ON GENERAL CONNECTIONS I

TOMINOSUKE ŌTSUKI

In a former paper [11][1], the author extended the concept of classical affine connections considering them as a sort of cross-sections of the tensor product bundle of the tangent bundle (of order 1) and the cotangent bundle of order 2 of a differentiable manifold, and defined naturally the covariant differentiation for any cross-section of this vector bundle over the manifold which he called a general connection. As an interesting example, the covariant differential operator for any tensor of type (1, 2), which is also an above mentioned cross-section, was the so-called trivial differential operator of the graded algebra of all tensor fields.

In the theory of general connections developed in [11], there exist some faults as follows : Let $\mathfrak{X}$ be a differentiable manifold[2] of dimension n. Let Γ[3] be any general connection of $\mathfrak{X}$ and P^i_j, Γ^i_{jk} be its components with respect to local coordinates u^i of $\mathfrak{X}$. For any tensor field, for instance, contravariant tangent vector field U^i, by the formula [11], (7. 4), its covariant differential is given by

$$DU^i=\left(P^i_j\frac{\partial U^j}{\partial u^k}+\Gamma^i_{jk}U^j\right)P^k_h\,du^h.$$

Accordingly, for a contravariant tangent vector field of $\mathfrak{X}$ which is given only along a curve C, $u^i=u^i(t)$, we can not, in general, define its covariant derivative with respect to t, DU^i/dt. If and only if the tangent vectors of C are eigen vectors of P^i_j, that is

$$P^i_j\frac{du^j}{dt}=\sigma\frac{du^i}{dt},$$

where σ in an eigen value of P^i_j, then we have

$$\frac{DU^i}{dt}=\sigma\left(P^i_j\frac{dU^j}{dt}+\Gamma^i_{jk}U^j\frac{du^k}{dt}\right).$$

We call such a curve an *eigen curve* belonging to the *eigen function* σ on $\mathfrak{X}$ which is an eigen value of $P=\lambda(\Gamma)$ at each point of $\mathfrak{X}$. But, as easily seen by [11], (6. 2—3), the forms

$$P^i_j\,dU^j+\Gamma^i_{jk}U^j\,du^k$$

1) The numbers in square brackets show the numbers of the references at the end of the present paper.

2) In the present paper, we deal with only manifolds, functions and transformations with suitable differentiabilities for our purpose.

3) We will make use of the notations in [11], with some exceptions.

are vectorial, acccordingly we can define a sort of covariant derivatives with respect to t for any tangent tensor field defined along any given curve by an analogous formula. According to the classical principle that the tangent vectors of a geodesic are parallelly displaced along itself, we can define geodesics only in the family of eigen curves. But the differential equations

$$P^i_j \frac{d^2u^j}{dt^2} + \Gamma^i_{jk} \frac{du^j}{dt}\frac{du^k}{dt} = \varphi \frac{du^j}{dt}$$

have a geometrical significance as mentioned above. This discordance will be removed.

In [11], § 9, the author could not define canonical mappings for any general connections which give an interesting interpretation of connections different from the ordinary ones. He treated with only regular general connections in a more arranged method than the one for classical affine connections in [10]. Introducing general frames of order 2, he will show that we can define also a canonical mapping for any general connection.

Furthermore, the universal affine connection of $\mathfrak{X}$ defined in [11], § 10, which is an affine connection naturally defined for the vector bundle over the space of all frames of order 2 of $\mathfrak{X}$ induced from the tangent bundle of $\mathfrak{X}$ by the projection of the associated principal bundle, has its significance only for regular general connections and their contravariant parts. But, making use of the space of all general frames of order 2 of $\mathfrak{X}$, the author will define a more generalized universal connection of $\mathfrak{X}$ which has a meaning for any general connection of $\mathfrak{X}$ such that the latter is induced from the former by the above mentioned canonical mapping of this general connection.

In the present paper, the author will study also the torsions and curvatures of general connections and some properties of eigen curves.

§ 1. Preliminary.

Let $\mathfrak{L}^2_n$ be the group of all generalized infinitesimal isotropies of order 2 at the origin of the n-dimensional coordinate space R^n, whose element is written as a set of real numbers (a^j_i, a^j_{ih}) such that $|a^j_i| \neq 0$ and whose multiplication is given by the following formulas : For any $\alpha, \beta \in \mathfrak{L}^2_n$

$$a^j_i(\alpha\beta) = a^j_k(\alpha)a^k_i(\beta), \qquad (1.1)$$

$$a^j_{ih}(\alpha\beta) = a^j_k(\alpha)a^k_{ih}(\beta) + a^j_{kl}(\alpha)\, a^k_i(\beta)a^l_h(\beta). \qquad (1.2)$$

We identify $L^1_n = GL(n, R)$ with the subgroup of $\mathfrak{L}^2_n$ which consists of all elements α such that $a^j_{ik}(\alpha) = 0$ and may also regard a^j_i as coordinates of L^1_n. Let σ be the natural homomorphism of $\mathfrak{L}^2_n$ onto L^1_n given by

$$a_i^j(\sigma(\alpha)) = a_i^j(\alpha). \tag{1.3}$$

Let $\mathfrak{N}_n^2$ be the kernel of σ and η, $\bar{\eta}$ be mappings of $\mathfrak{L}_n^2$ onto $\mathfrak{N}_n^2$ defined by

$$\eta(\alpha) = \sigma(\alpha^{-1})\alpha, \tag{1.4}$$

$$\bar{\eta}(\alpha) = \alpha\sigma(\alpha^{-1}). \tag{1.5}$$

An element α of $\mathfrak{L}_n^2$ can be written uniquely as a product of elements of L_n^1 and $\mathfrak{N}_n^2$ by

$$\alpha = \sigma(\alpha)\eta(\alpha) = \bar{\eta}(\alpha)\sigma(\alpha). \tag{1.6}$$

Now let $\mathfrak{X}$ be any n-dimensional differentiable manifold. With any coordinate neighborhood (U, u^j), where the local coordinates u^j are defined on the neighborhood U of $\mathfrak{X}$, we associate $n+n^2$ fields of vectors denoted by ∂u_i, $\partial^2 u_{ih}$. Let ∂v_i, $\partial^2 v_{ih}$ be the vector fields associated with another coordinate neighborhood (V, v^j). When $U \cap V \neq \emptyset$, we assume that they are related mutually on $U \cap V$ as

$$\partial u_i = \frac{\partial v^j}{\partial u^i} \partial v_j, \tag{1.7}$$

$$\partial^2 u_{ih} = \frac{\partial^2 v^j}{\partial u^h \partial u^i} \partial v_j + \frac{\partial v^j}{\partial u^i} \frac{\partial v^k}{\partial u^h} \partial^2 v_{jk}. \tag{1.8}$$

Thus we obtain at each point x of $\mathfrak{X}$ an $(n+n^2)$-dimensional vector space spanned by these $n+n^2$ vectors ∂u_i, $\partial^2 u_{ih}$, $x \in U$, but independent of coordinate neighborhood containing the point x, which is denoted by $\mathfrak{T}_x^2(\mathfrak{X})$. The union

$$\mathfrak{T}^2(\mathfrak{X}) = \bigcup_{x \in \mathfrak{X}} \mathfrak{T}_x^2(\mathfrak{X})$$

may be considered naturally as the total space of a vector bundle $\{\mathfrak{T}^2(\mathfrak{X}), \mathfrak{X}, \tau_2\}$ with the natural projection τ_2, whose structure group is $\mathfrak{L}_n^2$ (in fact $L_n^2 = \{\alpha \mid a_{ih}^j(\alpha) = a_{hi}^j(\alpha),\ \alpha \in \mathfrak{L}_n^2\}$ and the coordinate transformation $g_{VU} : U \cap V \to \mathfrak{L}_n^2$ is given by

$$a_i^j(g_{VU}) = \frac{\partial v^j}{\partial u^i}, \quad a_{ih}^j(g_{VU}) = \frac{\partial^2 v^j}{\partial u^h \partial u^i}. \tag{1.9}$$

We call any element of $\mathfrak{T}_x^2(\mathfrak{X})$ a *tangent vector of order 2 of* $\mathfrak{X}$ at x. For the sake of simplicity, we denote the vector bundle over $\mathfrak{X}$ by the same notation $\mathfrak{T}^2(\mathfrak{X})$ and call it *the tangent bundle of order 2 of* $\mathfrak{X}$. By means of (1.7), we may identify the vector ∂u_i with the tangent vector $\partial/\partial u^i$ in the ordinary sense. Accordingly the tangent bundle $T(\mathfrak{X})$ of $\mathfrak{X}$, which is to be called the tangent bundle of order 1 of $\mathfrak{X}$, with the projection $\tau : T(\mathfrak{X}) \to \mathfrak{X}$, may be considered as a subbundle of $\mathfrak{T}^2(\mathfrak{X})$.

Let $\{\mathfrak{B}^2(\mathfrak{X}), \mathfrak{X}, \pi_2\}$ be the associated principal bundle of $\mathfrak{T}^2(\mathfrak{X})$, which is called *the principal bundle of order 2 of* $\mathfrak{X}$. Any point $\bar{b}$ of $\mathfrak{B}^2(\mathfrak{X})$ is a frame of $\mathfrak{T}^2(\mathfrak{X})$ at $x = \pi_2(\bar{b})$, such that

$$\mathrm{e}_i(\bar{b}) = \partial u_j\, a_i^j(\bar{\beta}), \tag{1.10}$$

$$e_{ih}(b) = \partial u_j \, a^j_{ih}(\bar{\beta}) + \partial^2 u_{jk} \, a^j_i(\bar{\beta}) \, a^k_h(\bar{\beta}), \tag{1.11}$$

where $\bar{\beta} \in \mathfrak{L}^2_n$. For any $\bar{\alpha} \in \mathfrak{L}^2_n$, the right translation $r(\bar{\alpha})$ on $\mathfrak{B}^2(\mathfrak{X})$ is defined by

$$e_i(\bar{b}\bar{\alpha}) = e_j(\bar{b}) \, a^j_i(\bar{\alpha}), \tag{1.12}$$

$$e_{ih}(\bar{b}\bar{\alpha}) = e_j(\bar{b}) \, a^j_{ih}(\alpha) + e_{jk} \, (b) a^j_i(\bar{\alpha}) a^k_h(\bar{\alpha}), \tag{1.13}$$

where we simply denote $r(\bar{\alpha})(\bar{b})$ by $\bar{b}\bar{\alpha}$. Let $\{\mathfrak{B}(\mathfrak{X}), \mathfrak{X}, \pi\}$ be the associated principal bundle of $T(\mathfrak{X})$. As is well known, any point b of $\mathfrak{B}(\mathfrak{X})$ is a frame of $T(\mathfrak{X})$ at $x = \pi(b)$, such that

$$e_i(b) = \frac{\partial}{\partial u^j} \, a^j_i(\beta), \tag{1.14}$$

where $\beta \in \mathfrak{L}^1_n$. Since $T(\mathfrak{X}) \subset \mathfrak{T}^2(\mathfrak{X})$, we may make use of the same notation e_i for $\mathfrak{B}^2(\mathfrak{X})$ and $\mathfrak{B}(\mathfrak{X})$. For and $\alpha \in L^1_n$, the right translation $r(\alpha)$ on $\mathfrak{B}(\mathfrak{X})$ is defined by

$$e_i(b\alpha) = e_j(b) \, a^j_i(\alpha). \tag{1.15}$$

The natural homomorphism $\sigma = /\mathfrak{N}^2_n$: $\mathfrak{B}^2(\mathfrak{X}) \rightarrow \mathfrak{B}(\mathfrak{X})$ is given by

$$e_i(\sigma(\bar{b})) = e_i(\bar{b}). \tag{1.16}$$

For any $\bar{\alpha} \in \mathfrak{L}^2_n$, we have

$$\sigma \cdot r(\bar{\alpha}) = r(\sigma(\bar{\alpha})) \cdot \sigma. \tag{1.17}$$

For any coordinate neighborhood (U, u^i) and at each point $x \in U$, we associate an $(n+n^2)$-dimensional vector space which is spanned by $du^i \otimes du^h$ and *the differentials* d^2u^j *of order 2* which are assumed linearly independent mutually and of $du^i \otimes du^h$. We relate the two vector spaces corresponding to (U, u^i) and (V, v^i), at $x \in U \cap V$, with each other by

$$d^2v^j = \frac{\partial v^j}{\partial u^i} d^2u^i + \frac{\partial^2 v^j}{\partial u^h \partial u^i} \, du^i \otimes du^h. \tag{1.18}$$

Thus we obtain the cotangent vector space of order 2 of $\mathfrak{X}$ at x denoted by $\mathfrak{D}^2_x(\mathfrak{X})$ which is dual to $\mathfrak{T}^2_x(\mathfrak{X})$ and contain the tensor product $T^*_x(\mathfrak{X}) \otimes T^*_x(\mathfrak{X})$ of the cotangent space of $\mathfrak{X}$ at x. The base $\{du^i, du^i \otimes du^h\}$ is dual to the base $\{\partial u_i, \partial^2 u_{ih}\}$ of $\mathfrak{T}^2_x(\mathfrak{X})$. The union

$$\mathfrak{D}^2(\mathfrak{X}) = \bigcup_{x \in \mathfrak{X}} \mathfrak{D}^2_x(\mathfrak{X})$$

is the total space of *the cotangent bundle of order 2 of* $\mathfrak{X}$ which we denote by the same notation. $\mathfrak{D}^2(\mathfrak{X})$ contains the tensor product bundle $T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X})$ of the cotangent bundle $T^*(\mathfrak{X})$ as a subbundle.

In the following, for any vector bundle $\mathfrak{F} = \{\mathfrak{Z}, \mathfrak{X}, p\}$ over $\mathfrak{X}$ we shall denote generally by $\Psi(\mathfrak{F})$ the vector space consisting of all cross-sections of $\mathfrak{F}$ over the algebra $A(\mathfrak{X})$ of all scalar fields on $\mathfrak{X}$.

