ON SOME PROPERTIES OF GROUP CHARACTERS II

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Introduction. Let $\mathfrak{G}$ be a group of finite order g and let p be a fixed prime number. We set $g = p^a g'$ with $(p, g') = 1$. An arbitrary element G of $\mathfrak{G}$ can be written uniquely as a product RP of two commutative elements where R is a p-regular element, i. e. an element whose order is prime to p, while P is a p-element, i. e. an element whose order is a power of p. We shall call R the p-regular factor of G and P the p-factor of G. We define the p-regular section $S(R)$ of a p-regular element R as the set of all elements of $\mathfrak{G}$ whose p-regular factors are conjugate to R in $\mathfrak{G}$. Let $\mathfrak{K}_\nu$ be a class of conjugate elements in $\mathfrak{G}$ which contains an element whose p-regular factor is R. Then we see that $S(R)$ is the union of these classes $\mathfrak{K}_\nu$. Let $R_1 = 1, R_2, \cdots, R_t$ be a system of p-regular elements such that they all lie in different classes of conjugate elements in $\mathfrak{G}$, but that every p-regular element of $\mathfrak{G}$ is conjugate to one of them. Then all elements of $\mathfrak{G}$ are distributed into t p-regular sections $S(R_i)$.

We consider the representations of $\mathfrak{G}$ in the field of all complex numbers. Let $\chi_1, \chi_2, \cdots, \chi_n$ be the distinct irreducible characters of $\mathfrak{G}$ and let $\mathfrak{P}$ be a p-Sylow subgroup of $\mathfrak{G}$. Then the χ_i are distributed into a certain number of $\mathfrak{P}$-blocks $\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_s$ [7]. The main object of this note is to prove the following

Theorem 1. *Let $\mathfrak{B}$ be a $\mathfrak{P}$-block of $\mathfrak{G}$. If the elements G and H of $\mathfrak{G}$ belong to different p-regular sections of $\mathfrak{G}$, then*

$$\sum \chi_i(G)\bar{\chi}_i(H) = 0 \tag{1}$$

where the sum extends over all $\chi_i \in \mathfrak{B}$.

In § 1 we shall give, as preliminaries, some results concerning the induced characters of $\mathfrak{G}$([5], [8]). Applying Clifford's results on induced characters in § 1, we shall prove theorem 1 in § 2. This proof was obtained some years ago.

Let q be a prime number. As was shown in [4], the χ_i are distributed into a certain number of q-blocks $B_1, B_2, \cdots, B_r$. Recently K. Iizuka [6] pointed out that a $\mathfrak{P}$-block $\mathfrak{B}$ is a collection of q-blocks for any prime number q different from p. This, combined with Brauer's results ([1], [2], [3]), yields an alternative proof of theorem 1. We shall give this proof in § 3.

1. We set

$$\mathfrak{P}_0 = \bigcap_{G \in \mathfrak{G}} G^{-1}\mathfrak{P}G.$$

Then $\mathfrak{P}_0$ is the maximal normal p-subgroup of $\mathfrak{G}$. The irreducible characters θ_κ of $\mathfrak{P}_0$ are distributed into a certain number of classes of characters which are associated with regard to $\mathfrak{G}$; two characters θ_κ and θ_λ being associated if

$$\theta_\lambda(P_0) = \theta_\kappa(G^{-1}P_0G)$$

where P_0 is a variable element of $\mathfrak{P}_0$ and G is a fixed element of $\mathfrak{G}$. The totality of elements G of $\mathfrak{G}$ which satisfy $\theta_\kappa(P_0) = \theta_\kappa(G^{-1}P_0G)$ for any P_0 in $\mathfrak{P}_0$ constitutes a subgroup $\mathfrak{G}_\kappa$ of $\mathfrak{G}$ which will be called a subgroup of $\mathfrak{G}$ corresponding to θ_κ. Let T_μ ($\mu = 1, 2, \cdots, h$) be a complete residue system of $\mathfrak{G}$ (mod $\mathfrak{G}_\kappa$)

$$\mathfrak{G} = \mathfrak{G}_\kappa T_1 + \mathfrak{G}_\kappa T_2 + \cdots + \mathfrak{G}_\kappa T_h.$$

