SPECTRAL THEORY OF OPERATOR ALGEBRAS II

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This paper is a continuation of the preceding "Spectral Theory of Operator Algebras I" in this journal and consists of the latter three chapters.

Chapter 3. Extensions of Kaplansky Density Theorem and Gelfand-Naimark Representation Theory.

§ 1. An extension of Kaplansky Density Theorem.

(a). C*-algebra in general Banach space.

In what follows we consider C^*-algebras whose underlying spaces are not generally Hilbert spaces. Then we first define a C^*-algebra in a general Banach space.

Definition 1. 1. A topologico-algebraic homomorphism of a C^*-algebra A in an operator algebra in a certain Banach space $\mathfrak{B}$ is said to be a *representation* of A in $\mathfrak{B}$.

An operator algebra A on a Banach space $\mathfrak{B}$ is said to be a *C^*-algebra on* $\mathfrak{B}$ if A is isometric and isomorphic to a C^*-algebra.

The above definition is compatible with the ordinary definition of C^*-algebra whenever the underlying Banach space $\mathfrak{B}$ is a Hilbert space. In fact, let A be an operator algebra in a Hilbert space $\mathfrak{H}$ which contains the identity, where we do not need to assme that A is self-adjoint. If A is isometric and isomorphic to another C^*-algebra, then A is a uniformly closed self-adjoint algebra in the Hilbert space.

Proposition 1. 1.[1] *If a C^*-algebra A is represented in a Banach space $\mathfrak{B}$, then the representative algebra $\mathrm{A}_{\mathfrak{B}}$ is topologico-algebraically isomorphic to a C^*-algebra. If the representation $A \to A_{\mathfrak{B}}$ satisfies $|A_{\mathfrak{B}}| \leqq |A|$, where $|A_{\mathfrak{B}}|$ is the operator norm of $A_{\mathfrak{B}}$ in $\mathfrak{B}$, then $\mathrm{A}_{\mathfrak{B}}$ is a C^*-algebra on $\mathfrak{B}$.*

To prove the proposition we prepare the next lemma.

Lemma 1. 1. *Let A be an element of A, and put $B = (A^*A)^{\frac{1}{2}}$. Then two sequences $\{U_n\}$ and $\{V_n\}$ in A can be so chosen that $|U_n| \leqq 1$, $|V_n| \leqq 1$, $|U_nA - B| \to 0$ and $|V_nB - A| \to 0$.*

Proof. Assume that the uniform closure $\mathfrak{U}(A)$ of the set $(UA : U \in \mathrm{A}$

1) Kaplansky 8).

and $|U| \leqq 1$) does not contain $\mathfrak{B}$. Then a functional $f \in \mathrm{A}$ and a number $\delta > 0$ can be so chosen that $\mathfrak{Re}\ f(B)+\delta \leqq \mathfrak{Re}\ f(UA)$ for every $U \in \mathrm{A}$ with $|U| \leqq 1$. Consider a positive linear functional p on A with $f \in L^2(p)$ and a partially isometric operator $U_0 \in \mathrm{A}_p''$ in $L^2(p)$ such that $B=(A^*A)^{\frac{1}{2}} = U_0A$. U_0 is contained in the strong closure of the unit ball of A_p, and $f(B) = (Bf, p)_p = (U_0Af, p)_p$ hold. It is incompatible with the above Klein-Smulian's inequality. Then $\mathfrak{U}(A)$ contains B. Similarly $\mathfrak{U}(B)$ contains A. *Q.E.D.*

Proof of Proposition 1.1. Let $A \rightarrow A_\mathfrak{B}$ be the representation of A in $\mathrm{A}_\mathfrak{B}$. Then a number k can be so chosen as $|A_\mathfrak{B}| \leqq k|A|$. Consider an ideal $\mathrm{N}=(A \in \mathrm{A};\ A_\mathfrak{B}=0)$ of A. Then $|A_\mathfrak{B}| \leqq \inf_{B \in \mathrm{N}} k|A-B| = |A/\mathrm{N}|$. It is well-known that A/N is a C^*-algenra. Then to prove the proposition it is sufficient to show the inequality $k|A_\mathfrak{B}| \geqq |A/\mathrm{N}|$. Let A be a fixed element of A, and put $B=(A^*A)^{\frac{1}{2}}$. By Lemma 1.1 we have $k|B_\mathfrak{B}| \leqq |A_\mathfrak{B}| \leqq k^{-1}|B_\mathfrak{B}|$, and $k=1$ implies $|B_\mathfrak{B}| = |A_\mathfrak{B}|$. The smallest C^*-sub-algebra R of A which contains I and B is an abelian algebra. Let Ω denote the spectrum of the quotient algebra $\mathrm{R}/(\mathrm{N} \cup \mathrm{R})$. Ω is the totality of maximal ideals of R which contains $\mathrm{N} \cap \mathrm{R}$. And $|X/(\mathrm{N} \cap \mathrm{R})| = \sup_{\lambda \in \Omega} |X/\lambda|$ holds for $X \in \mathrm{R}$. On the other hand, $\mathrm{R}/\mathrm{N} \cap \mathrm{R}$ is a normed algebra with the norm $|||X/\mathrm{N} \cap \mathrm{R}||| = |X_\mathfrak{B}|$. By the Gelfand-Silov's Theorem we have

$$|X_\mathfrak{B}| \geqq |X_\mathfrak{B}^n|^{\frac{1}{n}} \rightarrow \sup_{\lambda \in \Omega} |X/\lambda| = |X/(\mathrm{N} \cap \mathrm{R})| \geqq |X/\mathrm{N}|.$$

Thus we have $|A/\mathrm{N}| = |B_\mathfrak{B}| \leqq k|A_\mathfrak{B}|$, from which the proposition follows.

(b). A generalization of the Kaplansky Density Theorem.

Consider a fixed C^*-algebra A and its left ideal N. The quotient space A/N is a Banach space with the quotient norm $|A/\mathrm{N}| = \inf_{B \in \mathrm{N}} |A-B|$, and A is represented as an operator algebra $\mathrm{A_N}$, so called the regular representation algebra of A, in A/N. The represented operator A_N of $A \in \mathrm{A}$ is an operator in A/N with $A_\mathrm{N}x = AX/\mathrm{N}$, where $x = X/\mathrm{N} \in \mathrm{A/N}$. By Proposition 1.1 $\mathrm{A_N}$ is a C^*-algebra on A/N.

A/N is a Banach space, and the strong topology is defined in the totality of bounded operators in it. Consider the strong closure H of the totality of Hermitian elements in $\mathrm{A_N}$ and the set $\mathrm{A_N}^q = (X+iY:\ X,\ Y \in \mathrm{H})$ of bounded operators in A/N. If $A = X+iY(X,\ Y \in \mathrm{H})$ belongs to $\mathrm{A_N}^q$, the operator $A^* = X-iY$ is called the *adjoint* of A.

As we shall observe below, $\mathrm{A_N}^q$ is a C*-algebra on A/N. The *Q*-topology* is defined as the self-adloint strong topology in $\mathrm{A_N}^q$. A *Q*-neighbourhood* (a quotient strong neighbourhood) of an $X \in \mathrm{A_N}^q$ is a set

$$U(X: x_1, x_2, \cdots x_n : \varepsilon) = (Y \in \mathrm{A_N}^q : \max_{1 \leqq i \leqq n} (|(X-Y)x_i|, |(X^*-Y^*)x_i|) < \varepsilon),$$

where $x_1, x_2, \cdots x_n$ are a fininite number of elements in A/N and ε is a positive number. A Hausdorf topology of $\mathrm{A_N}^q$, whose open base is the totality of Q^*-neighbourhoods of elements of $\mathrm{A_N}^q$, is said to be the Q^*-*topology* (quotient strong topology) of $\mathrm{A_N}^q$ divided by N. And a Q^*-closed $*$-sub-algebra of $\mathrm{A_N}^q$ is said to be a Q^*-*sub-algebra* of $\mathrm{A_N}^q$.

An *urtra-Q-*neighbourhood* (a quotient urtra-strong neighbourhood) of an $X \in \mathrm{A_N}^q$ is a set $\mathfrak{U}(X: \{x_i\}, \varepsilon) = (Y \in \mathrm{A_N}^q : \sup (|(X-Y)x_i|, |(X^*-Y^*)x_i|) < \varepsilon)$, where $\{x_i\}$ is a uniformly convergent sequence in A/N and ε is a positive number. A Hausdorff topology of $\mathrm{A_N}^q$, whose open base in the totality of urtra-Q^*-neighbourhoods of elements of $\mathrm{A_N}^q$, is said to be an *urtra-Q^*-topology* (quotient urtra-strong topology) of $\mathrm{A_N}^q$ devided by N.

The next Theorem 9 is a generalized Kaplansky Density Theorem relative to these Q^*- and urtra-Q^*-topologies. The extension problem of v. Neumann Density Theorem shall be dealt with in the next chapter.

Theorem 9. (Generalized Kaplansky Density Theorem). *Let* A *be a C^*-algebra,* N *its left ideal and* $\mathrm{A_N}$ *the representative algebra of* A *in* A/N *by the regular representation. Then* $\mathrm{A_N}^q$ *is a C^*-algebra on* A/N, *and*

(1). *The Q^*-closure and the urtra-Q^*-closure of a C^*-sub-algebra of* $\mathrm{A_N}^q$ *are an identical C^*-algebra.*

(2). *Let* R *be a C^*-subalgebra of $A_N{}^q$ and* R^q *be its Q^*-closure. Then the urtra-Q^*-closure of the unit ball of* R *is the unit ball of* R^q.

To prove the theorem, we need to prepare four sub-lemmas. The totality $\mathcal{S}(\mathrm{N})$ of states on A which vanish on N is a regularly convex subset of the dual space $\bar{\mathrm{A}}$ of A, and we have $|A/\mathrm{N}| = \sup_{p \in \mathcal{S}(\mathrm{N})} \|Ap\|_p$. If s is a state in $\mathcal{S}(\mathrm{N})$ and t is a state in $L^2(s)$, then t belongs to $\mathcal{S}(\mathrm{N})$ (cf. Lemma 4.1, 4.2 in Chapter 2), and the left ideal $\mathrm{N}(s) = (A \in \mathrm{A} : s(A^*A) = 0)$ contains the ideal N.

The dual space of A/N is the totality $\Phi(\mathrm{N})$ of functionals $\in$ A which vanish in N. We define the product of $f \in \Phi(\mathrm{N})$ and $x \in \mathrm{A/N}$ as follows: Let $x = A/\mathrm{N}$ (where $A \in \mathrm{A}$). Then $xf = Af$. xf is uniquely determined because $x = A/\mathrm{N} = B/\mathrm{N}$ imply $Af = Bf$. If X is a bounded operator in A/N and if $A \in \mathrm{A}$, then we define $X(A) = X(A/\mathrm{N})$.

Sub-lemma 1. *Let X be an operator in the unit ball of* $\mathrm{A_N}^q$ *and p a state in $\mathcal{S}(\mathrm{N})$. Then an operator X_p in $L^2(p)$ is determined* in such a way that $X_pAp = X(A)p$ $(A \in \mathrm{A})$ *and belongs to* $\mathrm{A_p}''$.

Proof. First we assume that p is a finite dimensional state in $\mathcal{S}(\mathrm{N})$. A_p' is a finite dimensional algebra. By Lemma 4.2 in Chapter 1 E_p is a finite dimensional projection, and p is a linear sum of finite number of pure states in $\mathcal{S}(\mathrm{N})$.

By Proposition 2.1 in Chapter 2, E_p is a regular projection in A_p'' and $A/\mathrm{N}(p) \leftrightarrow Ap$ is an isomorphism between $\mathrm{A}/\mathrm{N}(p)$ and $L^2(p)$. Hence, given any $x \in L^2(p)$, we can choose an $A \in \mathrm{A}$ with $x = Ap$.

By the assumption, X belongs to the unit ball of $\mathrm{A}_\mathrm{N}{}^q$. Then $|X(A)| \leqq |A/\mathrm{N}|$. Since $\mathrm{A}_\mathrm{N}{}^q$ is contained in the strong closure of the regular representation A_N of A, $A \in \mathrm{N}(p)$ implies $|X(A)| \in \mathrm{N}(p)/\mathrm{N}$ and $|X(A)/\mathrm{N}(p)| \leqq |A/\mathrm{N}(p)|$. A bounded operator X_p in $\mathrm{A}/\mathrm{N}(p)$ is so chosen as $X_p(A/\mathrm{N}(p)) = X(A)/\mathrm{N}(p)$, and its operator norm is $\leqq 1$. By the isomorphism between $L^2(p)$ and $A/\mathrm{N}(p)$, X_p is regarded as a bounded operator in $L^2(p)$ with $X(A)p = X_pAp$ (for every $A \in \mathrm{A}$) and belongs to A_p''. In fact, given any $A_1p, A_2p, \cdots, A_np \in L^2(p)$ and any $\varepsilon > 0$, we can choose a $B \in \mathrm{A}$ with $|(X(A_i) - B(A_i))/\mathrm{N}| < \varepsilon$. Then $\|(X_p - B)A_ip\|_p < \varepsilon$, and X_p belongs to the strong closure of A_p.

We shall show that the norm of X_p, as an element of A_p'', is $\leqq 1$.

The state p is extended to that of A_p'' with $p(A) = (Ap, p)_p$ (for $A \in \mathrm{A}_p''$). Consider a left ideal $\mathrm{N}''(p) = (X \in \mathrm{A}_p'' : p(X^*X) = 0)$ of the algebra A_p''. The algebra A_p'' is faithfully represented on an operator algebra on $\mathrm{A}_p''/\mathrm{N}''(p)$ by the regular representation. The operator norm of $A \in \mathrm{A}_p''$ and the norm of X_p in $L^2(p)$ are the norms as operators in $\mathrm{A}_p''/\mathrm{N}''(p)$. Since $\mathrm{A}_p''/\mathrm{N}''(p) = \mathrm{A}_p/\mathrm{N}(p)$ holds, the norm of X_p, as an element of A_p'', is that of X_p as an operator in $\mathrm{A}_p/\mathrm{N}(p)$ and satisfies $|X_p| \leqq 1$.

We shall now define the operator $X_p \in \mathrm{A}_p''$ for every state p in $\mathcal{S}(\mathrm{N})$. If p is of finite dimensional, then, as we already observed, X_p belongs to A_p'' and $|X_p| \leqq 1$ holds. Notice that $\mathcal{S}(\mathrm{N})$ contains finite dimensional states everywhere dense in it. Then we obtain

$$p(X(A)^*X(A)) \leqq p(A^*A)$$

and

$$X(A)p(B^*) = (X(A)p, Bp)_p = (Ap, X^*(B)p)_p = X^*(B)p(A)$$

for every $p \in \mathcal{S}(\mathrm{N})$ and $A, B \in \mathrm{A}$. Now X_p is an operator in $L^2(p)$ such that $|X_p| \leqq 1$ and $X_p(Ap) = X(A)p$ for $A \in \mathrm{A}$. Then it is sufficient to show the relation $X_p \in \mathrm{A}_p''$. Consider a state p in $\mathcal{S}(\mathrm{N})$ and a definite Hermitian operator $K \in \mathrm{A}_p'$ with $Kp(I) = 1$. Then $q = Kp$ belongs to $\mathcal{S}(\mathrm{N})$ and

$$
\begin{aligned}
(X_p(KAp),\ Bp)_p &= (KAp,\ X^*(B)p)_p \\
&= (Aq,\ X^*(B)q)_q = (X(A)q,\ Bq)_q \\
&= (KX_p(Ap),\ Bp)_p.
\end{aligned}
$$

Hence $X_pK = KX_p$ and $X_p \in \mathrm{A}_p''$.

Sub-lemma 2. $\mathrm{A_N}^q$ *is a* C^**-algebra in* $\mathrm{A/N}$.

Proof. Consider a product space $\mathfrak{H} = \sum_{p\in\mathcal{S}(\mathrm{N})} \oplus L^2(p)$. Each $X \in \mathrm{A_N}^q$ is represented as an operator $\sum \oplus X_p$ on $\mathfrak{H}$, where $|X| = \sup_{p\in\mathcal{S}(\mathrm{N})} |X_p|$ is the operator norm of $\sum \oplus X_p$ and the representation $X \to \sum \oplus X_p$ is isometric. Then $\mathrm{A_N}^q$ is a C^*-algebra.

Sub-lemma 3. *Let* R *be a* C^**-sub-algebra of* $\mathrm{A_N}^q$, R^q *its* Q^**-closure and* X *an Hermitian element in the unit ball of* R^q. *Then* X *belongs to the* Q^**-closure of the unit ball of* R^q.

Proof. R^q is a C^*-algebra, and the smallest C^*-subalgebra of R which contains X and I is abelian. Then $Y = X(I+(I-X^2)^{\frac{1}{2}})^{-1}$ is an Hermitian element in the unit ball of R^q, and $X = 2Y(I+Y^2)^{-1}$ holds. If B is an Hermitian element of R, $C = 2B(I+B^2)^{-1}$ belongs to the unit ball of R and we have

$$X - C = 2C(B-Y)X + 2(I+B^2)^{-1}(Y-B)(I+Y^2)^{-1}$$

Notice that $|C| \leqq 1$ and $|(I+B^2)^{-1}| \leqq 1$. Then when B converges to Y in the Q^*-topology, C converges to X and X belongs to the Q^*-closure of the unit ball of R^q.

Sub-lemma 4. *Consider a* C^**-sub algebra* R *of* $\mathrm{A_N}^q$, *and for each state* p *in* $\mathcal{S}(\mathrm{N})$ *let* R_p *denote the representative algebra* $(X_p\colon X \in \mathrm{R})$ *of* R *in* $L^2(p)$. *If* X *is an element of* $\mathrm{A_N}^q$ *with* $|X| \leqq 1$ *and each* X_p *belongs to the strong closure* R_p'' *of* R_p, *then* X *belongs to the urtra-*Q^**-closure of the unit ball of* R.

Proof. The totality $\mathfrak{C}$ of uniformly convergent suquences in A/N is a Banach space, where the norm of $x = \{x_i\} \in \mathfrak{C}$ is $|x| = \sup |x_i|$. If $x = \{x_n\}$ is an element of $\mathfrak{C}$, we put $x_\infty = \lim x_n$. The dual Banach space $\overline{\mathfrak{C}}$ of $\mathfrak{C}$ is determined as follows: Consider a sequence $\{f_n\}$ $(n = \infty, 1, 2, \cdots)$ of elements in the dual space $\Phi(\mathrm{N})$ of A/N with $\sum |f_i| < \infty$. Then $f(x) = \sum f_i(x_i) + f_\infty(x_\infty)$ is a bounded linear functional on $\mathfrak{C}$. $\overline{\mathfrak{C}}$ is the totality of these functionals, and the norm of $f \in \overline{\mathfrak{C}}$ is $|f| = |f_\infty| + \sum |f_i|$.

Consider the product space $\mathfrak{C}^2 = \mathfrak{C} \times \mathfrak{C}$ and its dual space $\overline{\mathfrak{C}}^2 = \overline{\mathfrak{C}} \times \overline{\mathfrak{C}}$, where the norm of $(x, y) \in \mathfrak{C}^2$ is $\max(|x|, |y|)$ and the norm of $(f, g) \in \overline{\mathfrak{C}}^2$ is $|f| + |g|$. Consider moreover the unit ball $U(\mathrm{R})$ of R and a fixed element $x = \{x_i\}$ of $\mathfrak{C}$. Then (Xx, X^*x) $(Xx = \{Xx_i\}, X^*x = \{X^*x_i\})$ is an element of $\mathfrak{C}^2$, and $\mathfrak{W} = (Bx, B^*x) : B \in \mathfrak{U}(\mathrm{R}))$ is a convex sub-set of $\mathfrak{C}^2$.