We define a natural differential operator

$$d: \begin{cases} \Psi(T(\mathfrak{X})) \to \Psi(\mathfrak{T}^2(\mathfrak{X}) \otimes T^*(\mathfrak{X})), \\ \Psi(T^*(\mathfrak{X})) \to \Psi(\mathfrak{D}^2(\mathfrak{X})) \end{cases}$$

by the equations :

$$d(V^i \partial u_i) = \partial^2 u_{ih} \otimes V^i du^h + \partial u_i \otimes dV^i, \tag{1.19}$$

$$d(V_i du_i) = V_i d^2 u^i + du^i \otimes dV_i. \tag{1.20}$$

Let $T(\mathfrak{X})^{\otimes(p,q)}$ be the vector bundle over $\mathfrak{X}$ which is the tensor product bundle of $T(\mathfrak{X})^{\otimes p}$ and $T^*(\mathfrak{X})^{\otimes q}$. We denote by

$$T^*(\mathfrak{X})^{\otimes(t-1)} \otimes \mathfrak{D}^2(\mathfrak{X}) \dot{\otimes} T^*(\mathfrak{X})^{\otimes(q-t)}$$

the vector bundle over $\mathfrak{X}$ which is obtained from $T^*(\mathfrak{X})^{\otimes(q+1)}$ by extending the tensor product bundle of its t-th component and $(q+1)$-th component to $\mathfrak{D}^2(\mathfrak{X})$. Then we can define generally the differential operator

$$\begin{aligned} d: \Psi(T(\mathfrak{X})^{\otimes(p,q)}) &\to \\ \Psi(\sum_{s=1}^{p} T(\mathfrak{X})^{\otimes(s-1)} &\otimes \mathfrak{T}^2(\mathfrak{X}) \otimes T(\mathfrak{X})^{\otimes(p-s)} \otimes T^*(\mathfrak{X})^{\otimes(q+1)} \\ + \sum_{t=1}^{q} T(\mathfrak{X})^{\otimes p} &\otimes T^*(\mathfrak{X})^{\otimes(t-1)} \otimes \mathfrak{D}^2(\mathfrak{X}) \dot{\otimes} T^*(\mathfrak{X})^{\otimes(q-t)}) \end{aligned} \tag{1.21}$$

by

$$\begin{aligned} &d(V^{i_1 \cdots i_p}_{j_1 \cdots j_q} \partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q}) \\ &= V^{i_1 \cdots i_p}_{j_1 \cdots j_q} d(\partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q}) \\ &+ \partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q} \otimes dV^{i_1 \cdots i_p}_{j_1 \cdots j_q}, \end{aligned}$$

and

$$\begin{aligned} &d(\partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q}) \\ &= \sum_{s=1}^{p} \partial u_{i_1} \otimes \cdots \otimes \partial u_{i_{s-1}} \otimes \partial^2 u_{i_s h} \otimes \partial u_{i_{s+1}} \otimes \cdots \otimes \partial u_{i_p} \otimes \\ &\qquad du^{j_1} \otimes \cdots \otimes du^{j_q} \otimes du^h \\ &+ \sum_{t=1}^{q} \partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_{t-1}} \otimes d^2 u^{j_t} \dot{\otimes} \\ &\qquad (du^{j_{t+1}} \otimes \cdots \otimes du^{j_q}), \end{aligned} \tag{1.23}$$

where we regard the summation in the bracket in the right hand side of (1.21) as follows : Any two of them contain $T(\mathfrak{X})^{\otimes(p,q+1)}$ as the common part and so we can construct naturally a sort of direct sum bundle of them by extending $T(\mathfrak{X})^{\otimes(p,q+1)}$ in the $(p+q)$ different methods. At each point $x \in \mathfrak{X}$, clearly

$$d(\partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q}),$$

$$\partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q} \otimes du^{j_{q+1}}$$

span an $n^{p+q}(1+n)$-dimensional subspace of the fibre over x of the above mentioned vector bundle over $\mathfrak{X}$ which is independent of the choice of local coordinates. We denote the vector bundle over $\mathfrak{X}$ with this subspace as its fibre over x by $T(\mathfrak{X})^{\bar{\otimes}(p,q+1)}$. Then the differential operator d of (1.21) is in fact as

$$d: \Psi(T(\mathfrak{X})^{\otimes(p,q)}) \to \Psi(T(\mathfrak{X})^{\bar{\otimes}(p,q+1)}). \tag{1.24}$$

§ 2. General connections and covariant differentiations.

We call any cross-section Γ of the vector bundle $T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})$ over $\mathfrak{X}$ *a general connection* of $\mathfrak{X}$ by definition. In a coordinate neighborhood, let Γ be written as

$$\Gamma = \partial u_i \otimes (P^i_j d^2 u^j + \Gamma^i_{jk} du^j \otimes du^k), \tag{2.1}$$

hence we have the mapping $f_U: U \to \mathfrak{L}^2_n$ by

$$a^i_j \cdot f_U = P^i_j, \quad a^i_{jk} \cdot f_U = \Gamma^i_{jk}. \tag{2.2}$$

Then, for any two coordinate neighborhood (U, u^i), (V, v^i), $U \cap V \neq \emptyset$, we have

$$(\sigma \cdot g_{VU}) f_U = f_V g_{VU}. \tag{2.3}$$

Since σ is a homomorphism of $\mathfrak{L}^2_n$ onto L^1_n, we get

$$(\sigma \cdot g_{VU})(\sigma \cdot f_U) = (\sigma \cdot f_V)(\sigma \cdot g_{VU}). \tag{2.4}$$

This shows that P^i_j are the components of a tangent tensor field of type (1, 1) of $\mathfrak{X}$ with respect to (U, u^i), which we denote by

$$\lambda(\Gamma) = \partial u_i \otimes P^i_j du^j = P. \tag{2.5}$$

Now, we define a homomorphism $\mu = \mu_\Gamma: \mathfrak{T}^2(\mathfrak{X}) \to T(\mathfrak{X})$ by the inner product

$$\mu(X) = \langle \Gamma, X \rangle, \quad X \in \mathfrak{T}^2(\mathfrak{X}),$$

that is

$$\mu(\partial u_j) = P^i_j \partial u_i, \quad \mu(\partial u_{jk}) = \Gamma^i_{jk} \partial u_i. \tag{2.6}$$

Since we have

$$T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X}) \subset \mathfrak{T}^2(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X}),$$

$$T(\mathfrak{X})^{\bar{\otimes}(1,2)} \subset \mathfrak{T}^2(X) \otimes T^*(\mathfrak{X})^{\otimes 2} + T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X}) \subset \mathfrak{T}^2(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X}),$$

we may consider as $d(\lambda(\Gamma)) \in \Psi(\mathfrak{T}^2(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X}))$. Then we can define a homomorphism $\varphi = \varphi_\Gamma: \mathfrak{D}^2(\mathfrak{X}) \to T^*(\mathfrak{X})^{\otimes 2}$ by the inner product

$$\varphi(\omega) = \langle \omega, d(\lambda(\Gamma)) - \Gamma \rangle, \quad \omega \in \mathfrak{D}^2(\mathfrak{X}),$$

that is

$$\begin{aligned} &\varphi(d^2 u^i) = -\Lambda^i_{jk} du^j \otimes du^k, \\ &\varphi(du^i \otimes du^h) = P^i_j du^j \otimes du^h, \end{aligned} \tag{2.7}$$

where we put

$$\Lambda^i_{jk} = \Gamma^i_{jk} - \frac{\partial P^i_j}{\partial u^k}.\text{[4]} \tag{2.8}$$

---

[4] See [11], § 6.

Furthermore we put generally

$$\varphi(du^i) = du^i, \tag{2.9}$$

$$\varphi(du^{i_1} \otimes \cdots \otimes du^{i_q} \otimes du^h) = P^{i_1}_{j_1} \cdots P^{i_q}_{j_q} du^{j_1} \otimes \cdots \otimes du^{j_q} \otimes du^h, \quad q > 1 \tag{2.10}$$

and

$$\varphi \mid \mathfrak{T}^2(\mathfrak{X}) = \mu, \tag{2.11}$$

then we get naturally homomorphisms of

$$T(\mathfrak{X})^{\otimes(s-1)} \otimes \mathfrak{T}^2(\mathfrak{X}) \otimes T(\mathfrak{X})^{\otimes(p-s)} \otimes T^*(\mathfrak{X})^{\otimes(q+1)},$$
$$T(\mathfrak{X})^{\otimes p} \otimes T^*(\mathfrak{X})^{\otimes(t-1)} \otimes \mathfrak{D}^2(\mathfrak{X}) \dot{\otimes} T^*(\mathfrak{X})^{\otimes(q-t)},$$
$$s = 1, 2, \cdots, p; \ t = 1, 2, \cdots, q,$$

into $T(\mathfrak{X})^{\otimes(p,q+1)}$ by the tensor products of these extended homomorphisms, which are coincide with each others on $T(\mathfrak{X})^{\otimes(p,q+1)}$. After the manner in which the tensor product symbol "$\dot{\otimes}$" is defined, we can define also a homomorphism of

$$\sum_{s=1}^{p} T(\mathfrak{X})^{\otimes(s-1)} \otimes \mathfrak{T}^2(\mathfrak{X}) \otimes T(\mathfrak{X})^{\otimes(p-s)} \otimes T^*(\mathfrak{X})^{\otimes(q+1)}$$
$$+ \sum_{t=1}^{q} T(\mathfrak{X})^{\otimes p} \otimes T^*(\mathfrak{X})^{\otimes(t-1)} \otimes \mathfrak{D}^2(\mathfrak{X}) \dot{\otimes} T^*(\mathfrak{X})^{\otimes(q-t)}$$

into $T(\mathfrak{X})^{\otimes(p,q+1)}$ by making use of the above homomorphisms on each terms. We denote this by the same symbol φ, thus we get naturally

$$\varphi = \varphi_\Gamma : \ T(\mathfrak{X})^{\bar{\otimes}(p,q+1)} \rightarrow T(\mathfrak{X})^{\otimes(p,q+1)} \quad p, q = 0, 1, 2, \cdots. \tag{2.12}$$

Now we define *the covariant differential operator* $D = D_\Gamma$ *of the general connection* Γ by

$$D = D_\Gamma = \varphi_\Gamma \cdot d \ : \ \Psi(T(\mathfrak{X})^{\otimes(p,q)}) \rightarrow \Psi(T(\mathfrak{X})^{\otimes(p,q+1)}). \tag{2.13}$$

In fact, for $V \in \Psi(T(\mathfrak{X})^{\otimes(p,q)})$,

$$V = V^{i_1 \cdots i_p}_{j_1 \cdots j_q} \partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q},$$

we get by (1. 22), (1. 23). (2, 7), (2. 9) and (2. 10)

$$DV = \partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q} \otimes DV^{i_1 \cdots i_p}_{j_1 \cdots j_q},$$
$$DV^{i_1 \cdots i_p}_{j_1 \cdots j_q} = V^{i_1 \cdots i_p}_{j_1 \cdots j_q, h} du^h, \tag{2.14}$$

$$V^{i_1 \cdots i_p}_{j_1 \cdots j_q, h} = P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} \frac{\partial V^{k_1 \cdots k_p}_{h_1 \cdots h_q}}{\partial u^h} P^{h_1}_{j_1} \cdots P^{h_q}_{j_q}$$
$$+ \sum_{s=1}^{p} P^{i_1}_{k_1} \cdots P^{i_{s-1}}_{k_{s-1}} \Gamma^{i_s}_{k_s h} P^{i_{s+1}}_{k_{s+1}} \cdots P^{i_p}_{k_p} V^{k_1 \cdots k_p}_{h_1 \cdots h_q} P^{h_1}_{j_1} \cdots P^{h_q}_{j_q}$$
$$- \sum_{t=1}^{q} P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} V^{k_1 \cdots k_p}_{h_1 \cdots h_q} P^{h_1}_{j_1} \cdots P^{h_{t-1}}_{j_{t-1}} \Lambda^{h_t}_{j_t h} P^{h_{t+1}}_{j_{t+1}} \cdots P^{h_q}_{j_q}. \tag{2.15}$$

The last formulas are identical with the ones taken away P^l_m from each term of the right hand side of [11], (7.4).

Now, we denote the dual homomorphism of $\mu_\Gamma \mid T(\mathfrak{X})$ by $\mu'_\Gamma : T^*(\mathfrak{X}) \to T^*(\mathfrak{X})$, that is

$$\mu'_\Gamma(du^i) = P^i_j\, du^j \tag{2.16}$$

and define a homomorphism $\bar\lambda = \bar\lambda_\Gamma : T(\mathfrak{X})^{\otimes(p,q)} \to T(\mathfrak{X})^{\otimes(p,q)}$ by

$$\bar\lambda_\Gamma = (\mu_\Gamma \otimes \cdots \otimes \mu_\Gamma) \otimes (\mu'_\Gamma \otimes \cdots \otimes \mu'_\Gamma), \tag{2.17}$$

for convenience putting $\bar\lambda_\Gamma \mid A(\mathfrak{X}) =$ the identity transformation.

For any two tangent tensor fields V, W of type (p, q), (a, b), with local components $V^{i_1\cdots i_p}_{j_1\cdots j_q}$, $W^{i_1\cdots i_a}_{j_1\cdots j_b}$ respectively, applying the formula (2.15) to the tensor product $V \otimes W$, we get

$$\begin{aligned}
&(V^{i_1\cdots i_p}_{j_1\cdots j_q} W^{i_{p+1}\cdots i_{p+a}}_{j_{q+1}\cdots j_{q+b}})_{,h} \\
&= V^{i_1\cdots i_p}_{j_1\cdots j_q,h} P^{i_{p+1}}_{h_{p+1}} \cdots P^{i_{p+a}}_{h_{p+a}} W^{h_{p+1}\cdots h_{p+a}}_{k_{q+1}\cdots k_{q+b}} P^{k_{q+1}}_{j_{q+1}} \cdots P^{k_{q+b}}_{j_{q+b}} \\
&+ P^{i_1}_{h_1} \cdots P^{i_p}_{h_p} V^{h_1\cdots h_p}_{k_1\cdots k_q} P^{k_1}_{j_1} \cdots P^{k_q}_{j_q} W^{i_{p+1}\cdots i_{p+a}}_{j_{q+1}\cdots j_{q+b},h},
\end{aligned}$$

that is

$$\begin{aligned}
&(V^{i_1\cdots i_p}_{j_1\cdots j_q} W^{i_{p+1}\cdots i_{p+a}}_{j_{q+1}\cdots j_{q+b}})_{,h} \\
&= V^{i_1\cdots i_p}_{j_1\cdots j_q,h} (\bar\lambda_\Gamma W)^{i_{p+1}\cdots i_{p+a}}_{j_{q+1}\cdots j_{q+b}} + (\bar\lambda_\Gamma V)^{i_1\cdots i_p}_{j_1\cdots j_q} W^{i_{p+1}\cdots i_{p+a}}_{j_{q+1}\cdots j_{q+b},h},
\end{aligned} \tag{2.18}$$

which are also written as

$$D(V \otimes W) = \varepsilon(DV \otimes \bar\lambda W) + \bar\lambda V \otimes DW, \tag{2.19}$$

where ε denotes the isomorphism of $T(\mathfrak{X})^{\otimes(p,q+1)} \otimes T(\mathfrak{X})^{\otimes(a,b)}$ onto $T(\mathfrak{X})^{\otimes(p,q)} \otimes T(\mathfrak{X})^{\otimes(a,b+1)}$ naturally defined from the above equations. This formulas shows that the covariant differential operator of general connection does not obey the classical rule if $\bar\lambda \neq$ the identity transformation.

