$, $[N: S^*]_l \geqq [V_R(S^*): V_R(N)]_r$ and $[S^*: N \cap H]_l \geqq [V_R(N \cap H): V_R(S^*)]_r = [V: V_R(S^*)]_r$ by Lemma 1.5 (i). If follows therefore $[S^*: N \cap H]_l = [V: V_R(S^*)]_r = [V: V_R(S'[\{e_{ij}\text{'s}\}])]_r$, for $R' \supseteq S'[F] \supseteq S'$. Since S^* contains F, the last fact implies at once $[R': H]_l \leqq [V: V_R(S'[\{e_{ij}\text{'s}\}])]_r < \infty$.

(ii) Let F be an arbitrary finite subset of R, and S' a $\mathfrak{G}$-shade of $\{g_{pq}\text{'s}\} \cup F$. We set here $V_0 = \{v \in V;\ S'\tilde{v} = S'\}$. Then, as is shown in the proof of Lemma 2.3, $V = [V_0]$. And so, noting that $R' = V_R(V_R(S'))$ is a simple ring containing $\{g_{pq}\text{'s}\}$ and finite over H by (i), we see that R' is a $\mathfrak{G}(H)$-shade of F.

Corollary 2.3. *Let R/S be $\mathfrak{G}$-locally Galois. If S' is a regular intermediate ring of R/S such that $[S': S]_l < \infty$ then $H' = V_R(V_R(S'))$ is simple and $[H': H]_l = [V: V_R(S')]_r$*

Proof. Let S'' be an arbitrary $\mathfrak{G}$-shade of $S'[\{e_{ij}\text{'s}\}, \{g'_{p'q'}\text{'s}\}]$, where $g'_{p'q'}$'s are matrix units of $V_R(S')$. Then, the proof of Corollary 2.2 (i) will yield at once $[H'': H]_l = [V: V_R(S'')]_r < \infty$, where $H'' = V_R(V_R(S''))$. Since, to be easily seen, R/S' is $\mathfrak{G}(S')$-locally Galois (whence H' is simple by Lemma 2.4) and S'' is a $\mathfrak{G}(S')$-shade, the same reason yields $[H'': H']_l = [V_R(S'): V_R(S'')]_r$. And so, $[H'': H']_l \cdot [H': H]_l = [H'': H]_l = [V: V_R(S'')]_r = [V: V_R(S')]_r \cdot [V_R(S'): V_R(S'')]_r$ implies $[H': H]_l = [V: V_R(S')]_r < \infty$.

Lemma 2.5. *If R/S is $\mathfrak{G}$-locally Galois then it is $\mathfrak{G}^*$-locally Galois for any regular Galois group $\mathfrak{G}$ of R/S.*

Proof. Let S' be a $\mathfrak{G}$-shade of an arbitrary finite subset of R. Then, by Lemma 2.3, ($H' = S' \cap H$ is simple and) $\infty > [S': H']_l = [V: V_R(S')]_r = [V_R(H'): V_R(S')]_r$. Accordingly, it follows that $\mathrm{Hom}_{H'_l}(S', R) = (\tilde{V} | S') R_r$ by Lemma 1.5. Since H'/S is outer Galois and $\mathfrak{G}^* | H$ is a Galois group of H/S, $\mathfrak{G}(H'/S) = \mathfrak{G}^* | H' = \{\sigma_1 | H', \cdots, \sigma_t | H'\}$ where $\sigma_i \in \mathfrak{G}^*$(cf. [8, Corollary 1.1]). And so, for each $\sigma \in \mathfrak{G}$ there exists some σ_j such that $\sigma | H' = \sigma_j | H'$, that is, $\sigma \sigma_j^{-1} \in \mathfrak{G}(H')$. Hence, $\sigma \sigma_j^{-1} | S' \in \mathrm{Hom}_{H'_l}(S', R) = (\tilde{V} | S') R_r$, whence $\sigma \sigma_j^{-1} | S' = \tilde{v} | S'$ $(\tilde{v} \in \tilde{V})$ by Corollary 1.1. It follows therefore $\sigma | S' = \tilde{v} \sigma_j | S' \in \mathfrak{G}^* | S'$. This proves evidently $\mathfrak{G} | S' = \mathfrak{G}^* | S'$. In particular, we obtain $\mathfrak{G}(S'/S) \subseteq \mathfrak{G}^* | S'$.

Theorem 2.3. *R/S is $\mathfrak{G}$-locally Galois if and only if R/S is Galois and locally Galois.*

Proof. The only if part is clear by Lemma 2. 1. Now, we shall prove the if part. Let S' be an arbitrary shade of $\{e_{ij}$'s$\}$, and $H'=S'\cap H$. Then, by the same way as in the proof of Lemma 2. 4, we can find a subset W of V such that $H'=V_{S'}(W)$. And then, for any shade S'' of $S'[W]$ we can see that H' coincides with the simple ring $S'\cap H''$, where $H''=V_{S''}(V_{S''}(S))$. Moreover, if we set $\mathfrak{G}'=\{\sigma\in\mathfrak{G}(S''/S);\ S'\sigma=S'\}$ then $\mathfrak{G}(S'/S)=\mathfrak{G}'|S'$ and $\mathfrak{G}'|H'(=\mathfrak{G}'|S'\cap H'')$ is a Galois group of H'/S. On the other hand, recalling that H is simple by Lemma 2. 4, it follows that $\mathfrak{G}(H'/S)=(\mathfrak{G}|H)|H'=\mathfrak{G}|H'$. And so, for each $\alpha\in\mathfrak{G}(S'/S)$ there exists some $\sigma\in\mathfrak{G}$ such that $\alpha|H'=\sigma|H'$, that is, $\alpha\sigma^{-1}$ is an H'-isomorphism. Since S''/S is trivially $\mathfrak{G}(S''/S)$-locally Galois, S' is still a shade of $\{e_{ij}$'s$\}$ in S'', and $S'\cap H''=H'$, as is cited in the proof of Lemma 2. 5, there holds $\mathrm{Hom}_{H'_l}(S', S'')=(\tilde{V}''|S')S''_r$ where $V''=V_{S''}(S)$. Accordingly, we have $\mathrm{Hom}_{H'_l}(S', R)=(\tilde{V}''|S')R_r$ by the same methods as in the proof of Theoem 2.2. Hence, by Corollary 1.1, $\alpha\sigma^{-1}=\tilde{u}|S'$ for some $u\in V(=V_R(H'))$, whence $\alpha=\tilde{u}\sigma|S'\in\mathfrak{G}|S'$. We have proved therefore that $\mathfrak{G}(S'/S)\subseteq\mathfrak{G}|S'$.

Theorem 2.4. *Let R/S be Galois and locally finite. If $[V: C_0]<\infty$ then R/S is $\mathfrak{G}$-locally Galois.*

Proof. Let S' be an intermediate ring with the properties cited in Corollary 1. 2. For an arbitrary finite subset F of R, we set $T=S'[F]$. Then, in virtue of Lemma 1. 3, there holds

$$\mathfrak{G}|T\subseteq(\mathfrak{G}|T)V_r=\textstyle\sum_1^t\oplus(\sigma_i|T)V_r\ (t\leqq[T:S]_l<\infty,\ \ \sigma_i\in\mathfrak{G}).$$

We set here $S^*=T[T\sigma_1,\cdots,T\sigma_t]$. Then, $H^*=V_R(V_R(S^*))$ is a simple ring containing V and $V=V_{H^*}(S)$ is finite over the center C^* of H^* (Corollary 1. 2). Noting that $V_R(S^*)=V_R(H^*)\subseteq V_R(V)\cap V_R(S^*)=V_R(S^*[V])$, we have $H^*\supseteq V_R(V_R(S^*[V]))$, whence $H^*=V_R(V_R(S^*[V]))$. As to be easily verified, $S^*[V]$ is $\mathfrak{G}$-normal, H^* is $\mathfrak{G}$-normal too. Hence, H^*/S is Galois. And then, H^*/S is $\mathfrak{G}(H^*/S)$-locally Galois by Lemma 2. 2. Accordingly, there exists a $\mathfrak{G}(H^*/S)$-shade of F. Hence, R/S is $\mathfrak{G}$-locally Galois by Theorem 2. 3.

Corollary 2.4. *Let R be Galois and locally finite over S, and $[V: C_0]<\infty$. For each finite subset F of R, there exists a $\mathfrak{G}$-normal intermediate simple ring H^* of $R/H[F, V]$ such that $[V_{H^*}(S): V_{H^*}(H^*)]<\infty$ and $H^*=V_R(V_R(H^*))$.*

Now, in virtue of Lemma 1. 9, Corollary 2. 1 and Lemma 2. 2, [4, Theorem 1] is true under the following condition:

(I) *R is Galois and 2-algebraic over S and $[V: C]<\infty$,* Moreover, by [8, Theorem 4.1][4], (I) is equivalent to the next:

(I′) *R/S is locally Galois and $[V: C] < \infty$.*

Thus, we obtain the following conclusion.

Conclusion 2.1. *The conditions* (I) *and* (I′) *are equivalent to each other. And if one of these is fulfilled, we obtain the following*:

(i) *For each regular intermediate rings R_1, R_2 of R/S, every S-(ring) isomorphism ρ of R_1 onto R_2 can be extended to an automorphism of R.*

(ii) *For each regular intermediate ring R' of R/S, R/S' is $\mathfrak{G}(R')$-locally Galois.*

Secondly, by the validity of Lemma 1.10, Theorems 2.1, 2.4 and Corollary 2.1, the condition (3) stated in [4, p. 191] is a consequence of the following:

(II) *R is Galois and 3-algebraic over S and $[V: C_0] < \infty$.*

And so, [4, Theorem 2], [13, Lemma] and Theorem 3.4 yield the next conclusion. (Cf. also [8, Theorem 4.4].)

Conclusion 2.2. *Let the condition* (II) *be satisfied and $[R: H]_l \leqq \aleph_0$ And let R_1, R_2 be f-regular intermediate rings of R/S.*

(i) *If ρ is an S-(ring) isomorphism of R_1 onto R_2 then ρ can be extended to an automorphism of R.*

(ii) *R/R_1 is $\mathfrak{G}(R_1)$-locally Galois.*

The second assertion in the following theorem is a particularly important property of $\mathfrak{G}$-locally Galois extensions.

Theorem 2.5. *Let R/S be $\mathfrak{G}$-locally Galois, and $\mathfrak{G}^*$ a regular Galois group of R/S.*

(i) *$\mathfrak{G}^*$ is dense in $\mathfrak{G}$.*

(ii) *For any regular subring S^* with $[S^*: S]_l < \infty$, there holds $J(\mathfrak{G}^*(S^*), R) = S^*$.*

Proof. (i) Let S' be an arbitrary $\mathfrak{G}$-shade. Then in the proof of Lemma 2.5, we have seen that $\mathfrak{G}|S' = \mathfrak{G}^*|S'$. This fact implies evidently the density of $\mathfrak{G}^*$.

(ii) At first, R/S is $\mathfrak{G}^*$-locally Galois by Lemma 2.5. Let $V_R(S^*) = \sum U^* g^*_{p^*q^*}$, where $U^* = V_{V_R(S^*)}(\{g^*_{p^*q^*}\text{'s}\})$ is a division ring. And consider a $\mathfrak{G}^*$-shade S' of $S^*[\{g^*_{p^*q^*}\text{'s}\}]$. Then, $J(\mathfrak{G}^*(S'), R) = S'$ by Corollary 2.1. Further, as $J(\mathfrak{G}(S'/S)(S^*), S') = S^*$ and $\mathfrak{G}(S'/S) \subseteq \mathfrak{G}^*|S'$, we readily see that $J(\mathfrak{G}^*(S^*), R) = S^*$.

Corollary 2.5. *If R/S is $\mathfrak{G}$-locally Galois, and R' a regular inter-*

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4) In the proof of [8, Theorem 4.1], we should remark that $\mathfrak{G}_\alpha | M_\beta \subseteq \mathfrak{G}_\beta$ if $M_\alpha \supseteq M_\beta$, which will be easily seen by [8, Corollary 1.1]. And, by the way, we remark here that the last part of the proof may be omitted. In fact, it is evident that σ is an automorphism of R.

mediate ring of R/H *such that* $[R':H]_l<\infty$, *then* $R'=V_R(V_R(R'))$.

Proof. $\widetilde{V}$ is evidently a regular Galois group of R/H, and R/H is $\mathfrak{G}(H)$-locally Galois by Corollary 2.2 (ii). And so, R/H is $\widetilde{V}$-locally Galois by Lemma 2.5. Now, in virtue of Theorem 2.5 (ii), we obtain $V_R(V_R(R'))=J(\widetilde{V}(R'), R)=R'$.

We shall state here the following topological properties of Galois groups as a theorem.

Theorem 2.6. *Let* R *be Galois and locally finite over* S.

(i) $\mathfrak{G}$ *is discrete if and only if* $[R:S]<\infty$.

(ii) $\mathfrak{G}$ *is compact if and only if* $\mathfrak{G}$ *is almost outer, or what is the same, if and only if* $\mathfrak{G}$ *is locally finite.*

(iii) $\mathfrak{G}$ *is locally compact if and only if* $[V:C]<\infty$.

Proof. Since (ii) is proved just before Lemma 1.7 and (iii) is contained in Lemma 1.7, it suffices to prove (i) only. At first, the if part is evident. And so, we shall prove the only if part. Since $\mathfrak{G}$ is locally compact, $[V:C]<\infty$ by (iii). And there exists a regular intermediate ring S' of R/S such that $[S':S]_l<\infty$ and $\mathfrak{G}(S')$ is compact. Combinig this with the assumption that $\mathfrak{G}$ is discrete, it follows that $\#\mathfrak{G}(S')<\infty$. Since R/S is $\mathfrak{G}$-locally Galois, R/S' is Galois by Theorem 2.5. And so, as is well-known, we have $[R:S']<\infty$, whence $[R:S]_l=[R:S']\cdot[S':S]_l<\infty$.

Our next task is concerned with the local-finite dimensionality.

Lemma 2.6. *If* R *is Galois and locally finite over* S *and* $[V:C]<\infty$, *then* $(\mathfrak{G}^*, R/S)$ *is l.f.d. for any finitely generated subgroup* $\mathfrak{G}^*=[\sigma_1,\cdots,\sigma_m]$ *of* $\mathfrak{G}$.