Now the sub-lemma is reduced to prove that (Xx, X^*x) belongs to the uniform closure, or rather, the weak closure of $\mathfrak{W}$ in $\mathfrak{C}^2$. Then it is sufficient to show that, for any given (f, g) (where $f = \{f_i\}$, $g = \{g_i\}$ $(i = \infty, 1, 2, \cdots)$) in $\overline{\mathfrak{C}}^2$ and for any given positive number ε, a $B \in \mathfrak{U}(\mathrm{R})$ can be so chosen that $|f(Xx - Bx)| < \varepsilon$ and $|g(X^*x - B^*x)| < \varepsilon$.

The absolute variations $p_i = f_i^{*v}$ and $q_i = g_i^{*v}$ of f_i^* and g_i^* belong to $\Phi(\mathrm{N})$ and satisfy:

$$|f| = \sum_{1 \leq i \leq \infty} |f_i| = \sum_{1 \leq i \leq \infty} p_i(I) < \infty, \quad |g| = \sum_{1 \leq i \leq \infty} |g_i| = \sum_{1 \leq i \leq \infty} q_i(I) < \infty.$$

Then

$$p = \sum_{1 \leq i \leq \infty} \alpha(p_i + q_i), \quad (\text{where } \alpha^{-1} = |f| + |g|)$$

belongs to $\mathcal{S}(\mathrm{N})$. Choose definite self-adjoint operators K_i and L_i in A_p' with $p_i = K_i^2 p$ and $q_i = L_i^2 p$, and choose partially isometric operators U_i and V_i in A_p'' such that

$$f_i^* = U_i p_i = U_i K_i^2 p, \quad g_i^* = V_i q_i = V_i L_i^2 p$$

respectively. Then

$$\sum_{1 \leq i \leq \infty} (\|K_i p\|_p^2 + \|L_i p\|_p^2) = \sum_{1 \leq i \leq \infty} (p_i(I) + q_i(I)) = |f| + |g| < \infty,$$

and for every $B \in \mathfrak{U}(\mathrm{R})$ we have

$$|f(Xx - Bx)| \leq \sum_{1 \leq i \leq \infty} |((X_p - B_p)A_i p, \ U_i K_i p)_p|$$

$$\leq (\sum_{1 \leq i \leq \infty} \|K_i p\|_p^2)^{\frac{1}{2}} (\sum_{1 \leq i \leq \infty} \|(X_p - B_p)A_i K_i p\|_p^2)^{\frac{1}{2}}$$

$$|g(X^*x - B^*x)| \leq (\sum_{1 \leq i \leq \infty} \|L_i p\|_p^2)^{\frac{1}{2}} (\sum_{1 \leq i \leq \infty} \|(X_p^* - B_p^*)A_i L_i p\|_p^2)^{\frac{1}{2}}.$$

By the assumption of the sub-lemma, X_p belongs to R_p'' and consequently to the urtra-strong closure of the unit ball of R_p. If we choose a suitable $B \in \mathfrak{U}(\mathrm{R})$, $|f(Xx - Bx)|$ and $|g(X^*x - B^*x)|$ are smaller than the given $\varepsilon > 0$, and (Xx, X^*x) belongs to the uniform closureof $\mathfrak{W}$.

Proof of Theorem 8. Let $\mathfrak{U}_1$ denote the urtra-Q^*-closure of $\mathfrak{U}(\mathrm{R})$, $\mathfrak{U}_2$ the totality of $X \in \mathrm{A}_{\mathrm{N}}^q$ with $|X| \leq 1$ and $X_p \in \mathrm{R}_p''$, $\mathfrak{U}_3$ the unit ball of the Q^*-closure R^q of R, $\mathfrak{U}_4$ the Q^*-closure of $\mathfrak{U}(\mathrm{R})$ and $\mathfrak{U}_5$ the unit ball of the urtra Q^*-closure of R. $\mathfrak{U}_3$, $\mathfrak{U}_4$ and $\mathfrak{U}_5$ contain $\mathfrak{U}_1$ and are contained in $\mathfrak{U}_2$ respectively. By Sub-lemma 2, $\mathfrak{U}_2$ is contained in $\mathfrak{U}_1$. Then they are identical with each other. Q. E. D.

We shall now observe some examples of Q^*-topologies.

Example (1). Let A be a C^*-algebra and N the 0-ideal. Then the Q^*-topology divided by N is the uniform topology of A in the ordinary sense.

(2). Let N be a maximal left ideal of A. Then the quotient space A/N is the Hilbert space $L^2(p)$, where p is a suitable pure state of A. The Q^*-topology divided by N is the self-adjoint strong topology in the total operator algebra on $L^2(p)$.

(3). Let A be a C^*-algebra in a Hilbert space $\mathfrak{H}$, B the total operator algebra on $\mathfrak{H}$, E a certain one-dimensional projection in $\mathfrak{H}$ and N(E) a left ideal N(E) $= (A \in$ B $: AE = 0)$ of B. The quotient space B/N(E) is the Hilbert space $\mathfrak{H}$ and the Q^*-closure of A is the W^*-closure of A in the ordinary sense.

(4). Consider a C^*-algebra A in a Hilbert space $\mathfrak{H}$. Let R be a C^*-algebra which contains A and a projection E and N(E) be a left ideal N(E) $=(A \in$ R$: AE=0)$ of R. The Algehra B^q is regarded as a C^*-algebra in the space $\mathfrak{H}$ (See the next sub-section (c)) and the Q^*-topology of B^q is weaker than the uniform topology but stronger than the self-adjoint strong topology. Let A_0 and A'' denote the uniform and the strong closures of the algebra A. Then $A_0 \subseteqq A^q \subseteqq A''$.

Roughly speaking, these examples show that a Q^*-topology has an intermediate strength between the uniform topology and the self-adjoint strong topology.

(c). Quotient space of a C^*-algebra devided by a projection.

Consider a C^*-algebra A in a Hilbert space $\mathfrak{H}$, a projection E in $\mathfrak{H}$ and the uniform closure A/E of the set $(AE : A \in$ A$)$. A/E is said to be the *quotient space* of A *divided by E*.

A is represented in an operator algebra in A/E. The represented operator of $A \in$ A in A/E: $X \in$ A/$E \rightarrow AX \in$ A/E is said to be the *regular representation* of A. The represented operator algebra of A in A/E is a C^*-algebra on A/E.

Lemma 1. 1. *Consider a C^*-algebra* A *in a Hilbert space* $\mathfrak{H}$ *and a projection E in* $\mathfrak{H}$. *Let* $\mathfrak{H}_0$ *denote the subspace of* $\mathfrak{H}$ *generated by E.* (*Namely,* $\mathfrak{H}_0$ *is the smallest unifomly closed linear set which contains the set* $(AEx : A \in$ A, $x \in \mathfrak{H})$). *Then the representative algebra of* A *in* A/E *is the induced algebra* A_E *of* A *in* $\mathfrak{H}_0$.

Consider the totality B(A/E) of bounded operators X in the space $\mathfrak{H}_0$ such that $B \in$ A/E implies XB, $X^*B \in$ A/E. B(A/E) is a C^*-algebra and contains the regular representation A_E of A in A/E. Now the Q^*-topology is defined in B(A/E).

A Q^*-*neighbourhood* of an $A \in$ B(A/E) is a set

$U(A: X_1, X_2, \cdots X_n;\ \varepsilon)$
$= (B \in \mathrm{B}(\mathrm{A}/E):\ \max(|(A-B)X_i|,\ |(A^*-B^*)X_i|) < \varepsilon),$

where $X_1, X_2, \cdots X_n$ are a finite number of elements in A/E and ε is a positive number. A Hausdorff topology of $\mathrm{B}(\mathrm{A}/E)$ whose open base is the totality of Q^*-neighbourhoods of elements of A/E is said to be the *Q^*-topology* of $\mathrm{B}(\mathrm{A}/E)$ divided by E.

An urtra-*Q^*-neighbourhood* of an $A \in \mathrm{B}(\mathrm{A}/E)$ is a set

$U(A: \{X_n\}, \varepsilon)$
$= (B \in \mathrm{B}(\mathrm{A}/E):\ \sup_i (|(A-B)X_i|,\ |(A^*-B^*)X_i| < \varepsilon),$

where $\{X_n\}$ is a uniformly convergent sequence in A/E and ε is a positive number. A Hausdorff topology of $\mathrm{B}(\mathrm{A}/E)$ whose open base is the totality of urtra-Q^*-neighbourhoods of elements of $\mathrm{B}(\mathrm{A}/E)$ is said to be the urtra-Q^*-topology of $\mathrm{B}(\mathrm{A}/E)$ divided by E.

Lemma 1. 2. *Let* A *be a C^*-algebra in a Hilbert space* $\mathfrak{H}$, *E a projection in* $\mathfrak{H}$ *and* $\mathrm{A} \cup E$ *the smallest C^*-algebra which contains* A *and* E. *Consider a left ideal* $\mathrm{N}(E) = (A \in \mathrm{A} \cup E:\ AE = 0)$ *of* $\mathrm{A} \cup E$, *the algebra* $\mathrm{B} = \mathrm{B}(\mathrm{A}/E)$ *and a left ideal* $\mathrm{N_B}(E) = (A \in \mathrm{B} \cup E:\ AE = 0)$ *of* $\mathrm{B} \cup E$. *Then*

$$(\mathrm{A} \cup E)/E = (\mathrm{B} \cup E)/E = (\mathrm{A} \cup E)/\mathrm{N}(E) = (\mathrm{B} \cup E)/\mathrm{N_B}(E).$$

Proof. $\mathrm{B} \cup E$ is the smallest uniformly closed linear set which contains those operators $Y = B_1 E B_2 E \cdots B_{n-1} E B_n$ with $B_i \in \mathrm{B}$. $B \in \mathrm{B}$ implies $BE \in \mathrm{A}/E$, and $Y \in \mathrm{B} \cup E$ implies $YE \in (\mathrm{A} \cup E)/E$. Then $(\mathrm{B} \cup E)/E = (\mathrm{A} \cup E)/E$. On the other hand $\mathrm{A} \cup E$ contains the projection E. Then

$$|A/\mathrm{N}(E)| = \inf_{B \in \mathrm{B}(E)} |A-B| = |AE| \quad \text{(for every } A \in \mathrm{A} \cup E).$$

Identifying $A/\mathrm{N}(E)$ with AE, we have $(\mathrm{A} \cup E)/\mathrm{N}(E) = (\mathrm{A} \cup E)/E$ and similarly $(\mathrm{B} \cup E)/\mathrm{N_B}(E) = (\mathrm{B} \cup E)/E$. Q. E. D.

The algebra $\mathrm{B} = \mathrm{B}(\mathrm{A}/E)$ has two kinds of underlying Banach spaces. One is $\mathrm{B}/E = \mathrm{A}/E$, another is $(\mathrm{B} \cup E)/\mathrm{N_B}(E) = (\mathrm{A} \cup E)/E$ and then two Q^*-topologies are defined in B. The Q^*-topology of B as an operator algebra on A/E is said to be an *E-Q^*-topology*, and the Q^*-topology of B as an operator algebra on $(\mathrm{A} \cup E)/E = (\mathrm{B} \cup E)/\mathrm{N_B}(E)$ is said to be an $\mathrm{N_B}(E)$-*Q^*-topology*.

Proposition 1. 2. (a). *Consider a C^*-algebra in a Hilbert space* $\mathfrak{H}$ *and a projection E in* $\mathfrak{H}$. *Then the algebra* $\mathrm{B} = \mathrm{B}(\mathrm{A}/E)$ *is* $\mathrm{N_B}(E)$-*Q^*-closed as an operator algebra in* $(\mathrm{A} \cup E)/E = (\mathrm{B} \cup E)/\mathrm{N_B}(E)$.

(b). *If* R *is a *-sub-algebra of* B, *then E-Q^*-closure of* R *is identical with the* $\mathrm{N_B}(E)$-*Q^*-closure of* R.

Proof. (a). It is sufficient to show that, if A is an operator in the $\mathrm{N}_{\mathrm{B}}(E)$-$Q^*$-closure of B with $|A| \leqq 1$, then $A \in \mathrm{B}$.

If A is such an operator, A is an operator in the space $(\mathrm{A} \cup E)/E$ and belongs to the $\mathrm{N}_{\mathrm{B}}(E)$-$Q$-*closure of the unit ball $U(\mathrm{B})$ of B. Then, given any $X_1, X_2, \cdots X_n \in \mathrm{A} \cup E$ and any $\varepsilon > 0$, an operator $B \in U(\mathrm{B})$ can be so chosen as $|(A-B)X_i E| < \varepsilon$.

$U(\mathrm{B})$ is a set of bounded operators in the Hilbert space $\mathfrak{H}$. Its weak closure is weakly compact and contains A. Then A is a bounded operator in $\mathfrak{H}$ and belongs to $\mathrm{B}(\mathrm{A}/E) = \mathrm{B}$.

(b). $\mathrm{N}_{\mathrm{B}}(E)$-$Q^*$-topology (the Q^*-topology of B as an operator algebra in $(\mathrm{A} \cup E)/E$) is clearly stronger than the E-Q^*-topology of B. Then, to prove (b), it is sufficient to show that, if K is an Hermitian element in the unit ball of the E-Q^*-closure Q of the given C^*-sub-algebra R, then it belongs to the $\mathrm{N}_{\mathrm{B}}(E)$-$Q^*$-closure of R.

Let K be such an Hermitian element in the unit ball of Q. Q^* is a C^*-algebra, and an Hermitian element T in Q is so chosen that $K = 2T(I+T^2)^{-1}$. Then $T = K(I-(I-K^2)^{\frac{1}{2}})^{-1}$ belongs to the unit ball of Q). Consider a filter of Hermitian elements A in R which converges to T in the E-Q^*-topology. Then $B = 2A(I+A^2)^{-1}$ converges to K because $|(I+A^2)^{-1}| \leqq 1$, $|B| \leqq 1$ and

$$K-B = 2(I+A^2)^{-1}(T-A)(I+T^2) + 2B(A-T)K.$$

Since $|B| \leqq 1$, K belongs to the E-Q^*-closure of the unit ball of R. Let $U(K: X_1, X_2, \cdots X_n, \varepsilon)$ be any $\mathrm{N}_{\mathrm{B}}(E)$-$Q^*$-neighbourhood of K, where X_i are elements of $(\mathrm{A} \cup E)/E$ and ε is a positive number. For each $X_i \in (\mathrm{A} \cup E)/E$, a Y_i with $|X_i - Y_i| < \varepsilon/3$ can be so chosen that Y_i is a sum of finite number of operators $B_1 E B_2 E \cdots B_{n-1} E B_n E$ with $B_i E \in \mathrm{A}/E$. K belongs to the E-Q^*-closure of the unit ball $U(\mathrm{R})$ of R, and an operator B in $U(\mathrm{R})$ can be so chosen that $|(B-K)Y_i| < \varepsilon/3 (1 \leqq i \leqq n)$. Then

$$|(K-B)X_i| \leqq (|K|+|B|)|X_i - Y_i| + |(K-B)Y_i| < \varepsilon$$

and $B \in U(K: X_1, X_2, \cdots X_n, \varepsilon)$. Hence K belongs to the $\mathrm{N}_{\mathrm{B}}(E)$-$Q^*$-closure of R. Q. E. D.

By Proposition 1.2, the Kaplansky Density Theorem is preserved in any subalgebra of $\mathrm{B}(\mathrm{A}/E)$.

Theorem 10. *Consider a C^*-algebra* A *in a Hilbert space* $\mathfrak{H}$, *a projection E in* $\mathfrak{H}$, *and a C^*-sub-algebra* R *of* $\mathrm{B}(\mathrm{A}/E)$. *Then the urtra E-Q^*-closure of the unit ball of* R *is the unit ball of the E-Q^*-closure* R^q *of* R.

The unit ball of the E-Q^*-closure ($\mathrm{N}_{\mathrm{B}}(E)$-$Q^*$-closure) of R *is the urtra-E-Q^*-closure of the unit ball of* R.

The urtra-E-Q^*-topology is weaker than the urtra-$\mathrm{N}_{\mathrm{B}}(E)$-$Q^*$-topology but stronger than the *E-Q^*-topology*. Then the theorem follows.

§ 2. Abelian representations of C^*-algebras.

Consider a C^*-algebra $\mathbf{A}$ in a Hilbert space $\mathfrak{H}$. A projection E is said abelian[1] relative to $\mathbf{A}$ if the reduced algebra $E\mathbf{A}E$ is abelian.

Definition 2. 1. A representation of $\mathbf{A}$ in a Hilbert space $\mathfrak{H}$ with a fixed generative abelian projection E is said an *abelian representation* of $\mathbf{A}$.

If $A \rightarrow A_\lambda$ is an abelian representation of $\mathbf{A}$ whose representative algebra is $\mathbf{A}_\lambda$ and the related abelian projection is E, then we call it an abelian representation $(\mathbf{A}_\lambda, E): A \rightarrow A_\lambda$.

A compoundly cyclic representation of a C^*-algebra is an abelian representation (cf. Lemma 4. 9 in Chapter 1), and there is a one-one correspondence between the totality of distribntions in the total state space $\mathcal{S}$ and the totality of compoundly cyclic representations up to the unitary equivalence. We shall assert that there is a one-one correspondence between the totality of compact subset of $\mathcal{S}$ and the totality of abelian representations up to the algebraic equivalence[2].

(a). Continuous vector fields in a compact set of states.

We use the notations and terms in Chapter 2.

Definition 2. 2. A vector field x in a compact set $\mathcal{W}$ of states on $\mathbf{A}$ is said to be *continuous* on $\mathcal{W}$ if it is a weakly continuous $\mathbf{A}$-valued function on $\mathcal{W}$ such that its norm function $\|x_\omega\|_\omega$ of the variable ω is bounded and continuous in $\mathcal{W}$.

If x is a vector field in $\mathcal{W}$, we define its norm by $\|x\|_{\mathcal{W}} = \sup_{\omega \in \mathcal{W}} \|x_\omega\|_\omega$. Notice that $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{K}$ are C^*-algebras of operator fields in $\mathcal{S}$. Then $\mathbf{A}_{\mathcal{W}}$, $\mathbf{C}_{\mathcal{W}}$ and $\mathbf{K}_{\mathcal{W}}$ denote the restrictions of those algebras $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{K}$ in the space $\mathcal{W}$ respectively.

Theorem 11. *Let $\mathcal{W}$ be a compact set of states on $\mathbf{A}$. If x and y are continuous vector fields, then $x+y$ is continuous. The totality $\mathfrak{F}_{\mathcal{W}}$ of continuous vector fields on $\mathcal{W}$ is a Banach space relative to the norm $\|x\|_{\mathcal{W}}$. It contains the set $(K\omega : K \in \mathbf{K})$ uniformly dense everywhere.*

Proof. Consider the smallest uniformly closed linear set $\mathfrak{E}$ of vector fields on $\mathcal{W}$ which contains $(fA\omega : f \in \mathbf{C},\ A \in \mathbf{A})$ relative to the norm $\|x\|_{\mathcal{W}}$. Then $\mathfrak{E}$ contains $(K\omega : K \in \mathbf{K})$ everywhere dense in it, and every field in $\mathfrak{E}$ is continuous. Hence it is sufficient to observe that $\mathfrak{E}$ contains $\mathfrak{F}_{\mathcal{W}}$.

1) Def. 2. 3 in Chapter 2.