Lastly, we consider the relation between covariant differentiations and contractions. For any tensor field V of type $(p+1,\ q+1)$, with local components $V^{i_1\cdots i_p i}_{j_1\cdots j_q j}$, by means of (2.15), we get

$$\begin{aligned}
\delta^j_i V^{i_1\cdots i_p i}_{j_1\cdots j_q j,m} &= P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} \frac{\partial V^{k_1\cdots k_p k}_{h_1\cdots h_q h}}{\partial u^m} (P^h_i P^i_k)\, P^{h_1}_{j_1} \cdots P^{h_q}_{j_q} \\
&+ \sum_{s=1}^{p} P^{i_1}_{k_1} \cdots P^{i_{s-1}}_{k_{s-1}} \Gamma^{i_s}_{k_s m} P^{i_{s+1}}_{k_{s+1}} \cdots P^{i_p}_{k_p} V^{k_1\cdots k_p k}_{h_1\cdots h_q h} (P^h_i P^i_k)\, P^{h_1}_{j_1} \cdots P^{h_q}_{j_q} \\
&+ P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} \Gamma^i_{km} V^{k_1\cdots k_p k}_{h_1\cdots h_q h} P^h_i P^{h_1}_{j_1} \cdots P^{h_q}_{j_q} \\
&- \sum_{t=1}^{q} P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} V^{k_1\cdots k_p k}_{h_1\cdots h_q h} (P^h_i P^i_k)\, P^{h_1}_{j_1} \cdots P^{h_{t-1}}_{j_{t-1}} \Lambda^{h_t}_{j_t m} P^{h_{t+1}}_{j_{t+1}} \cdots P^{h_q}_{j_q} \\
&- P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} P^i_k V^{k_1\cdots k_p k}_{h_1\cdots h_q h} \Lambda^h_{im} P^{h_1}_{j_1} \cdots P^{h_q}_{j_q}.
\end{aligned}$$

Putting

$$M^i_j = P^i_k P^k_j \tag{2.20}$$

and $M = P^2$, we have

$$\delta^j_i V^{i_1 \cdots i_p i}_{j_1 \cdots j_q j, m} = P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} \frac{\partial (V^{k_1 \cdots k_p k}_{h_1 \cdots h_q h} M^h_k)}{\partial u^m} P^{h_1}_{j_1} \cdots P^{h_q}_{j_q}$$
$$+ \sum_{s=1}^{p} P^{i_1}_{k_1} \cdots P^{i_{s-1}}_{k_{s-1}} \Gamma^{i_s}_{k_s m} P^{i_{s+1}}_{k_{s+1}} \cdots P^{i_p}_{k_p} (V^{k_1 \cdots k_p k}_{h_1 \cdots h_q h} M^h_k) P^{h_1}_{j_1} \cdots P^{h_q}_{j_q}$$
$$- \sum_{t=1}^{q} P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} (V^{k_1 \cdots k_p k}_{h_1 \cdots h_q h} M^h_k) P^{h_1}_{j_1} \cdots P^{h_{t-1}}_{j_{t-1}} \Lambda^{h_t}_{j_t m} P^{h_{t+1}}_{j_{t+1}} \cdots P^{h_q}_{j_q}$$
$$- P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} V^{k_1 \cdots k_p k}_{h_1 \cdots h_q h} \left[\frac{\partial P^h_i P^i_k}{\partial u^m} - P^h_i \Gamma^i_{km} + \Lambda^h_{im} P^i_k \right] \times$$
$$\times P^{h_1}_{j_1} \cdots P^{h_q}_{j_q}.$$

On the other hand, by (2. 15) we get

$$\delta^i_{j,h} = \Gamma^i_{kh} P^k_j - P^i_k \Lambda^k_{jh}, \tag{2. 21}$$

and by (2. 8)

$$\frac{\partial P^h_i P^i_k}{\partial u^m} - P^h_i \Gamma^i_{km} + \Lambda^h_{im} P^i_k$$
$$= P^h_i \left(- \Gamma^i_{km} + \frac{\partial P^i_k}{\partial u^m} \right) + \left(\frac{\partial P^h_i}{\partial u^m} + \Lambda^h_{im} \right) P^i_k$$
$$= \Gamma^h_{im} P^i_k - P^h_i \Lambda^i_{km} = \delta^h_{k,m}.$$

Hence we obtain the formula

$$\delta^j_i V^{i_1 \cdots i_p i}_{j_1 \cdots j_q j, m} = (V^{i_1 \cdots i_p k}_{j_1 \cdots j_q h} M^h_k)_{,m}$$
$$- P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} V^{k_1 \cdots k_p k}_{h_1 \cdots h_q h} \delta^h_{k,m} P^{h_1}_{j_1} \cdots P^{h_q}_{j_q}, \tag{2. 22}$$

where we understand that the covariant derivative of the first term of the right hand side is the one for the contracted tensor of type (p, q). It must be noted that *the covariant derivatives of the Kronecker's δ with respect to general connections do not always vanish.* We shall rewrite symbolically the formula (2. 22). For any tangent tensor tensor field N of type (1. 1) with local components N^i_j, we define the following contraction operator $N^{(s)}_{(t)}$ (s, $t=$ positive integers) : For any tensor V of type (p, q) $(s \leqslant p,\ t \leqslant q)$ with local components $V^{i_1 \cdots i_p}_{j_1 \cdots j_q}$, we denote by $N^{(s)}_{(t)} V$ the tensor field of type $(p-1,\ q-1)$ with local components

$$(N^{(s)}_{(t)} V)^{i_1 \cdots i_{p-1}}_{j_1 \cdots j_{q-1}} = V^{i_1 \cdots i_{s-1}\, k\, i_{s+1} \cdots i_{p-1}}_{j_1 \cdots j_{t-1}\, h\, j_{t+1} \cdots j_{q-1}} N^h_k \tag{2. 23}$$

and by $\Delta^{(s)}_{(t)} V$ the tensor field of type $(p-1,\ q)$ with local components

$$(\Delta^{(s)}_{(t)} V)^{i_1 \cdots i_{p-1}}_{j_1 \cdots j_q} = V^{i_1 \cdots i_{s-1}\, k\, i_{s+1} \cdots i_{p-1}}_{j_1 \cdots j_{t-1}\, h\, j_{t+1} \cdots j_{q-1}} \delta^h_{k, j_q}. \tag{2. 24}$$

Then we can write (2. 22) as

$$I^{(s)}_{(t)} DV = D M^{(s)}_{(t)} V - \varphi \Delta^{(s)}_{(t)} V$$

or

$$I^{(s)}_{(t)} \cdot D - D \cdot M^{(s)}_{(t)} + \varphi \cdot \Delta^{(s)}_{(t)} = 0, \tag{2. 25}$$

where I denotes the identity isomorphism of $T(\mathfrak{X})$ and $\varphi = \varphi_\Gamma$ is given by

(2. 12). From (2.25), we get immediately

Theorem 2. 1. *If and only if $P=\lambda(\Gamma)$ is involutive, that is $P^2=I$ and I is covariantly constant, then the covariant differentiation and the contraction by I are commutative as operators.*

Lastly, we denote explicitly (2. 3) in terms of components P^i_j, Γ^i_{jk}, for the sake of future purposes. Putting

$$a^i_j\cdot f_V=\bar{P}^i_j,\ a^i_{jk}\cdot f_V=\bar{\Gamma}^i_{jk},$$

we get immediately

$$\frac{\partial v^j}{\partial u^k}P^k_i=\bar{P}^j_k\frac{\partial v^k}{\partial u^i},$$

and

$$\frac{\partial u^j}{\partial u^k}\Gamma^k_{ih}=\bar{P}^j_k\frac{\partial^2 v^k}{\partial u^h\partial u^i}+\bar{\Gamma}^j_{kl}\frac{\partial v^k}{\partial u^i}\frac{\partial v^l}{\partial u^h},$$

that is

$$\bar{P}^j_i=\frac{\partial v^i}{\partial u^k}P^k_h\frac{\partial u^h}{\partial v^i} \tag{2. 26}$$

and

$$\begin{aligned}\bar{\Gamma}^j_{ih}&=\left(\frac{\partial v^j}{\partial u^k}\Gamma^k_{lm}-\bar{P}^j_k\frac{\partial^2 v^k}{\partial u^m\partial u^l}\right)\frac{\partial u^l}{\partial v^i}\frac{\partial u^m}{\partial v^h}\\&=\frac{\partial v^j}{\partial u^k}\left(P^k_l\frac{\partial^2 u^l}{\partial v^h\partial v^i}+\Gamma^k_{lm}\frac{\partial u^l}{\partial v^i}\frac{\partial u^m}{\partial v^h}\right).\end{aligned} \tag{2. 27}$$

From (2. 26), we get

$$\begin{aligned}\frac{\partial\bar{P}^j_i}{\partial v^h}=&\frac{\partial v^j}{\partial u^k}\frac{\partial P^k_l}{\partial u^m}\frac{\partial u^l}{\partial v^i}\frac{\partial u^m}{\partial v^h}\\&+\frac{\partial^2 v^j}{\partial u^m\partial u^k}P^k_l\frac{\partial u^l}{\partial v^i}\frac{\partial u^m}{\partial v^h}+\frac{\partial v^j}{\partial u^k}P^k_l\frac{\partial^2 u^l}{\partial v^h\partial v^i}\end{aligned}$$

hence, subtracting this from (2. 27), we get

$$\bar{\Delta}^j_{ih}=\left(-\frac{\partial^2 v^j}{\partial u^m\partial u^k}P^k_l+\frac{\partial v^j}{\partial u^k}\Delta^k_{lm}\right)\frac{\partial u^l}{\partial v^i}\frac{\partial u^m}{\partial v^h}. \tag{2. 28}$$

§ 3. Regular general connections and their contravariant and covariant parts.

A general connection Γ is said to be *regular* when $P=\lambda(\Gamma)$ is an isomorphism of $T(\mathfrak{X})$. Let Γ be a regular general connection, then, we get from (2. 3) and (1. 4)

$$\eta((\sigma\cdot g_{VU})f_U)=\eta(f_U),$$
$$\eta(f_V g_{VU})=(\sigma\cdot g_{UV})\eta(f_V)g_{VU},$$

hence

$$(\sigma\cdot g_{VU})\gamma(f_U)=\gamma(f_U)g_{VU}. \tag{3.1}$$

The equation shows that the system $\{\gamma(f_U)\}$ defines a general connection $'\Gamma$ whose components are

$$a^j_i(\gamma(f_U))=\delta^j_i,\quad a^j_{ih}(\gamma(f_U))=Q^j_k\Gamma^k_{ih}={}'\Gamma^j_{ih}, \tag{3.2}$$

where $Q=P^{-1}$. Since $\lambda('\Gamma)=I$, $'\Gamma$ is a classical affine connection, it is called *the contravariant part of* Γ. The induced connection $''\Gamma$ from $'\Gamma$ by the isomorphism Q of the tangent bundle $T(\mathfrak{X})$ is called *the covariant part of* Γ, which is also classical and whose local components $''\Gamma^j_{ih}$ are given, as is well known, by

$$''\Gamma^j_{ih}=P^j_k\Big(\frac{\partial Q^k_i}{\partial u^h}+{}'\Gamma^k_{lh}Q^l_i\Big)=\Big(\Gamma^j_{lh}-\frac{\partial P^j_l}{\partial u^h}\Big)Q^l_i,$$

that is

$$''\Gamma^j_{ih}=\Lambda^j_{kh}Q^k_i. \tag{3.3}$$

Then φ for $'\Gamma$ and $''\Gamma$ defined in § 2, writing respectively as

$$\varphi_{'\Gamma}=\varphi',\quad \varphi_{''\Gamma}=\varphi'',$$

we can write (2.6) and (2.7) as

$$\mu(\partial^2u_{jk})=P^i_h{}'\Gamma^h_{jk}\partial u_i=\varphi\varphi'(\partial u_{jk}),$$
$$\mu(d^2u^i)=-''\Gamma^i_{hk}P^h_j du^j\otimes du^k=\varphi\varphi''(\partial^2u_{jk}).$$

Since $'\Gamma$ and $''\Gamma$ are classical affine connections, φ' and φ'' are the identity transformation on each $T(\mathfrak{X})^{\otimes(p,q)}$. Hence we have

Theorem 3.1. *For any regular general connection* Γ, *the induced homomorphism* φ *of* Γ *can be written as product of the transformation* $\bar\varphi$ *which is the restriction of* φ *on tensor product bundles of order 1 and the homomorphism* $\bar\mu$ *which is the identity mapping on* $T(\mathfrak{X})$ *and* $T^*(\mathfrak{X})$, φ' *on* $T^2(\mathfrak{X})$ *and* φ'' *on* $\mathfrak{D}^2(\mathfrak{X})$, *that is*

$$\varphi=\bar\varphi\cdot\bar\mu.$$

We call $\bar\mu=\bar\mu_\Gamma$ *the basic homomorphism of the regular general connection* Γ. Putting

$$\bar D=\bar D_\Gamma=\bar\mu\cdot d, \tag{3.4}$$

we call this covariant differentiation *the basic covariant differentiation of* Γ. According to (2.13), we have

$$D=\bar\varphi\cdot\bar D. \tag{3.5}$$

In fact, for $V\in\Psi(T(\mathfrak{X})^{\otimes(p,q)})$, with local components $V^{i_1\cdots i_p}_{j_1\cdots j_q}$, analogous computations to (2.14) and (2.15) give

$$\bar D V^{i_1\cdots i_p}_{j_1\cdots j_q}=V^{i_1\cdots i_p}_{j_1\cdots j_q|h}\,du^h, \tag{3.6}$$

$$V^{i_1\cdots i_p}_{j_1\cdots j_q|h}=\frac{\partial V^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial u^h}+\sum_{s=1}^{p}{}'\Gamma^{i_s}_{kh}V^{i_1\cdots i_{s-1}ki_{s+1}\cdots i_p}_{j_1j_2\cdots\cdots\cdots\cdots j_q}$$

$$-\sum_{t=1}^{q} V^{i_1 i_2 \cdots\cdots\cdots i_p}_{j_1\cdots j_{t-1} k j_{t+1}\cdots j_q} {}''\Gamma^k_{j_t h} \tag{3.7}$$

and

$$V^{i_1\cdots i_p}_{j_1\cdots j_q, h} = P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} V^{k_1\cdots k_p}_{h_1\cdots h_q|h} P^{h_1}_{j_1} \cdots P^{h_q}_{j_q}. \tag{3.8}$$

On the basic covariant differentiation of a general regular connection Γ, we shall now investigate the ones analogous to (2. 19) and (2. 25). Firstly, for any tangent tensor fields V, W of type (p, q) and (a, b), by means of (3. 5), we get from (2. 19)

$$\begin{aligned}\bar{\varphi}\bar{D}(V \otimes W) &= \varepsilon(\bar{\varphi}\bar{D} V \otimes \bar{\lambda} W) + \bar{\lambda} V \otimes \bar{\varphi}\bar{D} W \\ &= \bar{\varphi}(\varepsilon(\bar{D} V \otimes W) + V \otimes \bar{D} W),\end{aligned}$$

since putting $\bar{\varphi}^p_q = \bar{\varphi} \mid T(\mathfrak{X})^{\otimes(p,q)}$, $\bar{\lambda}^p_q = \bar{\lambda} \mid T(\mathfrak{X})^{\otimes(p,q)}$, etc., as is easily proved,

$$\varepsilon\cdot(\bar{\varphi}^p_{q+1} \otimes \bar{\lambda}^a_b) = \bar{\varphi}^{p+a}_{q+b+1}\cdot\varepsilon, \quad \bar{\lambda}^p_q \otimes \bar{\varphi}^a_{b+1} = \bar{\varphi}^{p+a}_{q+b+1}.$$

Furthermore, $\bar{\varphi}$ is an isomorphism for any regular general connection, hence we get

$$\bar{D}(V \otimes W) = \varepsilon(\bar{D} V \otimes W) + V \otimes \bar{D} W, \tag{3.9}$$

which is identical with the well-known formula for the covariant differentiation of the tensor product of any two tensors with respect to a classical affine connection. By means of (3. 7), we have

$$\begin{aligned}\delta^j_i V^{i_1\cdots i_p i}_{j_1\cdots j_q j|h} &= \frac{\partial V^{i_1\cdots i_p i}_{j_1\cdots j_q i}}{\partial u^h} \\ &+ \sum_{s=1}^{p} {}'\Gamma^{i_s}_{kh} V^{i_1\cdots i_{t-1} k i_{t+1}\cdots i_p i}_{j_1 j_2 \cdots\cdots\cdots\cdots j_q i} - \sum_{t=1}^{q} V^{i_1 i_2 \cdots\cdots\cdots\cdots i_p i}_{j_1\cdots j_{t-1} k j_{t+1}\cdots j_q i} {}''\Gamma^k_{j_t h} \\ &+ {}'\Gamma^i_{kh} V^{i_1\cdots i_p k}_{j_1\cdots j_q i} - V^{i_1\cdots i_p i}_{j_1\cdots j_q k} {}''\Gamma^k_{ih} \\ &= (V^{i_1\cdots i_p i}_{j_1\cdots j_q j} \delta^j_i)_{|h} + V^{i_1\cdots i_p i}_{j_1\cdots j_q j}({}'\Gamma^j_{ih} - {}''\Gamma^j_{ih}).\end{aligned}$$

Since by (3. 7) we have

$$\delta^j_{i|h} = {}'\Gamma^j_{ih} - {}''\Gamma^j_{ih}, \tag{3.10}$$

hence the above equation can be written as

$$\delta^j_i V^{i_1\cdots i_p i}_{j_1\cdots j_q j|h} = (V^{i_1\cdots i_p i}_{j_1\cdots j_q j}\delta^j_i)_{|h} + V^{i_1\cdots i_p i}_{j_1\cdots j_q j}\delta^j_{i|h}. \tag{3.11}$$