Then there exist h characters θ_λ which are assosiated to θ_κ. As was shown in [7], $\mathfrak{P}$-blocks of $\mathfrak{G}$ are in $1-1$ correspondence with the associated classes of characters of $\mathfrak{P}_0$. Let $\mathfrak{B}$ be a $\mathfrak{P}$-block of $\mathfrak{G}$ corresponding to an associated class which contains θ_κ. Then we shall say that $\mathfrak{B}$ is a $\mathfrak{P}$-block determined by θ_κ. Let χ_i be any character belonging to a $\mathfrak{P}$-block $\mathfrak{B}$ determined by θ_κ. We have

$$\chi_i(P_0) = d_i \sum_\mu \theta_\kappa(T_\mu^{-1}P_0T_\mu) \qquad \text{(for } P_0 \in \mathfrak{P}_0) \tag{2}$$

where d_i is a positive rational integer. We shall assume that the irreducible characters of $\mathfrak{G}$ belonging to a $\mathfrak{P}$-block $\mathfrak{B}$ determined by θ_κ are given by $\chi_1, \chi_2, \cdots, \chi_l$. Then there exist exactly l irreducible characters $\chi_1^*, \chi_2^*, \cdots, \chi_l^*$ of $\mathfrak{G}_\kappa$ such that

$$\chi_i^*(P_0) = d_i\theta_\kappa(P_0) \qquad \text{(for } P_0 \in \mathfrak{P}_0). \tag{3}$$

If we denote by $\tilde{\chi}_i^*$ the character of $\mathfrak{G}$ induced from χ_i^* of $\mathfrak{G}_\kappa$, then we have

$$\chi_i(G) = \tilde{\chi}_i^*(G) \qquad (i = 1, 2, \cdots, l). \tag{4}$$

If we set $\chi_i^*(G) = 0$ for an element G of $\mathfrak{G}$ outside of $\mathfrak{G}_\kappa$, then

$$\tilde{\chi}_i^*(G) = \sum_\mu \chi_i^*(T_\mu G T_\mu^{-1}) \qquad \text{(for } G \in \mathfrak{G}). \tag{5}$$

In the following we shall denote by $\mathfrak{B}^*$ the collection of these l characters χ_i^* of $\mathfrak{G}_\kappa$. Let χ_i^* be any character in $\mathfrak{B}^*$. As was shown in [5], [8], we have

$$\chi_i^*(V) = \xi_\kappa(V)\zeta_i(V) \qquad \text{(for } V \in \mathfrak{G}_\kappa) \tag{6}$$

where ζ_i is the projective irreducible character of $\mathfrak{G}_\kappa$ with factor set $\{r_{S,T}\}$ and represents the elements of $\mathfrak{P}_0$ by the unit matrix and where ξ_κ is the projective irreducible character of $\mathfrak{G}_\kappa$ with factor set $\{r_{S,T}^{-1}\}$ such that $\xi_\kappa(P_0) = \theta_\kappa(P_0)$ for any P_0 in $\mathfrak{P}_0$. Here the factor set $\{r_{S,T}\}$ is determined by the irreducible representation of $\mathfrak{G}_\kappa$ belonging to θ_κ. The character ζ_i may be considered as the projective irreducible character of $\mathfrak{G}_\kappa/\mathfrak{P}_0$. On the other hand, if ζ_j is a projective irreducible character of $\mathfrak{G}_\kappa/\mathfrak{P}_0$ with factor set $\{r_{S,T}\}$, then $\xi_\kappa(V)\zeta_j(V)$ coincides with one of characters in $\mathfrak{B}^*$. This implies that $\mathfrak{G}_\kappa/\mathfrak{P}_0$ has exactly l projective irreducible characters ζ_i with factor set $\{r_{S,T}\}$ and that the χ_i in $\mathfrak{B}$ are in $1-1$ correspondence with the ζ_i of $\mathfrak{G}_\kappa/\mathfrak{P}_0$ with factor set $\{r_{S,T}\}$.