2) Def. 2. 3 in this Chapter.

Consider a fixed continuous vector field x in $\mathcal{W}$, a positive number ε and a distribution μ in the set $\mathcal{W}$. The coordinate field ω is cyclic in $L^2(\mu)$ relative to the algebra K (cf. Proosition 1. 2 in Chapter 2), and $y = \Sigma f_i A_i \omega$ (, where $f_i \in$ C, $A_i \in$ A) can be so chosen that

$$\| x - y \|_\mu^2 = \int \| x_\omega - y_\omega \|_\omega^2 \, d\mu(\omega) < \varepsilon^2.$$

A numerical function of the variable ω :

$$\|x_\omega - y_\omega\|_\omega^2 = \|x_\omega\|_\omega^2 - 2\,\Re e(\Sigma f_i(\omega) x_\omega(A_i^*)) \\ + \sum_i \sum_j f_i(\omega) f_j(\omega) \omega (A_j^* A_i)$$

is a continuous function in $\mathcal{W}$. The set $\mathfrak{B} = (\|x_\omega - y_\omega\|_\omega^2 : y \in \mathfrak{E})$ is a subset of the totality $C_\mathcal{W}$ of continuous functions on $\mathcal{W}$. The weak closure of $\mathfrak{B}$ and consequently the uniform convex span of $\mathfrak{B}$ in $C_\mathcal{W}$ contain the function 0 because the dual space of $C_\mathcal{W}$ is linearly spanned by distributions in $\mathcal{W}$.

Given any positive number ε, we can choose $y_1, y_2, \cdots y_n \in \mathfrak{E}$ and numbers $a_1, a_2, \cdots, a_n \geqq 0$ with

$$\Sigma\, a_i = 1 \quad \text{and} \quad \Sigma a_i \|x_\omega - y_{i\omega}\|_\omega^2 < \varepsilon^2.$$

Using the Schwarz's inequality, we have

$$\| x_\omega - (\Sigma\, a_i y_{i\omega}) \|_\omega^2 \leqq (\Sigma\, a_i \| x_\omega - y_{i\omega} \|_\omega)^2 \\ \leqq (\Sigma\, (a_i^{\frac{1}{2}})^2) (\Sigma\, a_i^{\frac{1}{2}} \| x_\omega - y_{i\omega} \|)^2) \\ \leqq \Sigma\, a_i \| x - y_i \|^2 < \varepsilon^2.$$

Then x belongs to $\mathfrak{E}$, and we have $\mathfrak{E} = \mathfrak{F}_\mathcal{W}$.

Consider a compact set $\mathcal{W}$ of states on A, a distribution μ on $\mathcal{W}$ and a summable and regularly weakly measurable field t^* on $\mathcal{W}$. t^* satisfies $\int \|t^*_\omega\|_\omega d\mu(\omega) < \infty$ and is weakly continuous removing an open set of any small mass from $\mathcal{W}$. Then

$$t(x) = \int (x_\omega, t^*_\omega) \, d\mu(\omega) = (x, t^*)_\mu$$

is a bounded linear functional on $\mathfrak{F}_\mathcal{W}$. The functional t is denoted by

$$t = \int t_\omega \, d\mu(\omega).$$

Theorem 12. *Consider a compact set $\mathcal{W}$ of states on* A *and the Banach space $\mathfrak{F}_\mathcal{W}$ of vector fields on $\mathcal{W}$. Then every bounded linear functional t in $\mathfrak{F}_\mathcal{W}$ is a weak integral*

$$t = \int t_\omega \, d\mu(\omega),$$

where μ is a distribution in $\mathcal{W}$, t^* is a summable and regularly and weakly measurable field in $\mathcal{W}$ with $t^*_\omega \neq 0 (\omega \in \mathcal{W})$ and

$$|t|_{\mathcal{W}} = \int \| t^*_\omega \|_\omega \, d\mu(\omega)$$

is the functional-norm of t.

The distribution μ and the field t^* are essentially uniquely determined up to those equivalent weak integral representations

$$t = \int s_\omega d\nu(\omega),$$

where $d\mu(\omega) = \varphi(\omega)d\nu(\omega)$, $s_\omega = \varphi(\omega)t_\omega$ and $\varphi > 0$.

Proof. Consider a functional f on $\mathfrak{F}_{\mathcal{W}}$ such that there is at least a distribution μ in $\mathcal{W}$ with

$$|f(x)| \leqq \int \| x_\omega \|_\omega \, d\mu(\omega) \quad (x \in \mathfrak{F}_{\mathcal{W}}).$$

The totality $\mathfrak{E}$ of such functionals f on $\mathfrak{F}_{\mathcal{W}}$ consists of a bounded regularly convex subset of the unit ball $\mathfrak{U}$ of the dual space of $\mathfrak{F}_{\mathcal{W}}$. Consider a state $\lambda \in \mathcal{W}$, a functional $y \in L^2(\lambda)$ with $\|y\|_\lambda \leqq 1$, and the functional y^* in $\mathfrak{F}_{\mathcal{W}}$ with $y^*(x) = (x_\lambda, y)_\lambda (x \in \mathfrak{F}_{\mathcal{W}})$. Then $\mathfrak{U}$ is the smallest regularly convex set in the dual space of $\mathfrak{F}_{\mathcal{W}}$ which contains all those functionals $(y^* : \|y\|_\lambda = 1, \ \lambda \in \mathcal{W})$. Let δ_λ be the point mass distribution on $\mathcal{W}$ which distributes its total mass 1 at $\lambda \in \mathcal{W}$. Then we have

$$|y^*(x)| \leqq \int \|x_\omega\|_\omega d\delta_\lambda(\omega),$$

and y^* belongs to $\mathfrak{E}$. Now we have $\mathfrak{U} \subseteqq \mathfrak{E}$, and every functional f in the unit ball $\mathfrak{U}$ of $\mathfrak{F}_{\mathcal{W}}$ has at least one distribution μ in $\mathcal{W}$ such that

$$|t(x)| \leqq \int \| x_\omega \|_\omega \, d\mu(\omega) \leqq \left(\int \| x_\omega \|_\omega^2 \, d\mu(\omega)\right)^{\frac{1}{2}}.$$

We consider a functional t on $\mathfrak{F}_{\mathcal{W}}$ with the functional norm 1 and a distribution μ on $\mathcal{W}$ which satisfies the above inequality. Then a vector field t^* in $L^2(\mu)$ can be so chosen that

$$t(x) = (x, t^*)_\mu = \int (x_\omega, t^*_\omega)_\omega d\mu(\omega).$$

t^* is regularly and weakly measurable, and

$$\begin{aligned} |t(fx)| &= \left| \int f(\omega)(x_\omega, t^*_\omega)_\omega \, d\mu(\omega) \right| \\ &\leqq \int |f(\omega)| \| x_\omega \|_\omega d\mu(\omega) \quad \text{(for every } f \in C). \end{aligned}$$

Then we have $|(x_\omega, t^*_\omega)_\omega| \leqq \|x_\omega\|_\omega (x \in \mathfrak{F}_{\mathcal{W}})$ and $\|t^*_\omega\|_\omega \leqq 1$ almost everywhere.

On the other hand, the functional norm of f is 1, and

$$|t(x)| = |\int (x_\omega, t^*_\omega)_\omega \, d\mu(\omega)|$$
$$\leqq \int \|x_\omega\|_\omega \|t^*_\omega\|_\omega \, d\mu(\omega) \leqq \|x\|_{\mathcal{W}} \int \|t^*_\omega\|_\omega \, d\mu(\omega).$$

Then we have $\int \|t^*_\omega\|_\omega \, d\mu(\omega) \geqq 1$ and $\|t^*_\omega\|_\omega = 1$ almost everywhere.

The uniqueness of the integral representation is shown as follows. Let μ be a distrinution in $\mathcal{W}$ and t^*_ω be a summable and regularly and weakly measurable vector field in $\mathcal{W}$ such that $t(x) = \int (x_\omega, t^*_\omega)_\omega \, d\mu(\omega)$ vanishes on $\mathfrak{F}_{\mathcal{W}}$. We can assume without loss of generality that $\|t^*_\omega\|_\omega$ is a constant in the carrier of μ. Then t^* belongs to $L^2(\mu)$ and

$$(K\omega, t^*)_\mu = \int (K\omega, t^*_\omega)_\omega \, d\mu(\omega) = 0 \text{ for every } K \in \mathrm{K}.$$

By Proposition 1. 3 in Chapter 2, t^* vanishes almost everywhere. Hence if we assume $\|t^*_\omega\|_\omega = \text{const}$ almost everywhere, the distribution μ and the field t^* are essentially uniquely determined.

Proposition 2. 1. *Consider a compact set $\mathcal{W}$ of compact states and a convex subset $\mathfrak{X}$ of $\mathfrak{F}_{\mathcal{W}}$. A continuous field x belongs to the uniform closure of $\mathfrak{X}$ if and only if x belongs to each uniform closure of $\mathfrak{X}$ in $L^2(\mu)$ such that μ is a distribution in $\mathcal{W}$ whenever we regard $\mathfrak{X}$ as a subset of $L^2(\mu)$.*

Proof. Assume that x does not belong to the uniform closure of $\mathfrak{X}$. Then a functional t in $\mathfrak{F}_{\mathcal{W}}$ and a positive number δ can be so chosen as

$$t(x) \geqq \delta + t(y) \quad (\text{for every } y \in \mathfrak{X}).$$

t is a weak integral $t(x) = \int (x_\omega, t^*_\omega)_\omega \, d\mu(\omega)$, where t^* belongs to $L^2(\mu)$ and satisfies $\|t^*_\omega\|_\omega = \text{const.}$ almost everywhere. $(x, t^*)_\mu \geqq \delta + (y, t^*)_\mu$ implies that x does not belong to the uniform closure of $\mathfrak{X}$ in the Hilbert space $L^2(\mu)$. Thus the proposition follows.

Lemma 2. 1. *Let $\mathcal{U}$ and $\mathcal{W}$ be compact spaces of states such that $\mathcal{U} \subseteqq \mathcal{W}$, $\mathfrak{M}$ a closed linear subspace of $\mathfrak{F}_{\mathcal{W}}$ such that $x \in \mathfrak{M}$ and $f \in \mathrm{C}$ imply $fx \in \mathfrak{M}$ and $\mathfrak{M}_{\mathcal{U}}$ the totality of restrictions of fields in $\mathfrak{M}$ as being in $\mathcal{U}$. Then $\mathfrak{M}_{\mathcal{U}}$ is a closed subspace of $\mathfrak{F}_{\mathcal{U}}$.*

1) Def. 1. 2, Chap. 2.

Proof. The set $\mathfrak{N} = (x \in \mathfrak{M} : x_\omega = 0$ for $\omega \in \mathcal{U})$ is a closed linear subspace of $\mathfrak{M}$, and we have

$$\| x \|_{\mathcal{U}} = \| x/\mathfrak{N} \|_{\mathcal{W}} = \inf_{y \in \mathfrak{N}} \| x - y \|_{\mathcal{W}}.$$

In fact, if x is a field in $\mathfrak{M}$ and f is a function $\mathcal{W}$ such that

$$f(\omega) = 0 \text{ if } \| x_\omega \|_\omega \leqq \| x \|_{\mathcal{U}}$$
$$f(\omega) = 1 - \| x \|_{\mathcal{U}} / \| x_\omega \|_\omega \text{ for } \| x_\omega \|_\omega > \| x \|_{\mathcal{U}}.$$

then $fx \in \mathfrak{N}$, $1 \geqq f(\omega) \geqq 0$, and

$$\| x/\mathfrak{N} \|_{\mathcal{W}} \leqq \| x - fx \|_{\mathcal{W}} \leqq \| x \|_{\mathcal{U}}.$$

Hence

$$\| x/\mathfrak{N} \|_{\mathcal{W}} = \| x \|_{\mathcal{U}}.$$

$\mathfrak{M}_{\mathcal{U}}$ is therefore isometric and isomorphic to the quotient Banach space $\mathfrak{M}/\mathfrak{N}$. Then it is closed in $\mathfrak{F}_{\mathcal{U}}$. D. E. D.

Now the next Lebesgue Extension Theorem of continuous fields follows.

Theorem 13. *Let $\mathcal{U}$ and $\mathcal{W}$ be compact sets of states on* A *with $\mathcal{U} \subseteqq \mathcal{W}$. Then every continuous field f in $\mathcal{U}$ is extended to a continuous field in $\mathcal{W}$ with $\| f \|_{\mathcal{U}} = \| f \|_{\mathcal{W}}$.*

For each $\omega \in \mathcal{W}$ and each $x_0 \in L^2(\omega)$ there exists a continuous field x in $\mathcal{W}$ with $x_\omega = x_0$ and $\| x \|_{\mathcal{W}} = \| x_0 \|_\omega$.

The reduced set $(\mathfrak{F}_{\mathcal{W}})_{\mathcal{U}}$ of $\mathfrak{F}_{\mathcal{W}}$ in $\mathcal{U}$ contains $(X_\omega : X \in \mathrm{K})$ and is closed in $\mathfrak{F}_{\mathcal{U}}$. Then we have $\mathfrak{F}_{\mathcal{U}} = (\mathfrak{F}_{\mathcal{U}})_{\mathcal{W}}$.

Proposition 2.2. *Let $\mathcal{W}$ be a compact set of states and $\mathfrak{M}$ be a closed linear set of continuous fields on $\mathcal{W}$ such that $x \in \mathfrak{M}$ and $f \in \mathfrak{C}$ imply $fx \in \mathfrak{M}$. Then $\mathfrak{M}$ contains every continuous field x in $\mathcal{W}$ such that $x_\omega \in \mathfrak{M}_\omega = (y_\omega : y \in \mathfrak{M})$ for each $\omega \in \mathcal{W}$.*

Proof. Let x be a continuous field in $\mathcal{W}$ with $x_\omega \in \mathfrak{M}_\omega (\omega \in \mathcal{W})$. To see $x \in \mathfrak{M}$, it is sufficient to show that x belongs to the uniform closure of $\mathfrak{M}$ in the space $L_2(\mu)$ whenever μ is a distribution in $\mathcal{W}$. Let μ be any distribution in $\mathcal{W}$ and E the projection in $L^2(\mu)$ whose range is the uniform closure of $\mathfrak{M}$ in $L^2(\mu)$. Then $z = x - Ex$ is orhtogonal to the space $\mathfrak{M}$ and $z_\omega = x_\omega - (Ex)_\omega$ belongs to $\mathfrak{M}_\omega$ almost everywhere ($\mathfrak{M}_\omega$ is closed). Now for every $y \in \mathfrak{M}$ and every $f \in \mathrm{C}$ we have $fy \in \mathfrak{M}$ and

$$(fy,\ z)_\mu = \int f(\omega)(y_\omega,\ z_\omega)_\omega d\mu(\omega) = 0$$

for each fixed $z \in \mathfrak{M}$. Then $(y_\omega,\ z_\omega)_\omega$ vanishes almost everywhere. Since z is weakly continuous removing an open set of any small open mass from $\mathcal{W}$ and z_ω is orthogonal to $\mathfrak{M}_\omega$ almost everywhere, we have $z = 0$. Hence

x belongs to the uniform closure of $\mathfrak{M}$ in $L^2(\mu)$.

(b). Abelian representation of C*-algebra and an extension of the Gelfand-Naimark Theorem.

Definition 2. 3. Consider a compact set $\mathcal{W}$ of states on **A**. An operator field X on $\mathcal{W}$ (Definition 1. 1 in Chapter 2) is said to be *continuous* if X and its adjoint field X^* are bounded operators in the space $\mathfrak{F}_{\mathcal{W}}$.

The reduced algebras $\mathbf{A}_{\mathcal{W}}$, $\mathbf{C}_{\mathcal{W}}$ and $\mathbf{K}_{\mathcal{W}}$ of **A**, **C** and **K** in the space $\mathcal{W}$ are C^*-algebras of continuous operator fields.

Lemma 2. 2. *The totality of continuous operator fields in $\mathcal{W}$ is a C^*-algebra in $\mathfrak{F}_{\mathcal{W}}$, and the norm of a continuous operator field X is* $|X|_{\mathcal{W}} = \sup_{\omega\in\mathcal{W}} |X_\omega|_\omega$.

Proof. Let k denote the operator norm of X as an operator in $\mathfrak{F}_{\mathcal{W}}$. Then $k \leqq |X|_{\mathcal{W}}$ follows from $\|Xx\|_{\mathcal{W}} \leqq |X|_{\mathcal{W}} \|x\|_{\mathcal{W}}$ $(x \in \mathfrak{F}_{\mathcal{W}})$.

On the other hand, given any $\omega \in \mathcal{W}$ and any $x_0 \in L^2(\omega)$ with $\|x_0\|_\omega = 1$, a contnuous field x in $\mathcal{W}$ can be so chosen that $\|x\|_{\mathcal{W}} = \|x_0\|_\omega = 1$ and $x_\omega = x_0$. Then $k \geqq \|Xx\|_{\mathcal{W}} \geqq \|X_\omega x_0\|_\omega$ and $k \geqq |X_\omega|$ for every $\omega \in \mathcal{W}$. Hence $k \geqq \sup |X_\omega| = |X|_{\mathcal{W}}$.

Lemma 2. 3. *Consider a compact set $\mathcal{W}$ of states on* **A** *and the coordinate projection field P*[1] *on $\mathcal{W}$. Then $(\mathbf{A}_{\mathcal{W}}, P)$ determines an abelian representation of* **A**. *(Namely, $\mathbf{A}_{\mathcal{W}} \cup P$ is regarded as a suitable C^*-algebra in a Hilbert space and P is a generative abelian projection relative to A).*

$\mathbf{C}_{\mathcal{W}}$ is the carrier algebra[2] *of P, and the primitive operator J_A*[3] *of $A \in$* **A** *is the primitive function*[4] *$J_A(\omega) = \omega(A)$ of A in $\mathcal{W}$.*

Proof. P is an operator field in $\mathcal{W}$ with $P_\omega x = (x_\omega, \omega)_\omega\, \omega (x \in \mathfrak{F}_{\mathcal{W}})$. Then

$$(PAP)_\omega = (A\omega, \omega)\, P_\omega = J_A(\omega) P_\omega$$

and $PAP = J_A P$. P is therefore abelian, and J_A is the primitive operator of A. Notice that $\mathbf{C}_{\mathcal{W}}$ (the totality of continuous functions on $\mathcal{W}$) is the smallest C^*-algebra which contains those primitive functions $(J_A : A \in \mathbf{A})$. Then $\mathbf{C}_{\mathcal{W}}$ is the carrier algebra of P in the sense of Definition 4. 6 in Chapter 1.

Proposition 2. 3. *Let $\mathcal{W}$ be a compact set of states on* **A**. *Then*

$$X\omega \leftrightarrow XP \ (X \in \mathbf{A}_{\mathcal{W}} \cup P)$$

determines an isometric isomorphism between the space $\mathfrak{F}_{\mathcal{W}}$ and the

1) Definition 1. 1 in Chapter 2.

2) and 3). Definition 4. 6 in Chapter 1.

4) Definition 1. 1 in Chapter 2.

quotient space $(\mathrm{A}_{\mathcal{W}} \cup P)/P$.

Proof. $\mathfrak{F}_{\mathcal{W}}$ is the uniform closure of $(X\omega : X \in \mathrm{A} \cup P)$, and every $X \in \mathrm{A} \cup P$ satisfies $(PX^*XP)_\omega = (X^*X\omega, \omega)_\omega P$. Then

$$|XP|_{\mathcal{W}}^2 = |PX^*XP|_{\mathcal{W}} = \sup_{\omega \in \mathcal{W}} (X^*X\omega, \omega) = \| X\omega \|_{\mathcal{W}}^2,$$

and $X\omega \leftrightarrow XP$ is an isometry between $\mathfrak{F}_{\mathcal{W}}$ and $(A \cup P)/P$.

