ON GENERAL CONNECTIONS II

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In a former paper [12],[1] the author investigated the tensor culculus of spaces with general connections and showed several formulas which are natural generalizations of ones in the spaces with classical affine connections.

In the present paper, he will investigate on the development of curves in spaces with general connections which satisfy a certain condition. In the classical theory, a curve in a space with an affine connection is developed into a curve in the affine space of the same dimensions and the developed curve become a straight line, if the curve is a geodesic. When a general connection Γ is regular, that is $P=\lambda(\Gamma)$[2] is regular, we can follow this idea of the development of curves by means of the contravariant part[3] of Γ which is a classical affine connection. But we shall fail in the same trial when Γ is not regular.

As spaces into which curves are developed, taking not only the affine space but also some spaces with general connections which may be considered as relatives of the affine space, he will show that the above mentined idea can be permitted so as developments of geodesics become geodesics in such spaces.

§ 1. Pseudo-affine spaces.[4]

Let A^n be an n-dimensional affine space and x^λ, $\lambda = 1, 2, \cdots, n$, be an affine coordinate system of A^n. We call a space $\mathfrak{X}$ with a general connection Γ such that A^n is the underlying manifold of $\mathfrak{X}$ and out of the components $(P^\lambda_\mu, \Gamma^\lambda_{\mu\nu})$ of Γ with respect to x^λ, the second components $\Gamma^\lambda_{\mu\nu} = 0$, a *pseudo-affine space* of n dimensions. This definition does not depend on the choice of affine coordinate systems of A^n, by virtue of (2. 27) of [12]. In the following, we denote the first components P^λ_μ of connections of pseudo-affine spaces by F^λ_μ.

When rank $(F^\lambda_\mu) = n$, we say that pseudo-affine spaces are *regular*. when $F^\lambda_\mu =$ constant, we call them *special pseudo-affine spaces.*

In a pseudo-affine space, the covariant differential of a contravariant vector field V^λ is given by

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1) The numbers in square brackets show the numbers of the references at the end of the present paper.

2) See [12], § 2.

3) See [12], § 3.

4) In the present paper, we deal with only manifolds, mappings with suitable differentiabilities for our purpose.

$$DV^\lambda = F^\lambda_\mu d V^\mu. \tag{1.1}$$

Since we have

$$\Lambda^\lambda_{\mu\nu} = -\frac{\partial F^\lambda_\mu}{\partial x^\nu} \tag{1.2}$$

for the space, the covariant differential of a covariant vector field W_μ is given by

$$D W_\mu = F^\lambda_\mu d W_\lambda + \frac{\partial F^\lambda_\mu}{\partial x^\nu} W_\lambda\, dx^\nu = d(F^\lambda_\mu W_\lambda). \tag{1.3}$$

Hence, we have immediately

Lemma 1. 1. *In a pseudo-affine space, in order that a contravariant vector field V and a covariant vector field W defined along a curve C: $x^\lambda = x^\lambda(t)$ are covariantly constant, it is necessary and sufficient that dV/dt and W are transformed to the zero vector and a constant vector respectively under the homomorphism F of the tangent bundle $T(A^n)$ which is defined by the components F^λ_μ.*

By the lemma, any constant contravariant vector, that is, with constant components with respect to affine coordinates, is covariantly constant also in any pseudo-affine spaces.

Now, by means of (4. 4) of [12], a geodesic in the pseudo-affine space is such a curve that $x^\lambda = x^\lambda(t)$ satisfy the equations:

$$\frac{D}{dt}\left(\frac{dx^\lambda}{dt}\right) = \psi\, F^\lambda_\mu \frac{dx^\mu}{dx}$$

with a function ψ of t. By (1. 1), these equations can be written as

$$\begin{cases} \dfrac{d^2x^\lambda}{dt^2} = \psi \dfrac{dx^\lambda}{dt} + V^\lambda, \\ F^\lambda_\mu V^\mu = 0. \end{cases} \tag{1.4}$$