2. We first consider the special case when $\mathfrak{G}_\kappa = \mathfrak{G}$. In this case we have for any χ_i in $\mathfrak{B}$ determined by θ_κ

(7) $$\chi_i(G) = \chi_i^*(G) = \xi_\kappa(G)\zeta_i(G) \qquad \text{(for } G \in \mathfrak{G}\text{)}.$$

Evidently every modular irreducible character of $\mathfrak{G}/\mathfrak{P}_0$ for p defines a modular irreducible character of $\mathfrak{G}$. On the other hand, every modular irreducible character of $\mathfrak{G}$ for p represents the elements of $\mathfrak{P}_0$ by the unit matrix, since the order of $\mathfrak{P}_0$ is a power of p and hence defines a modular irreducible character of $\mathfrak{G}/\mathfrak{P}_0$. Thus we see that the number of classes of conjugate p-regular elements in $\mathfrak{G}/\mathfrak{P}_0$ is equal to t. If we denote by $\bar{G}$ the residue class of $G(\text{mod } \mathfrak{P}_0)$, then this implies that any two $\bar{R}_i$ and $\bar{R}_j$ can not be conjugate in $\mathfrak{G}/\mathfrak{P}_0$.

Let the elements G and H belong to different p-regular sections of $\mathfrak{G}$ and let R_i and R_j be the p-regular factors of G and H respectively. Then residue classes $\bar{G}$ and $\bar{H}$ can not be conjugate in $\mathfrak{G}/\mathfrak{P}_0$, since $\bar{R}_i$ and $\bar{R}_j$ are not conjugate in $\mathfrak{G}/\mathfrak{P}_0$. Hence we have from the orthogonality relations for projective group characters [7]

$$\sum_{\chi_i \in \mathfrak{B}} \chi_i(G)\bar{\chi}_i(H) = \xi_\kappa(G)\bar{\xi}_\kappa(H) \sum_i \zeta_i(\bar{G})\bar{\zeta}_i(\bar{H}) = 0.$$

This shows that theorem 1 is true in the case when $\mathfrak{G}_\kappa = \mathfrak{G}$.

Now we shall consider the general case. We have from (4) and (5)

(8) $$\sum_{\chi_i \in \mathfrak{B}} \chi_i(G)\bar{\chi}_i(H) = \sum_\mu \sum_\nu \sum_{\chi_i^* \in \mathfrak{B}^*} \chi_i^*(T_\mu G T_\mu^{-1})\bar{\chi}_i^*(T_\nu H T_\nu^{-1}).$$

If one of two elements $T_\mu G T_\mu^{-1}$ and $T_\nu H T_\nu^{-1}$, say, $T_\mu G T_\mu^{-1}$ is not contained in $\mathfrak{G}_\kappa$, then the right side of (8) vanishes, since $\chi_i^*(T_\mu G T_\mu^{-1}) = 0$ for any χ_i^* in $\mathfrak{B}^*$. Now we assume that both $T_\mu G T_\mu^{-1}$ and $T_\nu H T_\nu^{-1}$ are contained in $\mathfrak{G}_\kappa$. Then the p-regular factors of $T_\mu G T_\mu^{-1}$ and $T_\nu H T_\nu^{-1}$ are given by $T_\mu R_i T_\mu^{-1}$ and $T_\nu R_j T_\nu^{-1}$ respectively. Since $T_\mu R_i T_\mu^{-1}$ is a power of $T_\mu G T_\mu^{-1}$, $T_\mu R_i T_\mu^{-1}$ is contained in $\mathfrak{G}_\kappa$. Similarly $T_\nu R_j T_\nu^{-1}$ belongs to $\mathfrak{G}_\kappa$. It follows that the elements $T_\mu G T_\mu^{-1}$ and $T_\nu H T_\nu^{-1}$ of $\mathfrak{G}_\kappa$ belong to different p-regular sections of $\mathfrak{G}_\kappa$. Hence

