

ON GALOIS THEORY OF DIVISION RINGS III

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Introduction. Throughout the present paper, we shall deal with a division ring K which is (left) locally finite over a division subring L , and use the following conventions: Let T be an arbitrary intermediate subring of K/L . Then, $\mathfrak{G}(T/L)$ means the totality of L -(ring) isomorphisms of T into K . (In particular, if T/L is Galois, the Galois group of T/L will be denoted by $\mathfrak{G}(T/L)$ as in [3].) And, if σ is a map defined on a given subset U of K containing T , then $\sigma|T$ means the contraction of σ to T . Similarly, if \mathfrak{S} is a set of maps defined on U , then $\mathfrak{S}|T = \{\sigma|T; \sigma \in \mathfrak{S}\}$. Finally, we set $H = V_{\kappa(x)(L)}$, and for other notations and terminologies used here we follow [3].

Recently, in [3], the notion of locally Galois extensions was introduced: If, for any finite subset F of K , there exists a division subring N containing $L[F]$ such that N/L is Galois and $[N:L]_t < \infty$, then K/L is said to be *locally Galois*. In the present paper, we shall consider a class of locally finite division ring extensions which contains locally Galois extensions and also Galois extensions. More precisely, we shall consider the following conditions:

- (I) For any subring L' of K such that $L' \supseteq L$ and $[L':L]_t < \infty$, $\mathfrak{G}(L'/L)$ contains an isomorphism different from the identity map: $\mathfrak{G}(L'/L) \neq 1$.
- (I') H/L is Galois.
- (II) $\mathfrak{G}(L_1/L)|_{L_2} = \mathfrak{G}(L_2/L)$ for any intermediate subrings L_1 and L_2 of K/L such that $L_1 \supseteq L_2 \supseteq L$ and $[L_1:L]_t < \infty$.
- (II') $\mathfrak{G}(T/L)|_H = \mathfrak{G}(H/L)$ for any intermediate subring T of K/H with $[T:H]_t < \infty$.

We introduce here the notion of quasi-Galois extensions: If K/L is locally finite and satisfies the conditions (I) and (II), the extension K/L will be called *quasi-Galois*. Needless to say, if K/L is locally Galois, then it is quasi-Galois. It is also clear that, if K/L is Galois, then it is quasi-Galois. The notion of quasi-Galois coincides with that of Galois when K is finite over H as is seen later. In this sense, one may regard the notion of quasi-Galois as a natural generalization of that of Galois.

We shall start our study with §1, which contains preliminary lemmas that will play important roles in the subsequent sections. Secondly, in §2, we shall see that K/L is quasi-Galois if and only if K/L is locally finite and either the conditions (I') and (II) or (I') and (II') are fulfilled,

and prove our principal theorem, which claims that, if K/L is quasi-Galois, then so is K/T for any division subring T containing L such that $[V_K(L) : V_K(T)]_r < \infty$. We shall conclude our study with § 3, which contains some supplementary remarks to the results cited in [3].

1. Preliminaries. In this section, we always assume that K is a division ring which is locally finite over a division subring L .

Lemma 1. *Let T be an arbitrary subring of H containing L .*

(i) *If the condition (II) is satisfied, then $T\mathfrak{G}'(T/L) \subseteq H$.*

(ii) *If the condition (I') is satisfied, then also $T\mathfrak{G}'(T/L) \subseteq H$. In particular, $\mathfrak{G}'(H/L) = \mathfrak{G}(H/L)$.*

Proof. It suffices to prove our lemma for the case $[T : L]_l < \infty$.

(i) Let σ be in $\mathfrak{G}'(T/L)$. If $T\sigma \not\subseteq H$, then there exist a non-zero element v in $V_K(L)$ and an element t in T such that $v^{-1}(t\sigma)v \neq t\sigma$. Since $\sigma^{-1} \in \mathfrak{G}'(T\sigma/L) = \mathfrak{G}'((T\sigma)[v]/L) | T\sigma$ by (II), there holds $\sigma^{-1} = \tau | T\sigma$ for some $\tau \in \mathfrak{G}'((T\sigma)[v]/L)$. Then, $v\tau$ being evidently contained in $V_K(L)$, we have $t = (v\tau)^{-1}t(v\tau) = (v^{-1}(t\sigma)v)\tau \neq t\sigma\tau = t$, which is a contradiction.