Therefore, we have

Theorem 1. 2. *In a pseudo-affine space, for any field of null direction V^λ: $F^\lambda_\mu V^\mu = 0$, there exists a geodesic which passes through a given point and has a given tangent direction at the point, and at each point of which it osculates to the tangent plane spanned by the tangent direction and the null direction.*

Remark. In general, if two field of null directions of F are different from each other, then the geodesics through the same point and with the same tangent direction at the point, corresponding to them, are different curves. When the pseudo-affine space is regular, geodesics are straight lines in the ordinary sense, since there exists no null direction of F at any point.

§ 2. **Normal tensor fields of type** (1, 1).

Let $\mathfrak{X}$ be a differentiable manifold of n dimensions. We say that any tensor field P of type (1, 1) is *normal*, if the homomorphism defined by P on the tangent space at each point of $\mathfrak{X}$ is an isomorphism on its image of the tangent space and

$$\dim P(T_x(\mathfrak{X})) = m, \quad x \in \mathfrak{X}, \tag{2. 1}$$

is constant. If U is a simply connected open neighborhood of $\mathfrak{X}$, we can choose a field of frame $\{V_\alpha, V_A\}$, $\alpha = 1, \cdots, n$, $A = m+1, \cdots n$, such that $\{V_1, \cdots, V_m\}$ is a base of $P(T_x(\mathfrak{X}))$ and $\{V_{m+1}, \cdots, V_n\}$ is a base of the space of tangent vectors transformed to the zero vector under P at each point $x \in U$. Then we have[5)]

$$\begin{cases} P(V_\alpha) = V_\beta W^\beta_\alpha, & |W^\beta_\alpha| \neq 0, \\ P(V_A) = 0. \end{cases} \tag{2. 2}$$

Let $\{U^\alpha, U^A\}$ be the dual base of $\{V_\alpha, V_A\}$, that is,

$$< V_\lambda, U^\mu > = \delta^\mu_\lambda. \tag{2. 3}$$

If we denote the dual mapping of P on the cotangent space $T^*_x(\mathfrak{X})$ at x by the same notation, we have

$$\begin{aligned} P(U^\beta) &= W^\beta_\alpha U^\alpha, \\ P(U^B) &= 0. \end{aligned} \tag{2. 4}$$

Let u^i be local coordinates defined on U and P^i_j, V^i_λ, U^λ_j be the components of P, V_λ, U^λ respectively. Then, the above equations are written as

$$P^j_i V^i_\alpha = V^i_\beta W^\beta_\alpha, \quad P^j_i V^i_A = 0, \tag{2. 5}$$

$$V^i_\lambda U^\mu_i = \delta^\mu_\lambda \tag{2. 6}$$

and

$$P^j_i U^\beta_j = W^\beta_\alpha U^\alpha_i, \quad P^j_i U^B_j = 0. \tag{2. 7}$$

We get easily from them

$$P^j_i = V^j_\beta W^\beta_\alpha U^\alpha_i. \tag{2. 8}$$

If we put

$$(S^\beta_\alpha) = (W^\beta_\alpha)^{-1}, \tag{2. 9}$$

then we can define a tensor field Q of type (1. 1) by

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5) The indices run as follows:

$\lambda, \mu, \nu, \cdots, i, j, h, \cdots = 1, 2, \cdots n,$

$\alpha, \beta, \gamma, \cdots = 1, 2, \cdots, m,$

$A, B, C, \cdots = m+1, \cdots, n.$

$$Q_i^j = V_\beta^j S_\alpha^\beta U_i^\alpha. \tag{2.10}$$

The tensor field Q is clearly independent of the choice of the base $\{V_\alpha, V_A\}$. It is the inverse of P on $P(T_x(\mathfrak{X}))$, and has the same kernel as P at each point x. We get easily

$$P_h^j Q_i^h = Q_h^j P_i^h = V_\alpha^j U_i^\alpha = A_i^j. \tag{2.11}$$

Hence, at each point $x \in U$, $A = PQ = QP$ is the projection of $T_x(\mathfrak{X})$, onto $P(T_x(\mathfrak{X}))$ and we have

$$A^2 = A, \quad AP = PA = P, \quad AQ = QA = Q. \tag{2.12}$$

§ 3. Developments of curves.