Proof. Let $\{r_1,\cdots,r_t\}$ be a linearly independent H-left basis of R. We set here $S_0=S[\{e_{ij}\text{'s}\},\{d_{hk}\text{'s}\},\{r_v\text{'s}\}]$ and $H_0=S_0\cap H$. Now, if $r=\sum h_v r_v$ $(h_v\in H)$ is an arbitrary element of R, then

$$r_u\sigma_w=\sum h_{uwv}r_v \quad (h_{uwv}\in H),$$
$$r_u\sigma_w^{-1}=\sum h'_{uwv}r_v \quad (h'_{uwv}\in H),\quad (w=1,\cdots,m).$$

Since H/S is outer Galois, there exists a $\mathfrak{G}(H/S)$-normal shade H' of $H_0[\{h_{uwv}\text{'s}\},\{h'_{uwv}\text{'s}\},\{h_v\text{'s}\}]$. And then, as is noted in the proof of Lemma 2.2. $N=\sum H'r_v(\supseteq S[r])$ is a ring finite over S. Moreover, as one can easily see that N is $\mathfrak{G}^*$-normal, N contains $S[\{r\}\mathfrak{G}^*]$. Hence, $(\mathfrak{G}^*, R/S)$ is l. f. d.

Corollary 2.6. *Let* R *be Galois and locally finite over* S, *and* $[V:C]<\infty$. *Then, there exists a Galois group* $\hat{\mathfrak{G}}$ *of* R/S *such that* $(\hat{\mathfrak{G}}[\mathfrak{F}], R/S)$ *is l.f.d. for each finite subset* $\mathfrak{F}$ *of* $\mathfrak{G}$.

Proof. Let $\hat{\mathfrak{G}}$ be the group considered in the proof of Lemma 2.2.

We have seen there that $(\hat{\mathfrak{G}}, R/S)$ is l. f. d. On the other hand, $([\mathfrak{F}], R/S)$ is l. f. d. by Lemma 2.6. Here, for an arbitrary finite subset F of R, we set $S_1 = S[S_0[F]\hat{\mathfrak{G}}]$ and $S_2 = S[S_0[F][\mathfrak{F}]]$, where S_0 is the ring mentioned in the proof of Lemma 2.2. Then, as is noted in the proof of Lemma 2.2, $S_i = \sum \oplus (H \cap S_i) r_v$ (i=1, 2). If H^* is a $\mathfrak{G}(H/S)$-normal shade of $(H\cap S_1) \cup (H\cap S_2)$ then $S^* = \sum \oplus H^* r_v$ is a $\hat{\mathfrak{G}}[\mathfrak{F}]$-normal subring finite over S. We have proved therefore that $(\hat{\mathfrak{G}}[\mathfrak{F}], R/S)$ is l. f. d.

Theorem 2.7. *If R is Galois and locally finite over S and $[V: C_0] < \infty$ then $(\mathfrak{G}^*, R/S)$ is l. f. d. for any finitely generated subgroup $\mathfrak{G}^*$ of $\mathfrak{G}$.*

Proof. Let r be an arbitrary element of R. Then, by Corollary 2.4, there exists a $\mathfrak{G}$-normal simple subring H^* containing $H[r, V]$ such that $V = V_{H^*}(S)$ is finite over the center of H^*. Since H^*/S is evidently Galois and $\mathfrak{G}^* | H^* \subseteq \mathfrak{G}(H^*/S)$, $(\mathfrak{G}^* | H^*, H^*/S)$ is l. f. d. by Lemma 2.6. Hence, $[S[\{r\}\, \mathfrak{G}^*] : S]_l < \infty$, which proves that $(\mathfrak{G}^*, R/S)$ is l. f. d.

Next, we shall prove that [15, Theorem 6] is still true under the assumption that R/S is $\mathfrak{G}$-locally Galois.

Lemma 2.7. *Let R be $\mathfrak{G}$-locally Galois over S, and $\mathfrak{H}$ a $(*_f)$-regular subgroup of $\mathfrak{G}$. If $R_0 = J(\mathfrak{H}, R)$ then ($H\cap R_0$ is simple and) $[R_0: H\cap R_0]_l < \infty$.*

Proof. Since H is simple and Galois over S by Lemma 2.4, there exists a $\mathfrak{G}(H/S)$-normal shade M of $\{d_{hk}$'s$\}$. To be easily seen (or by Theorem 2.5), R/M is Galois. We set here $\mathfrak{H}^* = \mathfrak{G}(R/M) \cap \mathfrak{H}$. Then, $\mathfrak{H}^*$ is a normal subgroup of $\mathfrak{H}$ and $V_{\mathfrak{H}} = V_{\mathfrak{H}^*}$. Moreover, noting that M/S is outer Galois, we readily see that $\infty > [M: S] \geqq (\mathfrak{H} : \mathfrak{H}^*)$. On the other hand, as $[V: V_{\mathfrak{H}}]_r < \infty$, we can find a finite subset $E \subseteq R_0$ such that $V_{\mathfrak{H}} = V_R(S[E])$. Now, let S' be a $\mathfrak{G}$-shade of $M[E]$, and set $R' = V_R(V_R(S'))$. Then, R' is simple and $[R': H]_l < \infty$ by Corollary 2.2. If $R^* = J(\mathfrak{H}^*, R)$ and $H^* = H \cap R^*$ then $V_{\mathfrak{H}^*} = V_{\mathfrak{H}} = V_R(S[E]) \supseteq V_R(S')$ and $\mathfrak{H} \supseteq \mathfrak{H}^*$ yield $R' \supseteq R^* \supseteq R_0$. Since R^* is $\mathfrak{H}$-normal, so is H^*. And so, noting here that H^* and $H\cap R_0$ are simple by [8, Theorem 1.1] and $J(\mathfrak{H} | H^*, H^*) = H \cap R_0$, we see that $H^*/H\cap R_0$ is outer Galois with a Galois group $\mathfrak{H} | H^*$. Hence, we have

$$\infty > (\mathfrak{H} : \mathfrak{H}^*) \geqq \#(\mathfrak{H} | H^*) = [H^* : H\cap R_0]. \tag{2.2}$$

Now let $R^* = \sum R^{**} d_{hk}$ with $R^{**} = V_{R^*}(\{d_{hk}\text{'s}\})$. And then, we can choose a linearly independent H^*-left basis $\{a_\iota$'s$\}$ of R^* from R^{**}. Suppose $\{a_\iota$'s$\}$ is linearly dependent over H, and $\sum_1^m h_\iota a_\iota = 0$ $(h_\iota \in H)$ a non-trivial relation of the shortest length. Here, without loss of generality, we may assume that $h_1 = 1$. If one of h_ι's, say h_2, is not contained in H^* then there exists some $\sigma \in \mathfrak{H}^*$ such that $h_2\sigma \neq h_2$. And so, $\sum h_\iota a_\iota - (\sum h_\iota a_\iota)\sigma = 0$, that is, $\sum_2^m (h_i - h_i\sigma) a_i = 0$ is a non-trivial relation. This contradiction shows that $\infty > [R' : H]_l \geqq [R^* : H^*]_l$. Now, combining this with (2.2), we have $\infty > [R^* : H^*]_l \cdot [H^* : H \cap R_0] = [R^* : H \cap R_0]_l \geqq [R_0 : H \cap R_0]_l$.

Corollary 2.7. *Let R be Galois and locally finite over S, and $[V : C] < \infty$. If $\mathfrak{H}$ is a $(*)$-regular subgroup of $\mathfrak{G}$ then it is regular.*

Proof. By Lemma 2.3, R/S is $\mathfrak{G}$-locally Galois. And so, $R_0 = J(\mathfrak{H}, R)$ is left-finite over $H_0 = H \cap R_0$ by Lemma 2.7. Since R/H_0 is Galois and $\mathfrak{G}(H_0)$-locally Galois by Conclusion 2.1, without loss of generality, we may assume from the beginning that $H_0 = S$. Now, let $\hat{\mathfrak{G}}$ a subgroup of $\mathfrak{G}$ with the property cited in Corollary 2.6, and set $R_1 = R_0[\{e_{ij}\text{'s}\}]$. Here, as $[R_1 : S]_l < \infty$ and $R_0 = J(\mathfrak{H}, R)$, there exists a finite subset $\mathfrak{F}$ of $\mathfrak{H}$ such that $J(\mathfrak{F}|R_1, R_1) = R_0$. We set here $\mathfrak{G}^* = \hat{\mathfrak{G}}[\mathfrak{F}]$, then $(\mathfrak{G}^*, R/S)$ is l. f. d. Let R_2 be a $\mathfrak{G}^*$-normal shade of $R_1[\{g'_{p'q'}\text{'s}\}]$, where $g'_{p'q'}$'s are matrix units of $V_{\mathfrak{H}}$. Then $\mathfrak{G}_2 = \{\sigma \in \mathfrak{G}(R_2/S) ; x\sigma = x \text{ for all } x \in R_0\}$ is a subgroup of $\mathfrak{G}(R_2/S)$ containing $\mathfrak{F}|R_2$. Hence, as R_2/R_1 is Galois and $J(\mathfrak{F}|R_1, R_1) = R_0$, we obtain $R_0 = J(\mathfrak{G}_2, R)$, whence $\mathfrak{G}_2$ is $(*)$-regular. Now, [15, Lemma] proves that R_0 is simple.

Theorem 2.8. *If R/S is $\mathfrak{G}$-locally Galois, then any $(*_f)$-regular subgroup of $\mathfrak{G}$ is f-regular.*

Proof. As to notations, we follow the proof of Lemma 2.7. And let S' be particularly a $\mathfrak{G}$-shade of $M[E, \{e_{ij}\text{'s}\}, \{g_{pq}\text{'s}\}]$. Then, R'/S is Galois and $[R' : H]_l < \infty$ implies $[V_{R'}(S) : V_{R'}(R')] < \infty$ (Corollary 2.2 and Lemma 1.5). And so, by Conclusion 2.1, we see that $R'/H \cap R_0$ is locally finite, whence $R^\S = R_0[\{e_{ij}\text{'s}\}]$ is a simple subring of R' left-finite over $H \cap R_0$ (Lemma 2.7). Accordingly, there exists a finite subset $\mathfrak{F}' = \{\sigma_1', \cdots, \sigma_m'\}$ of $\mathfrak{H}$ such that $J(\mathfrak{F}'|R^\S, R^\S) = R_0$. Now, let S_2 be a $\mathfrak{G}$-shade of $S'[\bigcup S'\sigma_k', \{g'_{p'q'}\text{'s}\}]$, where $g'_{p'q'}$'s are matrix units of $V_{\mathfrak{H}}$. And, we set $R_2 = V_R(V_R(S_2))$. Then, by the same reason as above, R_2/S is Galois and $[V_{R_2}(S) : V_{R_2}(R_2)] < \infty$. Moreover, $R_2 \supseteq V_R(V_R(S'\sigma_k')) = R'\sigma_k'$, whence $R_2 \supseteq R'[\bigcup R'\sigma_k']$. Recalling here that $R^\S$ and $R^\S\sigma_k'$ are regular intermediate rings of R_2/S, we have $\sigma_k'|R^\S = \sigma_k|R^\S$ for some $\sigma_k \in \mathfrak{G}(R_2/S) = \mathfrak{G}_2$ by Conclusion 2.1. We set here $\mathfrak{F} = \{\sigma_1, \cdots, \sigma_m\}$. Then, $R_2/R^\S$ being Galois again by Conclusion 2.1 and $J(\mathfrak{F}|R^\S, R^\S) = R_0$, it follows that $J(\mathfrak{G}_2(R_0), R_2) = R_0$. On the other hand, as $R_2 \supseteq \{g'_{p'q'}\text{'s}\}$, $V_{R_2}(R_0)$ is simple. Accordingly, $\mathfrak{G}_2(R_0)$ is a $(*)$-regular subgroup of $\mathfrak{G}_2$. Hence, $R_0 = J(\mathfrak{G}_2(R_0), R_2)$ is simple by Corollary 2.7.

The next is a generalization of [9, Lemma 1].

Corollary 2.8. *Let R/S be $\mathfrak{G}$-locally Galois. If S' is an intermediate ring of R/S with $[S' : S]_l < \infty$ such that R is S'-R-irreducible, then S' is a regular subring of R.*

Proof. At first, $J(\mathfrak{G}(S'), R) = S'$ by Corollary 2.1. As $[V :$

$V_R(J(\mathfrak{G}(S'), R))]_r = [V: V_R(S')]_r < \infty$ by Lemma 1.5 (i), $\mathfrak{G}(S')$ is a $(*_f)$-regular subgroup of $\mathfrak{G}$. Hence, our assertion is a direct consequence of Theorem 2.8.

In the rest of this section, we shall assume that R is Galois and locally finite over S and $\mathfrak{G}$ is locally compact. As $[R: H] = [V: C] < \infty$ by Theorem 2.6, there exists a linearly left-independent H-basis $\{r_1, \cdots, r_t\}$ of R. We set here $S_0 = S[\{e_{ij}\text{'s}\}, \{r_v\text{'s}\}]$, which is evidently a regular intermediate ring of R/S. And so, by Conclusion 2.1, $\mathfrak{G}_0 = \mathfrak{G}(S_0)$ is the Galois group of R/S_0. Now, under this situation, we shall present a necessary and sufficient condition that a closed N-regular subgroup of $\mathfrak{G}$ is regular.

Theorem 2.9. *Let R be Galois and locally finite over S, $\mathfrak{G}$ locally compact, and $\mathfrak{H}$ a closed N-regular subgroup of $\mathfrak{G}$. Then, in order that $\mathfrak{H}$ is regular it is necessary and sufficient that* $(\mathfrak{H}: (\mathfrak{G}_0 \cap \mathfrak{H}) \cdot \widetilde{I(\mathfrak{H})}) < \infty$.