Let $\bar{\Delta}^{(s)}_{(t)}$ be the operator defined by (2. 24) replaced δ^h_{k,i_q} by $\delta^h_{k|i_q}$, then we get generally the following formula

$$I^{(s)}_{(t)}\cdot\bar{D} - \bar{D}\cdot I^{(s)}_{(t)} - \bar{\Delta}^{(s)}_{(t)} = 0, \tag{3.12}$$

which shows that the basic covariant differentiation does not necessarily commute with the contraction operator by I and does so, when and only when $\delta^j_{i|h} = 0$ or ${}'\Gamma = {}''\Gamma$ by (3. 10). Now, by (3. 3) we get the equation

$${}''\Gamma^i_{kh} P^k_j = P^i_k {}'\Gamma^k_{jh} + P^i_k \frac{\partial Q^k_m}{\partial u^h} P^m_j$$

$$= P^i_k{}'\Gamma^k_{jh} - \frac{\partial P^i_j}{\partial u^h}$$

that is

$$\frac{\partial P^i_j}{\partial u^h} + {}''\Gamma^i_{kh} P^k_j - P^i_k{}'\Gamma^k_{jh} = 0. \tag{3.13}$$

Accordingly, we have

$$\begin{aligned} \delta^i_{j,h} &= P^i_k \delta^k_{l|h} P^l_j \\ &= P^i_k({}'\Gamma^k_{lh} - {}''\Gamma^k_{lh}) P^l_j \\ &= P^i_k\Big({}'\Gamma^k_{lh} P^l_j - P^k_l{}'\Gamma^l_{jh} + \frac{\partial P^k_j}{\partial u^h}\Big) \\ &= P^i_k{}'\nabla_h P^k_j \end{aligned} \tag{3.14}$$

and analogously

$$\delta^i_{j,h} = \Big(\frac{\partial P^i_l}{\partial u^h} + {}''\Gamma^i_{kh} P^k_l - P^i_k{}''\Gamma^k_{lh}\Big) P^l_j = P^l_j{}''\nabla_h P^i_l,$$

where we denote by ${}'\nabla_h$ and ${}''\nabla_h$ the covariant derivatives with respect to u^h for the classical affine connections ${}'\Gamma$ and ${}''\Gamma$ respectively. Hence we get the formulas :

$${}'\nabla_h P^i_j = \delta^i_{k|h} P^k_j \tag{3.15}$$

and

$${}''\nabla_h P^i_j = P^i_k \delta^k_{j|h}. \tag{3.16}$$

In connection with (3.13), we write explicitly $P^i_{j|h}$ by (3.7) :

$$P^i_{j|h} = \frac{\partial P^i_j}{\partial u^h} + {}'\Gamma^i_{kh} P^k_j - P^i_k{}''\Gamma^k_{jh}. \tag{3.17}$$

Thus we have proved the following

Theorem 3. 2. *In order that the basic covariant differentiation of a regular general connection Γ commutes with the contraction by I it is necessary and sufficient that one of the following conditions which are equivalent to each other is satisfied:*

i) *The Kronecker's δ = the identity isomorphism of $T(\mathfrak{X})$ is covariantly constant with respect to Γ,*

ii) *$P = (\Gamma)$ is covariantly constant with respect to the contravariant part ${}'\Gamma$ of Γ which is a classical affine connetion.*

iii) *$P = \lambda(\Gamma)$ is covariantly constant with respect to the covariant part ${}''\Gamma$ of Γ which is a classical affine connection.*

iv) *${}'\Gamma = {}''\Gamma$.*

Now, by virture of (3.8), we get from (2.22)

$$\begin{aligned} &\delta^j_i P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} P^i_k V^{k_1 \cdots k_p k}_{h_1 \cdots h_q h|m} P^{h_1}_{j_1} \cdots P^{h_q}_{j_q} P^h_j \\ &= P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} (V^{k_1 \cdots k_p k}_{h_1 \cdots h_q h} M^h_k)_{|m} P^{h_1}_{j_1} \cdots P^{h_q}_{j_q} \end{aligned}$$

$$- P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} V^{k_1\cdots k_p k}_{h_1\cdots h_q h} \delta^h_{k,m} P^{h_1}_{j_1} \cdots P^{h_q}_{j_q},$$

hence

$$V^{k_1\cdots k_p k}_{h_1\cdots h_q h|m} M^h_k = (V^{k_1\cdots k_p k}_{h_1\cdots h_q h} M^h_k)_{|m}$$
$$- V^{k_1\cdots k_p k}_{h_1\cdots h_q h} \delta^h_{k,m}. \quad (3.18)$$

Accordingly we have generally the formula

$$M^{(s)}_{(t)} \cdot \overline{D} - \overline{D} \cdot M^{(s)}_{(t)} + \Delta^{(s)}_{(t)} = 0. \quad (3.19)$$

Thus we obtain the following theorem analogous to Theorem 3. 2 :

Theorem 3. 3. *In order that the basic covariant differentiation of a regular general connection Γ commute with the contraction by $M = P^2$, it is necessary and sufficient that one of the conditions in Theorem 3. 2 is satisfied.*

Furthermore we get easily from the formulas (3. 12) and (3. 19) the following

Proposition 3. 4. *If $P^2 = cI$, where c is a constant $\neq 0$, then*

$$\delta^i_{j,h} + c\,\delta^i_{j|h} = 0. \quad (3.20)$$

Theorem 3. 5. *Let Γ be any general connection. A homomorphism N of $T(\mathfrak{X})$ such that*

$$P \cdot N = N \cdot P = cI, \quad (3.21)$$

where $P = \lambda(\Gamma)$ and c is a constant, is covariantly constant with respect to Γ.

Proof. By means of (2. 15), we have

$$N^i_{j,h} = P^i_k \frac{\partial N^k_l}{\partial u^h} P^l_j + \Gamma^i_{kh} N^k_l P^l_j - P^i_k N^k_l \Lambda^l_{jh}$$
$$= P^i_k \frac{\partial (N^k_l P^l_j)}{\partial u^h} + \Gamma^i_{kh} N^k_l P^l_j - P^i_k N^k_l \Gamma^l_{jh}.$$

Hence using (3. 21), we get easily

$$N^i_{j,h} = 0.$$

Corrollary 3. 6. *For any regular general connection Γ, the inverse P^{-1} of $P = \lambda(\Gamma)$ is covariantly constant with respect to Γ.*

§ 4. Covariant differentiation along curves and geodesics.

Let C be a curve of class C^1 with parameter t. Let be given a tangent tensor field V of type (p, q) along C with local components $V^{i_1\cdots i_p}_{j_1\cdots j_q}(t)$. Then we can define the covariant differentiation of V by

$$\frac{DV^{i_1\cdots i_p}_{j_1\cdots j_q}}{dt} = P^{i_1}_{k_1} \cdots P^{i_p}_{k_p} \frac{dV^{k_1\cdots k_p}_{h_1\cdots h_q}}{dt} P^{h_1}_{j_1} \cdots P^{h_q}_{j_q}$$

$$+\sum_{s=1}^{p} P^{i_1}_{k_1}\cdots P^{i_{s-1}}_{k_{s-1}}\Gamma^{i_s}_{k_s h}\frac{du^h}{dt}P^{i_{s+1}}_{k_{s+1}}\cdots P^{i_p}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_q}P^{h_1}_{j_1}\cdots P^{h_q}_{j_q}$$
$$-\sum_{t=1}^{q} P^{i_1}_{k_1}\cdots P^{i_p}_{k_p}V^{k_1\cdots k_p}_{h_1\cdots h_q}P^{h_1}_{j_1}\cdots P^{h_{t-1}}_{j_{t-1}}\Lambda^{h_t}_{j_t h}\frac{du^h}{dt}P^{h_{t+1}}_{j_{t+1}}\cdots P^{h_q}_{j_q}. \tag{4.1}$$

Hence, we have for the tangent vector of C the equation

$$\frac{D}{dt}\left(\frac{du^i}{dt}\right)=P^i_k\frac{d^2u^k}{dt^2}+\Gamma^i_{kh}\frac{du^k}{dt}\frac{du^h}{dt}. \tag{4.2}$$

If we take another parameter s for C, then we get

$$\frac{D}{ds}\left(\frac{du^i}{ds}\right)=P^i_k\frac{d^2u^k}{ds^2}+\Gamma^i_{kh}\frac{du^k}{ds}\frac{du^h}{ds}$$
$$=P^i_k\left(\frac{d^2u^k}{dt^2}\left(\frac{dt}{ds}\right)^2+\frac{du^k}{dt}\frac{d^2t}{ds^2}\right)+\Gamma^i_{kh}\frac{du^k}{dt}\frac{du^h}{dt}\left(\frac{dt}{ds}\right)^2,$$

that is

$$\frac{D}{ds}\left(\frac{du^i}{ds}\right)=\left(\frac{dt}{ds}\right)^2\frac{D}{dt}\left(\frac{du^i}{dt}\right)+\frac{d^2t}{ds^2}P^i_k\frac{du^k}{dt}. \tag{4.3}$$

This equation shows that the condition for a curve C with parameter t that the equations

$$\frac{D}{dt}\left(\frac{du^i}{dt}\right)=\psi P^i_j\frac{du^j}{dt} \tag{4.4}$$

are satisfied by a function ψ defined along C is intrinsic, that is a property of curves independent of the choice of its parameters. In fact, from (4.3) and (4.4), we get immediately

$$\frac{D}{ds}\left(\frac{du^i}{ds}\right)=\left\{\left(\frac{dt}{ds}\right)^2\psi+\frac{d^2t}{ds^2}\right\}\frac{ds}{dt}P^i_k\frac{du^k}{ds} \tag{4.5}$$

that is, ψ for t is to be replaced with

$$\left\{\left(\frac{dt}{ds}\right)^2\psi+\frac{d^2t}{ds^2}\right\}\frac{ds}{dt} \tag{4.6}$$

for another parameter s.

Definition 4.1. *A given curve C with parameter t that the equations (4.4) are satisfied with a function ψ defined along C is called a geodesic with respect to the general connection Γ.*

The parameter s for a geodesic such that

$$\frac{D}{ds}\left(\frac{du^i}{ds}\right)=0. \tag{4.7}$$

is called an *affine parameter* of the geodesic.

Theorem 4.2. *For a geodesic C such that at any point of C*

$$P^i_k\frac{du^k}{dt}\neq 0,$$

its affine parameters is determined uniquely affine transformations.

For a geodesic C such that at any point of C

$$P_k^i \frac{du^k}{dt} = 0, \tag{4.8}$$

any its parameter is an affine parameter.

Now, let us suppose that Γ is a regular general connection. Then we get immediately

$$\frac{D V_{j_1 \cdots j_q}^{i_1 \cdots i_p}}{dt} = P_{k_1}^{i_1} \cdots P_{k_p}^{i_p} \frac{\overline{D} V_{h_1 \cdots h_q}^{k_1 \cdots k_p}}{dt} P_{j_1}^{h_1} \cdots P_{j_q}^{h_q} \tag{4.9}$$

and

$$\frac{\overline{D} V_{j_1 \cdots j_q}^{i_1 \cdots i_p}}{dt} = \frac{d V_{j_1 \cdots j_q}^{i_1 \cdots i_p}}{dt} + \sum_{s=1}^{p} {'\Gamma}_{k h}^{i_s} \frac{du^h}{dt} V_{j_1 j_2 \cdots\cdots\cdots j_q}^{i_1 \cdots i_{s-1} k i_{s+1} \cdots i_p}$$
$$- \sum_{t=1}^{q} V_{j_1 \cdots j_{t-1} k j_{t+1} \cdots j_q}^{i_1 i_2 \cdots\cdots\cdots i_p} {''\Gamma}_{j_t h}^{k} \frac{du^h}{dt}. \tag{4.10}$$

Since P is an isomorphism, the condition (4.4) is equivalent to

$$\frac{\overline{D}}{dt}\left(\frac{du^i}{dt}\right) = \psi \frac{du^i}{dt}. \tag{4.11}$$

For contravariant tensors, $\overline{D}$ coincides with the covariant differentiation with respect to the contravariant part $'\Gamma$ of Γ, hence we have the following

Theorem 4.2. *A geodesic in $\mathfrak{X}$ with respect to a regular general connection Γ is also a geodesic with respect to the contravariant part $'\Gamma$ of Γ and the converse is also true.*

§ 5. Eigen functions and eigen curves of general connections.

Let Γ be any general connection. A function τ on $\mathfrak{X}$ is called an *eigen function* if τ is an eigen value of $P = \lambda(\Gamma)$ at each point of $\mathfrak{X}$. *A* curve C is called an *eigen curve* belonging to the eigen function τ, if its tangent vector is an eigen vector belonging to τ at each point, that is

$$P_j^i \frac{du^j}{dt} = \tau \frac{du^i}{dt}. \tag{5.1}$$

Now, let V be a contravariant tangent vector field with local components V^i such that it is an eigen vector belonging to an eigen function τ at each point of $\mathfrak{X}$, that is

$$P_j^i V^j = \tau V^i. \tag{5.2}$$

Let assume that V is also an eigen vector of another tensor field N with local components N_j^i at each point of $\mathfrak{X}$ such that

$$N_j^i V^j = \rho V^i. \tag{5.3}$$

By means of the formula (2.22), we get

$$\delta^j_k(N^i_j V^k)_{,m} = (N^i_j V^k M^j_k)_{,m} - P^i_h N^h_j V^k \delta^j_{k,m}.$$

By (2. 18), (5. 2) and (5. 3), the above equation can be written as

$$\text{the left hand side} = \delta^j_k \{ N^i_{j,m} P^k_l V^l + P^i_l N^l_h P^h_j V^k_{,m} \}$$
$$= N^i_{j,m} P^j_l V^l + P^i_l N^l_h P^h_j V^j_{,m}$$
$$= \tau N^i_{j,m} V^j + P^i_l N^l_h P^h_j V^j_{,m},$$

$$\text{the right hand side} = (\tau^2 \rho V^i)_{,m} - P^i_h N^h_j V^k \delta^j_{k,m}$$
$$= \tau^2 \rho V^i_{,m} + (\tau^2 \rho)_{,m} P^i_h V^h - P^i_h N^h_j V^k \delta^j_{k,m}$$

hence

$$\tau N^i_{j,m} V^j + P^i_l N^l_h P^h_j V^j_{,m}$$
$$= \tau^2 \rho V^i_{,m} + \tau (\tau^2 \rho)_{,m} V^i - P^i_h N^h_j V^k \delta^j_{k,m}. \tag{5. 4}$$

When N, V are defined only along a given curve C, we get analogously the equation

$$\tau \frac{DN^i_j}{dt} V^j + P^i_l N^l_h P^h_j \frac{DV^j}{dt}$$
$$= \tau^2 \rho \frac{DV^i}{dt} + \tau \frac{d(\tau^2 \rho)}{dt} V^i - P^i_h N^h_j V^k \frac{D\delta^j_k}{dt}. \tag{5. 5}$$

Here, we suppose that

$$NP = PN = cI, \tag{5. 6}$$

where c is a constant. If $V \neq 0$ at each point, it follows that

$$\rho \tau = c. \tag{5. 7}$$

By means of Theorem 3. 5, (5. 5) can be written as

$$cP^i_j \frac{DV^j}{dt} = c \left\{ \tau \left(\frac{DV^i}{dt} + \frac{d\tau}{dt} V^i \right) - \frac{D\delta^i_j}{dt} V^j \right\}.$$

If $c \neq 0$, we have

$$P^i_j \frac{DV^j}{dt} = \tau \left(\frac{DV^i}{dt} + \frac{d\tau}{dt} V^i \right) - \frac{D\delta^i_j}{dt} V^j. \tag{5. 8}$$

In fact, if $c \neq 0$, it must be that Γ is regular. But for any regular general connection Γ, we can take P^{-1} as N. Thus we have proved the following