Let Γ be a general connection[6] of $\mathfrak{X}$, which is written as

$$\Gamma = \partial u_i \otimes (P_j^i d^2 u^j + \Gamma_{jh}^i du^j \otimes du^h) \tag{3.1}$$

with respect to local coordinates u^i of $\mathfrak{X}$.

Definition. For a curve C: $u^i = u^i(t)$ in $\mathfrak{X}$, if there exists a curve $\bar{C}$: $x^\lambda = x^\lambda(t)$ in the affine space A^n and a field of frame $\{X_\lambda^i\}$ of $T(\mathfrak{X})$ along C such that

$$\frac{dx^\lambda}{dt} = Y_i^\lambda \frac{du^i}{dt}, \tag{3.2}$$

$$P_j^i \frac{DX_\lambda^j}{dt} = P_j^i\left(P_k^j \frac{dX_\lambda^k}{dt} + \Gamma_{kh}^j X_\lambda^k \frac{du^h}{dt}\right) = 0, \tag{3.3}$$

where $\{Y_i^\lambda\}$ is the dual base of $\{X_\lambda^i\}$, then $\bar{C}$ is called a *development* of C.

In this definition, $\bar{C}$ is considered as a curve in a pseudo-affine space, whose general connection's components F_μ^λ satisfy the condition

$$F_\mu^\lambda = Y_i^\lambda P_j^i X_\mu^j \qquad \text{on } \bar{C}. \tag{3.4}$$

But we have to restrict this intention to suitable subarcs of $\bar{C}$, when $\bar{C}$ has double points. It is easily seen that when $\bar{C}$ is a development of C, its image under any affine transformation of A^n is also a development of C.

A general connection Γ is called *normal*, when the tensor field $P = \lambda(\Gamma)$ with local components P_j^i is normal.

Theorem 3.1. *Let C be a curve in a space $\mathfrak{X}$ with a normal general connection such that $\dim P(T_x(\mathfrak{X})) = m$, then C has a development which depends on $n(n-m)$ arbitrary functions of the parameter of C.*

Proof. It is sufficient to prove that there exist n^2 functions X_λ^i of t

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6) See §2 of [12].

such that they satisfy (3. 3), $|X^i_\lambda| \neq 0$ and any initial condition.

we may take a field of frame $\{V_\alpha, V_A\}$ as in §2 in a neighborhood of C. Putting

$$X^i_\lambda = V^i_\mu \xi^\mu_\lambda,$$

we write the equations (3. 3) by means of the unknown functions ξ^μ_λ, that is

$$\begin{aligned} P^i_j DX^j_\lambda &= P^i_j \{P^j_k d(V^k_\mu \xi^\mu_\lambda) + \Gamma^j_{kh} V^k_\mu \xi^\mu_\lambda du^h\} \\ &= P^i_j \{P^j_k V^k_\mu d\xi^\mu_\lambda + (P^j_k dV^k_\mu + \Gamma^j_{kh} V^k_\mu du^h)\xi^\mu_\lambda\} \\ &= P^i_j \{P^j_k V^k_\alpha d\xi^\alpha_\lambda + DV^j_\mu \xi^\mu_\lambda\} = 0. \end{aligned}$$