Proof. Necessity. As $T = J(\mathfrak{H}, R)$ is a regular intermediate ring of R/S by Theorem 2.8, R/T is $\mathfrak{G}(T)$-locally Galois by Conclusion 2.1. And so, the closed $\mathfrak{H}$ coincides with $\mathfrak{G}(R/T)$ by Theorem 2.5. In virtue of [8, Theorem 1.1], $H[T]$ is a simple ring as an intermediate ring of $V_R(V_R(T))/T$. Hence, we have $V_R(I(\mathfrak{H})) = V_R(V_R(T)) = H[T]$. Now, let R_0 be a $\mathfrak{H}$-shade ($\mathfrak{G}(R/T)$-shade) of $T[S_0]$. Then, $H \cap R_0$ is evidently outer Galois and locally finite over $H \cap T$. As moreover $[T: H \cap T]_l < \infty$ by Lemma 2.7, we obtain $[H \cap R_0: H \cap T]_l \leqq [R_0: H \cap T]_l = [R_0: T]_l \cdot [T: H \cap T]_l < \infty$. We have seen therefore that $H \cap R_0$ is outer Galois and finite over $H \cap T$. On the other hand, as $\mathfrak{H}|H$ is a Galois group of $H/H \cap T$, $H \cap R_0$ is $\mathfrak{H}$-normal by [8, Corollary 1.1]. And so, we have $(\mathfrak{H}|H)|H \cap R_0 = \mathfrak{G}(H \cap R_0/H \cap T)$. As R/S_0 is outer Galois and locally finite, so is R/R_0. If we set $\mathfrak{H}_0 = \mathfrak{G}(R/R_0)$ $(\subseteq \mathfrak{G}_0 \cap \mathfrak{H})$, then $J(\mathfrak{H}_0|H, H) = H \cap R_0$. Now, it will be easy to see that $\mathfrak{H}_0|H = \mathfrak{H}(H/H \cap R_0)$. Accordingly, $\infty > [H \cap R_0: H \cap T] = (\mathfrak{H}|H: \mathfrak{H}_0|H) = (\mathfrak{H}|H[T]: \mathfrak{H}_0|H[T]) = (\mathfrak{H}: \mathfrak{H}_0 \cdot \widetilde{I(\mathfrak{H})}) \geqq (\mathfrak{H}: (\mathfrak{G}_0 \cap \mathfrak{H}) \cdot \widetilde{I(\mathfrak{H})})$.

Sufficiency. Set again $T = J(\mathfrak{H}, R)$. Then, $\mathfrak{H}|H$ is a Galois group of $H/H \cap T$. As $\mathfrak{H}_0 = \mathfrak{G}_0 \cap \mathfrak{H}$ is a closed subgroup of the outer Galois group $\mathfrak{G}_0 = \mathfrak{G}(R/S_0)$, $R_0 = J(\mathfrak{H}_0, R)$ is a simple intermediate ring of R/S_0 and $\mathfrak{G}(R/R_0) = \mathfrak{H}_0$ by [8, Corollary 1.4]. The assumption $(\mathfrak{H}: \mathfrak{H}_0 \cdot \widetilde{I(\mathfrak{H})}) < \infty$ implies evidently $\#(\mathfrak{H}|H \cap R_0) < \infty$. And so, $[H \cap R_0: H \cap T] < \infty$ by Lemma 1.8. As moreover $[R_0: H \cap R_0]_l < \infty$ by Lemma 2.7, we obtain eventually $[R_0: H \cap T]_l < \infty$. Noting here that $R_0 \supseteq T \supseteq H \cap T$, there exists a finite subset $\mathfrak{F}$ of $\mathfrak{H}$ such that $J(\mathfrak{F}|R_0, R_0) = T$. And then, $R/H \cap T$ being Galois and locally finite, in virtue of Corollary 2.6 we can find a Galois group $\hat{\mathfrak{H}}$ of $R/H \cap T$ such that $(\hat{\mathfrak{H}}[\mathfrak{F}], R/H \cap T)$ is l. f. d. Now, let $g'_{p'q'}$'s be matrix units of $I(\mathfrak{H})$, and R^* an arbitrary $\hat{\mathfrak{H}}[\mathfrak{F}]$-normal shade of $R_0[\{g'_{p'q'}\text{'s}\}]$. (One should remember $[R^*: H\cap T]<\infty$.) As $\mathfrak{G}^*=\mathfrak{G}(R/R^*)$ is (outer and so) compact, it will be easy to see that $\mathfrak{G}(H/H\cap R^*)=\mathfrak{G}^*|H$. We set here $\mathfrak{H}^*=\{\sigma\in\mathfrak{H}; R^*\sigma=R^*\}$, which contains evidently $\mathfrak{F}$. Moreover, as R^* is Galois and finite over R_0, [8, Corollary 1.1] will yield at once that R^* is $\mathfrak{H}_0$-normal. Accordingly, we obtain $\mathfrak{H}^*\supseteq\mathfrak{H}_0[\mathfrak{F}]$, whence it follows that $T\subseteq J(\mathfrak{H}^*|R^*, R^*)\subseteq J(\mathfrak{H}_0[\mathfrak{F}]|R^*, R^*)=J(\mathfrak{F}|R_0, R_0)=T$. Hence, $\mathfrak{H}^*|R^*$ is a Galois group of R^*/T.

Next, we shall prove that $\mathfrak{H}^*|R^*$ is an N-regular subgroup of $\mathfrak{G}(R^*/H\cap T)$. To this end, we shall consider

$$E=\{a\in V;\ \tilde{a}\in\mathfrak{H}\},\quad\text{and}$$
$$E^*=\{a^*\in V_{R^*}(S);\ \widetilde{a^*}|R^*\in\mathfrak{H}^*|R^*\}.$$

Take an arbitrary a^* of E^*. Then, $\widetilde{a^*}|R^*=\sigma|R^*$ for some $\sigma\in\mathfrak{H}^*$, whence $\sigma|H\cap R^*=\widetilde{a^*}|H\cap R^*=1$. Hence, $\sigma|H\in\mathfrak{G}(H/H\cap R^*)=\mathfrak{G}^*|H$, that is, $\sigma|H=\tau|H$ for some $\tau\in\mathfrak{G}^*(\subseteq\mathfrak{H}_0)$. Accordingly, $\sigma\tau^{-1}\in\mathfrak{H}\cap\tilde{V}=\tilde{E}$: $\sigma\tau^{-1}=\tilde{a}$ for some $a\in E$. Recalling here that $R^*\sigma=R^*$, we obtain

$$\tilde{a}|R^*=\sigma\tau^{-1}|R^*=\sigma|R^*=\widetilde{a^*}|R^*,$$

which implies $a^*a^{-1}\in V_R(R^*)=C$. It follows therefore $a^*\in C[E]=I(\mathfrak{H})$. As $C^*=V_{R^*}(R^*)=C\cap R^*$, it will be evident that $I(\mathfrak{H}^*|R^*)=C^*[E^*]\subseteq I(\mathfrak{H})$, whence it follows $I(\mathfrak{H}^*|R^*)\subseteq I(\mathfrak{H})\cap R^*$. As the converse inclusion is trivial, $I(\mathfrak{H}^*|R^*)$ coincides with the simple ring $I(\mathfrak{H})\cap R^*(\supseteq\{g'_{p'q'}\text{'s}\})$. Now, it will be easy to see that $\mathfrak{H}^*|R^*$ is an N-regular subgroup of $\mathfrak{H}(R^*/H\cap T)$.

As a consequence of finite Galois theory (cf. for instance [2, VI, §**12**] or [10]), we have $I(\mathfrak{H})\supseteq I(\mathfrak{H}^*|R^*)=V_{R^*}(T)$. Noting here that R^* can be chosen as large as we want, there holds $I(\mathfrak{H})\supseteq V_R(T)$, whence $I(\mathfrak{H})=V_R(T)$. Thus, we have proved that $\mathfrak{H}$ is regular.

Corollary 2.9. *Let R be Galois and left algebraic over S. If $\mathfrak{G}$ is almost outer then any closed N-regular subgroup $\mathfrak{H}$ of $\mathfrak{G}$ is regular.*

Proof. At first, R/S is locally finite by [3, Theorem 2]. And so, by Thenrem 2.6, $\mathfrak{G}$ is compact and locally finite. We set here $T=J(\mathfrak{H}, R)$. Then $T'=T[S_0\mathfrak{H}]$ is evidently a regular intermediate ring of R/S_0 and $\mathfrak{H}'=\mathfrak{H}\cap\mathfrak{G}(R/T')\subseteq\mathfrak{H}\cap\mathfrak{G}_0$. Now, recalling that $\mathfrak{H}$ is locally finite as a subgroup of $\mathfrak{G}$, we have $(\mathfrak{H}:\mathfrak{H}')=\#(\mathfrak{H}|T')<\infty$. Consequently, $(\mathfrak{H}:(\mathfrak{H}\cap\mathfrak{G}_0)\cdot\widetilde{I(\mathfrak{H})})\leq(\mathfrak{H}:\mathfrak{H}\cap\mathfrak{G}_0))\leq(\mathfrak{H}:\mathfrak{H}')<\infty$. And, our assertion is a direct consequence of Theorem 2.9.

3. Galois theory for hereditarily Galois extensions. In this section, one will penetrate what kinds of properties of extensions are essential in constructing Galois theory of simple rings. In fact, we shall show that all the Galois theories of simple rings and division rings appeared till now can be concentrated under the assumption that R/S is locally finite and hereditarily Galois.

Lemma 3.1. *Let R/S be Galois, H simple and left-algebraic over S, and $\mathfrak{G}^*$ a regular Galois group of R/S. If S' is an intermediate ring of R/S such that R is S'-R-irreducible and $[S': S]_l < \infty$, and if $[S': H \cap S']_l = [V: V_R(S')]_r$, then there holds the following*:

(i) $\mathrm{Hom}_{S_l}(S', R) = (\mathfrak{G}^* | S') R_r$.

(ii) $\mathfrak{G} | S' = \mathfrak{G}^* | S'$.

Proof. In virtue of Corollary 1.1, it suffices to prove (i) only. Set here $H' = H \cap S'$. Then, as H/S is (locally finite by [3, Theorem 2] and so) $(\mathfrak{G}^* | H)$-locally Galois, H is H'-H-irreducible by Lemma 1.2. Hence, in virtue of Theorem 2.2 and Lemma 1.3, it follows that

$$\mathrm{Hom}_{S_l}(H', H) = (\mathfrak{G}^* | H') H_r = \textstyle\sum_1^s \oplus (\sigma_i | H') H_r \text{ for some } \sigma_i \in \mathfrak{G}^*.$$

We obtain therefore $[H': S]_l = [\mathrm{Hom}_{S_l}(H', H): H_r]_r = s$. On the other hand, as $\sigma_i | H' \neq \sigma_j | H' (i \neq j)$, Lemma 1.3 (iv) proves that $(\sigma_i | S') R_r$ is not S'_r-R_r-isomorphic to $(\sigma_j | S') R_r$. And so, we have

$$\textstyle\sum_1^s (\tilde{V} \sigma_i | S') R_r = \sum_1^s \oplus (\tilde{V} \sigma_i | S') R_r.$$

Now, by Lemma 1.5 and our assumption, it follows that

$$[\textstyle\sum_1^s \oplus (\tilde{V} \sigma_i | S') R_r : R_r]_r = s \cdot [V: V_R(S')]_r = [H': S]_l \cdot [S': H']_l = [S': S]_l,$$

whence we have $\sum_1^s \oplus (\tilde{V} \sigma_i | S') R_r = \mathrm{Hom}_{S_l}(S', R)$. We have proved therefore $(\mathfrak{G}^* | S') R_r = \mathrm{Hom}_{S_l}(S', R)$.

Lemma 3.2. *Let R be locally finite over a regular subring S. And suppose that for each regular intermediate ring S' of R/S with $[S': S]_l < \infty$, R/S' is Galois and $V_R(V_R(S'))$ is simple.*

(i) *Let S_1 be a regular intermediate ring of $R/S[\{d_{hk}\text{'s}\}]$ with $[S_1 : S]_l < \infty$. If H_2 is an arbitrary subring of H containing $H_1 = H \cap S_1$, then $H_2[S_1]$ is simple and coincides with $H_2 \cdot S_1$, $H_2 = H \cap H_2 \cdot S_1$, and S_1 is linearly disjoint from H_2.*[5)]

(ii) *If R_1 is an arbitrary simple intermediate ring of R/S of capacity $n = [R | R]$ such that $H \cap R_1$ is of capacity $m = [H | H]$ then $H[R_1]$ is simple and R_1 is linearly disjoint from H.*

(iii) *R/H is locally finite.*

Proof. (i) Since $V_R(V_R(S_1))/S_1$ is locally finite and outer Galois, $H[S_1]$ is simple and Galois over S_1, whence we see that $H_2[S_1]$ is simple too. Evidently, $\mathfrak{G}(S_1) | H[S_1]$ is dense in $\mathfrak{G}_1 = \mathfrak{G}(H[S_1]/S_1)$. And so, $H\mathfrak{G}_1$

5) That is, every subset of S_1 linearly left-independent over $S_1 \cap H_2$ is linearly left-independent over H_2.

$=H$. Moreover, noting that $J(\mathfrak{G}_1|H, H)=H_1$ and $\mathfrak{G}_1$ is compact, one will readily see that $\mathfrak{G}_1|H=\mathfrak{G}(H/H_1)$.