(ii) Let N be an intermediate subring of H/T such that N/L is Galois and $[N : L]_l < \infty$ [2, Theorem 3]. Since $\text{Hom}_{L_l}(N, K) = \mathfrak{G}(N/L)K_r$, we obtain $\text{Hom}_{L_l}(T, K) = (\mathfrak{G}(N/L) | T)K_r = \mathfrak{G}'(T/L)K_r$. Accordingly, for each $\sigma \in \mathfrak{G}'(T/L)$, we can find some $\tau \in \mathfrak{G}(N/L) | T$ such that σK_r is $T_r - K_r$ -isomorphic to τK_r . And then, one will easily see that $\sigma = \tau\bar{v}$ with some $v \in V_K(L)$. Recalling here that $T\tau \subseteq N \subseteq H$, we obtain $\sigma = \tau$. Consequently, it follows $T\sigma \subseteq H$. The latter part is a consequence of [2, Theorem 11].

Corollary 1. *If K/L is quasi-Galois, then the condition (I') is satisfied.*

Proof. Let L' be an arbitrary subring of H containing L such that $[L' : L]_l < \infty$. Since $[\mathfrak{G}'(L'/L)K_r : K_r]_r < \infty$, we obtain $\mathfrak{G}'(L'/L)K_r = \sum_{i=1}^n \oplus \sigma_i K_r$ with some σ_i 's in $\mathfrak{G}'(L'/L)$. (One may remark here that each $\sigma_i K_r$ is $L'_r - K_r$ -irreducible.) Now, for any $\sigma \in \mathfrak{G}'(L'/L)$, there exists some σ_i such that σK_r is $L'_r - K_r$ -isomorphic to $\sigma_i K_r$. And then, we see that $\sigma = \sigma_i \bar{v}$ with some $v \in V_K(L)$. Since $L'\sigma_i \subseteq H$ by Lemma 1 (i), it follows $\sigma = \sigma_i$. Thus we have proved $\mathfrak{G}'(L'/L)$ is finite (and so is compact). Now let $\mathfrak{G}'(\ni I)$ be the inverse limit of the system $\{\mathfrak{G}'(L_\alpha/L); L_\alpha \text{ run over all the intermediate subrings of } H/L \text{ such that } [L_\alpha : L]_l < \infty\}$, which may be regarded as a set of L -(ring) isomorphisms of H into H . Then, since each $\mathfrak{G}'(L_\alpha/L)$ is finite and, by (II), $\mathfrak{G}'(L_\alpha/L) | L_\beta = \mathfrak{G}'(L_\beta/L)$ for $L_\alpha \supseteq L_\beta$, there holds $\mathfrak{G}' | L_\alpha = \mathfrak{G}'(L_\alpha/L)$ by [1, Corollary 3.9]. In virtue of this fact, we

readily see that, for arbitrary $\sigma \in \mathfrak{G}'$ and $x \in H$, there exists some positive integer n such that $x\sigma^n = x$, whence it follows that σ maps H onto H , that is, σ is an automorphism of H . Finally, if H contains an element a not contained in L such that $a\tau = a$ for all $\tau \in \mathfrak{G}'$, then $\mathfrak{G}'(L[a]/L) = \mathfrak{G}'|L[a] = 1$, which contradicts (I). Thus we have proved that H is Galois over L .

Corollary 2. *If K/L is locally Galois, then H/L is Galois.*

Proof. Since K/L is quasi-Galois as is noted in the introduction, our assertion is a direct consequence of Corollary 1.

Lemma 2. *Let L' be an intermediate subring of K/L such that $[L': L]_i < \infty$.*

- (i) *If (I') and (II) are satisfied, then $\text{Hom}_{L'}(L', K) = \mathfrak{G}'(L'/L)K_r$.*
- (ii) *If (I') and (II') are satisfied, then also $\text{Hom}_{L'}(L', K) = \mathfrak{G}'(L'/L)K_r$.*