Definition 2. 4. Consider two abelian representations $(\mathrm{A}_\lambda, E) : A \rightarrow A_\lambda$, and $(\mathrm{A}_\nu, F) : A \rightarrow A_\nu$ of A. These two representations are said to be *algebraically equivalent* if the correspondences

$$A_\lambda \leftrightarrow A_\nu \text{ (for } A \in \mathrm{A}) \text{ and } E \leftrightarrow F$$

determine an algebraic isomorphism between algebras $\mathrm{A}_\lambda \cup E$ and $\mathrm{A}_\nu \cup F$.

Theorem 15. *If* $(\mathrm{A}_\lambda, E) : A \rightarrow A_\lambda$ *is an abelian representation of* A, *then it is algebraically equivalent to a representation* $(\mathrm{A}_{\mathcal{W}}, P)$ *whose* $\mathcal{W}$ *is a suitable compact set of states on* A. *The space* $\mathcal{W}$ *is uniquely determined and is the spectrum of the carrier algebra* C_E *of* E. *The algebraic equivalence determines an isometric isomorphism between these spaces* $(\mathrm{A} \cup E)/E$ *and* $\mathfrak{F}_{\mathcal{W}}$.

Proof. Consider the carrier algebra C_E and the algebra $\mathrm{K}_E = \mathrm{A} \cup \mathrm{C}_E$. If $A \in \mathrm{A}$, then its primitive operator J_A belongs to C_E and satisfies $J_A E = EAE$. C_E is the smallest C^*-algebra which contains all those J_A. Consider the spectrum $\mathcal{W}$ of C_E and let ω_E denote a state on A with $\omega_E(A) = \omega(J_A)$ for each $\omega \in \mathcal{W}$. Notice that any two points in $\mathcal{W}$ are separated by a suitable J_A, then the weakly continuous mapping $\omega \in \mathcal{W} \rightarrow \omega_E$ is a homeomorphism. Identify each $\omega \in \mathcal{W}$ with the state ω_E on A, then $\mathcal{W}$ is a compact set of states on A, and the carrier algebra C_E is represented as the totality $\mathrm{C}_{\mathcal{W}}$ of continuous functions on $\mathcal{W}$.

If X is an operator $X = \sum A_i F_i$ with $A_i \in \mathrm{A}$ and $F_i \in \mathrm{C}_E = \mathrm{C}_{\mathcal{W}}$, then we denote by $X\omega$ a field in $\mathcal{W}$ with

$$X\omega = \sum F_i(\omega) A_i \omega.$$

Then

$$\begin{aligned} |XE|^2 &= |EX^*XE| = |(\sum F_j^* F_i J_{A_j^* A_i})E| \\ &= \sup_{\omega \in \mathcal{W}} (X\omega, X\omega) = \| X\omega \|_{\mathcal{W}}^2. \end{aligned}$$

The correspondence $XE \leftrightarrow X\omega$ is extended to an isometry between the quotient spaces K_E/E and $\mathfrak{F}_{\mathcal{W}} = \mathrm{K}_{\mathcal{W}}/P$. The isometry determines a spatial isomorhipsm between algebras K_E and $\mathrm{K}_{\mathcal{W}}$. Then the reminder of the theorem is reduced to prove the next lemma.

Lemma. *Consider two abelian representations* $(\mathrm{A}_\lambda, E) : A \rightarrow A_\lambda$

and $(\mathbf{A}_\nu, F): A \to A_\nu$. *Assume that the correspondences* $A_\lambda \leftrightarrow A_\nu$, $J_{A_\lambda} \leftrightarrow J_{A_\nu}$ *are extended to an algebraic isomorphism* $X_\lambda \leftrightarrow X_\lambda$ *between algebras* $\mathbf{K}_E$ *and* $\mathbf{K}_F$, *and assume* $|X_\lambda E| = |X_\nu F|$ *for every* $X \in \mathbf{K}_E$. *Then these two representations are algebraically equivelent.*

Proof of the lemma. The algebra $\mathbf{K}_E \cup E$ is the uniform closure of the set $(X_0 + X_1 E + EX_2: X_i \in \mathbf{K}_E)$. Then $(\mathbf{K}_E \cup E)/E = \mathbf{K}_E/E = (\mathbf{A}_E \cup E)/E$ and $(\mathbf{K}_F \cup F)/F = \mathbf{K}_F/F = (\mathbf{A}_F \cup F)/F$.

Consider two elements:

$W_\lambda = X_\lambda + Y_\lambda E + EZ_\lambda \in \mathbf{K}_E \cup E$ and $W_\nu = X_\nu + Y_\nu E + EZ_\nu \in \mathbf{K}_F \cup F$ such that $X_\lambda, Y_\lambda, Z_\lambda \in \mathbf{K}_E$ and $X_\nu, Y_\nu, Z_\nu \in \mathbf{K}_F$. Then

$$W_\lambda E \in \mathbf{K}_E/E, \quad W_\lambda F \in \mathbf{K}_F/F \text{ and } |W_\lambda E| = |W_\nu F|.$$

The isometry $W_\lambda E \leftrightarrow W_\lambda F (W_\lambda \in \mathbf{K}_E \cup E)$ determines a spatial isomorphism between algebras $\mathbf{K}_E \cup E$ and $\mathbf{K}_F \cup F$ as operator algebras on $\mathbf{K}_E/E$ and $\mathbf{K}_F/F$. The same spatial isomorphism dstermines the isomorphism between their subalgebras $\mathbf{A}_\lambda \cup E$ and $\mathbf{A}_\nu \cup F$. The lemma and the theorem are thus proved.

Corollary 1. *Consider an abelian representation* $(\mathbf{A}_\lambda, E): A \to A_\lambda$ *of* $\mathbf{A}$ *and the spectrum* $\mathcal{W}$ *of the carrier algebra* $\mathbf{C}_E$.

If $\mathcal{U}$ *is a compact subset of* $\mathcal{W}$, *then* $\mathbf{N}(\mathcal{U}) = (X \in \mathbf{A}_\lambda \cup E: \omega(J_{X^*X}) = 0$ *for every* $\omega \in \mathcal{U})$ *is a left ideal of* $\mathbf{A}_\lambda \cup E$ *which contains the left ideal* $\mathbf{N}(E) = (X \in A_\lambda \cup E: XE = 0)$.

Conversely, any left ideal $\mathbf{N}$ *of* $\mathbf{A}_\lambda \cup E$ *which contains* $\mathbf{N}(E)$ *is a left ideal* $\mathbf{N} = \mathbf{N}(\mathcal{U})$, *where* $\mathcal{U}$ *is a suitable compact subset of* $\mathcal{W}$.

Proof. We can assume without loss of generality that the given abelian representation is $(\mathbf{A}_\mathcal{W}, P)$ and E is the projection field P. Let $\mathbf{N}$ be a left ideal of $\mathbf{A}_\mathcal{W} \cup P$ which contains $\mathbf{N}(P) = (X \in \mathbf{A}_\mathcal{W} \cup P: AP = 0)$. The reduced algebra $P\mathbf{A}_\mathcal{W} P$ is a subalgebra of $\mathbf{A}_\mathcal{W} \cup P$, and $\mathbf{N} \cap P\mathbf{A}_\mathcal{W} P$ is an ideal of $P\mathbf{A}_\mathcal{W} P$. The induction $f \in \mathbf{C}_\mathcal{W} \to fP \in P\mathbf{A}_\mathcal{W} P$ is an isomorphism between $\mathbf{C}_\mathcal{W}$ and $P\mathbf{A}_\mathcal{W} P$. Then $\mathfrak{M} = (f \in \mathbf{C}_\mathcal{W}: fP \in \mathbf{N} \cap P\mathbf{A}_\mathcal{W} P)$ is an ideal of $\mathbf{C}_\mathcal{W}$. Consider a compact subset

$$\mathcal{U} = (\omega \in \mathcal{W}: f(\omega) = 0 \text{ for } f \in \mathfrak{M})$$

of $\mathcal{W}$. Then $f \in \mathbf{C}_\mathcal{W}$ and $X \in \mathbf{A}_\mathcal{W} \cup P$ imply

$$\mathfrak{M} = (f \in \mathbf{C}_\mathcal{W}: f(\omega) = 0 \text{ for } \omega \in \mathcal{U}),$$

$$|f/\mathfrak{M}| = \inf_{g \in \mathfrak{M}} |f - g| = \sup_{\omega \in \mathcal{U}} |f(\omega)| = |f|_\mathcal{U}$$

and

$$|X/\mathbf{N}|^2 = \inf_{Y \in \mathbf{N}} |(X - Y)P|^2 = \inf_{Y \in \mathbf{N}} |P(X^* - Y^*)(X - Y)P|$$

$$\geqq \inf_{Z \in \mathrm{N}} | PX^*XP - PZ | = | PX^*XP/(\mathrm{N} \cap P\mathrm{A}_{\mathcal{W}}P) |$$
$$= \sup_{\omega \in \mathcal{U}} (X^*X\omega, \omega)_\omega = \| X\omega \|^2_{\mathcal{U}}.$$

Couversely

$$\| X\omega \|_{\mathcal{U}} = \inf (\| (1 - f(\omega)) X\omega \|_{\mathcal{W}} : 0 \leqq f \leqq 1, f \in \mathrm{C}_{\mathcal{W}} \textit{ and } f(\omega) = 0 \textit{ for } \omega \in \mathcal{U})$$
$$= \inf (|(I - F)XP| : 0 \leqq FP \leqq I, FP \in \mathrm{N} \cup P\mathrm{A}_{\mathcal{W}}P)$$
$$\geqq \inf_{Y \in \mathrm{N}} | X - Y | = | X/\mathrm{N} |.$$

Hence we obtain

$$| X/\mathrm{N}(\mathcal{U}) | = \| X\omega \|_{\mathcal{U}} = | X/\mathrm{N} | \text{ (for } X \equiv \mathrm{A}_{\mathcal{U}} \cup P)$$

and $\mathrm{N}(\mathcal{U}) = \mathrm{N}$. Q. E. D.

When the algebra A is an abelian C^*-algebra, the identity I is an abelian projection. Apply the corollary to the algebra $\mathrm{A} \cup I = \mathrm{A}$ and the ideal $\mathrm{N}(I) = \{0\}$. Then we obtain the well-known correspondence between ideals of A and closed subspaces of the spectrum of A.

§ 3. Non-commutative extension of the Stone-Weierstruss Theorem.

If a property of an abelian repesentation of A is invariant under the algebraic equivalence, then the property may be characterized as a property of the corresponding compact set of states on A.

Consider a compact set $\mathcal{W}$ of pure states on A. If the algebra A is abelian, then for any continuous function f on $\mathcal{W}$, we can choose an element A of A with $f(\omega) = \omega(A)$ on $\mathcal{W}$.

Definition 3. 1. A compact set $\mathcal{W}$ of pure states on a C^*-algebra A is said to be a *subspectrum* if every continuous vector field x on $\mathcal{W}$ is a uniform limit of a sequence of fields $A_n\omega$ with $A_n \in \mathrm{A}$.

If the algebra is non-commutative, a compact set of pure states on A is not generally a subspectrum.

Definition 3. 2. A compact set $\mathcal{W}$ of states on A is said to be a *prespectrum* if the totality $\mathfrak{F}_{\mathcal{W}}$ of continuous fields on $\mathcal{W}$ is the uniform closure of $(A\omega : A \in \mathrm{A})$.

In Theorem 23 in Chapter 4 we shall assert that every compact set of pure traces is a prespectrum. This result may be a sort of non-commutative extension of the Gelfand-Stone-Weierstrass Theorem. We now observe some elementary properties of prespectrums and subspectrums.

Lemma 3. 1. *A compact set of states on* A *is a prespectrum if and only if, for any given positive number* ε *and any (numerical) continuous* *function f on $\mathcal{W}$, we can choose an $A \in \mathbf{A}$ such that*

$$\|f(\omega) - A\omega\|_\omega^2 = |f(\omega)|^2 - 2\Re(f(\omega)\omega(A^*)) + \omega(A^*A) < \varepsilon.$$

Theorem 14. (1). *Every distribution in a subspectrum is a spectral distribution. Convesely, if $\mathcal{W}$ is a compact set of states and if every distribution in $\mathcal{W}$ is spectral, then $\mathcal{W}$ is a subspectrum.*

(2). *Every distribution in a prespectrum is a prespectral distribution. Conversely, if $\mathcal{W}$ is a compact set of states and if every distribution in $\mathcal{W}$ is prespectral, then $\mathcal{W}$ is a prespectrum.*

Proof. We shall first prove (2). Consider a prespectrum $\mathcal{W}$. Then for every distribution μ in $\mathcal{W}$, the set $(A\omega : A \in \mathbf{A})$ is dense everywhere in the Hilbert space $L^2(\mu)$ and μ is a prespectral distribution (cf. Lemma 1. 8 in Chapter 2).

Conversely, assume that every distribution μ on $\mathcal{W}$ is prespectral. Then for every distribution μ in $\mathcal{W}$ the set $(A\omega : A \in \mathbf{A})$ is uniformly dense everywhere in $L^2(\mu)$. By Proposition 2. 1 $(A\omega : A \in \mathbf{A})$ is uniformly dense everywhere in $\mathfrak{F}_{\mathcal{W}}$, and $\mathcal{W}$ is a prespectrum.

Proof of (1). If $\mathcal{W}$ is a subspectrum, every state in $\mathcal{W}$ is pure and every distribution in $\mathcal{W}$ is spectral. Conversely, assume that every distribution μ in $\mathcal{W}$ is spectral. Then every point-mass distribution δ_λ in $\mathcal{W}$ is spectral and every point in $\mathcal{W}$ is a pure state.

Theorem 15. *When* $\mathbf{A}$ *is a uniformly separable C^*-algebra, the carrier of a spectral distribution on the total state space $\mathcal{S}$ is a subspectrum removing an open set of any small mass from it.*

Proof. Consider a fixed spectral distribution μ on $\mathcal{S}$. If x is a field in $L^2(\mu)$, a sequence $A_n \in \mathbf{A}$ can be so chosen as $\int \|A_n\omega - x_\omega\|_\omega^2 \, d\mu(\omega) < 2^{-n}$. Removing an open set of any small mass from $\mathcal{S}$, the sequence of numerical functions $\|A_n\omega - x_\omega\|_\omega$ of the variable ω tends to 0 uniformly. Then

$$\|A_n\omega - x\|_{\mathcal{W}-\mathcal{U}} \to 0 \ (n \to \infty).$$

Notice that the space $\mathfrak{F}_{\mathcal{W}}$ is uniformly separable and contains a countable subset $\{x_n\}$ which is dense everywhere in $\mathfrak{F}_{\mathcal{W}}$. Removing an open set $\mathcal{U}$ of any small mass from $\mathcal{W}$, $x_n (n = 1, 2, \cdots)$ belong to the uniform closure of the set $(A\omega : A \in \mathbf{A})$. Then $\mathcal{W} - \mathcal{U}$ is a prespectrum. Every state in the carrier of $\mathcal{W}$ is pure almost everywhere. Hence the theorem follows.

Chapter 4. Commutor Theory of Q*-algebra and the v. Neumann's Density Theorem.

§ 1. Q*-topologies in the algebra of continuous operator fields.

(a). C*-algebra on the Banach space of continuous vector fields.

Consider an abelian representation $(\mathbf{A}_\lambda, E)$: $A \rightarrow A_\lambda$ of $\mathbf{A}$ in a Hilbert space $\mathfrak{H}$. The algebra $\mathbf{B}((\mathbf{A}_\lambda \cup E)/E)$ is the totality of bounded operators X in $\mathfrak{H}$ such that X and X^* are both bounded operators in the space $(\mathbf{A}_\lambda \cup E)/E$. We shall first observe that the algebra $\mathbf{B}(\mathbf{A}_\lambda \cup E)/E)$ is invariant under the algebraic equvalence of the representation.

Definition 1.1. Consider a compact set $\mathcal{W}$ of states on $\mathbf{A}$. A bounded operator X on the Banach space $\mathfrak{F}_{\mathcal{W}}$ is said to be *adjointable* if there exists another bounded operator X^* (the adjoint of X) in $\mathfrak{F}_{\mathcal{W}}$, which satisfies

$$((Xx_\omega), y_\omega) = (x_\omega, (X^*y)_\omega) \quad (\text{where } x, y \in \mathfrak{F}_{\mathcal{W}}, \ \omega \in \mathcal{W}).$$

We denote by $\mathbf{B}(\mathfrak{F}_{\mathcal{W}})$ the totality of adjointable operators in the space $\mathfrak{F}_{\mathcal{W}}$.

Proposition 1.1. *Consider an abelian representation* $(\mathbf{A}_\lambda, E)$: $A \rightarrow A_\lambda$ *of* $\mathbf{A}$ *in a Hilbert space and the corresponding compact space* $\mathcal{W}$ *of states on* $\mathbf{A}$. *Then the representation is algebraically equivalent to the representation* $(\mathbf{A}_{\mathcal{W}}, P)$ *and the algebraic equivalence determines an isometry between* $(\mathbf{A} \cup E)$ *and* $\mathfrak{F}_{\mathcal{W}}$.

The isometry between $(\mathbf{A} \cup E)/$ *and* $\mathfrak{F}_{\mathcal{W}}$ *determines a spatial isomorphism between algebras* $\mathbf{B}((\mathbf{A} \cup E)/E$ *and* $\mathbf{B}(\mathfrak{F}_{\mathcal{W}})$.

To prove the proposition we need to prepare some sub-lemmas.

Sub-lemma 1.[1] *A C*-algebra* $\mathbf{A}$ *is linearly spanned by its unitary element.*

In fact, if A is an Hermitian element in $\mathbf{A}$ with $|A| \leqq 1$, then

$$A = \frac{U + U^*}{2}, \quad U = A + i(I - A^2)^{\frac{1}{2}} \text{ and } U^* = A - i(I - A^2)^{\frac{1}{2}},$$

where U and U^* are unitary elements in $\mathbf{A}$.

Sub-lemma 2. *$B(\mathfrak{F}_{\mathcal{W}})$ is a C*-algebra.*

Proof. It is sufficient to show $|X^*X| = |X|^2$ $(X \in \mathbf{B}(\mathfrak{F}_{\mathcal{W}}))$, where $|X|$ denotes the operator norm of X in $\mathfrak{F}_{\mathcal{W}}$.

Let $x, y \in \mathfrak{F}_{\mathcal{W}}$. Then

$$|((X^*x)_\omega, y_\omega)| = |(x_\omega, (Xy)_\omega)| \leqq \|x\|_{\mathcal{W}} \|Xy\|_{\mathcal{W}}$$
$$\leqq |X| \, \|x\|_{\mathcal{W}} \|y\|_{\mathcal{W}}.$$

Put $y = X^*x$. Then

$$\|X^*x\|^2_{\mathcal{W}} \leqq |X| \, \|x\|_{\mathcal{W}} \|X^*x\|_{\mathcal{W}},$$

and

1) Dixmier (1).

$$|X^*| \leqq |X|, \quad |X| = |X^{**}| \leqq |X^*| \text{ and } |X| = |X^*|.$$

Now

$$\| Xx \|_\omega^2 = |((X^*Xx)_\omega, x_\omega)_\omega| \leqq \| X^*Xx \|_{\mathcal{W}},$$

and

$$\| Xx \|_{\mathcal{W}}^2 \leqq | X^*X | \| x \|_{\mathcal{W}}^2.$$

Then

$$| X |^2 \leqq | X^*X |$$

and

$$| X |^2 = | X^*X |.$$

$B(\mathfrak{F}_{\mathcal{W}})$ is a $*$-Banach algebra with the identity and with the norm condition $| X |^2 = | X^*X |$. Then it is a C^*-algebra.