Now, let $\{d_1, \cdots, d_q\}$ be a linearly independent H_1-left basis of S_1. At first, we shall prove that $\{d_1, \cdots, d_q\}$ is linearly independent over H. To this end, it will suffice to prove the next:

$$\sum Hd_i = \sum Kd_{hk}d_i = \sum \oplus_{h,k,i}\ Kd_{hk}d_i.$$

If not, without loss of generality, we may assume that

$$d_{11}d_1 = \sum a_{hki}d_{hk}d_i \qquad (a_{hki} \in K)$$

is a non-trivial relation of the shortest length. Then, as there exists some $a_{h_0k_0i_0}$ not contained in $K\cap S_1$, we can find some $\sigma \in \mathfrak{G}(S_1)$ such that $a_{h_0k_0i_0} \neq a_{h_0k_0i_0}\sigma (\in K)$. Accordingly, we have a non-trivial relation of shorter length:

$$0 = d_{11}d_1 - (d_{11}d_1)\sigma = \sum (a_{hki} - a_{hki}\sigma)d_{hk}d_i.$$

This contradiction proves that for any intermediate ring M of H/H_1

$$(3.1) \qquad [M\cdot S_1 : M]_l = [S_1 : H_1]_l.$$

In particular, if M is an arbitrary $\mathfrak{G}(H/H_1)$-normal shade then

$$\mathfrak{G}(M[S_1]/S_1) = \mathfrak{G}_1|M[S_1] \cong \mathfrak{G}_1|M = \mathfrak{G}(M/H_1).$$

Hence, we have $[M[S_1]: S_1] = [M: H_1]$. And then, noting that $[M[S_1]: M]_l \cdot [M: H_1] = [M[S_1]: H_1]_l = [M[S_1]: S_1] \cdot [S_1 : H_1]_l$, we obtain

$$(3.2) \qquad [M[S_1]: M]_l = [S_1 : H_1]_l.$$

Now, by (3.1) and (3.2), it follows that $M\cdot S_1 = M[S_1]$. Since the $\mathfrak{G}(H/H_1)$-normal shade M can be choosen as large as we want, one will easily realize the validity of the fact $H[S_1] = H\cdot S_1 = \sum\oplus Hd_i$. Next, as $\mathfrak{G}_1(H_2)|H = \mathfrak{G}(H/H_2)$, it will be easy to see that $H_2[S_1] \subseteq J(\mathfrak{G}_1(H_2), \sum\oplus Hd_i) = \sum\oplus H_2d_i$. Hence, we have $H_2[S_1] = H_2\cdot S_1 = \sum\oplus H_2d_i$ and $H_2[S_1]\cap H = H_2$. Finally, recalling that $H_1 = H\cap S_1 = H_2\cap S_1$, we see that S_1 is linearly disjoint from H_2.

(ii) Without loss of generality, we may assume that $R_1 \supseteq \{e_{ij}\text{'s}\}$ and $\{d_{hk}\text{'s}\}$. And then, $R_1 = \cup_\lambda S_\lambda$ where S_λ runs over all the intermediate rings of $R_1/S[\{e_{ij}\text{'s}\}, \{d_{hk}\text{'s}\}]$ with $[S_\lambda: S]_l < \infty$. By (i), each $H[S_\lambda]$ is simple and S_λ is linearly disjoint from H. Hence, $H[R_1] = \cup_\lambda H[S_\lambda]$ is simple and R_1 is linearly disjoint from H.

(iii) Let F be an arbitrary finite subset of R. If we set $S_1 = S[\{d_{hk}\text{'s}\}, F]$, then $[H[S_1]: H]_l = [S_1 : H\cap S_1]_l < \infty$ by (i). Hence, R/H is locally finite.

Definition. Let S be a regular subring of R. R/S is said to be *hereditarily Galois* (abbr. *h-Galois*) if the following two conditions are fulfilled:

(a) For each regular intermediate ring S' of R/S with $[S':S]_l<\infty$, R/S' is Galois and $V_R(V_R(S'))$ is simple.

(b) For any regular intermediate ring R' of R/H with $[R':H]_l<\infty$, $R'=V_R(V_R(R'))$ and $[R':H]_l=[V:V_R(R')]_r$.

If a division ring R is Galois over S, R/S is h-Galois by [11. Lemma 2] and [7, Theorem 1]. Concerning simple rings, if R/S is $\mathfrak{G}$-locally Galois then Corollary 2.2, Corollary 2.3, Theorem 2.5 (ii) and Corollary 2.5 show that R/S is h-Galois.

Lemma 3.3. *If R/S is locally finite and h-Galois, then for each regular intermediate ring S' of R/S with $[S':S]_l<\infty$, $H[S']=V_R(V_R(S'))$ and $[V:V_R(S')]_r=[V_R(V_R(S')):H]_l<\infty$.*

Proof. Since R/H is locally finite by Lemma 3.2 (iii), $[H[S']:H]_l<\infty$. Next, $V_R(V_R(S'))/S'$ is locally finite and outer Galois. And so, $V_R(V_R(S'))\supseteq H[S']\supseteq S'$ yields the simplicity of $H[S']$, that is, $H[S']$ is a regular subring. Hence, by our assumption, it follows that $V_R(V_R(S'))=V_R(V_R(H[S']))=H[S']$ and $[V:V_R(S')]_r=[V_R(V_R(S')):H]_l<\infty$.

Corollary 3.1. *If R/S is locally finite and h-Galois then so is R/S' for each regular intermediate ring S' of R/S with $[S':S]_l<\infty$.*

Proof. We set $H'=V_R(V_R(S'))$. Then R/H is locally finite and $[V:V_R(S')]_r=[H':H]_l<\infty$ by Lemma 3.3. And so, for any regular intermediate ring R_1 of R/H' with $[R_1:H']_l<\infty$, we have $[R_1:H']_l\cdot[H':H]_l=[R_1:H]_l=[V:V_R(R_1)]_r=[V:V_R(S')]_r\cdot[V_R(S'):V_R(R_1)]_r$ by our assumption. Hence, combining this with $[V:V_R(S')]_r=[H':H]_l$, we obtain at once $[R_1:H']_l=[V_R(S'):V_R(R_1)]_r$. The rest of the proof will be almost evident.

Lemma 3.4. *Let R/S be locally finite and h-Galois. If $\mathfrak{G}^*$ is a regular Galois group of R/S, then $\mathfrak{G}^*R_r$ is dense in $Hom_{S_l}(R, R)$, and $\mathfrak{G}^*$ is dense in $\mathfrak{G}$.*

Proof. For an arbitrary finite subset F of R, we set $S_1=S[\{e_{ij}\text{'s}\}, \{d_{hk}\text{'s}\}, F]$. Then, $[S_1:S_1\cap H]_l=[H[S_1]:H]_l=[V:V_R(S_1)]_r$ by Lemma 3.2 (i) and Lemma 3.3. And so, $(\mathfrak{G}^*|S_1)R_r=\mathrm{Hom}_{S_l}(S_1, R)$ by Lemma 3.1, which proves the density of $\mathfrak{G}^*R_r$. The rest of the proof is an easy consequence of Corollary 1.1.

For an intermediate ring T of R/S, $\Gamma(T/S)$ will denote the totality of S-(ring) isomorphisms of T into R. Then, we have the next

Corollary 3.2. *Let R/S be locally finite and h-Galois. If R' is an intermediate ring of $R/S[\{e_{ij}\text{'s}\}]$, then $(R'\cap H)\,\Gamma(R'|S)\subseteq H$.*

Proof. For arbitrary $a\in H\cap R'$, we set $S'=S[\{e_{ij}\text{'s}\}, a](\subseteq R')$. Then, $(\mathfrak{G}|S')R_r=\mathrm{Hom}_{S_l}(S', R)$ by Lemma 3.4. Since R is S'-R-irreducible, by Corollary 1.1 we see that for each $\tau\in\Gamma(R'/S)$ there exists some $\sigma\in\mathfrak{G}$ such that $\tau|S'=\sigma|S'$. Hence, $a\tau=a\sigma\in H$.

Lemma 3.5. *Let R/S be locally finite and h-Galois. If R' is a regular intermediate ring of R/S such that $[H[R']:H]_l<\infty$, then $H[R']$ is outer Galois and locally finite over R', and $\mathfrak{G}(H[R']/R')\cong\mathfrak{G}(H/H\cap R')$ by the restriction map.*

Proof. There exists a simple intermediate ring S' of R'/S such that $H[R']=H[S']$ and $[S':S]_l<\infty$. Then, $H[R']$ $(=H[S'])$ coincides with the simple ring $H'=V_R(V_R(S'))$ by Lemma 3.3. Moreover, as H'/S' is outer Galois and $H'\supseteq R'\supseteq S'$, H'/R' is outer Galois and locally finite. And, noting that $\mathfrak{G}'=\mathfrak{G}(H'/S')$ is the totpological closure of $\mathfrak{G}(S')|H'$, we readily see that $H\mathfrak{G}'=H$. Now, the required isomorphism will be given by the restriction map ρ: $\mathfrak{G}(H'/R')\ni\sigma\rightarrow\sigma|H\in\mathfrak{G}(H/H\cap R')$. Here, one should remark that $\mathfrak{G}(H'/R')$ is compact and $\mathfrak{G}(H'/R')|H$ is dense in $\mathfrak{G}(H/H\cap R')$.

Theorem 3.1. (Transitivity Theorem). *Let R be locally finite and h-Galois over S, and R' an intermediate ring of R/S whose capacity coincides with that of R. If H' is an arbitrary intermediate ring of H/S such that H'/S is Galois, then $H'[R']$ is outer Galois and locally finite over R', and $\mathfrak{G}(H'[R']/R')\cong\mathfrak{G}(H'/H'\cap R)$ by the restriction map.*

Proof. To our end, it will suffice to prove our theorem for the case where $H'=H$(cf. [15, (m)]). And, without loss of generality, we may assume that $R'\supseteq\{e_{ij}\text{'s}\}$. Let $\mathfrak{G}'$ be the group of all the R'-(ring) automorphisms of $H[R']$. Then, $\mathfrak{G}'|H\subseteq\mathfrak{G}(H/H\cap R')$ by Corollary 3.2. We have $H[R']=\bigcup_\nu R_\nu$, where $R_\nu=H[\{e_{ij}\text{'s}\},F_\nu]$ and F_ν runs over all the finite subsets of R'. If we set $R'_\nu=R'\cap R_\nu(\supseteq\{e_{ij}\text{'s}\})$, then one will easily verify that $R_\nu=H[R'_\nu]$, $R'=\bigcup_\nu R'_\nu$ and $R'_\nu\cap H=R'\cap H$. Now, in virtue of Lemma 3.5, $\mathfrak{G}(R_\nu/R'_\nu)\cong\mathfrak{G}(H/H\cap R'_\nu)=\mathfrak{G}(H/R'\cap H)$ by the restriction map. Hence, for each $\tau\in\mathfrak{G}(H/R'\cap H)$ there exists a uniquely determined extension $\tau_\nu\in\mathfrak{G}(R_\nu/R'_\nu)$ of τ. Accordingly, one will easily see that if $R_\nu\supseteq R_\mu$ then $\mathfrak{G}(R_\nu/R'_\nu)|R_\mu=\mathfrak{G}(R_\nu/R'_\nu)|H[R'_\mu]=\mathfrak{G}(R_\mu/R'_\mu)$. By the light of this fact, we can define an extension $\bar{\tau}\in\mathfrak{G}'$ of τ by the rule $\bar{\tau}|R_\nu=\tau_\nu$. We have proved therefore that $\mathfrak{G}'|H=\mathfrak{G}(H/H\cap R')$. Moreover, by the way of the definition of $\bar{\tau}$, one will easily see that $\mathfrak{G}'|R_\nu(=\mathfrak{G}'|H[R'_\nu])=\mathfrak{G}(R_\nu/R'_\nu)$. Hence, it follows that $J(\mathfrak{G}',H[R'])=\bigcup_\nu J(\mathfrak{G}'|R_\nu,R_\nu)=\bigcup_\nu R'_\nu=R'$, which means that $H[R']/R'$ is outer Galois and $\mathfrak{G}'=\mathfrak{G}(H[R']/R')$, for $V_R(V_R(R'))\supseteq H[R']\supseteq R'$. Now, as it is easy to see that $\mathfrak{G}'$ is locally finite, $H[R']/R'$ is locally finite by [15, (a*)]. Finally, noting that $\mathfrak{G}'|H=\mathfrak{G}(H/H\cap R')$, our isomorphism will be evident.

Lemma 3.6. *Let R/S be locally finite and h-Galois, and S' a regular intermediate ring of R/S with$[S':S]_l<\infty$. If H^* is an arbitrary intermediate ring of $H/H\cap S'$ then $H^*[S']\cap H=H^*$.*

Proof. Let H_1 be an arbitrary intermediate ring of H/H' with $[H_1:H']_l<\infty$, where $H'=H\cap S'$. Then, $H_1=H'[a_1]$ with some a_1 by Lemma 1.8, and $S'[H_1]=S'[a_1]$, which is simple as an intermediate ring of $V_R(V_R(S'))/S'$. Here, noting that $\mathfrak{G}(S')|V_R(V_R(S'))$ and $\mathfrak{G}(S')|H$ are Galois groups of $V_R(V_R(S'))/S'$ and H/H' respectively, Lemma 1.8 yields $[S'[H_1]:S']_l=\#\{a_1\mathfrak{G}(S')\}=[H_1:H']_l$. Accordingly, $[S'[H_1]:H_1]_l\cdot[H_1:H']_l=[S'[H_1]:S']_l\cdot[S':H']_l$ yields at once $[S'[H_1]:H_1]_l=[S':H']_l$. In particular, if $H_1=S'[\{d_{hk}\text{'s}\}]\cap H$ then $S_1=S'[\{d_{hk}\text{'s}\}]=S'[H_1]$ evidently. Hence, we have $[S':H']_l=[S_1:H_1]_l=[S_1:S_1\cap H]_l=[H[S_1]:H]_l=[V:V_R(S_1)]_r=[V:V_R(S')]_r$ by Lemma 3.2 and Lemma 3.3. Now, let H_1 be again an arbitrary intermediate ring of H/H' with $[H_1:H']_l<\infty$. As $S'[H_1]$ is a regular subring left-finite over S, by the fact proved above, we have $[S'[H_1]:S'[H_1]\cap H]_l=[V:V_R(S'[H_1])]_r=[V:V_R(S')]_r=[S':H']_l=[S'[H_1]:H_1]_l$. Recalling here that $H_1\subseteq S'[H_1]\cap H$, we readily see that $H_1=S'[H_1]\cap H$. Evidently, there holds $H^*=\bigcup_\alpha H_\alpha$, where H_α runs over all the intermediate rings of H^*/H' with $[H_\alpha:H']_l<\infty$. Then $H^*[S']\cap H=(\bigcup_\alpha H_\alpha[S'])\cap H=\bigcup_\alpha(H_\alpha[S']\cap H)=\bigcup_\alpha H_\alpha=H^*$ by the fact cited just now.

Theorem 3.2. *Let R/S be locally finite and h-Galois, and R' an f-regular intermediate ring of R/S.*

(i) *For any intermediate ring H^* of $H/H\cap R'$, there holds $H^*[R']\cap H=H^*$.*

(ii) $[R':H\cap R']_l=[V:V_R(R')]_r$.