Proof. Let $U = \{u_1, \dots, u_n\}$ be a maximal subset of L' which is linearly left-independent over H . Then we can find an intermediate subring N of H/L such that $[N: L]_i < \infty$, N/L is Galois, and $M = \sum_{j=1}^n Nu_j \supseteq L'$. Since K/H is Galois, by Jacobson's theorem, there exist some $\xi_k \in \mathfrak{G}(K/H)K_r (k = 1, \dots, n)$ such that $u_j \xi_k = \delta_{jk}$ (Kronecker's δ). Here, we set $\xi_k = \sum_p \sigma_{kp}(a_{kp})_r$ where $\sigma_{kp} \in \mathfrak{G}(K/H)$ and $a_{kp} \in K$. On the other hand, if $\{x_1, \dots, x_m\}$ is a linearly left-independent L -basis of N , then there exist some $\rho_h \in \mathfrak{G}'(N/L)K_r (h = 1, \dots, m)$ such that $x_i \rho_h = \delta_{ih}$. We set $T = N[U]$ and $T' = T[(\bigcup_{k,p} T\sigma_{kp}) \cup \{a_{kp}'s\}]$, which are evidently left finite over L . Further, we set $T_0 = H[T]$ and $T^* = H[T']$, which are left finite over H by [4, Theorem 1].

(i) By (II), $\rho_h = \rho'_h|N$ for some $\rho'_h = \sum_q \tau_{hq}(b_{hq})_r \in \mathfrak{G}'(T'/L)K_r$ where $\tau_{hq} \in \mathfrak{G}'(T'/L)$ and $b_{hq} \in K$. Setting here $\gamma_{hk} = (\xi_k|T)\rho'_h$, one will readily see that $\gamma_{kh} = \sum_{p,q} (\sigma_{kp}|T)\tau_{hq}(a_{kp}\tau_{hq})_r(b_{hq})_r$ is contained in $\mathfrak{G}'(T/L)K_r$ and

$$x_i u_j \gamma_{kh} = \begin{cases} 1 & \text{if } j = k \text{ and } i = h, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain therefore $\text{Hom}_{L'}(M, K) = \mathfrak{G}'(T/L)K_r|M$, whence $\text{Hom}_{L'}(L', K) = \text{Hom}_{L'}(M, K)|L' = (\mathfrak{G}'(T/L)K_r)|L' \subseteq \mathfrak{G}'(L'/L)K_r$. Since the converse inclusion $\text{Hom}_{L'}(L', K) \supseteq \mathfrak{G}'(L'/L)K_r$ is clear, we obtain our assertion.

(ii) By (II') and Lemma 1 (ii), we can easily see that $\mathfrak{G}'(N/L) = \mathfrak{G}'(T^*/L)|N$. Consequently, $\rho_h = \rho^*_h|N$ for some $\rho^*_h \in \mathfrak{G}'(T^*/L)$. Then, as in the proof of (i), we can see that $\gamma^*_{kh} = (\xi_k|T_0)\rho^*_h$ is contained in $\mathfrak{G}'(T_0/L)K_r$ and

$$x_i u_j \tau_{kh}^* = \begin{cases} 1 & \text{if } j = k \text{ and } i = h, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have eventually $\text{Hom}_{\tau_1}(L', K) = \mathfrak{G}'(L'/L)K_r$.

2. Quasi-Galois extensions.

In this section, too, we shall assume that K/L is a locally finite division ring extension. To prove our first theorem, further several lemmas will be needed.

Lemma 3. *Let the condition (I') be satisfied, and L_1 and L_2 division subrings of K such that $L_1 \supseteq L_2 \supseteq L$ and $[L_1 : L]_i < \infty$.*

(i) *If (II) is satisfied, then $\mathfrak{G}'(L_1/L_2) \neq 1$.*

(ii) *If (II') is satisfied, then also $\mathfrak{G}'(L_1/L_2) \neq 1$.*

Proof. (i) Since $\text{Hom}_{L_1}(L_1, K) = \mathfrak{G}'(L_1/L)K_r$ by Lemma 2 (i), a similar argument as in the proof of [4, Theorem 2] will apply to see that $\text{Hom}_{\tau_2}(L_1, K) = \mathfrak{G}'(L_1/L) \cap \text{Hom}_{L_2}(L_1, K) K_r = \mathfrak{G}'(L_1/L_2)K_r$. Recalling that $[\text{Hom}_{\tau_2}(L_1, K) : K_r]_r = [L_1 : L_2]_i > 1$, we obtain $\mathfrak{G}'(L_1/L_2) \neq 1$.

(ii) Since $\text{Hom}_{\tau_1}(L_1, K) = \mathfrak{G}'(L_1/L)K_r$ by Lemma 2 (ii), the proof will proceed just as in (i).