Multiplying $Q^h_l Q^l_i$ and contracting with respect to i, we get the equations

$$\begin{aligned} &V^i_\alpha d\xi^\alpha_\lambda + A^i_k Q^k_j DV^j_\mu \xi^\mu_\lambda \\ &= V^i_\alpha (d\xi^\alpha_\lambda + U^\alpha_k Q^k_j DV^j_\mu \xi^\mu_\lambda) = 0, \end{aligned}$$

hence

$$\frac{d\xi^\alpha_\lambda}{dt} + U^\alpha_k Q^k_j \frac{DV^j_\mu}{dt} \xi^\mu_\lambda = 0$$

by (2. 11) and (2. 12), which are equivalent to (3. 3) by virtue of the properties of P, Q and A described in §2. Putting

$$K^\alpha_\mu = U^\alpha_k Q^k_j \frac{DV^j_\mu}{dt} \tag{3. 5}$$

which are known functions of t, ξ^i_λ must satisfy the equations

$$\frac{d\xi^\alpha_\lambda}{dt} + K^\alpha_\mu \xi^\mu_\lambda = 0. \tag{3. 6}$$

Therefore, we take any $n(n-m)$ auxiliary functions K^A_μ of t and consider the following equations

$$\frac{d\xi^A_\lambda}{dt} + K^A_\mu \xi^\mu_\lambda = 0 \tag{3. 7}$$

to be added to (3. 6). Then we can obtain a solution $\{\xi^\mu_\lambda(t)\}$ satisfying the equations (3. 6) and (3. 7) which is uniquely determined for given initial values

$$\xi^\mu_\lambda(t_0) = c^\mu_\lambda, \quad |c^\mu_\lambda| \neq 0.$$

It is easily seen that this solution depends on the initial values and the functions K^A_λ. The proof is finished. q. e. d.

Let $\bar{C}$ be the curve in Theorem 3. 1. If we extend the n^2 functions $F^\lambda_\mu = Y^\lambda_i P^i_j X^j_\mu$ considered as defined on $\bar{C}$ on the whole space A^n, then $\bar{C}$ is a curve in the pseudo-affine space whose connection's components are F^λ_μ. Now, we calculate the covariant derivative of the tangent vector of $\bar{C}$.

By virtue of (2. 11), (2. 12), we have

$$\begin{aligned}\frac{D}{dt}\frac{dx^\lambda}{dt} &= F^\lambda_\mu \frac{d^2x^\mu}{dt^2} = F^\lambda_\mu \frac{d}{dt}\Big(Y^\mu_i \frac{du^i}{dt}\Big) \\ &= F^\lambda_\mu\Big(Y^\mu_i \frac{d^2u^i}{dt^2} + \frac{dY^\mu_i}{dt}\frac{du^i}{dt}\Big) \\ &= Y^\lambda_j P^j_i\Big(\frac{d^2u^i}{dt^2} - \frac{dX^i_\mu}{dt} Y^\mu_h \frac{du^h}{dt}\Big) \\ &= Y^\lambda_j\Big(\frac{D}{dt}\frac{du^j}{dt} - \Gamma^j_{ih}\frac{du^i}{dt}\frac{du^h}{dt} - P^j_i \frac{dX^i_\mu}{dt} Y^\mu_h \frac{du^h}{dt}\Big),\end{aligned}$$

and

$$\begin{aligned}P^j_i \frac{dX^i_\mu}{dt} Y^\mu_h \frac{du^h}{dt} &= A^j_k P^k_i \frac{dX^i_\mu}{dt} Y^\mu_h \frac{du^h}{dt} \\ &= Q^j_l P^l_k P^k_i \frac{dX^i_\mu}{dt} Y^\mu_h \frac{du^h}{dt} \\ &= Q^j_l P^l_k\Big(\frac{DX^k_\mu}{dt} - \Gamma^k_{i\nu} X^i_\mu \frac{du^\nu}{dt}\Big) Y^\mu_h \frac{du^h}{dt} \\ &= -A^j_k \Gamma^k_{ih}\frac{du^i}{dt}\frac{du^h}{dt}.\end{aligned}$$

Hence we have the formula:

$$\frac{D}{dt}\frac{dx^\lambda}{dt} = Y^\lambda_j\Big\{\frac{D}{dt}\frac{du^j}{dt} - (\delta^j_k - A^j_k)\,\Gamma^k_{ih}\frac{du^i}{dt}\frac{du^h}{dt}\Big\}. \tag{3. 8}$$