$$\sum_{\chi_i^* \in \mathfrak{B}^*} \chi_i^*(T_\mu G T_\mu^{-1})\bar{\chi}_i^*(T_\nu H T_\nu^{-1}) = 0,$$

as was shown in the above special case. Consequently we have

$$\sum_{\chi_i \in \mathfrak{B}} \chi_i(G)\bar{\chi}_i(H) = 0$$

and theorem 1 is proved completely.

By the similar way as in [9], we can prove the following

Theorem 2. *Let $S(R)$ be a p-regular section of $\mathfrak{G}$. If the characters χ_i and χ_j of $\mathfrak{G}$ belong to different $\mathfrak{P}$-blocks, then*

$$\sum \chi_i(G)\bar{\chi}_j(G) = 0 \tag{9}$$

where the sum extends over all $G \in S(R)$.

We may assume that 1-character of $\mathfrak{G}$ belongs to $\mathfrak{B}_1$. Then it follows from theorem 2 that

$$\sum \chi_i(G) = 0 \qquad (\text{for } \chi_i \notin \mathfrak{B}_1) \tag{10}$$

where the sum extends over all $G \in S(R)$. In particular we find

$$\sum \chi_i(P) = 0 \qquad (\text{for } \chi_i \notin \mathfrak{B}_1) \tag{11}$$

where the sum extends over all p-elements P of $\mathfrak{G}$.

3. Let q be a prime number. We denote by $B_1, B_2, \cdots, B_r$ the q-blocks of $\mathfrak{G}$. If the elements G and H of $\mathfrak{G}$ belong to different q-sections of $\mathfrak{G}$, then we have ([2], [3])

$$\sum_{\chi_i \in B} \chi_i(G)\bar{\chi}_i(H) = 0. \tag{12}$$

On the other hand, if G and H have the same q-factor Q, that is, $G = QV = VQ$ and $H = QW = WQ$, then we have ([1], [2], [3])

$$\sum_{\chi_i \in B} \chi_i(QV)\bar{\chi}_i(QW) = \sum_{\chi_i^Q \in B^Q} \chi_i^Q(V)\bar{\chi}_i^Q(W) \tag{13}$$

where the χ_i^Q denote the irreducible characters of the normalizer $\mathfrak{N}(Q)$ of Q in $\mathfrak{G}$ and where B^Q denotes the collection of q-blocks of $\mathfrak{N}(Q)$ which determine the q-block B of $\mathfrak{G}$.

Using (12) and (13), we shall prove theorem 1 in the following. Let $q_1, q_2, \cdots, q_m$ be the distinct prime number which divide g'. Let R be the p-regular factor of G:

$$G = RP = PR.$$

Then R can be written uniquely as a product $Q_1 Q_2 \cdots Q_m$ of mutually commutative q_i-elements Q_i. Evidently each Q_i is the q_i-factor of G. Similarly we have for $H = R'P'$

$$R' = Q'_1 Q'_2 \cdots Q'_m.$$

We set $R_i = Q_i Q_{i+1} \cdots Q_m$ and $R'_i = Q'_i Q'_{i+1} \cdots Q'_m$. Let G and H belong to different p-regular sections, that is, R and R' be not conjugate in $\mathfrak{G}$.