Theorem 5. 1. *Let Γ be any regular general connection and V be a contravariant eigen vector field of $P = \lambda(\Gamma)$ belonging to an eigen function τ defined along a given cvrve C. Then, the covariant derivative of V satisfies the equation (5. 8).*

Corollary 5. 2. *Let V be a contravariant eigen vector field as in Theorem 5. 1. If I satisfies the equation*

$$\frac{D\delta^i_j}{dt} = \rho \delta^i_j \tag{5. 9}$$

or

$$\frac{D\delta^i_j}{dt} = \rho P^i_j \tag{5.10}$$

with a scalar field ρ *on C, then the covariant derivative* DV/dt *belongs to the eigen space of the eigen value* τ *at each point of C.*

Corollary 5.3. *Let V be a contravariant eigen vector field as in Theorem 5.1 and I satisfies (5.9) or (5.10) along C, then the tangent subspace spanned by V and* DV/dt *is invariant under the homomorphism P and the two-dimensional measure of the subspace is multiplied by* τ^2, *if V and* DV/dt *are linearly independent.*

Theorem 5.4. *Let* Γ *be a regular general connection and C be an eigen curve belonging to an eigen function* τ *of* $P=\lambda(\Gamma)$. *If* τ *is a simple eigen value at each point of C and I satisfies (5.9) or (5.10) along C, then C is a geodesic.*

Proof. Since we have

$$P^i_j \frac{du^j}{dt} = \tau \frac{du^i}{dt}$$

and

$$\frac{D\delta^i_j}{dt} = \rho\,\delta^i_j \text{ or } \rho P^i_j,$$

from (5.8) we get the equation

$$P^i_j \frac{D}{dt}\left(\frac{du^j}{dt}\right) = \tau\left\{\frac{D}{dt}\left(\frac{du^i}{dt}\right) + \frac{d\tau}{dt}\frac{du^i}{dt}\right\} - \left\{\rho\frac{du^i}{dt} \text{ or } \rho\tau\frac{du^i}{dt}\right\} \tag{5.11}$$

By the assumption that τ is a simple root of the characteristic equation of P, the above equation follows that

$$\frac{D}{dt}\left(\frac{du^i}{dt}\right) = \psi\frac{du^i}{dt} \tag{5.12}$$

with some function ψ defined along C. Since C is an eigen curve, the right hand side can be written as

$$\psi\frac{du^i}{dt} = \frac{\psi}{\tau} P^i_j \frac{du^j}{dt}.$$

Hence C is a geodesic. The proof is finished.

Furthermore, putting (5.12) in (5.11), we get easily the equations

$$\rho = \tau\frac{d\tau}{dt} \text{ or } \rho = \frac{d\tau}{dt}. \tag{5.13}$$

This follows immediately

Corollary 5.5. *Let* Γ *be a regular general connection and C be an eigen curve belonging to an eigen function* τ *of* $P=\lambda(\Gamma)$ *which is simple at each point of C and is not constant. Then I can not be covariantly constant along C.*

Let us suppose that an eigen function τ is simple at each point of a domain. Then at each point of the domain the eigen direction belonging to τ is uniquely determined. And so the domain can be simply covered by the eigen curves belonging to τ.

Theorem 5. 6. *If $P=\lambda(\Gamma)$ has a simple eigen function τ in a domain which is not constant everywhere to the eigen directions, then I is not covariantly constant in the domain.*

Lastly, we investigate eigen curves for a *regular general connection Γ such that I is covariantly constant with respect to Γ,* that is its contravariant part $'\Gamma$ is identical with its covariant part $''\Gamma$ by (3. 10) and (3. 14).

Let C be a given curve and V be a contravariant eigen vector field of $P=\lambda(\Gamma)$ defined along C belonging to an eigen function τ. Since I is covariantly constant, (5. 8) can be written as

$$P^i_j\frac{DV^j}{dt}=\tau\Big(\frac{DV^i}{dt}+\frac{d\tau}{dt}V^i\Big). \tag{5. 14}$$

Setting $N=P^{-1}$ and making use of (2. 22), we have

$$\delta^j_k\frac{D}{dt}\Big(N^i_j\frac{DV^k}{dt}\Big)=\frac{D}{dt}\Big(N^i_j\frac{DV^k}{dt}M^j_k\Big)-P^i_h M^h_j\frac{DV^k}{dt}\frac{D\delta^j_k}{dt}$$
$$=\frac{D}{dt}\Big(N^i_j\frac{DV^k}{dt}M^j_k\Big).$$

Making use of Corollary 3. 6 and (5. 14), this equation can be written as

$$\text{the left hand side }=\delta^j_k\Big\{\frac{DN^i_j}{dt}P^k_l\frac{DV^l}{dt}+P^i_l N^l_h P^h_j\frac{D^2V^k}{dt^2}\Big\}$$
$$=P^i_j\frac{D^2V^j}{dt^2},$$

$$\text{the right hand side }=\frac{D}{dt}\Big(P^i_j\frac{DV^j}{dt}\Big)$$
$$=\frac{D}{dt}\Big\{\tau\frac{DV^i}{dt}+\tau\frac{d\tau}{dt}V^i\Big\}$$
$$=\tau\frac{D^2V^i}{dt^2}+\frac{d\tau}{dt}P^i_j\frac{DV^j}{dt}+\tau\frac{d\tau}{dt}\frac{DV^i}{dt}+\frac{d}{dt}\Big(\tau\frac{d\tau}{dt}\Big)P^i_jV^j$$
$$=\tau\frac{D^2V^i}{dt^2}+\tau\frac{d\tau}{dt}\Big(\frac{DV^i}{dt}+\frac{d\tau}{dt}V^i\Big)+\tau\frac{d\tau}{dt}\frac{DV^i}{dt}+\tau\frac{d}{dt}\Big(\tau\frac{d\tau}{dt}\Big)V^i,$$

hence

$$P^i_j\frac{D^2V^j}{dt^2}=\tau\Big\{\frac{D^2V^i}{dt^2}+2\frac{d\tau}{dt}\frac{DV^i}{dt}+\Big(\tau\frac{d^2\tau}{dt^2}+2\frac{d\tau}{dt}\frac{d\tau}{dt}\Big)V^i\Big\}. \tag{5. 15}$$

Inductively, we suppose that

$$P^i_j\frac{D^pV^j}{dt^p}=\tau\Big\{\frac{D^pV^i}{dt^p}+\sum_{q=1}^{p}\binom{p}{q}\psi^p_q\frac{D^{p-q}V^i}{dt^{p-q}}\Big\} \tag{5. 16}$$

$$p = 1, 2, \dots, m,$$

where ψ_q^p are polynomials of τ and its derivatives with respect to t. And so, applying the formula (2. 22), we get, by caluculationś analogous to the above ones, the equation as follows :

$$\delta_k^j \frac{D}{dt}\left(N_j^i \frac{D^m V^k}{dt^m}\right) = \frac{D}{dt}\left(N_j^i \frac{D^m V^k}{dt^m} M_k^j\right).$$

We get easily

$$\text{the left hand side} = P_j^i \frac{D^{m+1} V^j}{dt^{m+1}},$$

$$\text{the right hand side} = \frac{D}{dt}\left(P_j^i \frac{D^m V^j}{dt^m}\right)$$

$$= \frac{D}{dt}\left(\tau \left\{ \frac{D^m V^i}{dt^m} + \sum_{p=1}^{m} \binom{m}{p} \psi_p^m \frac{D^{m-p} V^i}{dt^{m-p}} \right\}\right)$$

$$= \tau \left\{ \frac{D^{m+1} V^i}{dt^{m+1}} + \sum_{p=1}^{m} \binom{m}{p} \psi_p^m \frac{D^{m-p+1} V^i}{dt^{m-p+1}} \right\}$$

$$+ \frac{d\tau}{dt} P_j^i \frac{D^m V^j}{dt^m} + \sum_{p=1}^{m} \binom{m}{p} \frac{d}{dt}(\tau \psi_p^m) P_j^i \frac{D^{m-p} V^j}{dt^{m-p}}$$

$$= \tau \Bigg[\frac{D^{m+1} V^i}{dt^{m+1}} + \sum_{p=1}^{m} \binom{m}{p} \psi_p^m \frac{D^{m+1-p} V^i}{dt^{m+1-p}}$$

$$+ \frac{d\tau}{dt} \left\{ \frac{D^m V^i}{dt^m} + \sum_{p=1}^{m} \binom{m}{p} \psi_p^m \frac{D^{m-p} V^i}{dt^{m-p}} \right\}$$

$$+ \sum_{p=1}^{m} \binom{m}{p} \frac{d}{dt}(\tau \psi_p^m) \left\{ \frac{D^{m-p} V^i}{dt^{m-p}} + \sum_{q=1}^{m-p} \binom{m-p}{q} \psi_q^{m-p} \frac{D^{m-p-q} V^i}{dt^{m-p-q}} \right\} \Bigg],$$

hence

$$P_j^i \frac{D^{m+1} V^j}{dt^{m+1}} = \tau \left\{ \frac{D^{m+1} V^i}{dt^{m+1}} + \sum_{p=1}^{m+1} \binom{m+1}{p} \psi_p^{m+1} \frac{D^{m+1-p} V^i}{dt^{m+1-p}} \right\}$$

and

$$(m+1)\psi_1^{m+1} = m\psi_1^m + \frac{d\tau}{dt}, \tag{5. 17}$$

$$\binom{m+1}{r} \psi_r^{m+1} = \binom{m}{r} \psi_r^m + \binom{m}{r-1} \frac{d\tau}{dt} \psi_{r-1}^m$$
$$+ \sum_{p=1}^{r-1} \binom{m}{p} \binom{m-p}{r-1-p} \frac{d}{dt}\left(\tau \psi_p^m \right) \psi_{r-1-p}^{m-p}, \tag{5. 18}$$

$$r = 2, 3, \dots, m,$$

$$\psi_{m+1}^{m+1} = \frac{d\tau}{dt} \psi_m^m + \sum_{p=1}^{m} \binom{m}{p} \frac{d}{dt}(\tau \psi_p^m) \psi_{m-p}^{m-p}, \tag{5. 19}$$

where we put

$$\psi_0^p = 1, \quad p = 1. 2, \dots, m.$$

(5. 14) and (5. 15) show that

$$\psi_1^1 = \psi_1^2 = \frac{d\tau}{dt}, \quad \psi_2^2 = \tau\frac{d^2\tau}{dt^2} + 2\frac{d\tau}{dt}\frac{d\tau}{dt}.$$

Accordingly, we get from (5. 17)

$$\varphi_1 = \psi_1^m = \frac{d\tau}{dt}, \quad m=1, 2, 3, \ldots \tag{5. 20}$$

Analogously, we have

$$\binom{m+1}{2}\psi_2^{m+1} = \binom{m}{2}\psi_2^m + \binom{m}{1}\frac{d\tau}{dt}\frac{d\tau}{dt} + \binom{m}{1}\frac{d}{dt}\left(\tau\frac{d\tau}{dt}\right)$$
$$= \binom{m}{2}\psi_2^m + \binom{m}{1}\left(\tau\frac{d^2\tau}{dt^2} + 2\frac{d\tau}{dt}\frac{d\tau}{dt}\right)$$

hence

$$\varphi_2 = \psi_2^m = \tau\frac{d^2\tau}{dt^2} + 2\frac{d\tau}{dt}\frac{d\tau}{dt}, \quad m = 2, 3, \ldots. \tag{5. 21}$$

Furthermore we can prove that

$$\varphi_p = \psi_p^m, \quad m=p,\ p+1, \ldots, \tag{5. 22}$$

and ϕ_3, ϕ_4, ..., are determined inductively by

$$\phi_{m+1} = \frac{d\tau}{dt}\phi_m + \sum_{p=1}^{m}\binom{m}{p}\frac{d}{dt}(\tau\phi_p)\phi_{m-p}. \tag{5. 23}$$

In fact, putting $\phi_0 = 1$, ψ_r^{m+1}, $r = 1, 2, \ldots, m+1$, are determined uniquely by (5. 17), (5. 18) and (5. 19). If we put

$$\psi_r^{m+1} = \phi_r, \quad \psi_r^{m+1} = \phi_r, \quad \psi_{r-1}^m = \phi_{r-1},$$
$$\psi_{r-1-p}^{m-p} = \phi_{r-1-p}$$

in (5. 18), we obtain the equation

$$\binom{m+1}{r}\phi_r = \binom{m}{r}\phi_r + \binom{m}{r-1}\frac{d\tau}{dt}\phi_{r-1}$$
$$+ \sum_{p=1}^{r-1}\binom{m}{p}\binom{m-p}{r-1-p}\frac{d}{dt}(\tau\phi_p)\phi_{r-1-p}.$$

which is written as

$$\binom{m}{r-1}\phi_r = \binom{m}{r-1}\frac{d\tau}{dt}\phi_{r-1} + \sum_{p=1}^{r-1}\binom{m}{r-1}\binom{r-1}{p}\frac{d}{dt}(\tau\phi_p)\phi_{r-1-p},$$

that is

$$\phi_r = \frac{d\tau}{dt}\phi_{r-1} + \sum_{p=1}^{r-1}\binom{r-1}{p}\frac{d}{dt}(\tau\phi_p)\phi_{r-1-p}. \tag{5. 24}$$

On the other hand, if we put

$$\psi_{m+1}^{m+1} = \phi_{m+1}, \quad \psi_m^m = \phi_m, \quad \psi_p^m = \phi_p, \quad \psi_{m-p}^{m-p} = \phi_{m-p}$$

in (5. 19), we get the equation

$$\phi_{m+1} = \frac{d\tau}{dt}\phi_m + \sum_{p=1}^{m}\binom{m}{p}\frac{d}{dt}(\tau\phi_p)\phi_{m-p},$$

which is the one replaced r with $m+1$ in the above equation (5. 24). Therefore, the system of ψr_p^m defined by (5. 22) and (5. 23) are the solution of the system of equations (5. 17), (5. 18) and (5. 19). Thus we have obtained the following

Theorem 5. 7. *Let Γ be any regular general connection such that I is covariantly constant with respect to Γ. Let V be a contravariant eigen vector field of $P=(\Gamma)$ belonging to an eigen function τ defined along a given curve C. Then we have*

$$P_j^i V^j = \tau V^i,$$
$$P_j^i \frac{D^m V^j}{dt^m} = \tau \left\{ \frac{D^m V^i}{dt^m} + \sum_{p=1}^{m} \binom{m}{p} \phi_p \frac{D^{m-p} V^i}{dt^{m-p}} \right\}, \tag{5. 25}$$
$$m=1, 2, 3, \ldots,$$

where $\phi_0=1$ and ϕ_m, $m=1, 2, 3, \ldots$, are defined by (5. 20), (5. 21), (5. 23) and polynomials of

$$\tau, \frac{d\tau}{dt}, \ldots, \frac{d^m\tau}{dt^m}$$

of order m. At each point of C, the vectors

$$V^i, \frac{DV^i}{dt}, \frac{D^2V^i}{dt^2}, \ldots$$

belong to the eigen space of τ.

Corollary 5. 8. *Let Γ be a regular general connection as in Theorem 5. 7. Let C be an eigen curve belonging to an eigen function τ. Then the vectors*

$$\frac{du^i}{dt}, \frac{Ddu^i}{dt^2}, \frac{D^2du^i}{dt^3}, \ldots$$

belong to the eigen space of τ at each point of C.

§ 6. Torsion and curvature forms of general connections.

In this section, we shall introduce the torsion forms and the curvature forms of general connections so that they are natural generalizations of the classical ones.