Theorem 3. 2. *Let Γ be a normal general connection as in Theorem 3. 1. Developments of a geodesic in $\mathfrak{X}$ are geodesics also in pseudo-affine spaces respectively.*

Proof. Let C: $u^i = u^i(t)$ be a geodesic in $\mathfrak{X}$, that is, $u^i(t)$ satisfy the equations:

$$\frac{D}{dt}\frac{du^i}{dt} = \psi\, P^i_j \frac{du^j}{dt}$$

with a suitable function ψ of t, and $\bar{C}$ be a development of C. Then, we get by (3. 8) the equations:

$$\begin{aligned}\frac{D}{dt}\frac{dx^\lambda}{dt} &= \psi\, Y^\lambda_j P^j_i \frac{du}{dt} + Y^\lambda_j(\delta^j_k - A^j_k)\Big(P^k_i \frac{d^2u^i}{dt^2} - \psi P^k_i \frac{du^i}{dt}\Big) \\ &= \psi F^\lambda_\mu Y^\mu_i \frac{du^i}{dt},\end{aligned}$$

that is

$$\frac{D}{dt}\Big(\frac{dx^\lambda}{dt}\Big) = \psi\, F^\lambda_\mu \frac{dx^\mu}{dt}.$$

This shows that $\bar{C}$ is a geodesics in the corresponding pseudo-affine space.

§ 4. Developments into special pseudo-affine spaces.

Let Γ be a normal general connection as in Theorem 3. 1. Then, any curve C in $\mathfrak{X}$ is developable into a suitable pseudo-affine space. Using the same notations in § 3 and putting

$$Y_i^\lambda = \gamma_{/\mu}^\lambda U_i^\mu ,$$

we have

$$\frac{d\gamma_{/\mu}^\lambda}{dt} - \gamma_{/\nu}^\lambda K_\mu^\nu = 0 \tag{4. 1}$$

from (3. 6), (3. 7) and the relations

$$\gamma_{/\nu}^\lambda \xi_\mu^\nu = \delta_\mu^\lambda .$$

Then, we have

$$\begin{aligned}
\frac{d}{dt}(F_\mu^\lambda) &= \frac{d}{dt}(Y_i^\lambda P_j^i X_\mu^j) = \frac{d}{dt}(\gamma_{/\alpha}^\lambda U_i^\alpha P_j^i V_\beta^j \xi_\mu^\beta) \\
&= \frac{d}{dt}(\gamma_{/\alpha}^\lambda W_\beta^\alpha \xi_\mu^\beta) \\
&= \frac{d\gamma_{/\alpha}^\lambda}{dt} W_\beta^\alpha \xi_\mu^\beta + \gamma_{/\alpha}^\lambda W_\beta^\alpha \frac{d\xi_\mu^\beta}{dt} + \gamma_{/\alpha}^\lambda \frac{dW_\beta^\alpha}{dt} \xi_\mu^\beta \\
&= \gamma_{/\nu}^\lambda K_\alpha^\nu W_\beta^\alpha \xi_\mu^\beta - \gamma_{/\alpha}^\lambda W_\beta^\alpha K_\nu^\beta \xi_\mu^\nu + \gamma_{/\alpha}^\lambda \frac{dW_\beta^\alpha}{dt} \xi_\mu^\beta \\
&= \gamma_{/\alpha}^\lambda \left\{ \left(K_\gamma^\alpha W_\beta^\gamma - W_\gamma^\alpha K_\beta^\gamma + \frac{d}{dt} W_\beta^\alpha \right) \xi_\mu^\beta - W_\beta^\alpha K_A^\beta \xi_\mu^A \right\} \\
&\quad + \gamma_{/A}^\lambda K_\alpha^A W_\beta^\alpha \xi_\mu^\beta .
\end{aligned}$$