Since any $\mathfrak{P}$-block $\mathfrak{B}$ of $\mathfrak{G}$ is a collection of q_i-blocks of $\mathfrak{G}$, if Q_1 and Q'_1 are not conjugate in $\mathfrak{G}$, then we have from (12)

$$\sum_{\chi_i \in \mathfrak{B}} \chi_i(G)\bar{\chi}_i(H) = 0.$$

We consider the case when Q_1 and Q'_1 are conjugate in $\mathfrak{G}$. Replacing H by an element $S^{-1}HS$ if necessary, we may assume that the q_1-factor of H is Q_1:

$$H = Q_1 Q'_2 \cdots Q'_m.$$

If Q_2 and Q'_2 are conjugate in the normalizer $\mathfrak{N}(Q_1)$ of Q_1 in $\mathfrak{G}$, then replacing H by an element $V^{-1}HV$, $V \in \mathfrak{N}(Q_1)$, we may assume that the q_2-factor of H is Q_2. On applying the same argument successively, we can find the Q_k and Q'_k such that

$$Q_i = Q'_i \qquad (i = 1, 2, \cdots, k-1)$$

and that Q_k and Q'_k are not conjugate in the normalizer $\mathfrak{N}(Q_1 Q_2 \cdots Q_{k-1})$ of $Q_1 Q_2 \cdots Q_{k-1}$ in $\mathfrak{G}$, since R and R' are not conjugate in $\mathfrak{G}$. It follows from (13) that

$$\sum_{\chi_i \in \mathfrak{B}} \chi_i(G)\bar{\chi}_i(H) = \sum_{\chi_i^{(1)} \in \mathfrak{B}^{(1)}} \chi_i^{(1)}(R_1 P)\bar{\chi}_i^{(1)}(R'_1 P')$$

where the $\chi_i^{(1)}$ denote the irreducible characters of $\mathfrak{N}(Q_1)$ and where $\mathfrak{B}^{(1)}$ denotes the collection of q_1-blocks of $\mathfrak{N}(Q_1)$ which determine $\mathfrak{B}$. We see easily that $\mathfrak{B}^{(1)}$ is also the collection of q_i-blocks of $\mathfrak{N}(Q_1)$ for any prime number q_i. Thus applying (13) successively, we have

$$\text{(14)} \qquad \sum_{\chi_i \in \mathfrak{B}} \chi_i(G)\chi_i(H) = \sum_{\chi_i^{(k)} \in \mathfrak{B}^{(k)}} \chi_i^{(k)}(R_k P)\chi_i^{(k)}(R'_k P')$$

where the $\chi_i^{(k)}$ denote the irreducible characters of the normalizer $\mathfrak{N}(Q_1 Q_2 \cdots Q_{k-1})$ and where $\mathfrak{B}^{(k)}$ is the collection of q_i-blocks of $\mathfrak{N}(Q_1 Q_2 \cdots Q_{k-1})$ for any prime number q_i. Since R_k and R'_k belong to different q_k-sections of $\mathfrak{N}(Q_1 Q_2 \cdots Q_{k-1})$, we see from (12) that the right side of (14) vanishes. This completes the proof of theorem 1.

We see from our second proof that theorems 1 and 2 are also true for any collection of q_i-blocks of $\mathfrak{G}$ for any prime number q_i different from p.

REFERENCES

[1] R. BRAUER : On the connection between the ordinary and the modular characters of groups of finite order, Ann. of Math., 42 (1941), 926—935.
[2] ________ : On blocks of characters of groups of finite order II, Proc. Nat. Acad. Sci. U. S. A., 32 (1949), 215—219.
[3] ________ : Zur Darstellungstheorie der Gruppen endlicher Ordnung II, Math. Zeitschr., 72 (1959), 25—46.
[4] R. BRAUER and C. NESBITT : On the modular characters of groups, Ann. of Math., 42 (1941), 556—590.
[5] A. H. CLIFFORD : Representations induced in an invariant subgroup, Ann. of Math., 38 (1937), 533—550.
[6] K. IIZUKA : On Osima's blocks of group characters, Proc. Jap. Acad., 36 (1960), 392—396.
[7] M. OSIMA : On the representations of groups of finite order, Math. J. Okayama Univ., 1 (1952), 33—61.
[8] ________ : On the representations of the generalized symmetric group II, Math. J. Okayama Univ., 6 (1956), 81—97.
[9] ________ : On some properties of group characters, Proc. Jap. Acad., 36 (1960), 18—21.

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