Let be given a general connection Γ of $\mathfrak{X}$. In order to find the differential forms on $\mathfrak{B}(\mathfrak{X})$ for Γ, as in the theory of classical affine connec tions [2], we regard a_λ^i as functions defined on a coordinate neighborhood (U, u^i) and take b_i^λ such that

$$b_j^\lambda a_\mu^j = \delta_\mu^\lambda$$

for a moment. Then, for the contravariant tangent vector field

$$e_\lambda = a_\lambda^i \partial u_i,$$

we have by virtue of (2. 14) and (2. 15) the equation

$$De_\lambda = \partial u_i \otimes Da^i_\lambda = \partial u_i \otimes (P^i_j da^j_\lambda + \Gamma^i_{jh} a^j_\lambda du^h)$$
$$= e_\mu \otimes b^\mu_i (P^i_j da^j_\lambda + \Gamma^i_{jh} a^j_\lambda du^h).$$

Now, in the principal bundle $\{\mathfrak{B}(\mathfrak{X}), \mathfrak{X}, \pi\}$ of the first order, making use of local coordinates u^i, β, $\beta \in L^1_n$, of $\pi^{-1}(U)$, we define n^2 differential forms by

$$\theta^\mu_\lambda = b^\mu_i(\beta)\,\{P^i_j(u)\,da^j_\lambda(\beta) + \Gamma^i_{jh}(u)\,a^j_\lambda(\beta)\,du^h\}, \tag{6.1}$$

where b^i_j are functions on $GL(n, R) = L^1_n$ such that

$$b^i_j(\alpha) = a^i_j(\alpha^{-1}). \tag{6.2}$$

Lemma 6.1. *θ^μ_λ are defined on the whole space $\mathfrak{B}(\mathfrak{X})$ and are independent of the choice of local coordinates of $\mathfrak{X}$.*

Proof. For any point $b \in \pi^{-1}(U) \subset \mathfrak{B}(\mathfrak{X})$, putting

$$e_\lambda(b) = a^i_\lambda(\beta)\,\partial u_i,$$

we have

$$\beta = h_U:\ U \to L^1_n. \tag{6.3}$$

The system $\{h_U\}$ has the property as follows: For any two coordinate neighborhoods (U, u^i), (V, v^i), $U \cap V \neq \emptyset$,

$$(\sigma \cdot g_{VU})\,h_U = h_V \tag{6.4}$$

where g_{VU} is the mapping defined by (1.9). By (2.2), θ^μ_λ can be written as

$$\theta^\mu_\lambda = (b^\mu_i \cdot h_U)\,\{(a^i_j \cdot f_U) d(a^j_\lambda \cdot h_U) + (a^i_{jh} \cdot f_U)\,(a^j_\lambda \cdot h_U)\,du^h\} \tag{6.5}$$

or

$$\theta^\mu_\lambda = (a^\mu_i \cdot h_U^{-1} f_U)\,d\,(a^i_\lambda \cdot h_U) + (a^\mu_{ih} \cdot h_V^{-1} f_V)\,(a^i_\lambda \cdot h_U)\,du^h. \tag{6.6}$$

From (2.3) and (6.4), we have immediately

$$h_U^{-1} f_U = h_V^{-1} f_V g_{VU} \tag{6.7}$$

and putting this into (6.6) we get

$$\begin{aligned}
\theta^\mu_\lambda &= (a^\mu_i \cdot h_V^{-1} f_V g_{VU})\,d\,(a^i_\lambda \cdot (g_{UV} h_V)) \\
&\quad + (a^\mu_{ih} \cdot h_V^{-1} f_V g_{VU})\,(a^i_\lambda \cdot (g_{UV} \cdot h_V))\,du^h \\
&= (a^\mu_i \cdot h_V^{-1} f_V)\,d\,(a^i_\lambda \cdot h_V) \\
&\quad + (a^\mu_i \cdot h_V^{-1} f_V g_{VU})\,(a^i_{jk} \cdot g_{UV})\,(a^j_\lambda \cdot h_V)\,dv^k \\
&\quad + (a^\mu_{ih} \cdot h_V^{-1} f_V g_{VU})\,(a^i_j \cdot g_{UV})\,(a^j_\lambda \cdot h_V)\,(a^h_k \cdot g_{UV})\,dv^k \\
&= (a^\mu_i \cdot h_V^{-1} f_V)\,d\,(a^i_\lambda \cdot h_V) \\
&\quad + \{(a^\mu_i \cdot h_V^{-1} f_V g_{VU})\,(a^i_{jk} \cdot g_{UV}) + (a^\mu_{ih} \cdot h_V^{-1} f_V g_{VU})\,(a^i_j \cdot g_{VU})\,(a^h_k \cdot g_{UV})\} \\
&\quad \times (a^j_\lambda \cdot h_V)\,dv^k \\
&= (a^\mu_i \cdot h_V^{-1} f_V)\,d\,(a^i_\lambda \cdot h_V) + (a^\mu_{jk} \cdot h_V^{-1} f_V)\,(a^j_\lambda \cdot h_V)\,dv^k,
\end{aligned}$$

by means of (1.1) and (1.2). The last equations show that θ^μ_λ do not depend on the choice of local coordinates of $\mathfrak{X}$.

We call these differential form θ^μ_λ on $\mathfrak{B}(\mathfrak{X})$ *the connection forms on* $\mathfrak{B}(\mathfrak{X})$ *of* Γ. On $\mathfrak{B}(\mathfrak{X})$, as is well known, there are n differential forms:

$$\theta^\mu = b^\mu_i(\beta)\, du^i = (a^\mu_i \cdot h_U^{-1})\, du^i, \tag{6.8}$$

which do an important role for the developement of curves in the classical affine connections.

Lemma 6.2. *For a right translation* $r(\alpha)$ *of* $\mathfrak{B}(\mathfrak{X})$, $\alpha \in L^1_n$, *we have*

$$r(\alpha)^* \theta^\mu = a^\mu_\lambda(\alpha^{-1})\, \theta^\lambda, \tag{6.9}$$

$$r(\alpha)^* \theta^\mu_\lambda = a^\mu_\nu(\alpha^{-1})\, \theta^\nu_\sigma\, a^\sigma_\lambda(\alpha). \tag{6.10}$$

Proof. (6.9) is evident. For any $\alpha \in L^1_n$, we have

$$e_\lambda(r(\alpha)(b)) = e_\lambda(b\alpha) = a^i_\lambda(\beta\alpha)\, \partial u_i,$$

hence h_U: $U \to L^1_n$ is replaced by $h_U\alpha$. Therefore, by means of (6.6), we get

$$\begin{aligned} r(\alpha)^* \theta^\mu_\lambda &= (a^\mu_i \cdot (h_U\alpha)^{-1} f_U)\, d\,(a^i_\lambda \cdot h_U\alpha) \\ &\quad + (a^\mu_{ih} \cdot (h_U\alpha)^{-1} f_U)\,(a^i_\lambda \cdot h_U\alpha)\, du^h \\ &= a^\mu_\nu(\alpha^{-1})\,(a^\nu_i \cdot h_U^{-1} f_U)\, d\,(a^i_\sigma \cdot h_U)\, a^\sigma_\lambda(\alpha) \\ &\quad + a^\mu_\nu(\alpha^{-1})\,(a^\nu_{ih} \cdot h_U^{-1} f_U)\,(a^i_\sigma \cdot h_U)\, a^\sigma_\lambda(\alpha)\, du^h \\ &= a^\mu_\nu(\alpha^{-1})\, \{(a^\nu_i \cdot h_U^{-1} f_U)\, d\,(a^i_\sigma \cdot h_U) \\ &\quad + (a^\nu_{ih} \cdot h_U^{-1} f_U)\,(a^i_\sigma \cdot h_U)\, du^h\}\, a^\sigma_\lambda(\alpha) \\ &= a^\mu_\nu(\alpha^{-1})\, \theta^\nu_\lambda a^\sigma_\lambda(\alpha). \end{aligned}$$

Now, in the following, we shall denote the components of the tensor field $\pi\ominus P$[5], $P = \lambda(\Gamma)$, of the vector bundle $\pi^\#(T(\mathfrak{X}))$ over $\mathfrak{B}(\mathfrak{X})$, with respect to its n canonical cross-sections, by P^λ_μ with Greek indices, that is

$$P^\mu_\lambda = b^\mu_j(\beta)\, P^j_i(u)\, a^i_\lambda(\beta), \tag{6.11}$$

which can be written as

$$P^\mu_\lambda = a^\mu_\lambda \cdot (h_U^{-1} f_U\, h_U). \tag{6.12}$$

Then, we calculate the following differential forms:

$$\begin{aligned} & P^\mu_\lambda d\theta^\lambda + \theta^\mu_\lambda \wedge \theta^\lambda \\ &= (a^\mu_\lambda \cdot h_U^{-1} f_U h_U)\, d\,(b^\lambda_i \cdot h_U) \wedge du^i \\ &+ \{(a^\mu_i \cdot h_U^{-1} f_U)\, d\,(a^i_\lambda \cdot h_U) + (a^\mu_{ih} \cdot h_U^{-1} f_U)\,(a^i_\lambda \cdot h_U)\, du^h\} \wedge (b^\lambda_j \cdot h_U)\, du^j \\ &= (a^\mu_{ih} \cdot h_U^{-1} f_U)\, \delta^i_j du^h \wedge du^j \\ &= -(a^\mu_{ih} \cdot h_U^{-1} f_U)\, du^i \wedge du^h = -(b^\mu_j \cdot h_U)\,(a^j_{ih} \cdot f_U)\, du^i \wedge du^h. \end{aligned}$$

Therfore, putting

$$\Omega^j \equiv -(a^j_{ih} \cdot f_U)\, du^i \wedge du^h = -\Gamma^j_{ih} du^i \wedge du^h, \tag{6.13}$$

we have the equations

5) See [8], § 1 or [11], § 8.

$$\Theta^{\mu}\equiv P^{\mu}_{\lambda}d\theta^{\lambda}+\theta^{\mu}_{\lambda}\wedge\theta^{\lambda}=b^{\mu}_{j}(\beta)\Omega^{j}, \tag{6.14}$$

which shows that the vectorial differential forms Θ^{μ} of the second order defined on the whole space $\mathfrak{B}(\mathfrak{X})$ are induced ones by π from vectorial differential forms of the same kind on $\mathfrak{X}$, whose components with respect to u^i are Ω^j, according to the following lemma.

Lemma 6. 3. *Ω^j are vectorial differential forms of $\mathfrak{X}$.*

Proof. By means of (2. 3) and (6. 3), we have

$$\begin{aligned}
&(a^j_{ih}\cdot f_V)\,dv^i\wedge dv^h\\
&=(a^j_{ih}\cdot f_V)\,(a^i_l\cdot g_{VU})\,(a^h_m\cdot g_{VU})\,du^l\wedge du^m\\
&=\{(a^j_{lm}\cdot f_V g_{VU})-(a^j_i\cdot f_V)\,(a^i_{lm}\cdot g_{VU})\}\,du^l\wedge du^m\\
&=(a^j_{ih}\cdot f_V g_{VU})\,du^i\wedge du^h\\
&=(a^j_{ih}\cdot(\sigma g_{VU})f_U)\,du^i\wedge du^h=(a^j_k\cdot g_{VU})\,(a^k_{ih}\cdot f_U)\,du^i\wedge du^h,
\end{aligned}$$

that is

$$(a^j_{ih}\cdot f_V)\,dv^i\wedge dv^h=\frac{\partial v^i}{\partial u^k}(a^k_{ih}\cdot f_U)\,du^i\wedge du^h. \tag{6.15}$$

We call Ω^j *the torsion forms of the general connection Γ* and the corresponding tangent tensor of type (1, 2), with local components

$$T^j_{ih}=\Gamma^j_{ih}-\Gamma^j_{hi}, \tag{6.16}$$

the torsion tensor of Γ. we heve easily

$$\Omega^j=-\tfrac{1}{2}\;T^j_{ih}\,du^i\wedge du^h. \tag{6.17}$$

Nextly, we shall look for the curvature forms of Γ which are identical with the classical ones when Γ becomes a classical affine connection. At first, we have

$$\begin{aligned}
P^{\mu}_{\nu}d\theta^{\nu}_{\sigma}&=P^{\mu}_{\nu}d\{b^{\nu}_{j}(P^j_i da^i_{\sigma}+\Gamma^j_{ih}\,a^i_{\sigma}du^h)\}\\
&=P^{\mu}_{\nu}\{db^{\nu}_{j}\wedge P^j_i da^i_{\sigma}+db^{\nu}_{j}\wedge\Gamma^j_{ih}a^i_{\sigma}du^h\}\\
&\quad+P^{\mu}_{\nu}b^{\nu}_{j}\{dP^j_i\wedge da^i_{\sigma}+d\Gamma^j_{ih}\wedge a^i_{\sigma}du^h+\Gamma^j_{ih}da^i_{\sigma}\wedge du^h\},\\
\theta^{\mu}_{\nu}\wedge\theta^{\nu}_{\sigma}&=b^{\mu}_{j}\,(P^j_i da^i_{\nu}+\Gamma^j_{im}a^i_{\nu}\;du^m)\wedge b^{\nu}_{k}\,(P^k_h da^h_{\sigma}+\Gamma^k_{hl}\,a^h_{\sigma}du^l)\\
&=-\,P^{\mu}_{\nu}db^{\nu}_{k}\wedge P^k_h da^h_{\sigma}-P^{\mu}_{\nu}db^{\nu}_{k}\wedge\Gamma^k_{hl}a^h_{\sigma}du^l\\
&\quad+b^{\mu}_{j}\Gamma^j_{im}du^m\wedge P^i_h da^h_{\sigma}+b^{\mu}_{j}\Gamma^j_{im}du^m\wedge\Gamma^i_{hl}a^h_{\sigma}du^l,
\end{aligned}$$

and hence

$$\begin{aligned}
P^{\mu}_{\nu}d\theta^{\nu}_{\sigma}+\theta^{\mu}_{\nu}\wedge\theta^{\nu}_{\sigma}&=b^{\mu}_{j}\;\{P^j_l d\Gamma^l_{ik}\wedge du^k+\Gamma^j_{lh}du^h\wedge\Gamma^l_{ik}du^k\}\;a^i_{\sigma}\\
&\quad+b^{\mu}_{j}\{P^j_l dP^l_i\wedge da^i_{\sigma}-P^j_l\Gamma^l_{ih}du^h\wedge da^i_{\sigma}+\Gamma^j_{lh}P^l_i du^h\wedge da^i_{\sigma}\},
\end{aligned}$$

that is

$$\begin{aligned}
P^{\mu}_{\nu}d\theta^{\nu}_{\sigma}+\theta^{\mu}_{\nu}\wedge\theta^{\nu}_{\sigma}&=b^{\mu}_{j}\;\{P^j_l d\Gamma^l_{ik}\wedge du^k+\Gamma^j_{lh}du^h\wedge\Gamma^l_{ik}du^k\}\;a^i_{\sigma}\\
&\quad+b^{\mu}_{j}\,(\Gamma^j_{lh}du^h\,P^l_i-P^j_l\,A^l_{ih}du^h)\wedge da^i_{\sigma}
\end{aligned} \tag{6.18}$$

making use of (2. 8). Now, we put

$$\pi^{\mu}_{\lambda} \equiv \theta^{\mu}_{\lambda} - d P^{\mu}_{\lambda}, \tag{6.19}$$

which can be written as

$$\begin{aligned}\pi^{\mu}_{\lambda} &= b^{\mu}_{j} (P^{j}_{i} da^{i}_{\lambda} + \Gamma^{j}_{ih} a^{i}_{\lambda} du^{h}) - d (b^{\mu}_{j} P^{j}_{i} a^{i}_{\lambda}) \\ &= -db^{\mu}_{j} P^{j}_{i} a^{i}_{\lambda} + b^{\mu}_{j} (\Gamma^{j}_{ih} du^{h} - dP^{j}_{i}) a^{i}_{\lambda} \\ &= -db^{\mu}_{j} P^{j}_{i} a^{i}_{\lambda} + b^{\mu}_{j} \Lambda^{j}_{ih} du^{h} a^{i}_{\lambda},\end{aligned}$$

hence

$$\pi^{\mu}_{\lambda} = (-db^{\mu}_{j} P^{j}_{i} + b^{\mu}_{j} \Lambda^{j}_{ih} du^{h}) a^{i}_{\lambda} \tag{6.20}$$

or

$$\pi^{\mu}_{\lambda} = b^{\mu}_{j} (da^{j}_{\nu} P^{\nu}_{\lambda} + \Lambda^{j}_{ih} a^{i}_{\lambda} du^{h}). \tag{6.21}$$