Hence, in order that F_μ^λ are constants along $\bar{C}$, it is necessary and sufficient that

$$\begin{cases} K_\gamma^\alpha W_\beta^\gamma - W_\gamma^\alpha K_\beta^\gamma + \dfrac{d}{dt} W_\beta^\alpha = 0 , & (4.\ 2) \\ K_A^\alpha = 0, & (4.\ 3) \\ K_\alpha^A = 0. & (4.\ 4) \end{cases}$$

We rewrite these equations in intrinsic forms. Firstly, we have

$$\begin{aligned}
& K_\gamma^\alpha W_\beta^\gamma - W_\gamma^\alpha K_\beta^\gamma + \frac{d}{dt} W_\beta^\alpha \\
&= U_i^\alpha Q_j^i \frac{DV_\gamma^j}{dt} W_\beta^\gamma - W_\gamma^\alpha U_i^\gamma Q_j^i \frac{DV_\beta^j}{dt} + \frac{d}{dt}(U_i^\alpha P_j^i V_\beta^j)
\end{aligned}$$

$$= U_i^\alpha Q_j^i\Big(P_k^j\frac{dV_\gamma^k}{dt} + \Gamma_{kh}^j V_\gamma^k \frac{du^h}{dt}\Big) W_\beta^\gamma - U_i^\alpha \frac{DV_\beta^i}{dt} + \frac{d}{dt}(U_i^\alpha P_j^i V_\beta^j)$$

$$= U_i^\alpha \frac{dV_\gamma^i}{dt} W_\beta^\gamma + U_i^\alpha Q_j^i \Gamma_{kh}^j P_l^k V_\beta^l \frac{du^h}{dt}$$

$$- U_i^\alpha\Big(P_j^i\frac{dV_\beta^j}{dt} + \Gamma_{jh}^i V_\beta^j \frac{du^h}{dt}\Big)$$

$$+ \frac{dU_i^\alpha}{dt} V_\gamma^i W_\beta^\gamma + U_i^\alpha \frac{dP_j^i}{dt} V_\beta^j + U_i^\alpha P_j^i \frac{dV_\beta^j}{dt}$$

$$= U_i^\alpha\Big(\frac{dP_j^i}{dt} + Q_k^i \Gamma_{lh}^k P_j^l \frac{du^h}{dt} - \Gamma_{jh}^i \frac{du^h}{dt}\Big) V_\beta^j = 0.$$

Since $|W_\beta^\alpha| \neq 0$, the last equations are equivalent to

$$U_p^\alpha P_i^p \Big(\frac{dP_j^i}{dt} + Q_k^i \Gamma_{lh}^k P_j^l \frac{du^h}{dt} - \Gamma_{jh}^i \frac{du^h}{dt}\Big) P_q^j V_\beta^q$$

$$= U_p^\alpha\Big(P_i^p \frac{dP_j^i}{dt} P_q^j + \Gamma_{lh}^p P_j^l P_q^j \frac{du^h}{dt} - P_i^p \Gamma_{jh}^i P_q^j \frac{du^h}{dt}\Big) V_\beta^q$$

$$= U_p^\alpha\Big\{\frac{DP_q^p}{dt} + (P_i^p P_j^i A_{qh}^j - P_i^p \Gamma_{jh}^i P_q^j)\frac{du^h}{dt}\Big\} V_\beta^q$$

$$= U_p^\alpha\Big(\frac{DP_q^p}{dt} - P_i^p \delta_{q,h}^i \frac{du^h}{dt}\Big) V_\beta^q = 0,$$

where $\delta_{q,h}^i$ are the covariant derivatives of the Kronecker's δ_q^i. Furthermore, by means of (2.8), these are equivalent to the intrinsic eqations

$$P_h^i\Big(\frac{DP_k^h}{dt} - P_l^h \frac{D\delta_k^l}{dt}\Big) P_j^k = 0. \tag{4.5}$$