Proof of the proposition. Two abelian representations (A_λ, E) and $(A_{\mathcal{W}}, P)$ are algebraically equivalent. Then $A_\lambda \cup E$ and $A_{\mathcal{W}} \cup P$ are algebraically isomorphic. We consider these two algebras to be the same algebra. Therefore $A_{\mathcal{W}} \cup P$ is an operator algebra in a suitable Hilbert space $\mathfrak{H}$, and P is a generative abelian projection in $\mathfrak{H}$. Let A be any bounded operator in $\mathfrak{H}$ such that $X \in (A_{\mathcal{W}} \cup P)/P$ implies AX, $A^*X \in (A_{\mathcal{W}} \cup P)/P$. If $x, y \in \mathfrak{F}_{\mathcal{W}}$, then $X, Y \in (A_{\mathcal{W}} \cup P)/P$ can be so chosen that

$$x = X\omega, \quad y = Y\omega, \quad (Y^*X)\omega = (x_\omega, y_\omega)_\omega \omega.$$

Define $(AX)\omega = Ax$. Since $Y^*(AX) = (A^*Y)^*X$ and

$$((Ax)_\omega, y_\omega) = (x_\omega, (A^*y)_\omega) \; (\omega \in \mathcal{W}).$$

A is an adjointable operator in $\mathfrak{F}_{\mathcal{W}}$.

Conversely, let A be any adjointable operator in $\mathfrak{F}_{\mathcal{W}}$. Then A is considered to be a bounded operator in $(A_{\mathcal{W}} \cup P)/P$ with

$$Y^*(AX) = (A^*Y)^*X$$

for every $X, Y \in (A_{\mathcal{W}} \cup P)/P$.

Consider an unitary element U in $B(\mathfrak{F}_{\mathcal{W}})$. Then $U^*U = UU^* = I$ and

$$(U(XP))^*(U(YP)) = (XP)^*(YP)$$

hold for every $X, Y \in (A_{\mathcal{W}} \cup P)$.

Consider $X_1, X_2, \ldots. X_n \in A_{\mathcal{W}} \cup P$ and $x_1, x_2, \ldots. x_n \in \mathfrak{H}$. Then

$$\begin{aligned} \| \sum_i U(X_iP)x_i \|^2 &= \sum_{i,j} ((U(X_jP))^*(U(X_iP))x_i, x_j) \\ &= \sum_{i,j} ((X_jP)^*(X_iP)x_i, x_j) \\ &= \| \sum_{i,j} X_iPx_i \|^2. \end{aligned}$$

P is generative in the Hilbert space $\mathfrak{H}$, and $\mathfrak{H}$ is the smallest uniformly closed linear set which contains $(APx : A \in \mathrm{A}_{\mathcal{W}},\ x \in \mathfrak{H})$. Then an isometric operator U in $\mathfrak{H}$ is so determined that

$$(U(XP))x = U(XPx) \text{ (where } X \in \mathrm{A}_{\mathcal{W}} \cap P,\ x \in \mathfrak{H}).$$

Since $U^*U = UU^* = I$, U is an unitary operator in $\mathfrak{H}$ and belongs to $\mathrm{B}((\mathrm{A}_{\mathcal{W}} \cup P)/P)$. Q. E. D.

(b). C*-algebra of operator fields and its commutors.

In the algebra $\mathrm{B}(\mathfrak{F}_{\mathcal{W}})$ of adjointable operators in $\mathfrak{F}_{\mathcal{W}}$, the Q^*-and urtra-Q^*-topologies are defined as in (c) of § 1.

Definition 1.2. A *Q^*-neighbourhood* $U(X: x_1, x_2, \ldots x_n ; \varepsilon)$ of an adjointable operator X in $\mathfrak{F}_{\mathcal{W}}$ is a set $(Y \in \mathrm{B}(\mathfrak{F}_{\mathcal{W}}) :\ \max(\|(X-Y)x_i\|_{\mathcal{W}}, \|(X^*-Y^*)x_i\|_{\mathcal{W}}) < \varepsilon)$, where $x_1, x_2, \ldots . x_n$ are elements in $\mathfrak{F}_{\mathcal{W}}$ and ε is a positive number. The Hausdorff topology of $\mathrm{B}(\mathfrak{F}_{\mathcal{W}})$ which is determined by these Q^*-neighbourhoods of elements of $\mathrm{B}(\mathfrak{F}_{\mathcal{W}})$ is said to be the *Q^*-topology* of $\mathrm{B}(\mathfrak{F}_{\mathcal{W}})$.

An urtra-Q^*-neighoourhood $U(X: \{x_i\}, \underset{\varepsilon}{\varepsilon})$ of an adjointable operator X in $\mathfrak{F}_{\mathcal{W}}$ is a set $(Y \in \mathrm{B}(\mathfrak{F}_{\mathcal{W}}) :\ \sup(\|(X-Y)x_i\|_{\mathcal{W}}, \|(X^*-Y^*)x_i\|_{\mathcal{W}}) < \varepsilon)$, where $\{x_i\}$ is a uniformly convergent sequence in $\mathfrak{F}_{\mathcal{W}}$ and ε is a positive number.

A Q^*-closed $*$-algebra of adjointable operators in $\mathfrak{F}_{\mathcal{W}}$ is said to be a Q^*-algebra in $\mathfrak{F}_{\mathcal{W}}$. The Q^*-closure of a $*$-algebra R in $\mathfrak{F}_{\mathcal{W}}$ is denoted by R^q.

Definition 1. 3. If R is a C^*-algebra of adjointable operators in $\mathfrak{F}_{\mathcal{W}}$, then the totality of adjointable operators in $\mathfrak{F}_{\mathcal{W}}$ which commutes with every element of R is denoted by R′ and said to be the *commutor* of R. R′ is a Q^*-algebra.

Lemma 1. 1. *The totality of continuous operator fields on $\mathcal{W}$ is a C^*-algebra of adjointable operators in $\mathfrak{F}_{\mathcal{W}}$ and is the commutor $\mathrm{C}'_{\mathcal{W}}$ of the algebra $\mathrm{C}_{\mathcal{W}}$ (the totality of continuous functions on $\mathcal{W}$).*

Proof. A continuous operator field is adjointable in $\mathfrak{F}_{\mathcal{W}}$ and belongs to $\mathrm{C}'_{\mathcal{W}}$. Then it is sufficient to show that every adjointable operator in $\mathrm{C}'_{\mathcal{W}}$ is a continuous operator field.

If X is an adjointable operator in $\mathfrak{F}_{\mathcal{W}}$ which commutes with $\mathrm{C}_{\mathcal{W}}$, then

$$f(\omega)(Xx)_\omega = (Xfx)_\omega \quad (f \in \mathrm{C}_{\mathcal{W}}, \text{ and } x \in \mathfrak{F}_{\mathcal{W}})$$

and

$$\sup_{\omega \in \mathcal{W}} |f(\omega)|\ \|(Xx)_\omega\|_\omega \leq |X|\ \|fx\|_{\mathcal{W}},$$

where $|X|$ is the operator norm of X in $\mathfrak{F}_{\mathcal{W}}$.

Put

$$f_n(\omega) = \min(n, \|x_\omega\|_\omega^{-1}) \ (\omega \in \mathcal{W}).$$

Then

$$\|f_n x\|_{\mathcal{W}} \leqq 1 (n = 1, 2, \dots),$$
$$|f_n(\omega)| \, \|(Xx)_\omega\|_\omega \leqq |X| \ (n = 1, 2, \dots),$$

and

$$\|(Xx)_\omega\| \leqq |X| \, \|x_\omega\|_\omega \ (x \in \mathfrak{F}_{\mathcal{W}} \text{ and } \omega \in \mathcal{W}).$$

$x_\omega = y_\omega (x, y \in \mathfrak{F}_{\mathcal{W}})$ implies $(Xx)_\omega = (Xy)_\omega$, Then for each $\omega \in \mathcal{W}$ a bounded operator X_ω in $L^2(\omega)$ with $|X_\omega| \leqq |X|$ is determined in such a way that

$$(Xx)_\omega = X_\omega x_\omega \quad (x \in \mathfrak{F}_{\mathcal{W}}).$$

Then X is a continuous operator field in $\mathcal{W}$. Q. E. D.

By Theorem 10 the Kaplansky Density Theorem holds in every C^*-algebra of adjointable operators in $\mathfrak{F}_{\mathcal{W}}$. The remained problem is the extension of the v. Neumann Density Theorem which involves us in the study of the commutor theory of operator algebras in the Banach space $\mathfrak{F}_{\mathcal{W}}$. In what follows we shall devote ourselves to study the following two questions.

Consider a compact set $\mathcal{W}$ of states on A and a C^*-algebra R of continuous operator fields on $\mathcal{W}$ with $C_{\mathcal{W}}' \supseteq R \supseteq C_{\mathcal{W}}$. Then its commutor R′ is a Q^*-algebra of continuous operator fields on $\mathcal{W}$ with $C_{\mathcal{W}}' \supseteq R' \supseteq C_{\mathcal{W}}$.

(1). *If μ is a distribution in $\mathcal{W}$, is the W^*-closures of* R *and* R′ *(regarding as operator angebras) in $L^2(\mu)$ a commutor pair ?*

(2). *Is the bicommutor* R″ *of* R *the Q^*-closure* R^q *of* R *?*

Consider a compact set $\mathcal{W}$ of states on A and a C^*-algebra R of continuous operator fields in $\mathcal{W}$. For each $\omega \in \mathcal{W}$, we denote by R_ω the C^*-algebra $R_\omega = (A\ ;\ A \in R)$ in the Hilbert space $L^2(\omega)$.

If μ is a distribution in $\mathcal{W}$, then we denote by R_μ the representative algebra of R in the Hilbert space $L^2(\mu)$.

Theorem 15. *Consider a compact set $\mathcal{W}$ of states on* A *and a C^*-algebra* R *of continuous operator fields on $\mathcal{W}$ with $C_{\mathcal{W}}' \supseteq R \supseteq C_{\mathcal{W}}$. Assume that for each $\omega \in \mathcal{W}$ W^*-algebras $(R_\omega)''$ and $(R'_\omega)''$ in the Hilbert space $L^2(\omega)$ are a commutor pair. Then given any distribution μ in $\mathcal{W}$, the W^*-algebras $(R_\mu)''$ and $(R'_\mu)''$ on the space $L^2(\mu)$ are a commutor pair.*

To prove the theorem we prepare two sub-lemmas.

Sub-lemma 1. *Consider a C^*-algebra* R *of continuous operator fields as in Theorem 15 and a closed linear subset $\mathfrak{M}$ of $\mathfrak{F}_{\mathcal{W}}$ which is invariant under the algebra* R. *For each $\omega \in \mathcal{W}$, let $[\mathfrak{M}_\omega]_\omega$ denote the uniform closure of the set $\mathfrak{M}_\omega = (x_\omega : x \in \mathfrak{M})$ and E_ω the projection in $L^2(\omega)$ whose range is $[\mathfrak{M}_\omega]_\omega$. Then the projection field E is a measurable operator field in $L^2(\mu)$* (cf. Def. 1.4 in Chapter 2), *and it is a projection in $L^2(\mu)$*

whose range is the uniform closure $[\mathfrak{M}]_\mu$ *of* $\mathfrak{M}$ *in* $L^2(\mu)$.

Proof of the sub-lemma. It is sufficient to show the following

(1). If $x \in [\mathfrak{M}]_\mu$, then $E_\omega x_\omega = x_\omega$ ($\omega \in \mathcal{W}$) almost everywhere.

(2). If x is a field in $L^2(\mu)$ which is orthogonal to $[\mathfrak{M}]_\mu$, then $E_\omega x_\omega = 0$ ($\omega \in \mathcal{W}$) almost everywhere.

By Lemma 1.4 in Chapter 2, the set $(x \in L^2(\mu) : x_\omega \in [\mathfrak{M}_\omega]_\omega)$ is a uniformly closed linear subspace of $L^2(\mu)$ and contains $\mathfrak{M}$. Then it contains $[\mathfrak{M}]_\mu$ and hence (1) follows.

Next, let x be a field in $L^2(\mu)$ which is orthogonal to $\mathfrak{M}$. Then

$$(x, fz)_\mu = \int f(\omega)(x_\omega, z_\omega)d_\mu(\omega) = 0,$$

where $z \in \mathfrak{M}$ and $f \in C_{\mathcal{W}}$. Now (x_ω, z_ω) vanishes almost everywhere for each fixed $z \in \mathfrak{M}$. x_ω is regularly and weakly measurable, and removing an open set of any small mass, it is weakly continuous. Then x_ω is orthogonal to $[\mathfrak{M}_\omega]_\omega$ almost everywhere and hence (2) follows.

Sub-lemma 2. *Consider the compact space* $\mathcal{W}$ *and the algebra* R *as in Theorem* 15 *and Sub-lemma* 1. *If* $\mathfrak{M}$ *is a closed linear subspace of* $\mathfrak{F}_{\mathcal{W}}$ *which is invariant under* R, *we denote by* $E\mathfrak{M}$ *the projection field in* $\mathcal{W}$ *such that each value* $(E\mathfrak{M})_\omega$ *is a projection in* $L^2(\omega)$ *with the range* $[\mathfrak{M}_\omega]_\omega$. *If* μ *is a distribution in* $\mathcal{W}$, *then the commutor* $(\mathrm{R}_\mu)'$ *of* R_μ *in the Hilbert space* $L^2(\mu)$ *is the smallest* W^**-algebra which contains all those projections* $E\mathfrak{M}$.

Proof. For each $x \in L^2(\mu)$ we consider the cyclic projection E^{R}_x. It is a projection in $L^2(\mu)$ whose range is the uniform closure $[\mathrm{R}x]_\mu$ of the set $(Ax : A \in \mathrm{R})$. The algebra $(\mathrm{R}_\mu)'$ becomes the smallest W^*-algebra in $L^2(\mu)$ which contains all those projections $(E^{\mathrm{R}}_x : x \in L^2(\mu))$. Then it is sufficient to show that each $E^{\mathrm{R}}_x (x \in L^2(\mu))$ is a strong limit of a sequence of projection fields $E^{\mathrm{R}}_{y_n} (n = 1, 2, \dots)$ with $y_n \in \mathfrak{F}_{\mathcal{W}}$. Consider a fixed $x \in L^2(\mu)$. x is regularly and weakly measurable. Then it is continuous removing an open set of any small mass from $\mathcal{W}$. Now given any $\varepsilon > 0$, there is a compact subset $\mathcal{U}$ of $\mathcal{W}$ with $\mu(\mathcal{W} - \mathcal{U}) < \varepsilon$ such that x is continuous in $\mathcal{U}$. By the Lebesgue extension theorem (Theorem 13), this restricted field in $\mathcal{U}$ is extended to a continuous field y in $\mathcal{W}$ and, when the mass $\mu(\mathcal{W} - \mathcal{U})$ tends to 0, the projection E^{R}_y tends to E^{R}_x strongly. Hence the smallest W^*-algebra which contains $(E^{\mathrm{R}}_y : y \in \mathfrak{F}_{\mathcal{W}})$ contains $(E^{\mathrm{R}}_x : x \in L^2(\mu))$ and its W^*-envelop $(\mathrm{R}_\mu)'$.

Proof of Theorem 15. Consider a fixed distribution μ in $\mathcal{W}$, and two closed linear subspaces $\mathfrak{W}$ and $\mathfrak{N}$ in $\mathfrak{F}_{\mathcal{W}}$ which are invariant under R and its commutor R′ respectively.

Projection fields $E_{\mathfrak{M}}$ and $E_{\mathfrak{N}}$ are measurable operator fields in $L^2(\mu)$ and belong to $(\mathrm{R}_\mu)'$ and $(\mathrm{R}'_\mu)'$ respectively. The values $(E_{\mathfrak{M}})_\omega$ and $(E_{\mathfrak{N}})_\omega$ of $E_{\mathfrak{M}}$ and $E_{\mathfrak{N}}$ at $\omega \in \mathcal{W}$ are projections in $L^2(\omega)$ whose ranges are $[\mathfrak{M}_\omega]_\omega$ and $[\mathfrak{N}_\omega]_\omega$ respcctively. Then we have $(E_{\mathfrak{M}})_\omega \in (\mathrm{R}_\omega)'$ and $(E_{\mathfrak{N}})_\omega \in (\mathrm{R}'_\omega)'$.

By the assumption in the theorem every pair of W^*-algebras $(\mathrm{R}_\omega)''$ and $(\mathrm{R}'_\omega)''$ are a commutor pair in $L^2(\mu)$. Then $(E_{\mathfrak{M}})_\omega \in (\mathrm{R}_\omega)' = (\mathrm{R}'_\omega)''$, $(E_{\mathfrak{N}})_\omega \in (\mathrm{R}'_\omega)' = (\mathrm{R}_\omega)''$. $(E_{\mathfrak{M}})_\omega (E_{\mathfrak{N}})_\omega = (E_{\mathfrak{N}})_\omega (E_{\mathfrak{M}})_\omega$ and $E_{\mathfrak{M}} E_{\mathfrak{N}} = E_{\mathfrak{N}} E_{\mathfrak{M}}$. By Sub-lemma 2, the algebra $(\mathrm{R}_\mu)'$ is spanned by the projections $E_{\mathfrak{M}}$. Similarly the algebra $(\mathrm{R}'_\mu)'$ is spanned by $E_{\mathfrak{N}}$ and hence $(\mathrm{R}_\mu)''$ and $(\mathrm{R}'_\mu)''$ are a commutor pair in $L^2(\mu)$.

By Theorem 15 we obtain an extension of the v. Neumann's Density Theorem.

Theorem 16. *Consider a compact ret $\mathcal{W}$ of states on* A *and a $*$-algebra* R *of continuous operator fields in $\mathcal{W}$ with* $\mathrm{C}_{\mathcal{W}} \supseteq \mathrm{R} \supseteqq \mathrm{C}_{\mathcal{W}}$. *Assume that each pair of W^*-algebras $(\mathrm{R}_\omega)''$ and $(\mathrm{R}'_\omega)''$ $(\omega \in \mathcal{W})$ is a commutor pair in $L^2(\omega)$. Then the bicommutor* R'' *of* R *is the Q^*-closure* R^q *of* R *in the space $\mathfrak{F}_{\mathcal{W}}$.*

Proof. Let μ be any distribution in $\mathcal{W}$, Then the representative algebra $(\mathrm{R}'')_\mu$ of R'' in $L^2(\mu)$ is contained in $(\mathrm{R}'_\mu)'$ and consequently in the W^*-closure $(\mathrm{R}_\mu)''$ of R_μ, in $L^2(\mu)$.