If we regard b^{μ}_{j} as components of a covariant tangent tensor, then we have

$$Db^{\mu}_{i} = db^{\mu}_{j} P^{j}_{i} - b^{\mu}_{j} \Lambda^{j}_{ih} du^{h}.$$

Therefore $-\pi^{\mu}_{\lambda}$ are the differential forms on $\mathfrak{B}(\mathfrak{X})$ derived from the above forms on $\mathfrak{X}$. Now, we have

$$\begin{aligned}\theta^{\mu}_{\nu} P^{\nu}_{\sigma} - P^{\mu}_{\nu} \pi^{\nu}_{\sigma} &= b^{\mu}_{j} (P^{j}_{i} da^{i}_{\nu} + \Gamma^{j}_{ih} a^{i}_{\nu} du^{h}) P^{\nu}_{\sigma} \\ &\quad - P^{\mu}_{\nu} b^{\nu}_{j} (da^{j}_{\rho} P^{\rho}_{\sigma} + \Lambda^{j}_{ih} a^{i}_{\sigma} du^{h}) \\ &= b^{\mu}_{j} (\Gamma^{j}_{lh} du^{h} P^{l}_{i} - P^{j}_{l} \Lambda^{l}_{ih} du^{h}) a^{i}_{\sigma},\end{aligned}$$

and substituting these into (6.18) we get

$$\begin{aligned}P^{\mu}_{\nu} d\theta^{\nu}_{\sigma} + \theta^{\mu}_{\nu} \wedge \theta^{\nu}_{\sigma} &= b^{\mu}_{j} \{ P^{j}_{l} d\Gamma^{l}_{ik} \wedge du^{k} + \Gamma^{j}_{lh} du^{h} \wedge \Gamma^{l}_{ik} du^{k} \} a^{i}_{\sigma} \\ &\quad + (\theta^{\mu}_{\nu} P^{\nu}_{\rho} - P^{\mu}_{\nu} \pi^{\nu}_{\rho}) \wedge b^{\rho}_{i} da^{i}_{\sigma}.\end{aligned}$$

Comparing the second term of the right hand side of the above equation with (6.21), we have

$$\begin{aligned}&(P^{\mu}_{\nu} d\theta^{\nu}_{\sigma} + \theta^{\mu}_{\nu} \wedge \theta^{\nu}_{\sigma}) P^{\sigma}_{\lambda} \\ &= b^{\mu}_{j} \{ P^{j}_{l} d\Gamma^{l}_{mk} \wedge du^{k} + \Gamma^{j}_{lh} du^{h} \wedge \Gamma^{l}_{mk} du^{k} \} P^{m}_{i} a^{i}_{\lambda} \\ &+ (\theta^{\mu}_{\nu} P^{\nu}_{\rho} - P^{\mu}_{\nu} \pi^{\nu}_{\rho}) \wedge (\pi^{\rho}_{\lambda} - b^{\rho}_{j} \Lambda^{j}_{ih} du^{h} a^{i}_{\lambda}).\end{aligned}$$

Therefore, we define differential forms of the second order on $\mathfrak{B}(\mathfrak{X})$ and in $\mathfrak{X}$ respectively by

$$\Theta^{\mu}_{\lambda} \equiv (P^{\mu}_{\nu} d\theta^{\nu}_{\sigma} + \theta^{\mu}_{\nu} \wedge \theta^{\nu}_{\sigma}) P^{\sigma}_{\lambda} - (\theta^{\mu}_{\nu} P^{\nu}_{\rho} - P^{\mu}_{\nu} \pi^{\nu}_{\rho}) \wedge \pi^{\rho}_{\lambda} \tag{6.22}$$

and

$$\begin{aligned}\Omega^{j}_{i} &\equiv (P^{j}_{l} d\Gamma^{l}_{mk} \wedge du^{k} + \Gamma^{j}_{lh} du^{h} \wedge \Gamma^{l}_{mk} du^{k}) P^{m}_{i} \\ &\quad - (\Gamma^{j}_{lh} du^{h} P^{l}_{m} - P^{j}_{l} \Lambda^{l}_{mh} du^{h}) \wedge \Lambda^{m}_{ik} du^{k},\end{aligned} \tag{6.23}$$

which, using (2.21), can be written as

$$\begin{aligned}\Omega^{j}_{i} &= (P^{j}_{l} d\Gamma^{l}_{mk} \wedge du^{k} + \Gamma^{j}_{lh} du^{h} \wedge \Gamma^{l}_{mk} du^{k}) P^{m}_{i} \\ &\quad - D\delta^{j}_{m} \wedge \Lambda^{m}_{ik} du^{k}.\end{aligned} \tag{6.24}$$

Then, the above equations can be written as

$$\Theta^{\mu}_{\lambda} = b^{\mu}_{j} \Omega^{j}_{i} a^{i}_{\lambda}, \tag{6.25}$$

which shows that Ω_i^j are tensorial differential forms of the second order and of type (1, 1) in $\mathfrak{X}$. If $\lambda(\Gamma) = P = I$, Ω_i^j are clearly the curvature forms in the classical sense for the classical affine connection Γ. And so, we call Ω_i^j *the curvature forms of* Γ and the corresponding tensor of type (1, 3) with local components R_{ihk}^j by

$$\Omega_i^j = \tfrac{1}{2} R_i{}^j{}_{hk} du^h \wedge du^k, \tag{6.26}$$

where

$$R_i{}^j{}_{hk} = - R_i{}^j{}_{kh}, \tag{6.27}$$

the curvature tensor of Γ. In fact, $R_i{}^j{}_{hk}$ are given by

$$\begin{aligned} R_i{}^j{}_{hk} = {} & P_l^j \left(\frac{\partial \Gamma_{mk}^l}{\partial u^h} - \frac{\partial \Gamma_{mh}^l}{\partial u^k} + \Gamma_{lh}^j \Gamma_{mk}^l - \Gamma_{lk}^j \Gamma_{mh}^l \right) P_i^m \\ & - \delta_{m,h}^j A_{ik}^m + \delta_{m,k}^j A_{ih}^m, \end{aligned} \tag{6.28}$$

where

$$\delta_{i,h}^j = \Gamma_{lh}^j P_i^l - P_l^j A_{ih}^l.$$

§ 7. Curvature forms of regular general connections.

In this section, *let assume that* Γ *is a regular general connection.* Let $'\Gamma$ and $''\Gamma$ be its contravariant part and covariant part respectively.

Substituting (3. 2) and (3. 14)

$$\Gamma_{ih}^j = P_l^j \, '\Gamma_{ih}^l,$$
$$\delta_{i,h}^j = P_l^j \, '\nabla_h P_i^l$$

into (6. 24), we have

$$\begin{aligned} \Omega_i^j = {} & \{P_l^j d(P_t^l \, '\Gamma_{mk}^t) \wedge du^k + P_l^j \, '\Gamma_{th}^l du^h \wedge P_s^t \, '\Gamma_{mk}^s du^k\} P_i^m \\ & - P_l^j \, 'DP_m^t \wedge (P_s^m \, '\Gamma_{ik}^s du^k - dP_i^m) \\ = {} & \{P_l^j dP_t^l \wedge '\Gamma_{mk}^t du^k + M_l^j d\,'\Gamma_{mk}^l \wedge du^k + P_l^j \, '\Gamma_{th}^l du^h \wedge P_s^t \, '\Gamma_{mk}^s du^k\} P_i^m \\ & + P_l^j \, 'DP_m^t \wedge ('DP_i^m - '\Gamma_{ik}^m du^k P_i^l) \\ = {} & \{M_l^j d\,'\Gamma_{mk}^l \wedge du^k + P_l^j dP_t^l \wedge '\Gamma_{mk}^t du^k + P_l^j \, '\Gamma_{mh}^l du^h P_t^m \wedge '\Gamma_{mk}^t du^k \\ & - P_l^j \, 'DP_t^l \wedge '\Gamma_{mk}^t du^k\} P_i^m + P_l^j \, 'DP_m^t \wedge 'DP_i^m \\ = {} & \{M_l^j d\,'\Gamma_{mk}^l \wedge du^k + P_l^j P_s^l \, '\Gamma_{th}^s du^h \wedge '\Gamma_{mk}^t du^k\} P_i^m \\ & + P_l^j \, 'DP_m^t \wedge 'DP_i^m, \end{aligned}$$

hence

$$\begin{aligned} \Omega_i^j = {} & M_l^j \{d\,'\Gamma_{mk}^l \wedge du^k + '\Gamma_{th}^l du^h \wedge '\Gamma_{mk}^t du^k\} P_i^m \\ & + P_l^j \, 'DP_m^l \wedge 'DP_i^m. \end{aligned}$$

Denoting the curvature forms of the contravariant part $'\Gamma$ by

$$'\Omega_i^j = d\,'\omega_i^j + '\omega_l^j \wedge '\omega_i^l, \quad '\omega_i^j = '\Gamma_{ih}^j du^h, \tag{7.1}$$

the above equations can be written as

$$\Omega_i^j = M_l^j {}'\Omega_m^l P_i^m + P_l^j {}'DP_m^l \wedge {}'DP_i^m. \tag{7.2}$$

With regard to the covariant part $''\Gamma$, we have from (3.3)

$$''\omega_i^j = ''\Gamma_{ih}^j du^h = P_k^j (dQ_i^k + {}'\omega_l^k Q_i^l). \tag{7.3}$$

Denoting the curvature forms of the covariant part $''\Gamma$ by

$$''\Omega_i^j = d\,''\omega_i^j + ''\omega_m^j \wedge ''\omega_i^m, \tag{7.4}$$

we get

$$\begin{aligned} ''\Omega_i^j &= dP_k^j \wedge dQ_i^k + dP_k^j \wedge {}'\omega_l^k Q_i^l + P_k^j d\,{}'\omega_l^k Q_i^l - P_k^j {}'\omega_l^k \wedge dQ_i^l \\ &\quad + (P_l^j dQ_m^l + P_l^j {}'\omega_s^l Q_m^s) \wedge (P_s^m dQ_i^s + P_s^m {}'\omega_h^s Q_i^h) \\ &= P_k^j (d\,{}'\omega_h^k + {}'\omega_l^k \wedge {}'\omega_h^l) Q_i^h = P_k^j {}'\Omega_h^k Q_i^h, \end{aligned}$$

that is

$$''\Omega_i^j = P_k^j {}'\Omega_h^k Q_i^h. \tag{7.5}$$

Then, we have

$$\begin{aligned} ''DP_i^j &= dP_i^j + ''\omega_l^j P_i^l - P_l^j ''\omega_i^l \\ &= dP_i^j + (P_k^j dQ_l^k + P_k^j {}'\omega_h^k Q_l^h) P_i^l \\ &\quad - P_l^j (P_m^l dQ_i^m + P_m^l {}'\omega_h^m Q_i^h) \\ &= P_k^j (dP_h^k + {}'\omega_l^k P_h^l - P_l^k {}'\omega_h^l) Q_i^h = P_k^j {}'DP_h^k Q_i^h, \end{aligned}$$

that is

$$''DP_i^j = P_k^j {}'DP_h^k Q_i^h \text{[6)]}. \tag{7.6}$$

Substituting (7.5) and (7.6) into (7.2), we get

$$\begin{aligned} \Omega_i^j &= M_l^j Q_h^l ''\Omega_k^h P_m^k P_i^m + P_l^j Q_h^l (''DP_t^h) P_m^t \wedge Q_s^m (''DP_k^s) P_i^k \\ &= P_h^j ''\Omega_k^h M_i^k + ''DP_h^j \wedge ''DP_k^h P_i^k, \end{aligned}$$

that is

$$\Omega_i^j = P_h^j ''\Omega_k^h M_i^k + ''DP_h^j \wedge ''DP_k^h P_i^k. \tag{7.7}$$

Furthermore, let us denote the curvature forms on $\mathfrak{B}(\mathfrak{X})$ of $'\Gamma$ and $''\Gamma$ by $'\theta_\lambda^\mu$ and $''\theta_\lambda^\mu$ respectively, then we have

$$'\theta_\lambda^\mu = b_j^\mu {}'\Omega_i^j a_\lambda^i, \quad ''\theta_\lambda^\mu = b_j^\mu ''\Omega_i^j a_\lambda^i,$$

therefore, we get from (7.2), (7.7) and (6.25)

$$\theta_\lambda^\mu = M_\rho^\mu {}'\theta_\sigma^\rho P_\lambda^\sigma + P_\rho^\mu {}'DP_\nu^\rho \wedge {}'DP_\lambda^\nu \tag{7.8}$$

$$= P_\rho^\mu ''\theta_\sigma^\rho M_\lambda^\sigma + ''DP_\rho^\mu \wedge ''DP_\nu^\rho P_\lambda^\nu \tag{7.9}$$

where M_μ^λ are the components of $\pi^{\ominus} M$, $M = P^2$.

Lastly, we shall investigate the Ricci formula for the basic covariant differentiation of a regular general connection Γ.

For instance, we take a tensor field of type (1, 1) with local components U_i^j. By means of (3.7), we have

6) Formulas (7.6) and (7.7) are special ones which generally hold good for the induced connection derived from another classical connection by a bundle mapping.

$$U^j_{i|k} = \frac{\partial U^j_i}{\partial u^h} + {}'\Gamma^j_{lh} U^l_i - U^j_l {}''\Gamma^l_{ih},$$

$$\begin{aligned} U^j_{i|h|k} &= \frac{\partial U^j_{i|h}}{\partial u^k} + {}'\Gamma^j_{mk} U^m_{i|h} - U^j_{m|h} {}''\Gamma^m_{ik} - U^j_{i|m} {}''\Gamma^m_{hk} \\ &= \frac{\partial^2 U^j_i}{\partial u^k \partial u^h} + \frac{\partial {}'\Gamma^j_{lh}}{\partial u^k} U^l_i + {}'\Gamma^j_{lh} \frac{\partial U^l_i}{\partial u^k} - \frac{\partial U^j_l}{\partial u^k} {}''\Gamma^l_{ih} \\ &\quad - U^j_l \frac{\partial {}''\Gamma^l_{ih}}{\partial u^k} + \Gamma^j_{mk}\left(\frac{\partial U^m_i}{\partial u^h} + {}'\Gamma^m_{th} U^t_i - U^m_t {}''\Gamma^t_{ih}\right) \\ &\quad - \left(\frac{\partial U^j_m}{\partial u^h} + {}'\Gamma^j_{th} U^t_m - U^j_t {}''\Gamma^t_{mh}\right) {}''\Gamma^m_{ik} \\ &\quad - U^j_{i|m} {}''\Gamma^m_{hk}. \end{aligned}$$