Proof. There exists an intermediate simple ring S' of R'/S such that $V_R(S')=V_R(R')$ and $[S':S]_l<\infty$. Evidently, $V_R(V_R(S'))/S'$ is outer Galois and $V_R(V_R(S'))\supseteq R'\supseteq S'$. Accordingly, each intermediate ring of R'/S' is a regular subring of R.

(i) We have $R'=\bigcup_\alpha S'_\alpha$, where S'_α runs over all the intermediate rings of R'/S' with $[S'_\alpha:S']_l<\infty$. Then, by Lemma 3.6, there holds $H^*[R']\cap H=(\bigcup_\alpha H^*[S'_\alpha])\cap H=\bigcup_\alpha(H^*[S'_\alpha]\cap H)=\bigcup_\alpha H^*=H^*$.

(ii) Let N be a $\mathfrak{G}(H/S)$-normal shade of $\{d_{hk}\text{'s}\}$. Since $H/H\cap R'$ is outer Galois and $R'[N]\cap H=((R'\cap H)[N])[R']\cap H=(R'\cap H)[N]$ by (i), we can easily see that $[R'[N]\cap H:H\cap R']_l=[(H\cap R')[N]:H\cap R']_l=[N:N\cap(H\cap R')]_l=[N:N\cap R']_l<\infty$. On the other hand, noting that $V_R(V_R(S'))\supseteq R'[N]\supseteq R'\supseteq S'$ and $\mathfrak{G}(R'[N]/R')|N$ is the Galois group of $N/N\cap R'$(cf. Lemma 3.5), we obtain $[R'[N]:R']_l=[N:N\cap R']_l$. And so, it follows that $[R'[N]:R']_l=[R'[N]\cap H:R'\cap H]_l<\infty$. Moreover, by Lemmas 3.5 and 3.2 (i), one will easily see that $R'[N]=(R'[N]\cap H)[S']$. Hence, we obtain $[R'[N]:R'[N]\cap H]_l=[(R'[N]\cap H)[S']:R'[N]\cap H]_l=[H[S']:H]_l=[V:V_R(S')]_r=[V:V_R(R')]_r$ by Lemma 3.2 (i) and Lemma 3.3. Now, noting that $[R'[N]:R']_l\cdot[R':R'\cap H]_l=[R'[N]:R'[N]\cap H]_l\cdot$

$[R'[N]\cap H: R'\cap H]_l$ and $[R'[N]: R']_l=[R'[N]\cap H: R'\cap H]_l<\infty$, we readily obtain $[R': R'\cap H]_l=[V: V_R(R')]_r$.

Lemma 3.7. *Let R/S be locally finite and h-Galois.*

(i) *If R_1 is an intermediate ring of $R/H[\{e_{ij}$'s$\}]$ with $[R_1: H]_l<\infty$ then $\mathfrak{G}(H/S)=\Gamma(R_1/S)|H$.*

(ii) *If H' is an intermediate ring of H/S, then R/H' is locally finite.*

(iii) *Let R be Galois over an intermediate ring H' of H/S. If $\mathfrak{H}$ is a regular Galois group of R/H' then $\mathfrak{H}$ is dense in $\mathfrak{G}(R/H')$. If moreover $\mathfrak{H}$ is closed then $\mathfrak{H}=\mathfrak{G}(R/H')$.*

Proof. (i) By Corollary 3.2, it will be easy to see that $\Gamma(R_1/S)|H\subseteq\mathfrak{G}(H/S)$. Now, we shall prove the converse inclusion. As $[V: V_R(R_1)]_r<\infty$, there exists an intermediate ring S_1 of $R_1/S[\{e_{ij}$'s$\}, \{d_{hk}$'s$\}]$ such that $[S_1: S]_l<\infty$ and $V_R(R_1)=V_R(S_1)$. We set here $H_1=H\cap S_1$. Then, noting that $R_1=V_R(V_R(R_1))=V_R(V_R(S_1))$, one can readily see that $\mathfrak{G}(R_1/S_1)|H=\mathfrak{G}(H/H_1)$. On the other hand, $\mathfrak{G}|H$ is dense in $\mathfrak{G}(H/S)$. Accordingly, for each $\sigma\in\mathfrak{G}(H/S)$ there exists some $\tau\in\mathfrak{G}$ such that $\sigma|H_1=\tau|H_1$. As evidently $\sigma\tau^{-1}\in\mathfrak{G}(H/H_1)=\mathfrak{G}(R_1/S_1)|H$, we obtain eventually $\sigma\in\Gamma(R_1/S)|H$.

(ii) Let R_1 be an arbitrary intermediate ring of $R/H[\{e_{ij}$'s$\}]$ with $[R_1: H]_l<\infty$. Then, under the same notations as in the proof of (i), we set $H^*=H'[H_1]$. As H/H' is locally finite, $[H^*: H']_l<\infty$ evidently. By Lemma 3.2 (i), there hold $H^*[S_1]=H^*\cdot S_1$ and $[H^*\cdot S_1: H^*]_l=[S_1: H_1]_l<\infty$, whence it follows $[H^*[S_1]: H']_l<\infty$. On the other hand, R_1/S_1 being outer Galois and locally finite, $R_1\supseteq H^*[S_1]\supseteq S_1$ implies the local finiteness of $R_1/H^*[S_1]$, whence R_1/H' is locally finite. Recalling here that by Lemma 3.2 (iii) R_1 can be chosen as large as we want, we readily see that R/H' is locally finite.

(iii) Let F be an arbitrary finite subset of R, and $S_1=S[\{e_{ij}$'s$\}, \{d_{hk}$'s$\}, F]$. We set here $H_1=S_1\cap H$ and $H^*=H'[H_1]$. Then, by Lemma 3.2 (i), $[S_1: H_1]_l=[H^*[S_1]: H^*]_l=[H[S_1]: H]_l=[V: V_R(H[S_1])]_r$, $H^*=H^*[S_1]\cap H$ and $V_R(H')=V$. Hence, we have $[H^*[S_1]: H^*[S_1]\cap V_R(V_R(H'))]_l=[V_R(H'): V_R(H^*[S_1])]_r$. On the other hand, as $[H^*: H']_l<\infty$ by (ii), $[H^*[S_1]: H']_l<\infty$. Accordingly, $\mathfrak{H}|H^*[S_1]=\mathfrak{G}(R/H')|H^*[S_1]$ by Lemma 3.1 and $H^*[S_1]\supseteq H'[F]$, which proves the density of $\mathfrak{H}$ in $\mathfrak{G}(R/H')$.

Corollary 3.3. *Let R/S be locally finite and h-Galois.*

(i) *If R' is an f-regular intermediate ring of R/S then R/R' is locally finite.*

(ii) *If $\mathfrak{H}$ is an f-regular subgroup of $\mathfrak{G}$ then $\mathfrak{H}$ is dense in $\mathfrak{G}(R/J(\mathfrak{H}, R))$. If moreover $\mathfrak{H}$ is closed then $\mathfrak{H}=\mathfrak{G}(R/J(\mathfrak{H}, R))$.*

Proof. (i) There exists a simple intermediate ring S' of R'/S such that $V_R(R')=V_R(S')$ and $[S':S]_l<\infty$. Since R/S' is locally finite and h-Galois by Corollary 3.1 and $V_R(V_R(S'))\supseteq R'\supseteq S'$, R/R' is locally finite by Lemma 3.7 (ii).

(ii) In the proof of (i), if we set $R'=J(\mathfrak{H}, R)$ then our assertion is a direct consequence of Lemma 3.7 (iii).

Lemma 3.8. *Let R/S be locally finite and h-Galois. If $R_1\supseteq R_2$ are intermediate rings of $R/H[\{e_{ij}\text{'s}\}]$ with $[R_1:H]_l<\infty$, then $\Gamma(R_2/S)=\Gamma(R_1/S)|R_2$.*

Proof. There exists an in intermediate ring S_2 of $R_2/S[\{e_{ij}\text{'s}\}]$ such that $[S_2:S]_l<\infty$ and $V_R(R_2)=V_R(S_2)$. Then, R/S_2 is locally finite and h-Galois by Corollary 3.1. Since $V_R(V_R(S_2))=R_2\subseteq R_1$, $\mathfrak{G}(R_2/S_2)=\Gamma(R_1/S_2)|R_2$ by Lemma 3.7 (i). On the other hand, $\Gamma(R_2/S)|S_2\subseteq(\mathfrak{G}|S_2)R_r$ by Lemma 3.4. And so, by Corollary 1.1, for each $\sigma\in\Gamma(R_2/S)$ there exists some $\tau\in\mathfrak{G}$ such that $\sigma|S_2=\tau|S_2$. We obtain therefore $\sigma\tau^{-1}\in\Gamma(R_2/S_2)=\mathfrak{H}(R_2/S_2)=\Gamma(R_1/S_2)|R_2$ by Lemma 3.7 (i). Hence we have $\sigma\in\Gamma(R_1/S)|R_2$.

Lemma 3.9. *Let R be locally finite and h-Galois over S, and $[R:H]_l\leqq\aleph_0$. If T is an intermediate simple ring of R/H of capacity $n=[R|R]$ such that $[T:H]_l<\infty$ then $\Gamma(T/S)=\mathfrak{G}|T$.*

Proof. We may assume $[R:H]_l=\aleph_0$. Let $\{d_1, d_2, \cdots\}$ be a countable linearly independent H-left basis of R, and set $R_k=H[d_1,\cdots, d_k]\,(k=1, 2, \cdots)$. Now, let τ be an arbitrary element of $\Gamma(T/S)$. Then, as $H\tau=H$ by Corollary 3.2, there holds $\infty>[T\tau:H\tau]_l=[T\tau:H]_l$. And so, there exists some h_1 such that $T\tau\subseteq R_{h_1}$. Recalling here that $[T\tau|T\tau]=[T|T]=[R|R]$, by Lemma 3.8 there exists some $\sigma\in\Gamma(R_{h_1}/S)$ such that $\tau^{-1}=\sigma|T\tau$. Now, repeating the same argument for σ instead of τ, we can find a positive integer $k_1>h_1$ and $\tau_1\in\Gamma(R_{k_1}/S)$ such that $R_{h_1}\sigma\subseteq R_{k_1}$ and $\sigma^{-1}=\tau_1|R_{h_1}\sigma$. Here, one will easily see that $\tau=\tau_1|T$ and $T\tau\subseteq R_{h_1}\subseteq R_{k_1}\tau_1$. Repeating the above arguement for τ_1 instead of τ, we can find positive integers $k_2>h_2>k_1$ and $\tau_2\in\Gamma(R_{k_2}/S)$ such that $R_{k_1}\tau_1\subseteq R_{h_2}\subseteq R_{k_2}\tau_2$ and $\tau_1=\tau_2|R_{k_1}$. Continuing the same procedures, we can find inductively positive integers k_i, h_i and $\tau_i\in\Gamma(R_{k_i}/S)$ such that

$$k_{i-1}<h_i<k_i,$$
$$R_{k_{i-1}}\tau_{i-1}\subseteq R_{h_i}\subseteq R_{k_i}\tau_i, \quad \text{and}$$
$$\tau_{i-1}=\tau_i|R_{k_{i-1}} \quad (i=2, 3, \cdots).$$

Now, we can define an extension $\bar{\tau}\in\Gamma(R/S)$ by the rule

$$\bar{\tau}|R_{k_i}=\tau_i \qquad (i=1, 2, \cdots).$$

Since $R\bar{\tau}\supseteq\bigcup R_{h_i}=R$, $\bar{\tau}$ is evidently an automorphism of R, that is, $\bar{\tau}\in\mathfrak{G}$.

Corollary 3.4. *If R is locally finite and h-Galois over S, and $[R:H]_l\leqq\aleph_0$, then $\mathfrak{G}(H/S)=\mathfrak{G}|H$.*

Proof. Set $T=H[\{e_{ij}\text{'s}\}]$. Then, $\mathfrak{G}(H/S)=\Gamma(T/S)|H=\mathfrak{G}|H$ by Lemma 3.7 and Lemma 3.9.

The next contains evidently [12, Theorem 4].

Corollary 3.5. *If a division ring R is locally finite and quasi-Galois over S,*[6] *and $[R:H]_l\leqq\aleph_0$, then R/S is Galois, whence it is h-Galois.*

Proof. By [12, Theorem 2], R/H is quasi-Galois. And so, noting the validity of [12, Lemma 1], we can apply the same arguement as in the proof of Lemma 3.9 to see that $\Gamma(T/S)=\mathfrak{G}|T$ for every intermediate ring T of R/H with $[T:H]_l<\infty$. Now, as H/S is Galois and $\Gamma(T/S)|H=(\Gamma(H/S)=)\mathfrak{G}(H/S)$, we obtain $\mathfrak{G}|H=\mathfrak{G}(H/S)$. Hence, we readily see that $J(\mathfrak{G}, R)=S$.

Theorem 3.3. *Let R be locally finite and h-Galois over S, and $[R:H]_l\leqq\aleph_0$ If R' is an f-regular intermediate ring of R/S then R/R' is locally finite and h-Galois.*

Proof. At first, R/R' is locally finite by Corollary 3.3 (i). Next, we shall prove that R/R' is Galois. There exists a simple intermediate ring S' of R'/S such that $V_R(R')=V_R(S')$ and $[S':S]_l<\infty$. Since R/S' is locally finite and h-Galois by Corollary 3.1 and $V_R(V_R(S'))\supseteqq R'\supseteqq S'$, we may assume, from the beginning, that $H\supseteqq R'\supseteqq S$. And then, $R'=J(\mathfrak{G}(H/S)(R'), H)=J(\mathfrak{G}(R')|H, H)=J(\mathfrak{G}(R'), R)$ by Corollary 3.4. If R'' is an arbitrary regular intermediate ring of R/R' with $[R'':R']_l<\infty$, then $V=V_R(R')$ secures the existence of such a simple intermediate ring S'' of R''/S that $V_R(R'')=V_R(S'')$ and $[S'':S]_l<\infty$. And then $[V:V_R(R'')]_r=[V:V_R(S'')]_r<\infty$ by Lemma 3.3, which means R'' is f-regular. Accordingly, the rest of the proof will be easily seen.