Nextly, since

$$K_A^\beta = U_i^\beta Q_j^i \frac{DV_A^j}{dt} = S_\gamma^\beta U_j^\gamma \frac{DV_A^j}{dt},$$

(4.3) is equivalent to

$$U_j^\alpha \frac{DV_A^j}{dt} = 0,$$

and hence

$$P_j^i \frac{DV_A^j}{dt} = 0$$

by virtue of (2.8). The left hand side of the last equation can be written as

$$P_j^i\Big(P_k^j \frac{dV_A^k}{dt} + \Gamma_{kh}^j V_A^k \frac{du^h}{dt}\Big)$$

$$= P^i_j \frac{d}{dt}(P^j_k V^k_A) + P^i_j \Lambda^j_{kh} V^k_A \frac{du^h}{dt}$$

$$= P^i_j \Lambda^j_{kh} V^k_A \frac{du^h}{dt}$$

$$= -(\Gamma^i_{jh} P^j_k - P^i_j \Lambda^j_{kh}) V^k_A \frac{du^h}{dt}$$

$$= -\frac{D\delta^i_k}{dt} V^k_A.$$

Accordingly, (4. 3) can be replaced with the intrinsic equations :

$$\frac{D\delta^i_k}{dt} V^k_A = 0. \tag{4. 6}$$

Thus, we obtain the following theorem.

Theorem 4. 1. *In order that a curve C in $\mathfrak{X}$ with a normal general connection Γ has a development $\bar{C}$ in a special pseudo-affine space, it is necessary and sufficient that it holds good that*

$$\text{i)} \quad P^i_h\left(\frac{DP^h_k}{dt} - P^h_l \frac{D\delta^l_k}{dt}\right) P^k_j = 0,$$

$$\text{ii)} \quad \text{if } P^i_j V^j = 0, \text{ then } \frac{D\delta^i_j}{dt} V^j = 0$$

along C and we put the auxiliary functions K^A_α of t as

$$K^A_\alpha = 0, \quad A = m+1, \cdots, n; \quad \alpha = 1, \cdots, m.$$

Theorem 4. 2. *In order that any curve in $\mathfrak{X}$ with a normal general connection Γ is developable into one and the same special pseudo-affine space, it is necessary and sufficient that it holds good that*

$$P^i_h(P^h_{k,q} - P^h_l \delta^l_{k,q}) P^k_j = 0 \tag{4. 7}$$

and

$$\text{if } P^i_j V^j = 0, \quad \text{then } \delta^i_{j,h} V^j = 0. \tag{4. 8}$$

§ 5. Developments of vector fields.

Let C: $u^i = u^i(t)$ be a given curve in $\mathfrak{X}$ with a normal general connection Γ and $\bar{C}$ be a development of C in a pseudo-affine space. For a contravariant vector field and a covariant vector field with local components V^i and W_i respectively defined along C, we call the contravariant vector field and the covariant vector field defined along $\bar{C}$ by

$$\bar{V}^\lambda = Y^\lambda_i V^i \tag{5. 1}$$

and

$$\overline{W}_\lambda = X^i_\lambda W_i \tag{5.2}$$

their *developments along* $\bar{C}$.

Using the notations in §3, we have

$$\begin{aligned} D\overline{V}^\lambda &= F^\lambda_\mu d\overline{V}^\mu \\ &= Y^\lambda_i P^i_j X^j_\mu (Y^\mu_h dV^h + dY^\mu_h V^h) \\ &= Y^\lambda_i (P^i_j dV^j - P^i_j dX^j_\mu Y^\mu_h V^h) \\ &= Y^\lambda_i (DV^i - \Gamma^i_{jh} V^j du^h) - Y^\lambda_i (DX^i_\mu Y^\mu_h V^h - \Gamma^i_{jh} V^j du^h) \\ &= Y^\lambda_i DV^i - Y^\lambda_i DX^i_\mu Y^\mu_h V^h \\ &= Y^\lambda_i DV^i - Y^\lambda_i (A^i_j + (\delta^i_j - A^i_j)) DX^j_\mu \overline{V}^\mu. \end{aligned}$$