Consider a fixed continuous operator field X in R''. Given any $\varepsilon > 0$, any $x_1, x_2, \ldots. x_n$ in $\mathfrak{F}_{\mathcal{W}}$ and any distribution μ in $\mathcal{W}$, an operator field A in R can be so chosen that

$$\|(X-A)x_i\|_\mu^2 = \int \|(X_\omega - A_\omega)x_{i\omega}\|_\omega^2 \, d_\mu(\omega) < \varepsilon^2,$$

where

$$\varphi_A(\omega) = \sum_i \|(X_\omega - A_\omega)x_{i\omega}\|_\omega^2$$

is a continuous function of the variable ω. The weak closure of the set of those functions $(\varphi_A : A \in \mathrm{A})$ and consequently its uniform convex span in the space $\mathrm{C}_{\mathcal{W}}$ contain the function 0. Then bor any $\varepsilon > 0$ we can choose positive numbers $a_1, a_2, \ldots, a_n$ and elements $B_1, B_2, \ldots\ldots, B_n$ in R such that

$$\sum_j a_j \Big(\sum_i \|(X_\omega - B_{j\omega})x_{i\omega}\|_\omega^2\Big) < \varepsilon$$

for every $\omega \in \mathcal{W}$. Using the Schwarz's inequality we have

$$\sum_i \sum_j \|X_\omega x_{i\omega} - \Big(\sum a_j B_{j\omega}\Big)x_{i\omega}\|_\omega^2 < \varepsilon$$

and

$$\| Xx_i - (\sum_j a_j B_j) x_i \|_{\mathcal{W}} < \varepsilon \quad (i = 1, 2, \ldots, n).$$

Hence X belongs to the Q^*-closure R^q of R. Q. E. D.

§ 3. Extension Theorem of continuousm operator fields.

Consider a compact set $\mathcal{W}$ of states on A. A vector field p on $\mathcal{W}$ is said to be a *positive field* if each value p_ω is a positive definite functional in A.

Definition 3. 1. If x is a vector field in $\mathcal{W}$, then we denote by x^v a vector field in $\mathcal{W}$ whose each value $(x^v)_\omega$ is the absolute variation of the value x_ω of x. We call it the *absolute variation* of the field x.

Definition 3. 2. A continuous vector field x in $\mathcal{W}$ is said to be *absolutely continuous* if the function $|x_\omega|$ of the variable ω (where $|x_\omega|$ denotes the functional norm of the value x_ω as a bounded linear functional in A) is a continuous function in $\mathcal{W}$.

Proposition 3. 1. *A continuous vector field x in a compact state space $\mathcal{W}$ is absolutely continuous if and only if its absolute variation x^v is a continuous field in $\mathcal{W}$.*

Proof. If x is a continuous field in $\mathcal{W}$ such that x^v is continuous, then $|x_\omega| = x^v(I)$ is continuous and x is absolutely continuous.

Conversely, assume that x is absolutely continuous in $\mathcal{W}$ but x^v is not continuous at a point λ in $\mathcal{W}$. Since $\|x^v{}_\omega\|_\omega (= \|x_\omega\|_\omega)$ is continuous in $\mathcal{W}$, x^v is not weakly continuous at λ. x^v is bounded in $\mathcal{W}$ and there is at least a filter F in $\mathcal{W}$ which converges to λ and induces the weak convergence of the value x_ω^v to a positive functional $y \neq x_v^\lambda$ $\omega \to \lambda$. Since $|x_\omega| (= x_\omega^v(I))$ is continuous and $|x_\omega| x_\omega^v(A^*A) \geqq |x_\omega(A^*)|^2$ holds in $\mathcal{W}$, we have $y(I) = |x_\lambda|$ and $y(A^*A)|x_\lambda| \geqq |x_\lambda(A^*)|^2$.

By the corollary of Theorem 1 in Chapter 1 we have $y = x_\lambda^v$. Hence x^v is continuous in $\mathcal{W}$.

Definition 3. 3. A compact set $\mathcal{W}$ of states on A is said to be *absolutely continuous* if every field $A\omega (A \in A)$ is absolutely continuous in $\mathcal{W}$.

Lemma 3. 1. *If a compact state space $\mathcal{W}$ is absotutely continuous, then every continuous field in $\mathcal{W}$ is absolutely continuous.*

Proof. Consider a field

$$x = \sum f_i A_i \omega \quad (f_i \in C_{\mathcal{W}} \text{ of } A_i \in A).$$

We shall show that x is absolutely continuous. Let λ be a fixed point in $\mathcal{W}$ and A an element $A = \sum f_i(\cdot) A_i$ of A. Then

$$x_\omega - A\omega = \sum (f_i(\omega) - f_i(\lambda)) A_i \omega$$

and

$$\| |x_\omega| - |A\omega| \| \leqq \sum | f_i(\omega) - f_i(\lambda) | \|A_i\|.$$

$|A\omega|$ is a continuous function and $|x_\omega| - |A\omega|$ tends to 0 when $\omega (\in \mathcal{W})$ tends to λ. Then $|x_\omega|$ is continuous at λ.

Let $\{x_n\}$ be a sequence of absolutely continuous fields in $\mathcal{W}$ which converges to a continuous field x in $\mathcal{W}$. Then

$$\| |x_{n\omega}| - |x_\omega| \| \leqq \| x_{n\omega} - x_\omega \|_\omega \leqq \| x_n - x \|_{\mathcal{W}} \to 0$$

and $|x_\omega|$ is a continuous function in $\mathcal{W}$. Hence x is absolutely continuous.

Lemma 3. 2. *Consider an absolutely continuous compact set $\mathcal{W}$ of states on* A *and its compact subset $\mathcal{U}$. Then every contiuous positive field p in $\mathcal{U}$ is extensible to a continuous positive field in $\mathcal{W}$.*

Proof. p is extensible to a continuous field f in $\mathcal{W}$. Then its absolute variation $q = f^v$ is a desired positive extension of p.

Lemma 3. 3. *Consider a compact set $\mathcal{W}$ of states on* A. *Let q be a positive continuous field in $\mathcal{W}$ such that each $q_\omega + \omega$ is a cyclic element in the space $L^2(\omega)$ (for each $\omega \in \mathcal{W}$).*

Let Q_ω denote a definite self-adjoint operator in $L^2(\omega)$ which $\eta \mathrm{A}_\omega'$ and which is determined by $q_\omega = Q_\omega \omega$, and let f be any real continuous function in the half real-line $0 \leqq x \leqq \infty$, where we assume that f is continuous at ∞ (i. e., $\lim_{x \to \infty} f(x) = f(\infty)$). Then the operator field $f(Q)$, whose value at $\omega \in \mathcal{W}$ is $f(Q_\omega)$, is a continuous operator field in $\mathcal{W}$ and commutes with every operator in $\mathrm{K}_{\mathcal{W}} = \mathrm{A}_{\mathcal{W}} \cup \mathrm{C}_{\mathcal{W}}$.

Proof. Consider the set $\mathfrak{M} = (K(q+\omega) : K \in \mathrm{K})$. $q_\omega + \omega$ is cyclic in $L^2(\omega)$ (for each $\omega \in \mathcal{W}$) and $\mathfrak{M}_\omega (= (A(q_\omega + \omega) : A \in \mathrm{A}) = (x_\omega : x \in \mathfrak{M}))$ is uniformly dense everywhere in $L^2(\omega)$.

Then by Proposition 2. 2 in Chapter 3, the uniform closure $[\mathfrak{M}]_{\mathcal{W}}$ of $\mathfrak{M}$ in $\mathfrak{F}_{\mathcal{W}}$ is the space $\mathfrak{F}_{\mathcal{W}}$.

Notice that

$$\| X(q + \omega) \|_{\mathcal{W}} \geqq \| X\omega \|_{\mathcal{W}} \quad (X \in \mathrm{K}).$$

Then $X(q+\omega) \to X\omega$ is extended to a bounded operator T in $\mathfrak{F}_{\mathcal{W}}$ with

$$T(X(q+\omega)) = X\omega \quad (X \in \mathrm{K}).$$

T is a continuous Hermitian operator field in $\mathcal{W}$. Its value T_ω at $\omega \in \mathcal{W}$ is determined by $T_\omega = (Q_\omega + I)^{-1}$ and belongs to A_ω'. T is a definite Hermitian in $(\mathrm{K}_{\mathcal{W}})'$ and $|T| \leqq 1$. Let f be a real continuous function in the half-line $0 \leqq x \leqq \infty$. $g(x) = f((1-x)/x)$ is a continuous function in the closed interval $0 \leqq x \leqq 1$, the operator $g(T)$ is a continuous operator field in $(\mathrm{K}_{\mathcal{W}})'$ and we have $g(T_\omega) = f(Q_\omega)$ $(\omega \in \mathcal{W})$. Hence the lemma follows.

Lemma 3. 4. *Let p be a state and A an element of A_p''. Then $p +$*

$(Ap)^v$ is cyclic in $L^2(p)$.

Proof. Set $(Ap)^v = UAp = Kp$, where U is a partially isometric operator in A_p'' and K is a definite self-adjoint operator in $L^2(p)$ with $K \eta A_p'$. Notice that $(UA)^m p = K^m p \in L^2(p)$ and $(1+\alpha UA)^{-1}p = (1+\alpha K)^{-1}p \in L^2(p)$ for $|\alpha| < |A|^{-1}$. Then the uniform closure of $(A(1+K)p : A \in \mathbf{A})$ in $L^2(p)$ contains $(1+\alpha K)^{-1}(1+K)p$ and the range of the operator $(1+\alpha K)^{-1}(1+K)$ which is invertible in $L^2(p)$. Q. E. D.

Lemma 3. 5. *Let p be a state and x any element of $L^2(p)$. If A_n is a sequence in $\mathbf{A}$ with $\|A_n p - x\|_p \to 0$, then $\|(A_n p)^v - x^v\|_p \to 0$.*

Proof. By Lemma 2. 5 in Chapter 1, $(A_n p)^v$ converges to x^v in the point weak topology of $\bar{\mathbf{A}}$. Since $\|(A_n p)^v\|_p = \|A_n p\|_p \to \|x\|_p = \|x^v\|_p$, $(A_n p)^v$ converges to x^v in the weak topology of $L^2(p)$. Hence

$$\|(A_n p)^v - x^v\|_p^2 = \|A_n p\|_p^2 - \|x\|_p^2 - 2\Re((A_n p)^v - x^v, x^v) \to 0.$$

Lemma 3. 6. *Let μ be a distribution in the total state space $\mathcal{S}$ and x an element of $L^2(\mu)$. Regard x as a functional on $\mathbf{K}$ such that $x(K) = (Kx, \omega)_\mu$. Then the absolute veriation x^v of x is a field in $\mathcal{S}$ which satisfies $(x^v)_\omega = (x_\omega)^v (\omega \in \mathcal{S})$.*

Proof. Choose two sequences U_n and V_n of operators in $\mathbf{K}$ such that $|U_n| \leqq 1$, $|V_n| \leqq 1$, $\|U_n x - x^v\|_\mu \leqq 2^{-n}$ and $\|V_n(x^v) - x\|_\mu \leqq 2^{-n}$. Then we have $\|U_{n\omega} x_\omega - (x^v)_\omega\|_\omega \to 0$, $\|V_{n\omega}(x^v)_\omega - x_\omega\|_\omega \to 0$ and consequently $(x^v)_\omega = (x_\omega)^v$ almost everywhere.

Lemma 3.7. *Let $\mathcal{W}$ be any absolutely continuous compact set of states, x a continuous field in $\mathcal{W}$ and K_n a sequence in $\mathbf{K}$ with $\|K_n\omega - x\|_{\mathcal{W}} \to 0$. Then $(K_n\omega)^v$ converges to x^v in the point weak topology of $\mathfrak{F}_{\mathcal{W}}$.*

Proof. Let μ be any distribution in $\mathcal{W}$. Then $(K_n\omega)^v$ converges to x^v in $L^2(\mu)$, from which the lemma follows.

Lemma 3.8. *Let $\mathcal{W}$ be an absolutely continuous compact set of states, $\mathcal{U}$ its closed subset and q a continuous positive field in $\mathcal{U}$. If $\omega - q$ is positive in $\mathcal{U}$, then q is extended to a continuous positive field in $\mathcal{W}$ preserving the positivity of $\omega - q$.*

Proof. q is extended to a continuous positive field r in $\mathcal{W}$. We choose a sequence $r_n (= K_n\omega$ with $K_{n\omega} \in (\mathbf{A}_\omega)'')$ of continuous positive fields in $\mathcal{W}$ which converges to r uniformly in $\mathfrak{F}_{\mathcal{W}}$ and which is contained in the linear span of the set $((K\omega)^v$: $K \in \mathbf{K})$. By Lemma 3. 4 each $r_{n\omega} + \omega (\omega \in \mathcal{W})$ is cyclic in $L^2(\omega)$. Let R_n denote the operator field in $\mathcal{W}$ with $R_n\omega = r_n$ and Q the continuous operator field in $\mathcal{U}$ with $Q\omega = q$. Now $(I+R_n)^{-1}$ are continuous operator fields in $\mathfrak{F}_{\mathcal{W}}$ and the following relations hold in $\mathfrak{F}_{\mathcal{U}}$.

$$((I+R_n)^{-1} - (I+Q)^{-1})(q+\omega) = (I+R_n)^{-1}(r_n - q) \to 0.$$

$(K(q+\omega): K \in K) = (K(I+Q)\omega : K \in K)$ is uniformly dense everywhere in $\mathfrak{H}_{\mathcal{U}}$. Then $(I+R_n)^{-1}$ converges to $(I+Q)^{-1}$ in the Q^*-topology of $B(\mathfrak{H}_{\mathcal{U}})$. Consider the function $f(x) = \min(x, 1)$ $(0 \leqq x \leqq \infty)$ and set $q_n = f(R_n)\omega$. Then q_n and $\omega - q_n$ are positive and continuous in $\mathfrak{H}_{\mathcal{W}}$ and $\|q_n - q\|_{\mathcal{U}} \to 0$. We can assume without loss of generality that $\|q_n - q\|_{\mathcal{U}} \leqq 2^{-n-1}(n=1, 2, \cdots)$ holds. Choose a sequence f_n of continuous functions in $\mathcal{W}$ which satisfies the next conditions

(1). $1 = f_0 \geqq f_1 \geqq f_2 \cdots\cdots \geqq 0.$

(2). $|f_n(\omega)| \|q_{n+1\omega} - q_{n\omega}\|_\omega \leqq 2^{-n}(\omega \in \mathcal{W}).$

(3). $f_n(\omega) = 1$ for $\omega \in \mathcal{U}.$

Then $q' = q_1 + \sum f_n(\omega)(q_{n+1} - q_n)$ converges uniformly in $\mathfrak{H}_{\mathcal{W}}$. q' is the desired positive continuous extension of q in $\mathcal{W}$ because $\omega - q'$ is positive in $\mathcal{W}$.

Theorem 17. *Consider an absolutely continuous compact set $\mathcal{W}$ of states on* A *and its closed subset $\mathcal{U}$. If K is a continuous operator field in* $(K_{\mathcal{U}})'$, *then K is extended to a continuous operator field in* $(K_{\mathcal{W}})'$.

Proof. If K is a definite Hermitian in $K_{\mathcal{U}}'$ with $|K| \leqq 1$, then $K\omega$ and $\omega - K\omega$ are positive continuous fields in $\mathcal{U}$ and $K\omega$ is extensible to a positive continuous field q in $\mathcal{W}$ preserving the positivity of $\omega - q$. Choose a definite Hermitian Q in $(K_{\mathcal{W}})'$ with $q = Q\omega$. Then Q is a desired extension of K.

Corollary 1. *Consider an absolutely continuous compact set $\mathcal{W}$ of states on* A *and the commutor* $(K_{\mathcal{W}})'$ *of the algebra* $K_{\mathcal{W}}$. *Then*

$$(K_{\mathcal{W}})'_\omega = (X_\omega : X \in (K_{\mathcal{W}})')$$

is a W^*-algebra which is the commutor $(A_\omega)'$ *of the representative algebra* A_ω *of* A *in* $L^2(\omega)$.

By Theorem 15, 16 and 17 we obtain :

Theorem 18. *Consider an absolutely continuous compact set $\mathcal{W}$ of states on* A. *Then given any distribution μ in $\mathcal{W}$, W^*-closures* K_μ'' *and* $((K_{\mathcal{W}})'_\mu)''$ *of algebras* K_μ *and* $(K_{\mathcal{W}})'_\mu$ *in* $L^2(\mu)$ *are a commutor pair in* $L^2(\mu)$.

Theorem 19. *Let $\mathcal{W}$ be given as in Theorem* 18. *Then the bi-commutor* $(K_{\mathcal{W}})''$ *of* $K_{\mathcal{W}}$ *in* $\mathfrak{H}_{\mathcal{W}}$ *is the Q^*-closure* $(K_{\mathcal{W}})^q$ *of* $K_{\mathcal{W}}$.

§ 4. Abselute continuity of states and the Reduction Theory.

Definition 4.1. (1). A state p is said to be *absolutely continuous* if every numerical function $|A\omega|$ $(A\in \mathrm{A})$ of the variable ω is continuous at p in the total state space $\mathcal{S}$.

(2). Consider a compact set $\mathcal{W}$ of states on A. A state p in $\mathcal{W}$ is said to be absolutely continuous in $\mathcal{W}$ if every numerical function $|A\omega|(A\in\mathrm{A})$ of the variable ω is continuous at p in the space $\mathcal{W}$.

(3). Let p be any state. An absolute weak neighbourhood $V(p: A_1, A_2, \cdots A_n, \varepsilon)$ of p is a subset of $\mathcal{S}$:

$$(q\in\mathcal{S}:\ \max(|p(A_i)-q(A_i)|,\ \|A_ip| - |A_iq\|)<\varepsilon).$$

A Hausdorff topology of the total state space $\mathcal{S}$ which is determined by the absolute weak neighbourhood of elements of $\mathcal{S}$ is said to be the *absolute weak topology* of $\mathcal{S}$.

Theorem 20. *Every pure state is absolutely continuous. The totality $\mathcal{S}_p$ of pure states is the totality of absolutely continuous states in the weak closure of $\mathcal{S}_p$.*

Proof. Consider a fixed pure state p. If q is any state, then $|Aq|\leqq \|Aq\|_q$ holds. Let F be any filter in the total state space $\mathcal{S}$ which converges to p. Then

$$\overline{\lim_{q\in F}}\ |Aq|\leqq\|Ap\|_p\ (A\in\mathrm{A}).$$

p is a pure state, and every $A\in\mathrm{A}$ satisfies $\|Ap\|_p=|Ap|$. Since $|Aq|=\sup_{|U|\leqq 1}|q(AU)|$ is lower semicontinuous on $\mathcal{S}$, we have

$$\underline{\lim_{q\in F}}\ |Aq|\geqq|Ap|\ (A\in\mathrm{A}).$$

Then we have $\lim_{q\in F}|Aq|=|Ap|$, and p is an absolutely continuous state.

Using the similar arguments of Proposition 3. 1, we obtain

Proposition 4. 1. *Consider a compact set of states on* A *and a continuous field x in $\mathcal{W}$. Then the absolute variation x^v is weakly continuous at every absolutely continuous state in $\mathcal{W}$.*

In the following propositions 4. 2—4. 5 we assume that the algebra A is separable.

Proposition 4. 2. *Let* A *be a separable C^*-algebra and $\mathcal{S}$ its total state space. Choose a countable subset $\{A_n\}$ of the unit ball* U *of* A *which is everywhere dense in* U *and consider a numerical function on $\mathcal{S}$:*

$$d^v(\omega)=\sum_n 2^{-n}|A_n\omega|.$$

Then d^v is a lower semicontinuous function on $\mathcal{S}$. A state p is absolutely continuous in a compact subset $\mathcal{W}$ of $\mathcal{S}$ if and only if d^v (as a function in $\mathcal{W}$) is continuous at p.