Now, combining Theorem 3.3 with Corollary 3.3 (ii), we readily obtain the following fundamental theorem.

Theorem 3.4 (Fundamental Theorem). *If R is locally finite and h-Galois over S, and $[R:H]_l\leqq\aleph_0$, then there exists a* 1-1 *dual correspondence between closed f-regular subgroups of $\mathfrak{G}$ and f-regular intermediate rings of R/S, in the usual sense of Galois theory.*

Our fundamental theorem yields at once the following principal theorem in [18].

Corollary 3.6 (Walter). *If a division ring R is Galois and locally finite over S, and $[R:H]_l\leqq\aleph_0$, then there exists a* 1—1 *dual correspondence between closed f-regular subgroup of $\mathfrak{G}$ and f-regular intermediate rings of R/S, in the usual sense of Galois theory.*

Lemma 3.10. *Let R be locally finite and h-Galois over S, and S' a regular intermediate ring of R/S with $[S':S]_l<\infty$. R is S'-R-homogen-*

6) See [12, p. 67].

eously completely reducible and the length of its composition series coincides with the capacity of $V_R(S')$. In particular, R is S'-R-irreducible if and only if $V_R(S')$ is a division ring.

Proof. Since R/S' is locally finite and h-Galois by Corollary 3.1, it suffices to prove our lemma for the case $S'=S$. Now, let $S=\sum S_0 f_{ij}$ where $S_0=V_S(\{f_{ij}\text{'s}\})$ is a division ring. Then, $R=\sum R_0 f_{ij}$ with $R_0=V_R(\{f_{ij}\text{'s}\})$ and, to be easily seen, R_0/S_0 is locally finite and h-Galois. By Lemma 3.4, $\mathfrak{G}(R_0/S_0)R_{0r}$ is dense in $\mathrm{Hom}_{S_{0l}}(R_0, R_0)$, whence R_0 is $(\mathfrak{G}|R_0)R_{0r}$-irreducible. (Here, it will be evident that $\mathfrak{G}(R_0/S_0)=\mathfrak{G}|R_0$.) Accordingly, it will be clear that R is $S_l\cdot\mathfrak{G}R_r$-irreducible. Hence, for an arbitrary S-R-irreducible submodule N of R there holds $R=\sum_{\sigma\in\mathfrak{G}} N\sigma$. As evidently $N\sigma$ is S-R-irreducible, $R=\sum_{i=1}^{s}\oplus N\sigma_i(\sigma_i\in\mathfrak{G},\ s\leqq[R|R])$ is completely reducible. Moreover, noting that $V_{\mathrm{Hom}(R,R)}(S_l\cdot R_r)=V_l$ is simple, R is homogeneously completely reducible. And the rest of the proof is trivial.

Lemma 3.11. *Let R/S be locally finite and h-Galois, R_1 an intermediate ring of R/S with $[R_1:S]_l<\infty$, and R_2 a regular intermediate ring of R/S such that $[R_2:S]_l<\infty$ and $V_R(R_2)$ is a division ring. If ρ is an S-(ring) homomorphism of R_1 onto R_2 then ρ is contained in $\mathfrak{G}|R_1$.*

Proof. By Lemma 3.4, $\mathfrak{G}R_r$ is dense in $\mathrm{Hom}_{S_l}(R, R)$. And so, there holds $\mathrm{Hom}_{S_l}(R_1, R)=(\mathfrak{G}|R_1)R_r=\sum_1^s(\sigma_i|R_1)R_r$ with some σ_i's in $\mathfrak{G}$. Now, ρ can be represented as a linear combination of these $(\sigma_i|R)$'s with coefficients in R_r. Without loss of generality, we may assume here that

$$\rho=\sum_1^t(\sigma_i|R_1)a_{ir}\quad(a_i\neq 0\in R)$$

is a representation of the shortest length. As R is R_2-R-irreducible by Lemma 3.10, if $u\in R$ is non-zero then there holds $R_{1r}\rho u_r R_r=\rho(R_2uR)_r=\rho R_r\ni\rho$. Thus, we see that

$$\text{(3.3)}\qquad \rho u_r\notin\sum_2^t(\sigma_i|R_1)R_r\qquad\text{if } u\neq 0\in R.$$

In particular, we readily see that a_1 is a regular element. Accordingly, we have

$$\rho a_{1\ r}^{-1}=(\sigma_1|R_1)+\sum_2^t(\sigma_i|R_1)(a_i a_1^{-1})_r.$$

Now, for an arbitrary $x\in R_1$, there holds

$$\rho(x\rho\cdot a_1^{-1}-a_1^{-1}\cdot x\sigma_1)_r=x_r\rho a_{1\ r}^{-1}-\rho a_{1\ r}^{-1}(x\sigma_1)_r$$
$$=\sum_2^t(\sigma_i|R_1)\{x\sigma_i\cdot a_i a_1^{-1}-a_i a_1^{-1}\cdot x\sigma_1\}_r.$$

And so, by (3.3) it follows that $x\rho\cdot a_1^{-1}-a_1^{-1}x\sigma_1=0$, which means $\rho a_{1\ r}^{-1}=(\sigma_1|R_1)a_{1\ l}^{-1}$. We have proved therefore $\rho=\sigma_1\widetilde{a_1^{-1}}|R_1$, where a_1 is evidently contained in V. Hence, $\rho\in\mathfrak{G}|R_1$.

Corollary 3.7. *Let R/S be locally finite and h-Galois, S_1 and S_2 regular intermediate rings of R/S left-finite over S. If σ is an S-(ring) isomorphism of S_1 onto S_2 then σ is contained in $\mathfrak{G}|S_1$.*

Proof. Let $V_1 = V_R(S_1) = \sum_1^{m_1} U_1 g_{p'q'}^{(1)}$ and $V_2 = V_R(S_2) = \sum_1^{m_2} U_2 g_{p''q''}^{(2)}$ be respective matrix representations of V_1 and V_2 over division rings U_1 and U_2. Here, without loss of generality, we may assume $m_1 \geqq m_2$. (In case $m_1 < m_2$, treat σ^{-1} instead of σ.) We consider here the following two rings:

$$R_1 = \sum_1^{m_2} S_1 g_{p'q'}^{(1)} + S_1 g, \quad \text{where } g = \sum_{m_2+1}^{m_1} g_{p'q'}^{(1)},$$
$$R_2 = \sum_1^{m_2} S_2 g_{p'q'}^{(2)}.$$

Evidently, R_1 is an intermediate ring of R/S_1 with $[R_1:S]_l < \infty$, R_2 an intermediate simple ring of R/S with $[R_2:S]_l < \infty$, and $V_R(R_2) = U_2$ a division ring. As $\{g_{p'q'}^{(1)}\text{'s}\}$ and $\{g_{p''q''}^{(2)}\text{'s}\}$ are linearly independent over S_1 and S_2 respectively, we can define a map ρ of R_1 onto R_2 by

$$\begin{cases} (S_1 g)\rho = 0, \\ (\sum_1^{m_2} s_{p'q'}^{(1)} g_{p'q'}^{(1)})\rho = \sum_1^{m_2} (s_{p'q'}^{(1)}\sigma) g_{p'q'}^{(2)}, \quad s_{p'q'}^{(1)} \in S_1. \end{cases}$$

Then, one can easily verify that ρ is a ring homomorphism and $\sigma = \rho | S_1$. As $\rho \in \mathfrak{G} | R_1$ by Lemma 3.11, $\sigma \in \mathfrak{G} | S_1$ of course.

Corollary 3.8. *Let R be locally finite and h-Galois over S, and $[R:H]_l \leqq \aleph_0$. If R_1 is an f-regular intermediate ring of R/S whose capacity coincides with that of R, and σ an S-(ring) isomorphism of R_1 into R, then σ is contained in $\mathfrak{G} | R_1$.*

Proof. Without loss of generality, we may assume that $R_1 \supseteq \{e_{ij}\text{'s}\}$. There exists an intermediate ring S_1 of $R_1/S[\{e_{ij}\text{'s}\}]$ such that $V_R(R_1) = V_R(S_1)$ and $[S_1:S]_l < \infty$. As $[S_1\sigma | S_1\sigma] = [S_1 | S_1] = [R | R]$, $S_1\sigma$ is a regular intermediate ring of R/S with $[S_1\sigma : S]_l < \infty$. And so, in virtue of Corollary 3.7, there exists some $\tau \in \mathfrak{G}$ such that $\sigma | S_1 = \tau | S_1$, whence $\sigma\tau^{-1} \in \Gamma(R_1/S_1)$. Since R/S_1 is locally finite and h-Galois by Corollary 3.1 and $H_1 = V_R(V_R(S_1)) \supseteq R_1 \supseteq S_1$, $R_1\sigma\tau^{-1} \subseteq H_1$ by Corollary 3.2. Accordingly, we see that $\sigma\tau^{-1} \in \mathfrak{G}(H_1/S_1) | R_1 = (\mathfrak{G}(R/S_1) | H_1) | R_1 = \mathfrak{G}(R/S_1) | R_1$ by Corollary 3.4. Now, it will be easy to see that $\sigma \in \mathfrak{G} | R_1$.

Theorem 3.5. (Extension Theorem). *Let R be locally finite and h-Galois over S, and $[R:H]_l \leqq \aleph_0$. And let R_1 and R_2 be f-regular intermediate rings of R/S. If σ is an S-(ring) isomorphism of R_1 onto R_2 then σ is contained in $\mathfrak{G} | R_1$.*

Proof. There exists an intermediate ring S_0 of R_1/S such that $[S_0:S]_l < \infty$ and $V_R(R_1) = V_R(S_0)$. Since $[S_0\sigma : S]_l < \infty$, there exists a simple intermediate ring S_2 of $R_2/S_0\sigma$ such that $V_R(R_2) = V_R(S_2)$ and $[S_2:S]_l < \infty$. And then, we set $S_1 = S_2\sigma^{-1}$. Evidently, S_1 is an intermediate simple ring of R_1/S_0 and $V_R(R_1) = V_R(S_1)$. By Corollary 3.7, there exists some $\tau \in \mathfrak{G}$ such that $\sigma | S_1 = \tau | S_1$, whence $\sigma\tau^{-1} \in \Gamma(R_1/S_1)$. Since $V_R(V_R(S_1)) \supseteq R_1 \supseteq S_1$, $R_1\sigma\tau^{-1} \subseteq V_R(V_R(R_1\sigma\tau^{-1})) = V_R(V_R(R_2))\tau^{-1} = V_R(V_R(S_2))\tau^{-1} = V_R(V_R(S_1\sigma\tau^{-1})) =$

$V_R(V_R(S_1))$, and R is locally finite and h-Galois over S_1 by Corollary 3.1, we obtain $\sigma_1^{-1} \in \mathfrak{G}(V_R(V_R(S_1))/S_1) | R_1 = \mathfrak{G}(R/S_1) | R_1$ by Corollary 3.4. Now, it will be easy to see that $\sigma \in \mathfrak{G} | R_1$.

Theorem 3.6. (Normality Theorem). *Let R be locally finite and h-Galois over S, and $[R:H]_l \leqq \aleph_0$. And let T be an f-regular intermediate ring of R/S. If $\mathfrak{T}^*$ is the composite of $\mathfrak{T} = \{\sigma \in \mathfrak{G};\ T\sigma = T\}$ and $\widetilde{V}_{\mathfrak{T}}$, and $\mathfrak{H}$ the group of all S-automorphisms of T, then $J(\mathfrak{H}, T) = S$ if and only if $\mathfrak{T}^*$ is dense in $\mathfrak{G}$.*

Proof. At first, we suppose $J(\mathfrak{H}, T) = S$. Then, as $\mathfrak{T} | T = \mathfrak{H}$ by Theorem 3.5 and R/T is Galois by Theorem 3.4, it follows that $J(\mathfrak{T}, R) = S$. Hence, $\mathfrak{T}^* = \mathfrak{T} \cdot \widetilde{V}$ is dense in $\mathfrak{G}$ by Lemma 3.4. Conversely, suppose $\mathfrak{T}^*$ is dense in $\mathfrak{G}$. Then, again recalling that R/T is Galois, it follows that $J(\mathfrak{T} | T, T) = J(\mathfrak{T}^*, R) = J(\overline{\mathfrak{T}^*}, R) = S$.

As an easy consequence of Theorem 3.4, under the assumption that R is locally finite and h-Galois over S and $[R: H]_l \leqq \aleph_0$, an f-regular intermediate ring T of R/S is $\mathfrak{G}$-normal if and only if $\mathfrak{G}(T)$ is an invariant subgroup of $\mathfrak{G}$. Moreover, we obtain the next:

Corollary 3.9. *Let R be locally finite and h-Galois over S, $[R:H]_l \leqq \aleph_0$, and T an f-regular intermediate ring of R/S that is w-Galois over S. If either $T \subseteq H$ or $V \subseteq T$ then T is $\mathfrak{G}$-normal.*

Proof. In case T is contained in H, T/S being outer Galois, for each $t \in T$ there exists a $\mathfrak{G}(T/S)$-normal shade N of $\{t\}$. By Theorem 3.6, N is $\mathfrak{G}(H/S)$-normal, whence $\mathfrak{G}$-normal. On the other hand, in case $V \subseteq T$, we set $\mathfrak{T} = \{\alpha \in \mathfrak{G};\ T\alpha = T\}$. Then, $\mathfrak{T}$ contains $\widetilde{V}$, and so $\mathfrak{T}$ is dense in $\mathfrak{G}$ by Theorem 3.6. It follows therefore that $\mathfrak{T} | S[t] = \mathfrak{G} | S[t]$ for every $t \in T$. Hence, $t\mathfrak{G} = t\mathfrak{T} \subseteq T$, which proves evidently T is $\mathfrak{G}$-normal.

We shall conclude this section with the following theorem.