As a direct consequence of Corollary 1 and Lemma 3 (i), it will be easily seen that K/L is quasi-Galois if and only if K/L is locally finite and the conditions (I') and (II) are satisfied. Moreover, Corollary 1 and Lemma 3 (i) yield at once the first and second assertion of the next

Corollary 3. *If K/L is quasi-Galois and L' a subring of K containing L with $[L' : L]_i < \infty$, then K/L' is quasi-Galois, $H' = V_{\kappa}(V_{\kappa}(L'))$ is Galois over L' , and $\mathfrak{G}'(H'/L')$ is (topologically) isomorphic to $\mathfrak{G}'(H/H \cap L')$ by the restriction map.*

Proof. Recalling that $\mathfrak{G}'(H'/L')$ is compact and $\mathfrak{G}'(H/L) = \mathfrak{G}'(H/L)$, our last assertion will be easily obtained. (Cf. the proof of [2, Lemm 9 (iii)].)

Lemma 4. *Let K/L be quasi-Galois, and L' a division subring containing L with $[L' : L]_i < \infty$. If $H' = V_{\kappa}(V_{\kappa}(L'))$, then $\mathfrak{G}'(L'/L) = \mathfrak{G}'(H'/L)|_{L'}$.*

Proof. Since H'/L' is outer Galois by Corollary 3, $H' = \bigcup L'_\alpha$ where L'_α run over all the $G(H'/L')$ -normal intermediate subrings of H'/L' such that $[L'_\alpha : L']_i < \infty$ (Cf. [2]). Now let ρ be an arbitrary element of $\mathfrak{G}'(L'/L)$, and denote by \mathfrak{E}_α the set of all the extensions of ρ to L -isomorphisms of L'_α , which is evidently non-empty by (II). And then, we denote by \mathfrak{E} the inverse limit of the system $\{\mathfrak{E}_\alpha\}$. If σ and τ are in \mathfrak{E}_α , then $\sigma =$

$\tau\varepsilon$ with some $\varepsilon \in \mathfrak{G}'(L'_{\alpha\tau}/L'\rho)$. But $\mathfrak{G}'(L'_{\alpha\tau}/L'\rho) (\cong \mathfrak{G}'(L'_{\alpha}/L') = \mathfrak{G}'(L'_{\alpha}/L'))$ is finite, that is, \mathfrak{G}_{α} is finite. Consequently, the inverse limit \mathfrak{G} is non-empty by [1, Theorem 3.6], and is a subset of $\mathfrak{G}'(H'/L)$.

Now we are at the position to present the following theorem.

Theorem 1. *For a locally finite division ring extension K/L , the following three conditions are equivalent :*

- (1) K/L is quasi-Galois.
- (2) (I') and (II) are satisfied.
- (3) (I') and (II') are satisfied.

Proof. The equivalence of (1) and (2) was already remarked immediately after the proof of Lemma 3.

(3) *implies* (2). If L_1 and L_2 are division subrings such that $L_1 \supseteq L_2 \supseteq L$ and $[L_1 : L]_i < \infty$, then $\mathfrak{G}'(L_2/L)K_r = \text{Hom}_{L_2}(L_2, K) = \text{Hom}_{L_2}(L_1, K)|_{L_2} = (\mathfrak{G}'(L_1/L)|_{L_2})K_r$ by Lemma 2 (ii). In virtue of this fact, it will be easy to see that $\mathfrak{G}'(L_2/L) = \mathfrak{G}'(L_1/L)|_{L_2}$. (Cf. the proof of Lemma 1 (ii).)

(2) *implies* (3). If T is a division subring containing H such that $[T : H]_i < \infty$, then by [4, Lemma 2], $T = H[L'] = V_{\mathbf{x}}(V_{\mathbf{x}}(L'))$ with some intermediate subring L' of T/L such that $[L' : L]_i < \infty$. Hence, by Lemma 1 (ii) and Corollary 3, we obtain $\mathfrak{G}'(T/L')|_H = \mathfrak{G}'(T/L')|_H = \mathfrak{G}'(H/H \cap L') = \mathfrak{G}'(H/H \cap L')$. On the other hand, since $\mathfrak{G}'(T/L)|_{L'} = \mathfrak{G}'(L'/L)$ by Lemma 4, (II) yields $\mathfrak{G}'(T/L)|_{H \cap L'} = \mathfrak{G}'(H \cap L'/L)$. Now, combining these with Lemma 1 (ii), one can easily obtain $\mathfrak{G}'(T/L)|_H = \mathfrak{G}'(H/L) (= \mathfrak{G}'(H/L))$.

From (3) of this theorem, we can conclude that, if K/L is quasi-Galois such that $[K : H]_i < \infty$, then K/L is Galois as was declared in the introduction.