Proposition 4. 3. *Consider a metric m in $\mathcal{S}$ which induces the weak topology of $\mathcal{S}$. Then the absolute weak topology of $\mathcal{S}$ is the topology induced by the metric*

$$v(p, q) = m(p, q) + |d^v(p) - d^v(q)|.$$

The totality $\mathcal{S}_P$ of pure states in A *is its closed subspace.*

Proposition 4. 4. *A weakly compact set of states in* A *contains absolutely continuous states in it everywhere dense.*

Proposition 4. 5. *Let μ be a distribution in the total state space $\mathcal{S}$. Then, removing an open set $\mathcal{U}$ of any small mass from $\mathcal{S}$, the set $\mathcal{S}-\mathcal{U}$ is absolutely continuous.*

Consider a sequence $\{s_n\}$ of states in A which converges to a state s by the absolute weak topology. Then $\mathcal{W} = \{s_n, s\}$ is an absolutely continuous compact set of states in A and given any bounded operator K in A_s' we can choose a sequence K_n of bounded operators in A_{s_n}' such that $|K_n| \leq |K|$ and $K_n s_n$ converges weakly to K_s. Roughly speaking, the algebra A_s' is approximated by a sequence of algebras $\{A_{s_n}'\}$.

Finally we notice that Propositions 4. 5 and Theorems 18, 19 include completely the v. Neumann's reduction theorem.

Let A be a C^*-algebra A in a Hilbert space $\mathfrak{H}$, A$'$ its commutor and A$''$ its bicommutor. Consider the center Z $(=A'' \cap A')$ of A$''$ and its commutor Z$'$. It is well-known that:

Lemma.

(1). Z *is abelian and* Z$'$ *contains at least a generative abelian projection E.*

(2). *If the Hilbert space $\mathfrak{H}$ is separable, then the algebras* A$''$ *and* A$'$ *are products* $A'' = A_1 \times A_2$, $A' = A_1' \times A_2'$ *of algebras* A_1, A_2 *and their commutors* A_1', A_2' *respectively,* where (A_1, A_1') and (A_2, A_2') are commutor-pairs on closed supspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ of $\mathfrak{H}$ such that the projection E on $\mathfrak{H}$ with range $E = \mathfrak{H}_1$ belongs to the center Z of A$''$. *The algebra* $Z_1' = (A_1 \cup A_1')''$ *contains at least an abelian projection E_1 which is generative in $\mathfrak{H}_1$ relative to the algebra* A_1 *and the algebra* $Z_2' = (A_2 \cup A_2')''$ *contains at least one abelian projection E_2 which is generative in $\mathfrak{H}_2$ relative to the algebra* A_2'.

Even if A is inseparable and E is an abelian generative projection in the algebra Z$'$, we consider the smallest uniformly closed linear space $\mathfrak{H}_1$ which contains $(AEx: A \in \mathrm{A},\ x \in \mathfrak{H})$. The induction of the algebra A$''$ in $\mathfrak{H}_1$ is an isomorphism and E is generative in $\mathfrak{H}_1$ relative to the algebra A.

Consider an abelian representation (A_λ, E): $A \rightarrow A_\lambda$ of A in a Hilbert sgace $\mathfrak{H}$ whose projection E is abelian relative to the commutor Z$'$ $(=(A \cup A')'')$ of the center Z of A$''$. We call such an abelian representation a central abelian representation. Then the v. Neumann's reduction theory is essentially the reduction theory of central abelian representations.

Assume that the algebra A and the Hilbert space $\mathfrak{H}$ are separable. A central abelian representation $(A_\lambda, E): A \rightarrow A_\lambda$ in the space $\mathfrak{H}$ has a suitable compoundly cyclic element g relative to the algebra A and the center Z. Then g is cyclic relative to A. This compoundly cyclic representation is unitary equivalent to the representation of A in $L^2(\mu)$ which is defined by a suitable pre-spectral distribution μ in the total state space $\mathcal{S}$, where the carrier of μ is the carrier of the projection E. Apply Theorem 20 to this distribution μ, then a reformed v. Neumann reduction theorem follows.

Theorem. *If* $(A_\lambda, E): A \rightarrow A_\lambda$ *is a central abelian representation of a separable* C^**-algebra* A, *then there is a sequence* Z_n *of projections in the center* Z *of* A'' *such that* Z_n *converges to* I *strongly and the carrier of each* Z_nE *is absolutely continuous.*

Next we observe the central abelian representation of a W^*-algebra.

Theorem 21. *Consider a* W^**-algebra* A *in a Hilbert space* $\mathfrak{H}$ *with an generative abelian projection* E, *which is abelian relative to the algebra* Z' $=(A \cup A')''$. *Then the center* Z *is the carrier algebra of* A *reduced by* E, *and the spectrum* $\mathcal{W}$ *of* Z, *as a compact state space on* A, *is absolutely continuous.*

To prove the theorem we first observe the next lemma.

Lemma 4. 1. *Let* A, Z', E *and* $\mathfrak{H}$ *be given as in Theorem* 21. *Then for every* $A \in$ A *a definite self-adjoint operator* $K \eta$ A' *and a partially isometric operator* $U \in$ A *can be so chosen as* $AE = KUE$.

Proof. Consider a fixed $A \in$ A, $g \in$ Range E and the uniform closure $[Ag]$ of the set $(Bg : B \in A)$. By Lemmas 2. 2, 2. 3 in Chapter 1, a partially isometric operator U_g in A and a definite self-adjoint operator $K_g \eta$ A' can be so chosen that

$$Ag = U_g K_g g, \ \text{Range } K_g \subseteqq [Ag] \text{ and } U_g = U_g Z_g,$$

where Z_g is the least projection in the center Z of A whose range contains $[Ag]$. U_g, K_g and Z_g are uniquely determined by A and g. Z_gE is a projection whose range is the uiform closure of the set $(Fg: F \in Z)$. Notice that $AFg = U_g K_g Fg$ $(F \in Z)$. Then we have

$$AZ_gE = U_g K_g E.$$

The system $(Z_g: g \in \text{Range } E)$ is a directed system of projections and converges strongly to the identity I. Since $h \in$ Range Z_gE implies $U_h = U_g Z_h$ and $K_h = K_g Z_h$, the system $(U_g: g \in \text{Range } E)$ converges strongly to a suitable partially isometric operator U in A when Z_g converges to I. Similarly the system $(K_g: g \in \text{Range } E)$ determines a definite self-adjoint operator $K \eta$ A' with $K_g = KZ_g$ $(g \in \text{Range } E)$.

Proof of the theorem. (A, E) is a faithful abelian representation of A and algebraically equivalent to an abelian representation ($A_{\mathcal{W}}$, P) which is determined by a compact state space $\mathcal{W}$. The projection E is therefore identified with the coordinate projection field P in $\mathcal{W}$, where $C_{\mathcal{W}}$ is the center of the algebra $A_{\mathcal{W}}$.

Now given any $A \in$ A, a partially isometric operator U in A and a definite self-adjoint operator K in A$'$ can be so chosen that $AP = UKP$ and $U^*AP = KP$. Then we obtain

$$A\omega = U(K\omega) \text{ and } U^*A\omega = K\omega.$$

$K\omega$ is a positive field in $\mathcal{W}$. In fact $K_{\mathcal{W}}$ and A are identical,

$$P(B^*B)^* KP \geqq 0 \ (B \in \mathrm{A}),$$

and

$$(B^*B\omega, K\omega)_\omega \geqq 0 \text{ for every } B \in \mathrm{A}.$$

Hence $K\omega$ is the absolute variation of the field $A\omega$ and is continuous.

§ 5. A non-commutaive extension of the Gelfand-Stone-Weierstrass Theorem in a compact space of pure traces.

(a). Absolute continuity of compact spaces of traces.

A state t on A is a *trace* if $t(AB) = t(BA)$ holds.

Lemma 5.1. *Consider a fixed trace t on* A. *Then*

(1). *$L^2(t)$ is a two-sided invariant self-adjoint subspace of the dual space* $\bar{\mathrm{A}}$ *of* A.

(2). *$x \to x^*$ is a conjugate linear and isometric automorphism of $L^2(t)$.*

(3). *Let* A_l *denote the algebra of left multiplies (A_l: $A \in$* A*) of* A *in* A *where $(A_l x)(B) = x(AB)$ holds for every A, $B \in$* A *and $x \in$* A*). Then* A_t'' *and* $(\mathrm{A}_l)_t''$ *are a commutor pair in $L^2(t)$.*

We denote by $\mathcal{I}$ the totality of traces on A. If $\mathcal{W}$ is a compact space of traces on A and x is a vector field in $\mathcal{W}$, then we denote by x^* a field $(x^*)_\omega = (x_\omega)^*$ on $\mathcal{W}$. If X is an operator field in K, each value X_ω of X belongs to A. We denote by X_l an operator field in $\mathcal{W}$ with $(X_l)_\omega = (X_\omega)_l$ ($\omega = \mathcal{W}$).

Lemma 5.2. *Consider a fixed distribution μ in $\mathcal{I}$. Then*:

(1). *$x \in L^2(\mu)$ and $K \in$* K *implie Kx, $K_l x \in L^2(\mu)$.*

(2). *$x \to x^*$ is a reflexive, conjugate linear and isometric automorphism of the Hilbert space $L^2(\mu)$.*

(3). *Strong closures of the representative algebras* $(\mathrm{K}_\mu)''$ *and* $(\mathrm{K}_{l\mu})''$ *are a commutor pair in $L^2(\mu)$.*

These two lemmas follow immediately from the well-known trace theory. Further we obtain the next proposition.

Proposition 5.1. *Consider a compact set $\mathcal{W}$ of traces on* A. *Then*

(1). *$\mathcal{W}$ is absolutely continuous.*

(2). *$\mathfrak{F}_{\mathcal{W}}$ is $*$-invariant and invariant under algebras* K *and* K_ι.

(3). *$x \rightarrow x^*$ is reflexive, conjugate linear and isometric automorphism of the Banach space $\mathfrak{F}_{\mathcal{W}}$.*

(4). *Q^*-closures $(K\mathcal{W})_q$ and $(K_\iota\mathcal{W})_q$ of $K\mathcal{W}$ and $K_\iota\mathcal{W}$ in the Banach space $\mathfrak{F}_{\mathcal{W}}$ are a commutor pair.*

Proof. (1). Consider a trace t and an element A of A. Then $At = U(A^*A)^{\frac{1}{2}}t$, where U is a partially isometric operator in A_t'' and $(A^*A)^{\frac{1}{2}}t$ is a positive definite functional. The absolute variation of At is $(A^*A)^{\frac{1}{2}}t$ and the field $A\omega$ is absolutely continuous in the total trace space $\mathfrak{T}$.

Hence (1) follows.

Since (2) and (3) are obvious, we prove (4) only. $\mathcal{W}$ is absolutely continuous and $(K\mathcal{W})^q$ is the bicommutor of $K\mathcal{W}$ in $\mathfrak{F}_{\mathcal{W}}$. The conjugate linear isometric automorphism $x \leftrightarrow x^*$ in $\mathfrak{F}_{\mathcal{W}}$ determines a conjugate linear spatial isomorphism $X \leftrightarrow (X^*)_\iota$ because $(Xx^*)^* = (X^*)_\iota x$. Then $(K\mathcal{W}_\iota)^q$ is the bicommutor of $K\mathcal{W}_\iota$. Since $(K\mathcal{W})^q$ and $(K_\iota\mathcal{W})^q$ commute with each other, to prove (4) it is sufficient to show that $(K\mathcal{W})'$ and $(K_\iota\mathcal{W})'$ commute with each other. Let $X \in (K\mathcal{W})'$ and $Y \in (K_\iota\mathcal{W})'$. X is a continuous operator field which commutes with $K\mathcal{W}$ and whose each value $X\omega$ ($\omega \in \mathcal{W}$) belongs to $(K_\omega)' = (A_\omega)' = (A_\omega)_\iota''$. Similarly Y is a continuous operator field whose each value Y_ω belongs to A_ω'' and which satisfies $X_\omega Y_\omega = Y_\omega X_\omega$. Then we have $XY = YX$ and hence $((K\mathcal{W})^q, (K_\iota\mathcal{W})^q)$ is a commutor pair in $\mathfrak{F}_{\mathcal{W}}$.

(b). $\natural$-mapping in the dual space of C*-algebra.

Hereafter we shall use the following notations.

(1). *If $A \in$ A, then U_A is the smallest uniformly closed convex subset of* A *which contains* (U^*AU: *U are unitary operators in* A).

(2). U *denotes the smallest uniformly closed convex set of bounded operators in the dual space* A *of* A *which contains the set* ($U^*{}_\iota U$: *U are unitary operators in* A), *where $y = U^*{}_\iota Ux$ implies $y(A) = x(U^*AU)$.*

(3). *If x is a functional in* A, *then $[Ux]$ is the uniform closure of the set* (Tx: $T \in$ U) *in the dual space of* A.

(4). *If μ is a distribution in the total trace space $\mathfrak{T}$ and x is a field in $L^2(\mu)$, then $[Ux]_\mu$ is the uniform closure of the set* (Tx: $T \in$ U) *in $L^2(\mu)$.*

(5). If $\mathcal{W}$ is a compact set of traces in A and x a continuous field in $\mathcal{W}$, then $[Ux]_{\mathcal{W}}$ is the uniform closure of the set (Tx: $T \in$ U) in $\mathfrak{F}_{\mathcal{W}}$.

we first prepare some lemmas and sub-lemmas to define the $\natural$-applications of functionals in A.

Sub-lemma 1. *S, $T \in$ U implies $ST \in$ U.*

Sub-lemma 2. *Let t be a trace and x an element of* $L^2(t)$. *Then* $[\mathrm{U}x]$ *is the uniform closure of* $(Tx: T\in\mathrm{U})$ *in* $L^2(t)$.

Lemma 5. 3. *Consider a fixed trace t on* A.

(1). $x\in L^2(t)$ *and* $y\in[\mathrm{U}x]$ *imply* $[\mathrm{U}y]\subseteqq[\mathrm{U}x]$.

(2). *If* $x, y\in L^2(t)$ *and* $x_0\in[\mathrm{U}x]$, *then a* $y_0\in[\mathrm{U}y]$ *can be so chosen that*

$$\|x_0-y_0\|_t\leqq\|x-y\|_t \text{ and } x_0+y_0\in[\mathrm{U}(x+y)].$$

(3). *Let* x_n *and* y_n *be two sequences in* $L^2(t)$ *such that*

$$\|x_n-x\|_t\to 0,\ \|y^n-y\|_t\to 0 \text{ and } y_n\in[\mathrm{U}x_n].$$

Then $y\in[\mathrm{U}x]$

Proof. (1). $T\in\mathrm{U}$ implies $T[\mathrm{U}x]\subseteqq[\mathrm{U}x]$. Then $y\in[\mathrm{U}x]$ implies $(Ty: T\in\mathrm{U})\subseteqq[\mathrm{U}x]$ and $[\mathrm{U}y]\subseteqq[\mathrm{U}x]$. Hence (1) follows.

(2). If x, $y\in L^2(t)$ and $x_0\in[\mathrm{U}x]$, then a sequence $\{x_n\}(x_n=T_nx)$ with $T_n\in\mathrm{U}$ can be so chosen that $\|x_n-x_0\|_t\to 0$. Now $\{y_n\}(y_n=T_ny)$ is a sequence in $[\mathrm{U}y]$ and has at least a sub-sequential weak limit y_0 because of the weak compactness of $[\mathrm{U}y]$. Since $x_n+y_n\in[\mathrm{U}(x+y)]$ and

$$\|x_n-y_n\|_t=\|T_n(x-y)\|_t\leqq\|x-y\|_t,$$

the desired relations in (2) are satisfied.

(3) follows immediately from (2).

Similarly as Lemma 5. 3, we have the followng lemma.

Lemma 5. 4. *Consider a fixed distribution* μ *in the total trace space* $\mathfrak{T}$. *Then*

(1). $x\in L^2(\mu)$ *and* $y\in[\mathrm{U}x]_\mu$ *imply* $[\mathrm{U}y]_\mu\subseteqq[\mathrm{U}x]_\mu$.

(2). *If* $x, y\in L^2(\mu)$ *and* $x_0\in[\mathrm{U}x]_\mu$, *then a* $y_0\in[\mathrm{U}y]_\mu$ *can be so chosen that*

$$\|x_0-y_0\|_\mu\leqq\|x-y\|_\mu \text{ and } x_0+y_0\in[\mathrm{U}(x+y)]_\mu.$$

(3). *Let* x_n *and* y_n *be two sequences in* $L^2(t)$ *such that*

$$\|x_n-x\|_\mu\to 0,\ \|y_n-y\|_\mu\to 0 \text{ and } y_n\in[\mathrm{U}x_n]_\mu.$$

Then $y\in[\mathrm{U}x]_\mu$.

Proposition 5. 2. *Consider a trace t and a functional x in* $L^2(t)$.

Then there is one and only one functional x in $L^2(t)$ *such that*

$$x^\natural\in[\mathrm{U}x] \text{ and } x^\natural(AB)=x^\natural(BA)\ (A, B\in\mathrm{A}).$$

Proof. A_t'' is a W^*-algebra of finite type in which the Dixmier's $\natural$-*application* is defined. *The $A^\natural$ of $A \in \mathrm{A}$ is contained in the common part of the center of A_t'' and the strong closure of U_A in A_t''.* In fact, $A^\natural$ is contained in the uniform convex span of the set $(U^*AU$: U are unitary operators in $\mathrm{A}_t'')$. Each U is a strong limit of unitary operators in A. Since $U^*{}_tU(At) = UAU^*t(A, U \in \mathrm{A})$, $[\mathrm{U}(At)]$ is the uniform closure of the set $(Tt\colon T \in \mathrm{U}_A)$ and contains $A^\natural t$.

Hence putting $x = At$ and $x^\natural = A^\natural t$ we have the relations

$$x^\natural(AB) = x^\natural(BA) \text{ and } (U^*{}_tUx)^\natural = x^\natural \in [\mathrm{U}x] \text{(for unitary } U \text{ in } A).$$

The mapping $At \to (At)^\natural = A^\natural t$ is extended to a bounded linear mapping $x \to x^\natural$ in $L^2(t)$ which preserves above relations in virtue of Lemma 5.3. Finally we prove the uniqueness of $x^\natural$ in the proposition. Suppose that there is another functional y in $[\mathrm{U}x]$ such that $y(AB) = y(BA)$. Since we have $(Tx)^\natural = x^\natural(T \in \mathrm{U})$, $y \in [\mathrm{U}x]$ implies $y^\natural = x^\natural$. Hence $Ty = y(T \in \mathrm{U})$ implies $y = y^\natural = x^\natural$ and the uniqueness of $x^\natural$. Q. E. D.

Now we observe the $\natural$-application of pure traces. A trace t is said *pure* if the algebra A_t' is a factor. We have immediately:

Lemma 5. 5. *A trace t is pure if and only if $x \in L^2(t)$ implies $x^\natural = x(I)t$.*

We next define the $\natural$-applications of vector fields and operator fields.

(1). If $\mathcal{W}$ is a compact set of taces and x is a vector field in $\mathcal{W}$, then we denote by $x^\natural$ the field $(x^\natural)_\omega = (x_\omega)^\natural$ in $\mathcal{W}$.

(2). If t is a trace, then the $\natural$-application of $A \in \mathrm{A}$ inA_t'' is denoted by $(A_t)^\natural$.

(3). Consider the totality $\mathcal{T}$ of traces in A. If $A \in \mathrm{A}$, then we denote by $A^\natural$ the operator field in $\mathcal{T}$ such that $(A^\natural)_\omega = (A_\omega)^\natural\,(\omega \in \mathcal{T})$.

(4). If μ is a distribution in $\mathcal{T}$, then $(A_\mu)''$ is a W^*-algebra of finite type in $L^2(\mu)$. The $\natural$-application of $A \in \mathrm{A}$ in $(\mathrm{A}_\mu)''$ is denoted by $(A_\mu)^\natural$.