By means of (2.11) and (3.3), since we have

$$A^i_j DX^j_\mu = Q^i_h P^h_j DX^j_\mu = 0$$

and

$$\begin{aligned} U^A_j DX^j_\mu \overline{V}^\mu &= U^A_j (P^j_i dX^i_\mu + \Gamma^j_{ih} X^i_\mu du^h) \overline{V}^\mu \\ &= U^A_j \Gamma^j_{ih} V^i du^h \\ &= U^A_j (P^j_i dV^i + \Gamma^j_{ih} V^i du^h) = U^A_j DV^j, \end{aligned}$$

the above equations can be written as

$$\begin{aligned} D\overline{V}^\lambda &= Y^\lambda_i DV^i - Y^\lambda_i V^i_A U^A_j DX^j_\mu \overline{V}^\mu \\ &= Y^\lambda_i (DV^i - V^i_A U^A_j DV^j) \\ &= Y^\lambda_i (\delta^i_j - V^i_A U^A_j) DV^j = Y^\lambda_i A^i_j DV^j, \end{aligned}$$

hence we have the formula :

$$\frac{D\overline{V}^\lambda}{dt} = Y^\lambda_i A^i_j \frac{DV^j}{dt}. \tag{5.3}$$

Then, we have analogously

$$\begin{aligned} D\overline{W}_\lambda &= d(F^\mu_\lambda \overline{W}_\mu) \\ &= d(X^i_\lambda P^j_i W_j) \\ &= dX^i_\lambda P^j_i W_j + X^i_\lambda d(P^j_i W_j) \\ &= (DX^j_\lambda - \Gamma^j_{ih} X^i_\lambda du^h) W_j + X^i_\lambda (DW_i + \Gamma^j_{ih} W_j du^h) \\ &= DX^j_\lambda W_j + X^i_\lambda DW_i \\ &= DX^i_\lambda (A^j_i + (\delta^j_i - A^j_i)) W_j + X^i_\lambda DW_i \\ &= DX^i_\lambda (\delta^j_i - A^j_i) W_j + X^i_\lambda DW_i. \end{aligned}$$

Since

$$DX^i_\lambda (\delta^j_i - A^j_i) = DX^i_\lambda U^A_i V^j_A$$

$$= (P^i_k dX^k_\lambda + \Gamma^i_{kh} X^k_\lambda du^h) U^A_i V^j_A$$
$$= X^k_\lambda \Gamma^i_{kh} du^h U^A_i V^j_A$$

and

$$DU^A_i = d(U^A_j P^j_i) - U^A_j \Gamma^j_{ih} du^h = -U^A_j \Gamma^j_{ih} du^h, \tag{5.4}$$

we obtain the formulas :

$$\frac{D\overline{W}_\lambda}{dt} = X^i_\lambda \Big(\frac{DW_i}{dt} - \frac{DU^A_i}{dt} (V^j_A W_j) \Big). \tag{5.5}$$

Definition. A general connection Γ is called *proper*, if any covariant vector field transformed to zero under $P = \lambda(\Gamma)$ is covariantly constant, that is,

if $U_i P^i_j = 0$, then $DU_i = 0$.

From the formula (5. 3), we obtain the following theorem.

Theorem 5. 1. *The development of a covariantly constant contravariant vector field defined along a curve C in $\mathfrak{X}$ with a normal general connection Γ is also covariantly constant along the development $\bar{C}$ of C in the pseudo-affine space.*

From the formula (5. 5), we obtain the following theorem.

Theorem 5. 2. *The development of a covariantly constant covariant vector field defined along a curve C in $\mathfrak{X}$ with a normal general connection Γ is also covariantly constant along the development $\bar{C}$ of C in the pseudo-affine space, if Γ is proper or the vector field is invariant under the projection A of $T(\mathfrak{X})$.*

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