Theorem 3.7. *Let R be locally finite and h-Galois over S, and $[R:H]_l \leqq \aleph_0$. If an f-regular intermediate ring T of R/S with simple $V_T(S)$ is $\mathfrak{G}$-normal then $\mathfrak{G}(T/S)$ is (algebraically and topologically) isomorphic to $\mathfrak{G}/\mathfrak{G}(T)$.*

Proof. Since $\mathfrak{G} | T = \mathfrak{G}(T/S)$ by Theorem 3.5, the restriction map $\rho : \sigma \rightarrow \sigma | T$ is a continuous homomorphism of $\mathfrak{G}$ onto $\mathfrak{G}(T/S)$. In what follows, we shall prove that ρ is an open map. Now, let S' be a simple intermediate ring of T/S such that $[S':S]_l < \infty$ and $V_R(T) = V_R(S')$, and set $H' = V_R(V_R(S'))$. For an arbitrary finite subset F of R we set $S'' = S'[\{e_{ij}\text{'s}\}, F]$, which is evidently a regular intermediate ring of R/S with $[S'':S]_l < \infty$. Since R/S' is locally finite and h-Galois by Corollary 3.1, $[R: H'] \leqq \aleph_0$, $H' \supseteq T \supseteq S'$ and T/S' is Galois, we obtain $\mathfrak{G}(T[S'']/S'') | T$

$=\mathfrak{G}(T/T\cap S'')$ by Theorem 3.1. And, again by Theorem 3.5, $\mathfrak{G}(T[S'']/S'')=\mathfrak{G}(S'')|T[S'']$. Hence, it follows that $\mathfrak{G}(T/T\cap S'')=\mathfrak{G}(S'')|T\subseteq\mathfrak{G}(F)|T$. As $\mathfrak{G}(T/T\cap S'')$ is open in $\mathfrak{G}(T/S)$, the last fact proves that ρ is open. Now, our assertion is a direct consequence of this fact.

4. Further results for $\mathfrak{G}$-locally Galois extensions. At first, we shall prove the next

Theorem 4.1. *Let R/S be $\mathfrak{G}$-locally Galois. If $\mathfrak{H}$ is a $(*_f)$-regular subgroup of $\mathfrak{G}$ then $R_0=J(\mathfrak{H}, R)$ is f-regular, R/R_0 is $\mathfrak{G}(R_0)$-locally Galois, and $\mathfrak{G}(R_0)$ coincides with the closure $\bar{\mathfrak{H}}$ of $\mathfrak{H}$.*

Proof. At first, $\mathfrak{H}$ is f-regular by Theorem 2.8. There exists a finite subset F of R_0 such that $V_R(S[F])=V_R(R_0)=V_{\mathfrak{H}}$. Now for an arbitrary finite subset $F'\subseteq R$, choose a $\mathfrak{G}$-shade S' of $F\cup F'\cup\{g_{pq}\text{'s}\}\cup\{g'_{p'q'}\text{'s}\}\cup\{e_{ij}\text{'s}\}$, where $g'_{p'q'}$'s are matrix units of $V_{\mathfrak{H}}$. Then, $R'=V_R(V_R(S'))\supseteq V_R(V_R(S[F, F']))=V_R(V_R(R_0[F']))\supseteq R_0[F']$. And moreover, R' is simple and $[R':H]_l<\infty$ by Corollary 2.3. It follows therefore that $[V_{R'}(S):V_{R'}(R')]<\infty$ by Lemma 1.5. Since R' is Galois and locally finite over S and R_0 is a regular subring of R', R'/R_0 is $\mathfrak{G}(R'/R_0)$-locally Galois by Conclusion 2.1. And so, there exists a $\mathfrak{G}(R'/R_0)$-shade of F'. Thus, we have proved that R/R_0 is locally Galois. Recalling here that R/R_0 is Galois, it is $\mathfrak{G}(R_0)$-locally Galois by Theorem 2.3. Now, the rest of the proof is clear by Corollary 3.3 (ii).

Corollary 4.1. *Let R/S be $\mathfrak{G}$-locally Galois, and $[R:H]_l\leqq\aleph_0$. If R' is an f-regular intermediate ring of R/S then R/R' is $\mathfrak{G}(R')$-locally Galois.*

Proof. Since R/S is h-Galois, R/R' is Galois by Theorem 3.3. Accordingly, R/R' is $\mathfrak{G}(R')$-locally Galois by Theorem 4.1.

Now, combining Theorem 3.4 with Theorem 2.8, we obtain

Theorem 4.2. *If R/S is $\mathfrak{G}$-locally Galois and $[R:H]_l\leqq\aleph_0$ then there exists a 1—1 dual correspondence between closed $(*_f)$-regular subgroups of $\mathfrak{G}$ and f-regular intermediate rings of R/S, in the usual sense of Galois theory.*

Corollary 4.2. *If R is Galois and locally finite over S, and $\mathfrak{G}$ is locally compact, then there exists a 1—1 dual correspondence between closed $(*)$-regular subgroups of $\mathfrak{G}$ and regular intermediate rings of R/S.*

Lemma 4.1. *Let R/S be locally Galois. If T is an intermediate ring of R/S with the capacity $n=[R|R]$, then $(H\cap T)\sigma\subseteq H$ for each $\sigma\in\Gamma(T/S)$. In particular, if an intermediate ring N of $H\cap T/S$ is Galois over S, then $N\sigma=N$.*

Proof. There exists an intermediate ring S' of T/S with the capacity

n and $[S':S]_l<\infty$. Let a be an arbitrary element of $H\cap T$. Then, for each $\sigma\in\Gamma(T/S)$ there holds $[S'[a]\sigma:S]_l<\infty$ and $[S'[a]\sigma|S'[a]\sigma]=[S'[a]|S'[a]]=n$. And so, if S'' is an arbitrary shade of $S'[a]\cup S'[a]\sigma$, then $\sigma|S'[a]=\tau|S'[a]$ for some $\tau\in\mathfrak{G}(S''/S)$. As $(H\cap S'')\tau\subseteq H$ by [8, Lemma 4.1], we obtain $a\sigma\in H$, which is our first assertion. Next, concerning the second assertion, it will suffice to remark that H is simple and (outer) Galois over S by Lemma 2.4 and N is $\mathfrak{G}(H/S)$-normal by Theorem 3.6.

Lemma 4.2. *Let R/S be locally Galois. For each finite subset F of R, there exists an intermediate smple ring R' of $R/H[F]$ with $[R':H]_l<\infty$ such that R'/S is Galois (and $[V_{R'}(S):V_{R'}(R')]<\infty$).*

Proof. Let S' be a shade of $S[F, \{e_{ij}\text{'s}\}, \{g_{pq}\text{'s}\}]$, and set $R'=H[S']$. Now, let N be an arbitrary $\mathfrak{G}(H/S)$-normal shade of $H\cap S'$. And then, we shall prove that $N'=N[S']$ is Galois (and finite) over S. Choose here a shade N'' of N'. If we set $\mathfrak{G}''=\{\tau\in\mathfrak{G}(N''/S);\ S'\tau=S'\}$, then for each $\tau\in\mathfrak{G}''$ we have $N'\tau=N[S']\tau=(N\tau)[S']=N[S']=N'$ by Lemma 4.1. Accordingly, noting that N''/S' is Galois and $\mathfrak{G}(S'/S)\subseteq\mathfrak{G}(N''/S)|S'$, we readily see that $J(\mathfrak{G}''|N', N')=S$. And so, $V_{N'}(S)$ being simple, N'/S is Galois. Moreover, as N'/S' is outer Galois and $N\mathfrak{G}(N'/S)=N$ again by Lemma 4.1, it follows that $\mathfrak{G}(N'/S')\cong\mathfrak{G}(N/N\cap S')$, whence $[N':S']=[N:N\cap S']$. And so, $[N':S']\cdot[S':N\cap S']_l=[N':N]_l\cdot[N:N\cap S']$ yields at once $[N':N]_l=[S':N\cap S']_l=[S':H\cap S']_l$ Now, there holds $R'=\bigcup_\alpha N'_\alpha$, where $N'_\alpha=N_\alpha[S']$ and N_α runs over all the $\mathfrak{G}(H/S)$-normal shades of $H\cap S'$. Since arbitrary finite subset $\{a_1,\cdots,a_m\}$ of R' is contained in some N'_α, R'/S is evidently locally Galois. And, in particular if $\{a_1,\cdots,a_m\}$ is linearly left-independent over H then $[N'_\alpha:N_\alpha]_l=[S':H\cap S']_l$ yields $m\leqq[S':H\cap S']_l$, which proves $[R':H]_l\leqq[S':H\cap S']_l<\infty$. Accordingly, $[V_{R'}(S):V_{R'}(R')]=[V_{R'}(H):V_{R'}(R')]<\infty$ by Lemma 1.5. Hence, R'/S is Galois by [8, Theorem 4.1].

Now we can prove the next which contains Corollary 2.2 and Corollary 2.3.

Theorem 4.3. *Let R/S be locally Galois.*

(i) *R/H is $\mathfrak{G}(H)$-locally Galois.*

(ii) *If S' is a regular intermediate ring of R/S such that $[S':S]_l<\infty$ then $V_R(V_R(S'))=H[S']$ is simple and $[V_R(V_R(S')):H]_l=[V:V_R(S')]_r<\infty$.*

Proof. (i) Let F be an arbitrary finite subset of R. Then, by Lemma 4.2, there exists an intermediate simple ring R' of $R/H[F]$ with $[R':H]_l<\infty$ such that R'/S is Galois and $[V_{R'}(S):V_{R'}(R')]<\infty$. And so, R'/H is Galois by Conclusion 2.1, whence we see that R/H is locally Galois. Hence, R/H is $\mathfrak{G}(H)$-locally Galois by Theorem 2.3.

(ii) As R/S' is locally Galois, $H'=V_R(V_R(S'))$ is simple and outer Galois over S'. And so, noting that R/H is locally finite by (i), we see that $H[S']$ is a regular intermediate ring of R/H with $[H[S']:H]_l<\infty$. Since R/H is $\mathfrak{G}(H)$-locally Galois, as is noted in §3, R/H is (locally finite and) h-Galois. Hence, our assertion is clear by Lemma 3.3.

As another consequence of Lemma 4.2, we shall prove the following generalization of [12, Theorem 4].

Theorem 4.4. *If R/S is locally Galois and $[R:H]_l\leqq\aleph_0$ then R/S is Galois, whence $\mathfrak{G}$-locally Galois.* (Cf. Theorem 4.5 and Lemmas 4.5—4.7.)

Proof. Let $\{x_1, x_2, \cdots\}$ be a countable linearly independent H-left basis of R. Then, in virtue of Lemma 4.2, we can construct inductively an ascending chain of simple rings

$$R_1\subseteq R_2\subseteq R_3 \cdots\cdots$$

such that $R_1\supseteq H[\{e_{ij}\text{'s}\}, x_1]$, $R_{i+1}\supseteq R_i[x_{i+1}]$, $[R_i:H]_l<\infty$, $[V_{R_i}(S): V_{R_i}(R_i)]<\infty$ and that R_i/S is Galois. As $\mathfrak{G}(R_i/S)\subseteq\mathfrak{G}(R_{i+1}/S)|R_i$ by Conclusion 2.1, we can readily see that $\mathfrak{G}(H/S)=\mathfrak{G}|H$. Combining this with $H=J(\widetilde{V}, R)$ ([8, Lemma 4.3]), it will be easy to see that R/S is Galois.

Let R be Galois and finite over S, and T a $\widetilde{V}$-normal intermediate simple ring of R/S. And let $v\in V$ be regular and quasi-regular. If $\{1, v\}$ is linearly left-independent over T, then for arbitrary $t\in T$, $vt=t'v$ and $(v-1)t=t''(v-1)$ $(t', t''\in T)$ yield $(t'-t'')v+(t''-t)=0$, whence it follows that $t'=t''=t$. Hence, we obtain $v\in V_R(T)$. On the other hand, if $\{1, v\}$ is linearly dependent then it will be easy to see that $v\in T$, that is, $v\in V_T(S)$. If V is neither a division ring nor the complete 2×2 matrix ring over $GF(2)$ then we can prove that each element of V is represented as a finite sum of regular elements which are quasi-regular as well.[7] And so, for such V described above, we obtain $V=V_R(T)+V_T(S)$. Recalling here that $[V: V_S(R)]<\infty$, [1, Lemma 9] yields at once the following:

Lemma 4.3. *Let R be Galois and finite over S, and V neither a division ring nor the complete 2×2 matrix ring over $GF(2)$. If a regular intermediate ring T of R/S with simple $V_T(S)$ is $\widetilde{V}$-normal then $V\subseteq T$ or $T\subseteq H$.*

For the case where V is the complete 2×2 matrix ring over $GF(2)$, the following example shows that Lemma 4.3 is not true: If R is the complete 2×2 matrix ring over $GF(2)$ and $S=GF(2)$ then $V=R$. And then $T=\{1, 0, e_{11}+e_{12}+e_{21}, e_{22}+e_{12}+e_{21}\}$ is a commutative subfield of V and $\widetilde{V}$-normal.

Lemma 4.4. *Let R/S be locally Galois, T an f-regular intermediate*

7) This fact is due to Prof. G. Azumaya.

ring of R/S with simple $V_T(S)$, and V not the complete 2×2 matrix ring over $GF(2)$. If T is $\tilde{V}$-normal then $V\subseteq T$ or $T\subseteq H$.

Proof. In case V is a division ring, our assertion is clear by [8, Lemma 3.5]. And so, in what follows, we shall restrict our attention to the case where V is not a division ring. Assume here that $T\not\subseteq H$, that is, there exists an element $t\in T$ such that $tv\neq vt$ for some $v\in V$. Our present task is to prove that $V\subseteq T$. Since T is f-regular, there exists a finite subset E of T such that $V_R(T)=V_R(S[E])$. Let $e'_{i'j'}$'s, $g'_{p'q'}$'s and $g^*_{p''q''}$'s be matrix units of T, $V_R(T)$ and $V_T(S)$ respectively. Now for an arbitrary subset W of V we consider a shade N of $\{t, v\}\cup\{e'_{i'j'}\text{'s}\}\cup\{g'_{p'q'}\text{'s}\}\cup\{g^*_{p''q''}\text{'s}\}\cup E\cup W$, and set $M=T\cap N$. Then, to be easily seen, M, $V_M(S)=M\cap V_T(S)$ and $V_N(M)=N\cap V_R(M)$ are all simple, and M is $(\widetilde{V_N(S)}\,|\,N)$-normal by our assumption. Moreover, as $v\in V_N(S)$ and $t\in M$, $M\not\subseteq V_N(V_N(S))$. Finally, if W is sufficiently large, $V_N(S)\,(\supseteq W)$ is neither a division ring nor the complete 2×2 matrix ring over $GF(2)$. Hence, by Lemma 4.3, we obtain $W\subseteq V_N(S)\subseteq M\subseteq T$, which evidently proves our assertion.