Lemma 5. 6. *Consider a distribution μ in the total trace space $\mathcal{T}$. Then the $A^\natural$ of each $A \in \mathrm{A}$ is a measurable operator field in $L^2(\mu)$ and identical with the operator $(A_\mu)^\natural$.*

Proof. $(A^\mu)^\natural$ is contained in the common part of the cenfer of $(\mathrm{A}_\mu)''$ and the strong closure of U_A in $(\mathrm{A}_\mu)''$. If f is a continuous function in $\mathcal{T}$, $(A_\mu)^\natural f$ belongs to the center of K_μ''. Then

$$(BC(A_\mu)^\natural f\,\omega, \omega)_\mu = (CB(A_\mu)^\natural f\,\omega, \omega)\ (B, C \in \mathrm{A}).$$

and

$$\int f(\omega)((A)^\natural{}_\mu\omega)_\omega(BC)\,d\mu(\omega) = \int f(\omega)((A_\mu)^\natural\omega)_\omega(CB)\,d\mu(\omega).$$

For each fixed B, C in A the equality

$$((A_\mu)^\natural\omega)_\omega(BC)=((A'_\mu)^\natural_\omega)\omega(CB)$$

holds almost everywhere. $(A_\mu)^\natural\omega$ is regularly and weakly measurable. Then, removing an open set of any small mass from $\mathfrak{T}$, it is weakly continuous. Hence the above equality is valid for every B, C in A removing a fixed null sup-set of $\mathfrak{T}$.

On the other hand $(A_\mu)^\natural$ is contained in the strong closure of U_A in $(A_\mu)''$ and a sequence T_n in U_A can be so chosen that

$$\| T_n\omega-(A_\mu)^\natural\omega\|_\mu^2=\int\|T_n\omega-((A_\mu)^\natural\omega)_\omega\|_\omega\, d\mu\,(\omega)\leqq 4^{-n}.$$

Now we have

$$\|T_n\omega-((A_\mu)^\natural\omega)_\omega\|_\omega\to 0\ (n\to\infty) \text{ and } ((A_\mu)^\natural\omega)_\omega\in[U(A\omega)]$$

almost everywhere, By these relations we have $((A_\mu)^\natural\omega)_\omega=(A^\natural)\omega$ almost everywhere, and the equality in $L^2(\mu)$:

$$(A_\mu)^\natural x = A^\natural x$$

is valid for every $x=\sum_1^n f_i A_i$ with $f_i\in C$ and $A_i\in A$. Then the same equality is valid for every $x\in L^2(\mu)$ and hence $A^\natural$ is a measurable operator field which is identical with $(A_\mu)^\natural$ in $L^2(\mu)$. Hence the lemma follows.

We now obtain the following proposition.

Proposition 5. 3. *Consider a distribution μ in the total trace space $\mathfrak{T}$ and a vector field x in $L^2(\mu)$. Then $[Ux]_\mu$ contains the field $x^\natural$.*

Proof. Consider a subset of $L^2(\mu)$:

$$\mathfrak{M}=(x\in L^2(\mu):\ y\in[Ux]_\mu \textit{ implies } x^\natural=y^\natural\in[Uy]_\mu).$$

We show that $\mathfrak{M}$ satisfies the following (1)—(4).

(1). *$\mathfrak{M}$ is uniformly closed.*

(2). $(A\omega:\ A\in A)\subseteqq\mathfrak{M}$.

(3). *$f\in C$ and $x\in\mathfrak{M}$ imply $fx\in\mathfrak{M}$.*

(4). *$x, y\in\mathfrak{M}$ imply $x+y\in\mathfrak{M}$.*

(1) follows from (3) of Lemma 5. 4. (2) follows from what $T\in U_A$ implies $T^\natural=A^\natural$ and $(A^\natural\omega)=(T^\natural\omega)\in[U(T\omega)]_\mu$. We now prove (3). Let $f\in C$ and $x\in\mathfrak{M}$. Every $y=Tfx$ with $T\in U$ satisfies the relation:

$$y^\natural=fx^\natural\in f[UTx]_\mu\subseteqq[U(y)]_\mu.$$

Then the same relations are satisfied by every $y\in[U(fx)]_\mu$ and hence $fx\in\mathfrak{M}$. We finally prove (4). Let $x, y\in\mathfrak{M}$. By $x^\natural\in[Ux]_\mu$ we can choose a suitable $y_0\in[Uy]_\mu$ such that $x^\natural+y_0\in[U(x+y)]_\mu$. Now

$$x^{\natural}+[\mathrm{U}y_o]_\mu \subseteqq [\mathrm{U}(x+y)]_\mu$$

and

$$(x^{\natural}+y)^{\natural}=x^{\natural}+y_o^{\natural}\in x^{\natural}+[\mathrm{U}y_o]_\mu \subseteqq [\mathrm{U}(x+y)]_\mu.$$

Notice that $T\in \mathrm{U}$ implies $(Tx)^{\natural}=x^{\natural}$, $(Ty)^{\natural}=y^{\natural}$ and Tx, $Ty\in \mathfrak{M}$.
Then every $z=T(x+y)$ ($T\in \mathrm{U}$) satisfies the relations:

$$(x+y)^{\natural}=z^{\natural}\in [\mathrm{U}z]_\mu.$$

The same relations are satisfied for every $z\in [\mathrm{U}(x+y)]_\mu$ and hence $x+y$ belongs to $\mathfrak{M}$.

By (1)$\cdots$(4) $\mathfrak{M}$ contains the set $(Af\colon\ A\in \mathrm{A}$ and $f\in \mathrm{C})$ and its uniform linear span $L^2(\mu)$, from which the proposition follows.

(c). A generalized Stone-Weierstruss Theorem in a pure state space.

To study the problem which is mentioned in the section 3 of Chapter 3, we introduce here the following notations.

(1). We denote by $\mathrm{A}^{\natural}$ the smallest C^*-algebra of operator fields in the total trace space $\mathfrak{T}$ which contains the set $(A^{\natural}\colon\ A\in \mathrm{A})$.

(2). If μ is a distribution in the total trace space $\mathfrak{T}$, then we denote by $[\mathrm{A}\omega]_\mu$ the uniform closure of the set $(A\omega\colon\ A\in \mathrm{A})$ in $L^2(\mu)$.

Lemma 5. 3. *Consider a distribution μ in the total trace space $\mathfrak{T}$. Then the set $(X\omega\colon\ X\in \mathrm{A}^{\natural})$ is contained in $[\mathrm{A}\omega]_\mu$.*

Proof. Let E denote the projection in $L^2(\mu)$ whose range is $[\mathrm{A}\omega]_\mu$. Then E commutes with every operator in the strong closure $(\mathrm{A}_\mu)''$ of A in $L^2(\mu)$. The $A^{\natural}$ of each $A\in \mathrm{A}$ belongs to $(\mathrm{A}_\mu)''$ and commutes with E. Hence every operator X in $\mathrm{A}^{\natural}$ commutes with E. Since ω is contained in the range of E, we have

$$(X\omega\colon\ X\in \mathrm{A}^{\natural})\subseteqq \text{Range } E=[\mathrm{A}\omega]_\mu. \quad \text{Q. E. D.}$$

Consider the totality $\mathfrak{T}_p$ of pure traces on A. If the algebra A is abelian, then $\mathfrak{T}_p$ is the spectrum of A and there is a one-one correspondence between the totality $\mathfrak{T}$ of traces and the totality of distributions in $\mathfrak{T}_p$.

Consider a prespectral distribution μ in $\mathfrak{T}_p$ in the sense of Definition 1.7 in Chapter 2 and set $p=\int \omega d\mu$. Then the Fourier induction

$$x\in L^2(\mu)\rightarrow \int x_\omega d\mu(\omega)$$

is an isometry between $L^2(\mu)$ and $L^2(p)$. If A is abelian, every distribution μ in $\mathfrak{T}_p$ is prespectral because $L^2(\mu)$ is the Hilbert space of square summable and measurable functions in $\mathfrak{T}_p$. We now show that these essential properties of distributions in $\mathfrak{T}_p$ are preserved even if A is non-abelian.

Theorem 22. *Let μ be a distribution in the total trace space $\mathfrak{T}$ whose carrier $\mathcal{W}$ consists of pure traces except for a null set. Then μ is prespectral and the center of $(\mathrm{A}_\mu)''$ is the totality $\mathrm{M}(\mu)$ of bounded measurable functions in $\mathfrak{T}_p$.*

Proof. Notice that the operator field $A^\natural$ of $A \in \mathrm{A}$ satisfies

$$(A^\natural)_\omega = (A_\omega)^\natural = \omega(A)I \quad (\omega \in \mathfrak{T}_p).$$

Then $\natural$-application $(A_\mu)^\natural$ of A in $L^2(\mu)$ is the primitive function J_A of A in $\mathcal{W}$ which is defined by $J_A(\omega) = \omega(A)$ in $\mathcal{W}$. The smallest C^*-algebra of operator fields in $\mathcal{W}$ which contains $(J_A: A \in \mathrm{A})$ is the totality $\mathrm{C}_\mathcal{W}$ of continuous functions in $\mathcal{W}$. Then we have $\mathrm{A}^\natural = \mathrm{C}_\mathcal{W}$ and $(f\omega : f \in \mathrm{C}_\mathcal{W}) \subseteq [\mathrm{A}\omega]_\mu$. By Lemma 1. 8 in Chapter 2 μ is a prespectral distribution. The center of $(\mathrm{A}_\mu)''$ is the strong closure of $\mathrm{A}^\natural (= \mathrm{C}_\mathcal{W})$ in $L^2(\mu)$ and is the algebra $\mathrm{M}(\mu)$.

The space $\mathfrak{T}_p$ is not generally compact, but its compact subspaces have the following properties.

Theorem 23. *A compact set $\mathcal{W}$ of pure traces is a prespectrum. If x is a continuous field in $\mathcal{W}$, then for any $\varepsilon > 0$ we can choose a $T = \sum \alpha_i U_{ii}^* U_i \in \mathrm{U}$ such thvt*

$$\sup_{\omega \in \mathcal{W}} \| T x_\omega - (x_\omega)(I)\omega \|_\omega < \varepsilon.$$

Proof. Every distribution μ in $\mathcal{W}$ is prespectral, and by Theorem 14 $\mathcal{W}$ is a pre-spectrum. Next, consider a fixed $x \in \mathfrak{F}_\mathcal{W}$ and the set $[\mathrm{U}x]_\mathcal{W}$ (= the uniform closure of the set $(Tx: T \in \mathrm{U})$ in $\mathfrak{F}_\mathcal{W}$). If μ is a distribution in $\mathcal{W}$, the field x is contained in its uniform closure $[\mathrm{U}x]_\mu$ in $L^2(\mu)$. Since $x^\natural (= x_\omega(I)\omega$ in $\mathcal{W})$ belongs to $\mathfrak{F}_\mathcal{W}$, by Proposition 2. 1 in Chapter 3 $x_\omega(I)\omega$ belongs to $[\mathrm{U}x]_\mathcal{W}$ and hence the theorem follows.

Consider a compact set $\mathcal{W}$ of traces in A. A trace $t \in \mathcal{W}$ is said $\natural$-*continuous* in $\mathcal{W}$ if every numerical function $\| A^\natural \omega \|_\omega (A \in \mathrm{A})$ of the variable ω is continuous in $\mathcal{W}$.

Proposition 5. 3. *Consider a compact set $\mathcal{W}$ of traces on A and a continuous field in $\mathcal{W}$. If x is $\natural$-continuous in $\mathcal{W}$, then $x^\natural$ and the numerical function $\| (x^\natural)_\omega \|_\omega$ of the variable ω are weaky continuous in $\mathcal{W}$ respectively.*

Proof. If $A, B \in \mathrm{A}$, then the function of the variable ω:

$$A^\natural \omega(B) = (B\omega, A^{*\natural}\omega)_\omega = (B^\natural \omega, A^{*\natural}\omega)_\omega$$

is continuous in $\mathcal{W}$. Therefore every $A^\natural \omega (A \in \mathrm{A})$ is weakly continuous in

$\mathcal{W}$. Consider a field $x=\Sigma f_i A_i\omega$ ($f_i \in \mathrm{C}$, $A_i \in \mathrm{A}$ and its $\natural$-application $x^\natural = \Sigma f_i A_i^\natural \omega$. Then $x^\natural$ and the numerical function of the variable ω:

$$\|x^\natural{}_\omega\|_\omega^2 (= \Sigma \overline{f_j(\omega)} f_i(\omega)(A_j{}^{*\natural}\omega)(A_i))$$

are weakly continuous and continuous in $\mathcal{W}$ respectively.

Proposition 5. 4. *Every pure trace is $\natural$-continuous in $\mathcal{T}$. The totality $\mathcal{T}_p$ of pure traces is the totality of $\natural$-conuous points in the weak closure of $\mathcal{T}_p$. If the algebra* A *is separable, then $\mathcal{T}_p$ is a G_δ-subspace of $\mathcal{T}$ and has a complete metric which induces the wevk topology.*

Proof. If $A \in \mathrm{A}$, then the function of the variable ω:

$$\|A^\natural \omega\|_\omega (= \inf_{T \in \mathrm{U}} \|T(A\omega)\|_\omega)$$

is upper semicontinuous in $\mathcal{T}$, and $\mathcal{B}_p$ is the set

$$\mathcal{T}_p = \bigcap_{A \in \mathrm{A}} (t \in \mathcal{T} : \|A^\natural t\|_t = |t(A)|).$$

Let $\mathcal{T}_p$ be the weak closure of $\mathcal{T}_p$. Then a trace in $\mathcal{T}_p$ is $\natural$-continuous in $\mathcal{T}_p$ if and only if it belongs to $\mathcal{T}_p$.

Assume that A is separable and choose a countable subset $\{A_n\}$ of A which is dense verywhere in the unit ball of A. Then we have

$$\mathcal{T}_p = \bigcap_{n=1}^{\infty} (t \in \mathcal{T} : \|A_n{}^\natural t\|_t = |t(A_n)|)$$

and $\mathcal{T}_p$ is a G_δ-subset of $\mathcal{T}$. Q. E. D.

Assume that A is separable and consider the sequence A_n in the unit ball of A which is everywhere dense in it. Then

$$d^\natural(t, s) = \sum_{n=1}^{\infty} 2^{-n} |\, \|A_n{}^\natural t\|_t - \|A_n{}^\natural s\|_s |$$

is a metric in $\mathcal{T}$. We call it a *$\natural$-metric* in $\mathcal{T}$ and its induced topology a *$\natural$-weak topology* of $\mathcal{T}$.

If a distribution μ in $\mathcal{T}$ vanishes outside of $\mathcal{T}_p$, it is regarded as a distribution in $\mathcal{T}_p$ such that every Borel set in $\mathcal{T}_p$ is measurable and, removing a suitable set of any small mass from $\mathcal{T}_p$, each measurable set is compact. Such a distribution in $\mathcal{T}_p$ is said merely a distribution in $\mathcal{T}_p$. Consider the totality $\mathrm{D}(\mathcal{T}_p)$ of distributions in $\mathcal{T}_p$ and the totality $\mathrm{C}(\mathcal{T}_p)$ of continuous functions in $\mathcal{T}_p$. $\mathrm{D}(\mathcal{T}_p)$ is a subset of the dual space of $\mathrm{C}(\mathcal{T}_p)$ in which the weak topology is defined.

Theorem 24. *If* A *is separable, every trace t on* A *is a mean*

$$t = m_\mu = \int \omega d\mu(\omega)$$

of a suitable distribution μ in $\mathcal{T}_p$. The mapping

$$\mu \in \mathrm{D}(\mathcal{T}_p) \to m_\mu \in \mathcal{T}$$

is a homeomorphism between $\mathrm{D}(\mathcal{T}_p)$ *and* $\mathcal{T}_p$ *in their weak and* $\natural$*-weak topologies.*

Proof. Let t be a trace and Z the center of $(\mathrm{A}_t)''$. By Proposition 1. 7 in Chapter 2, a prespectral distribution μ in the total state space $\mathcal{S}$ is so uniquely chosen that two representations $(\mathrm{A}_t, \mathrm{Z}, L^2(t))$ and $(\mathrm{A}_\mu, \mathrm{M}(\mu), L^2(\mu))$ are unitary equivalent. It is easy to show that the carrier of μ is contained in $\mathcal{T}$ and consists of pure traces almost everywhere. Since every distribution in $\mathcal{T}_p$ is prespectral, the mapping $\mu \to m_\mu$ is one-one between $\mathrm{D}(\mathcal{T}_p)$ and $\mathcal{T}$. We show that the mapping is a homeomorphism. Since $(A_\omega)^\natural = \omega(A) I = J_A(\omega) I$ holds for $A \in \mathrm{A}$ and $\omega \in \mathcal{T}_p$, $\mathrm{C}(\mathcal{T}_p)$ is regarded as the smallest C^*-algebra of operator fields in $\mathcal{T}_p$ which contains $(A^\natural : A \in \mathrm{A})$. Let $\mu \in \mathrm{D}(\mathcal{T}_p)$, $x \in L_2(\mu)$, $t = m_\mu = \int \omega d\mu$ and $m_x = \int x_\omega d\mu(\omega) \in L^2(t)$. Then we have

$$(A_t)^\natural m_x = \int (A_\omega)^\natural x_\omega d\mu(\omega) = \int J_A(\omega) x_\omega d\mu(\omega).$$

For every operator field $X \in \mathrm{A}^\natural$ in $\mathcal{T}$ a continuous function f_X in $\mathcal{T}_p$ is so determined that

$$(X_t t, t)_t = \int f_X(\omega) d\mu(\omega).$$

By the mapping $\mu \to m_\mu$ the weak topology in $\mathrm{D}(\mathcal{T}_p)$ is induced to the weakest topology in $\mathcal{T}$ such that each numerical function $(X_t t, t)_t$ (where $X \in \mathrm{A}^\natural$) of the variable t is continuous in $\mathcal{T}$. The $\natural$-weak topology of $\mathcal{T}$ is the weakest topology such that each $((A_t)^\natural t, (A_t)^\natural t)_t$ $(A \in \mathrm{A})$ is continuous. Then it is weaker than the former induced weak topology. Conversely, let t_n be a sequence in $\mathcal{T}$ which converges to $t \in \mathcal{T}$ in the $\natural$-weak topology. Then $(A^\natural t_n)(B) = (B^\natural t_n, A^{*\natural} t_n)_{tn} \to (A^\natural t)(B)$ and $t_n(A) = (A^\natural t_n, I^\natural t_n)_{tn} \to t(A)$ when $n \to \infty$. Therefore $\mathcal{W} = \{t, t_1, t_2, \cdots\}$ is a weakly compact sub-set of $\mathcal{T}$ in which every $A^\natural \omega$ $(A \in \mathrm{A})$ is a continuous field. Now $A^\natural(K\omega) = KA^\natural \omega \in \mathfrak{F}_{\mathcal{W}}$ $(K \in \mathrm{K})$ imply that every $A^\natural (A \in \mathrm{A})$ and consequently every $X (\in \mathrm{A}^\natural)$ are continuous operator fields in $\mathcal{W}$, so that we have $(X_{tn} t_n, t_n)_{tn} \to (X_t t, t)_t$ $(X \in \mathrm{A}^\natural)$ and t_n converges to t by the induced weak topology. Hence $\mu \leftrightarrow m_\mu$ is a homeomorphism between $\mathrm{D}(\mathcal{T}_p)$ and $\mathcal{T}$.

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