In virtue of the validity of Corollary 3, a similar argument as in the proof of [5, Proposition 2] will apply to obtain the following lemma, whose proof may be left to readers.

Lemma 5. *If K/L is quasi-Galois and T an intermediate subring of H/L , then K/T is locally finite.*

We shall conclude this section with the principal theorem.

Theorem 2. *If K/L is quasi-Galois, then so is K/T for any division subring T containing L such that $[V_{\mathbf{x}}(L) : V_{\mathbf{x}}(T)]_r < \infty$.*

Proof. Let L' be an intermediate division subring of T/L such that $[L' : L]_i < \infty$ and $V_{\mathbf{x}}(T) = V_{\mathbf{x}}(L')$. Then K/L' is quasi-Galois by Corollary 3. Since $H' = V_{\mathbf{x}}(V_{\mathbf{x}}(L')) \supseteq T \supseteq L'$, K/T is locally finite by Lemma 5. Moreover, noting that $V_{\mathbf{x}}(V_{\mathbf{x}}(T)) = H'$, K/T satisfies (II') by Theorem 1,

and H'/T is Galois by [2, Theorem 9]. It follows, again by Theorem 1, K/T is quasi-Galois.

3. Locally Galois extensions.

In this section, we shall consider a locally Galois division ring extension K/L . Our first theorem of this section is the next one, which asserts [3, Theorem 2] is still valid without assuming that K/L is Galois.

Theorem 3. *If K/L is locally Galois, and T a division subring containing L such that $[V_{\kappa}(L) : V_{\kappa}(T)]_r < \infty$, then K/T is locally Galois.*

Proof. Let L' be an intermediate subring of T/L such that $V_{\kappa}(L') = V_{\kappa}(T)$ and $[L' : L]_i < \infty$. Then, for an arbitrary finite subset F of K , there exists a division subring L^* containing $L'[F]$ such that L^*/L is Galois and $[L^* : L]_i < \infty$. And then, by [4, Lemma 2], $H^* = V_{\kappa}(V_{\kappa}(L^*))$ coincides with $H[L^*]$ (which is left finite over H by [4, Theorem 1]) and contains $T[F]$. Since K/L is evidently quasi-Galois, there holds $\mathcal{G}(L^*/L) = \mathcal{G}(H^*/L)|_{L^*}$ by Lemma 4. Hence each L -automorphism of L^* can be extended to an isomorphism of H^* . Further, recalling that $\mathcal{G}(H/L) = \mathcal{G}(H/L)$ (Lemma 1 (ii)), one will easily see that the extended isomorphism is in fact an automorphism of $H^* = H[L^*]$. Combining this with the fact that H^*/L^* is Galois by Corollary 3, it will be easily seen that H^*/L is Galois. Moreover, by [3, Lemma 1 (i)], there holds $[V_{H^*}(L) : V_{H^*}(H^*)]_r < \infty$, whence it follows that H^*/T is locally Galois by [3, Theorem 1]. Since $H^* \supseteq T[F]$, we have proved our assertion.

Now from the proof of Theorem 3, one will penetrate the necessary part of the following

Corollary 4. *K/L is locally Galois, if and only if K/L is locally finite and, for each finite subset F of K , there exists a division subring T containing $H[F]$ such that $[T : H]_i < \infty$ and T/L is Galois.*

Proof. We shall prove here the sufficiency only. Since $\infty > [T : H]_i \geq [V_r(H) : V_r(T)]_r = [V_r(L) : V_r(T)]_r$ by [3, Lemma 1], T/L is locally Galois again by [3, Theorem 1].

In [3], Galois theory of division rings was undertaken under the assumption that the division ring extension K/L considered is Galois and locally Galois and $[K : H]_i \leq \aleph_0$. Our next theorem contains Corollary 1, and enables us to exclude the assumption cited just now that K/L is Galois.

Theorem 4. *If K/L is locally Galois and $[K : H]_i \leq \aleph_0$, then K/L is Galois.*

Proof. By the validity of Corollary 4, the same argument as in the

proof of [3, Lemma 4] will apply to see that $\mathfrak{G}(H/L) = \mathfrak{G}^*|H$, where \mathfrak{G}^* is the group of all L -automorphisms of K . Since $\mathfrak{G}^* \supseteq V_{\mathfrak{z}}(L)$, we obtain eventually $J(\mathfrak{G}^*, K) = L$.

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