Combining Lemma 4.4. with Corollary 3.9, we obtain at once

Theorem 4.5. *Let R/S be locally Galois, $[R:H]_l\leqq\aleph_0$, and T an f-regular intermediate ring of R/S with simple $V_T(S)$. Then T is $\mathfrak{G}$-normal if and only if T/S is Galois and either $T\subseteq H$ or $V\subseteq T$, provided V is not the complete 2×2 matrix ring over $GF(2)$.*

In what follows, T_i be an intermediate ring of R/S. We consider here the following conditions, where $D(T_i,\ R/S')$ denotes the set of all the derivations of T_i into R vanishing on S', and in particular, we write $D(R/S')$ for $D(R, R/S')$.

(a*) $D(T_1,\ R/T_1\cap T_2)=0$.

(a$_0^*$) $D(T_2,\ R/T_1\cap T_2)=0$.

(b*) Every compatible pair[8] $(\delta^{(1)},\ \delta^{(2)})$ $(\delta^{(i)}\in D(T_i,\ T/S))$ has an extension $\delta\in D(R/S)$.

(d$_1$) $V_R(T_1\cap T_2)=V_R(T_1)$.

(d$_2$) $V_R(T_1\cap T_2)=V_R(T_2)$.

Theorem 4.6. *Let R/S be locally Galois, and T an f-regular intermediate ring of R/S. If $\delta\in D(T,\ R/S)$ then δ can be extended to an inner derivation δ_a induced by an element a of V.*

Proof. At first, we shall show that if T is contained in H then $\delta=0$. Let $T=\bigcup_\alpha T_\alpha$, where T_α runs over all the simple subrings of T finite over S. Since δ may be regarded as an S-(left) linear transformation of T

8) Following [1], $(\delta^{(1)},\ \delta^{(2)})$ is said to be compatible if $\delta^{(1)}|T_1\cap T_2=\delta^{(2)}|T_1\cap T_2$.

into R, there exists a shade N_α of $T_\alpha \cup T_\alpha \delta$. And then, by [1, (xi)], $\delta | T_\alpha = \delta_b | T_\alpha$ for some $b \in V_N(S) \subseteq V = V_R(H)$, whence it follows that $\delta_b | T_\alpha = 0$. We obtain therefore that $\delta = 0$.

Now let T be not necessarily contained in H. Then, we can find an intermediate simple ring S' of T/S such that $V_R(S') = V_R(T)$ and $[S': S]_l < \infty$. Since R/S is locally Galois, by [1, (xi)], we can find an element $a \in V$ such that $\delta | S' = \delta_a | S'$. Moreover, R/S' is still locally Galois and $V_R(V_R(S')) \supseteq T \supseteq S'$. Hence, the fact mentioned above will yield at once $\delta - (\delta_a | T) = 0$, that is, $\delta = \delta_a | T$.

Now, we shall prove the next

Theorem 4.7. *Let R be Galois and locally finite over S, and $[V: C_0] < \infty$. If T_1 and T_2 are f-regular intermediate rings of R/S, then* (a*) *or* (a_0^*) $\Longleftrightarrow$ (b*) $\Longleftrightarrow$ (d_1) *or* (d_2).

Proof. (b*) $\Rightarrow$ (d_1) or (d_2). Let $\delta \in D(R/T_1 \cap T_2)$. Then, by (b*), there exists a derivation $\delta^* \in D(R/T_1)$ such that $\delta^* | T_2 = \delta | T_2$. Now, $\delta = \delta^* + (\delta - \delta^*)$ and $\delta - \delta^* \in D(R/T_2)$. Thus, $D(R/T_1 \cap T_2) = D(R/T_1) + D(R/T_2)$. We consider here an arbitrary element $c \in V_R(T_1 \cap T_2)$. Since $\delta_c \in D(R/T_1 \cap T_2)$, there holds $\delta_c = \delta' + \delta''$ with some $\delta' \in D(R/T_1)$ and $\delta'' \in D(R/T_2)$. By Theorem 4.6, $\delta' = \delta_a$ and $\delta'' = \delta_b$ with some $a \in V_R(T_1)$ and $b \in V_R(T_2)$. We obtain therefore $c = a + b + z$ for some $z \in C$, which proves that $V_R(T_1 \cap T_2) = V_R(T_1) + V_R(T_2)$. Recalling here again that T_1 and T_2 are f-regular, by Corollary 2.4 there exists a simple subring H^* containing T_1, T_2 and V such that $V_{H^*}(S)$ is finite over the center of H^*. And so, there holds (d_1) or (d_2) by [1, Lemma 9].

(d_1) or (d_2) $\Rightarrow$ (b*). Suppose $V_R(T_1 \cap T_2) = V_R(T_1)$ and $(\delta^{(1)}, \delta^{(2)})$ is compatible. By Theorem 4.6, there exists an element $a \in V$ such that $\delta^{(1)} = \delta_a | T_1$. Since $\delta^{(2)} | T_1 \cap T_2 = \delta^{(1)} | T_1 \cap T_2 = \delta_a | T_1 \cap T_2$, we obtain $\delta^{(2)} - (\delta_a | T_2) \in D(T_2, R/T_1 \cap T_2)$. Hence, again by Theorem 4.6, $\delta^{(2)} - \delta_a | T_2 = \delta_b | T_2$ with some $b \in V_R(T_1 \cap T_2) = V_R(T_1)$. Now, one will readily see that δ_{a+b} is our desired extension.

(a*) or (a_0^*) $\Longleftrightarrow$ (d_1) or (d_2). These implications are easy consequences of Theorem 4.6, too.

Lemma 4.5. *Let R/S be locally Galois, $[R: H]_l \leqq \aleph_0$, and let $S \subseteq U \subseteq T \subseteq R$, where T is an f-regular subring of R. If T/U is w-Galois and $T \subseteq V_R(V_R(U))$ then U is f-regular, T is outer Galois and locally finite over U, and T is $\mathfrak{G}(U)$-normal.*

Proof. At first, $U \subseteq T \subseteq V_R(V_R(U))$ implies $V_R(U) = V_R(T)$. Since each U-automorphism of T can be extended to an automorphism of R by Theorem 3.5, $U = J(\mathfrak{G}(U), R)$ and $\mathfrak{G}(U)$ is a $(*_f)$-regular subgroup of $\mathfrak{G}$. Hence, U is f-regular by Theorem 2.8. Further, $V_R(V_R(U))/U$ is outer

Galois and locally finite by Corollary 3.3. And so, noting that $V_R(V_R(U))$ is $\mathfrak{G}(U)$-normal, the rest of the proof is a direct consequence of Theorem 3.6.

Lemma 4.6. *Let R/S be locally Galois and $[R:H]_l \leq \aleph_0$. And let T, T' be f-regular intermediate rings of R/S such that T is w-Galois over $U=T\cap T'$. If $V_R(U)=V_R(T)$ then $\mathfrak{G}(T, R/U)=\mathfrak{G}(T')|T$, where $\mathfrak{G}(T, R/U)$ means the totality of U-isomorphisms of T onto f-regular intermediate rings of R/S.*

Proof. At first, by Theorem 3.5, there holds $\mathfrak{G}(T, R/U)=\mathfrak{G}(U)|T$. And so, to our end, it suffices to prove that $\mathfrak{G}(U)|T=\mathfrak{G}(T')|T$. Now, as $T\subseteq V_R(V_R(T))=V_R(V_R(U))$, by Lemma 4.5, we see that T is $\mathfrak{G}(U)$-normal, outer Galois and locally finite over the f-regular U. Since there holds $V_R(V_R(T'[T]))=V_R(V_R(T')\cap V_R(T))=V_R(V_R(T')\cap V_R(U))=V_R(V_R(T'))$, we have $T'\subseteq T'[T]\subseteq V_R(V_R(T'))$. And so, by Theorem 3.3, we readily see that the simple ring $T'[T]$ is outer Galois and locally finite over T', whence $\mathfrak{G}(T'[T]/T')$ is compact. Moreover, $\mathfrak{G}(T'[T]/T')=\mathfrak{G}(T')|T'[T]$ by Theorem 3.6 and $\mathfrak{G}(T')|T$ is a Galois group of T/U. Hence, as usual, we see that $\mathfrak{G}(T')|T=\mathfrak{G}(T/U)=\mathfrak{G}(U)|T$.

We consider here the following conditions where T_1 and T_2 are f-regular intermediate rings of R/S.

(a) $\mathfrak{G}(T_1)|T_2=\mathfrak{G}(T_2, R/T_1\cap T_2)$.

(a_0) $\mathfrak{G}(T_2)|T_1=\mathfrak{G}(T_1, R/T_1\cap T_2)$.

(b) Every compatible pair[9] (σ_1, σ_2) $(\sigma_i\in\mathfrak{G}(T_i, R/S))$ has an extension $\sigma\in\mathfrak{G}$.

(c) $\mathfrak{G}(T_1\cap T_2)=\mathfrak{G}(T_1)\cdot\mathfrak{G}(T_2)$ $(=\mathfrak{G}(T_2)\cdot\mathfrak{G}(T_1))$.

Lemma 4.7. *Let R/S be locally Galois and $[R:H]_l\leq\aleph_0$. Then, the four conditions* (a), (a_0), (b) *and* (c) *are equivalent to each other.*

Proof. (a)$\Rightarrow$(b). Let (σ_1, σ_2) be compatible. By Theorem 3.5, we can find some $\tau_1\in\mathfrak{G}$ such that $\tau_1|T_1=\sigma_1$. Thus, $\gamma=\sigma_2\tau_1^{-1}\in\mathfrak{G}(T_2, R/T_1\cap T_2)=\mathfrak{G}(T_1)|T_2$ by (a). And so, $\gamma=\tau|T_2$ for some $\tau\in\mathfrak{G}(T_1)$. Now, it is evident that $\tau\tau_1$ is our desired extension.

(b)$\Rightarrow$(c). If σ is in $\mathfrak{G}(T_1\cap T_2)$, then $\sigma|T_2=\tau|T_2$ for some $\tau\in\mathfrak{G}(T_1)$ by (b). It follows therefore $\sigma=(\sigma\tau^{-1})\tau$ and $\sigma\tau^{-1}\in\mathfrak{G}(T_2)$.

(c)$\Rightarrow$(a). By Theorem 3.5 and (c), we obtain $\mathfrak{G}(T_2, R/T_1\cap T_2)=\mathfrak{G}(T_1\cap T_2)|T_2=\mathfrak{G}(T_2)\cdot\mathfrak{G}(T_1)|T_2=\mathfrak{G}(T_1)|T_2$.

(a)$\Longleftrightarrow$(a_0). By the symmetry of the condition (b), this equivalence will be evident.

We shall conclude our study with the following theorem.

9) Following [1], we say that (σ_1, σ_2) is compatible if $\sigma_1|T_1\cap T_2=\sigma_2|T_1\cap T_2$.

Theorem 4.8. *Let R be Galois and locally finite over S,* $[V: C_0] < \infty$, *and* $[R: H]_l \leqq \aleph_0$. *If* T_1 *and* T_2 *are f-regular intermediate rings of* R/S *such that* $T_i/T_1 \cap T_2 (i=1, 2)$ *is w-Galois then* (b*) *implies* (b).

Proof. In virtue of Theorem 4.7, without loss of generality, we may assume that $V_R(T_1 \cap T_2) = V_R(T_1)$. Then, by Lemma 4.6, we have $\mathfrak{G}(T_2)|T_1 = \mathfrak{G}(T_1, R/T_1 \cap T_2)$, which is the condition (a_0). Hence, our assertion is a direct consequence of Lemma 4.7.

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Added in proof: Recently, we have seen that Lemmas 1.9 and 1.10 are valid even when R/S is (Galois and) algebraic and 2-algebraic respectively. Moreover, we can prove Theorem 2.1 without assuming the local finiteness of R/S, and Theorem 3.1 is still true for any regular intermediate ring R' of R/S (even under a somewhat weaker assumption). The proofs of these facts will be given in the forthcomming paper "*Some Theorems on Galois Theory of Simple Rings*", to appear in J. Fac. Sci. Hokkaido Univ., Ser. I.

ERRATA:

ON GALOIS THEORY OF SIMPLE RINGS

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Page 102, line 18.	For "$\mathfrak{G}(H'/H'\cap R)$" read "$\mathfrak{G}(H'/H'\cap R')$".
Page 106, line 6.	The proof contains a gap. This gap is filled in "On quasi-Galois extensions of division rings", J. Fac. Sci. Hokkaido Univ., Ser. I, 17 (1963), 73—78.
Page 108, line 6.	For "$g=\sum_{m_2+1}^{m_1} g_{p'q'}^{(1)}$" read "$g=\sum_{m_2+1}^{m_1} g_{p'p'}^{(1)}$".
Page 113, line 29.	For "$D(T_i, T/S)$" read "$D(T_i, R/S)$".
Page 114, lines 19—21.	For "By Theorem 4. 6, $\cdots\cdots V_R(T_1\cap T_2)=V_R(T_1)+V_R(T_2)$" read "Now, let S_i $(i=1, 2)$ be simple intermediate rings of T_i/S such that $[S_i : S]_l<\infty$ and $V_R(S_i)=V_R(T_i)$, and let N be a shade of $S_1[S_2]$ (Theorem 2. 4). Then, by Theorem 4. 6, $\partial'\mid N=\partial_a\mid N$ and $\partial''\mid N=\partial_b\mid N$ with some $a\in V_R(S_1)=V_R(T_1)$ and $b\in V_R(S_2)=V_R(T_2)$. We obtain therefore $c=a+b+z$ for some $z\in V_R(N)\subseteq V_R(S_1[S_2]) = V_R(T_1)\cap V_R(T_2)$, which proves that $V_R(T_1\cap T_2)=V_R(T_1)+V_R(T_2)$".