Now, let the components of the curvature tensors of $'\Gamma$ and $''\Gamma$ be $'R^j_{ihk}$ and $''R^j_{ihk}$ respectively, that is

$$\begin{aligned} {}'\Omega^j_i &= \tfrac{1}{2}\, {}'R^j_{ihk} \mathrm{d}u^h \wedge \mathrm{d}u^k, \\ {}''\Omega^j_i &= \tfrac{1}{2}\, {}''R^j_{ihk} \mathrm{d}u^h \wedge \mathrm{d}u^k, \end{aligned} \tag{7.10}$$

$$\begin{aligned} {}'R^j_{ihk} &= \frac{\partial {}'\Gamma^j_{ik}}{\partial u^h} - \frac{\partial {}'\Gamma^j_{ih}}{\partial u^k} + {}'\Gamma^j_{lh} {}'\Gamma^l_{ik} - {}'\Gamma^j_{lk} {}'\Gamma^l_{ih}, \\ {}''R^j_{ihk} &= \frac{\partial {}''\Gamma^j_{ik}}{\partial u^h} - \frac{\partial {}''\Gamma^j_{ih}}{\partial u^k} + {}''\Gamma^j_{lh} {}''\Gamma^l_{ik} - {}''\Gamma^j_{lk} {}''\Gamma^l_{ih} \end{aligned} \tag{7.11}$$

and the components of the torsion tensor of $'\Gamma$ and $''\Gamma$ be $'T^j_{ih}$ and $''T^j_{ih}$ respectively, that is

$$\begin{aligned} {}'\Omega^j &= {}'\omega^j_i \wedge du^i = -\tfrac{1}{2}\, {}'T^j_{ih} du^i \wedge du^h, \\ {}''\Omega^i &= {}''\omega^j_i \wedge du^i = -\tfrac{1}{2}\, {}''T^j_{ih} du^i \wedge du^h \end{aligned} \tag{7.12}$$

and

$${}'T^j_{ih} = {}'\Gamma^j_{ih} - {}'\Gamma^j_{hi}, \quad {}''T^j_{ih} = {}''\Gamma^j_{ih} - {}''\Gamma^j_{hi}. \tag{7.13}$$

Then, we get easily

$$\begin{aligned} 2U^j_{i|[h|k]} &= U^j_{i|h|k} - U^j_{i|k|h} \\ &= {}'R^j_{lkh} U^l_i - U^j_l {}''R_i{}^l{}_{kh} - U^j_{i|m} {}''T^m_{hk}. \end{aligned}$$

As is easily proved, on the basic covariant differentiation, we have the formulas

$$\begin{aligned} (V^{i_1\cdots i_p}_{j_1\cdots j_q} W^{i_{p+1}\cdots i_{p+a}}_{j_{q+1}\cdots j_{q+b}})_{|h} &= V^{i_1\cdots i_p}_{j_1\cdots j_q|h} W^{i_{p+1}\cdots i_{p+a}}_{j_{q+1}\cdots j_{q+b}} + \\ &+ V^{i_1\cdots i_p}_{j_1\cdots j_q} W^{i_{p+1}\cdots i_{p+a}}_{j_{q+1}\cdots j_{q+b}|h}, \end{aligned} \tag{7.14}$$

which are identical with the classical ones on the covariant differentiation of the tensor product of two tensors. Making use of the formulas, we can generally obtain

Lemma 7.1. *For any tensor field with local components* $V^{i_1\cdots i_p}_{j_1\cdots j_q}$, *the* *following Ricci formulas*[7] *hold good*:

$$2V^{i_1\cdots i_p}_{j_1\cdots j_q|[h|k]} = V^{i_1\cdots i_p}_{j_1\cdots j_q|h|k} - V^{i_1\cdots i_p}_{j_1\cdots j_q|k|h}$$
$$= -\sum_{s=1}^{p} {}'R^{i_s}_{l\,hk} V^{i_1\cdots l\cdots i_p}_{j_1 j_2\cdots j_q} + \sum_{t=1}^{q} V^{i_1 i_2\cdots i_p}_{j_1\cdots l\cdots j_q} {}''R^{l}_{j_t hk}$$
$$- V^{i_1\cdots i_p}_{j_1\cdots j_q|l} {}''T^{l}_{hk}. \tag{7.15}$$

In connection with curvature tensors and Corollary 5. 2, we shall prove the following theorems.

Theorem 7. 2. *Let Γ be a regular connection. In order that DI/dt is similar to I or P along any curve in $\mathfrak{X}$, P must be commutative with each element of the infinitesimal homogeneous holonomy groups*[8] *of the contravariant part $'\Gamma$ and the covariant part $''\Gamma$ of Γ.*

Proof. In order that DI/dt is similar to I along any curve, there must be a covariant vector field with local components ρ_h such that

$$\delta^i_{j,h} = \delta^i_j \rho_h. \tag{7.16}$$

By means of (3. 15) and (3. 16), we get

$$P^i_k {}'\nabla_h P^k_j = P^k_j {}''\nabla_h P^i_k = \delta^i_j \rho_h.$$

Denoting P^{-1} by Q, the above equation can be written as

$${}'\nabla_h P^i_j = Q^i_j \rho_h \text{ or } {}''\nabla_h P^i_j = Q^i_j \rho_h, \tag{7.17}$$

from which we get

$$\rho_h = \frac{1}{n} P^j_i {}'\nabla_h P^i_j = \frac{1}{2n} {}'\nabla_h (P^j_i P^i_j) = \frac{1}{2n} {}''\nabla_h (P^j_i P^i_j),$$

since $'\Gamma$ and $''\Gamma$ are classical affine connections. Hence ρ_h must be a gradient vector field such that

$$\rho_h = \frac{1}{2n} \frac{\partial}{\partial u^h} (\text{trace } M). \tag{7.18}$$

The first of (7. 17) is clearly equivalent to

$${}'\nabla_h Q^i_j = - Q^i_l Q^l_m Q^m_j \rho_h. \tag{7.19}$$

Covariantly differentiate (7. 17) and using (7. 19), we get

$${}'\nabla_k {}'\nabla_h P^i_j = Q^i_j {}'\nabla_k \rho_h + \rho_h {}'\nabla_k Q^i_j$$
$$= Q^i_j {}'\nabla_k \rho_h - \rho_k \rho_h Q^i_l Q^l_m Q^m_j.$$

By the Ricci formula of $'\Gamma$, we have

$$2{}'\nabla_{[k} {}'\nabla_{h]} P^i_j = - {}'R_l{}^i{}_{hk} P^l_j + P^i_l {}'R_j{}^l{}_{hk} - {}'\nabla_l P^i_j {}'T^l_{hk}$$
$$= - {}'R_l{}^i{}_{hk} P^l_j + P^i_l {}'R_j{}^l{}_{hk} - Q^i_j \rho_l {}'T^l_{hk}.$$

---

7) When Γ is not regular, we can not introduce its curvature tensor in this manner, because the singularity of P produces an obstruction to obtain a tensor of type (1, 3) to be called its curvature tensor.

8) See A. Nijenhuis [5].

Since ρ_h are the components of a gradient vector, we have

$$2'\nabla_{[k}\rho_{h]} = -\rho_l{}'T^l_{hk}.$$

Accordingly, from these equations, we get immediately

$$P^i_l{}'R_j{}^l{}_{hk} - 'R_l{}^i{}_{hk}P^l_j = 0. \tag{7.20}$$

Now, we assume inductively that the equations

$$P^i_l{}'R_j{}^l{}_{hk;m_1\cdots m_r} - 'R_l{}^i{}_{hk;m_1\cdots m_r}P^l_j = 0 \tag{7.21}$$

hold good for $r = 1, 2, \cdots, p$, where we denote by the symbol "$_{;m_1m_2\cdots m_r}$" the covariant derivative by $u^{m_1}, u^{m_2}, \cdots, u^{m_r}$ in this order with respect to the connection $'\Gamma$. From these equations, we get

$$Q^i_l{}'R_j{}^l{}_{hk;m_1\cdots m_r} - 'R_l{}^i{}_{hk;m_1\cdots m_r}Q^l_j = 0$$

$$r = 1. 2, \cdots, p.$$

Covariantly differentiate the above equations for $r=p$, we get

$$\begin{aligned} &P^i_l{}'R_j{}^l{}_{hk;m_1\cdots m_p m} - 'R_l{}^i{}_{hk;m_1\cdots m_p m}P^l_j \\ &= -P^i_{l;m}{}'R_j{}^l{}_{hk;m_1\cdots m_p} + 'R_l{}^i{}_{hk;m_1\cdots m_p}P^l_{j;m} \\ &= -\rho_m\{Q^i_l{}'R_j{}^l{}_{hk;m_1\cdots m_p} - 'R_l{}^i{}_{hk;m_1\cdots m_r}Q^l_j\} = 0. \end{aligned}$$

Thus we have seen that (7. 21) holds good for all $r=0, 1, 2, \cdots$. Since $R_j{}^i{}_{hk;m_1\cdots m_r}$, $r=0, 1, \cdots$, are the generating elements of the Lie algebra of the infinitesimal homogeneous holonomy group of $'\Gamma$, P must be commutative with each element of this group at each point of $\mathfrak{X}$.

In the next place, in order that DI/dt is similar to P along any curve, there must be a covariant vector field with local components ρ_h such that

$$\delta^i_{j.h} = P^i_j\rho_h, \tag{7.22}$$

from which we get the equivalent equations

$$'\nabla_h P^i_j = \delta^i_j\rho_h \textit{ and } ''\nabla_h P^i_j = \delta^i_j\,\rho_h. \tag{7.23}$$

Hence we have

$$\rho_h = \frac{1}{n}\frac{\partial}{\partial u^h}\text{trace } P.$$

In this case, ρ_h must be also a gradient vector. From the first of (7. 23), we get easily

$$'\nabla_k{}'\nabla_h P^i_j = \delta^i_j{}'\nabla_k\rho_h.$$

Since we have

$$\begin{aligned} 2'\nabla_k{}'\nabla_h P^i_j &= -'R_l{}^i{}_{hk}P^l_j + P^i_l{}'R_j{}^l{}_{hk} - '\nabla_l P^i_j{}'T^l_{hk} \\ &= -'R_l{}^i{}_{hk}P^l_j + P^i_l{}'R_j{}^l{}_{hk} - \delta^i_j\rho_l{}'T^l_{hk}, \end{aligned}$$

we get easily

$$P^i_l{}'R_j{}^l{}_{hk} - 'R_l{}^i{}_{hk}P^l_j = 0.$$

Once more, assume (7.21) for $r=1, 2, \cdots, p$.

$$P^i_l{}'R_j{}^l{}_{hk;m_1\cdots m_p m} - {}'R_l{}^i{}_{hk;m_1\cdots m_p m} P^l_j$$
$$= - P^i_{l;m}{}'R_j{}^l{}_{hk;m_1\cdots m_p} + {}'R_l{}^i{}_{hk;m_1\cdots m_p} P^l_{j;m}$$
$$= -\rho_m ({}'R_j{}^i{}_{hk;m_1\cdots m_p} - {}'R_j{}^i{}_{hk;m_1\cdots m_p}) = 0.$$

Thus, we obtain the same result of the first part.

It is clear that the same fact must be hold good for the covariant part $''\Gamma$ of Γ as easily seen from the above calculations.

Corollary 7.3. *Let Γ be a regular general connection such that the infinitesimal homogeneous holonomy groups of its contravariant (covariant) part $'\Gamma$ ($''\Gamma$) is irreducible at each point of $\mathfrak{X}$. If DI/dt is a homothety along any curve, then $P=\lambda(\Gamma)$ is everywhere a homothety and its magnification is constant if and only if $'\Gamma = ''\Gamma$.*

Proof. Making use of Theorem 7.2 and the Schur's lemma, it must be

$$P^i_j = \varphi \delta^i_j. \tag{7.24}$$

Substituting this into (7.17), we get

$$\rho_h = \varphi \frac{\partial \varphi}{\partial u^h}.$$

On the other hand, from (3.2) and (3.3) we get

$$'\Gamma^j_{ih} = Q^j_k \Gamma^k_{ih} = \frac{1}{\varphi} \Gamma^j_{ih},$$

$$''\Gamma^j_{ih} = \Lambda^j_{kh} Q^k_i = \frac{1}{\varphi} \Lambda^j_{ih}$$
$$= \frac{1}{\varphi}\left(\Gamma^j_{ih} - \frac{\partial P^j_i}{\partial u^h}\right) = {}'\Gamma^j_{ih} - \delta^j_i \frac{\partial \log \varphi}{\partial u^h},$$

that is

$$\delta^j_{i|h} = {}'\Gamma^j_{ih} - {}''\Gamma^j_{ih} = \delta^j_i \frac{\partial \log \varphi}{\partial u^h} \tag{7.25}$$

Hence, $'\Gamma = ''\Gamma$ if and only if φ is constant.

§ 8. The semi-group $\tilde{\mathfrak{L}}^2_n \supset \mathfrak{L}^2_n$.

In the paper [11], the author could not succeed in making the canonical mapping[9] of a general connection Γ when it is not regular. In the following three sections, he will show that making use of an associated bundle of $T(\mathfrak{X})$ containing its associated principal bundle of the second order $\mathfrak{B}^2(\mathfrak{X})$, we can define the canonical mapping of Γ which goes over into

9) See [11], § 9

the one in [10] when Γ is a classical affine connection.

At first, we prepare some concepts. Let $\tilde{\mathfrak{L}}_n^2$ be a semi-group whose any element is written as a set of real numbers $(a_i^j,\ a_{ih}^j,\ p_i^j)$ such that $|a_i^j| \neq 0$ and its multiplication is given by the following formulas: For any $\alpha, \beta \in \tilde{\mathfrak{L}}_n^2$,

$$a_i^j(\alpha\beta) = a_k^j(\alpha)a_i^k(\beta), \tag{8.1}$$

$$a_{ih}^j(\alpha\beta) = a_k^j(\alpha)a_{ih}^k(\beta) + a_{kl}^j(\alpha)p_i^k(\beta)a_h^l(\beta), \tag{8.2}$$

$$p_i^j(\alpha\beta) = p_k^j(\alpha)p_i^k(\beta), \tag{8.3}$$

where we regard a_i^j, a_{ih}^j, p_i^j as coordinates of $\tilde{\mathfrak{L}}_n^2$ and so $\tilde{\mathfrak{L}}_n^2$ may be a differentiable semi-group[10]. It is clear that $\tilde{\mathfrak{L}}_n^2$ satisfies the associative law $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, because $\tilde{\mathfrak{L}}_n^2$ has a natural representation ψ: $\tilde{\mathfrak{L}}_n^2 \to GL(n+n^2, R)$ defined by

$$\psi(\alpha) = \begin{pmatrix} a_i^j(\alpha) & a_{ih}^j(\alpha) \\ 0 & p_i^j(\alpha)a_h^k(\alpha) \end{pmatrix}. \tag{8.4}$$

The group $\mathfrak{L}_n^2$ of all generalized infinitesimal isotropies of order 2 at the origin of the n-dimensional coordinate space R^n, whose any element is written as a set of real number $(a_i^j,\ a_{ih}^j)$ such that $|a_i^j| \neq 0$, can be considered as subgroup of $\tilde{\mathfrak{L}}_n^2$ which is the set of all elements α of $\tilde{\mathfrak{L}}_n^2$ such that $a_i^j(\alpha) = p_i^j(\alpha)$. Analogously, we may regard $L_n^1 = GL(n, R)$ as subgroup of $\tilde{\mathfrak{L}}_n^2$, for L_n^1 may be identified with the subgroup of $\mathfrak{L}_n^2$ which consists of all elements α such that $a_{ih}^j(\alpha) = 0$.

Let σ be the natural mapping of $\tilde{\mathfrak{L}}_n^2$ onto L_n^1 given by

$$\sigma(a_i^j,\ a_{ih}^j,\ p_i^j) = (a_i^j), \tag{8.5}$$

(8.1) shows that σ is a homomorphism. Let $\tilde{\mathfrak{N}}_n^2$ be the kernel of σ, that is the semi-subgroup consisting of all elements α such that $a_i^j(\alpha) = \delta_i^j$. Then

$$\mathfrak{N}_n^2 = \tilde{\mathfrak{N}}_n^2 \cap \mathfrak{L}_n^2$$

is a subgroup of $\mathfrak{L}_n^2$. Then we define two mappings γ, $\bar{\gamma}$: $\tilde{\mathfrak{L}}_n^2 \to \tilde{\mathfrak{N}}_n^2$ by

$$\gamma(\alpha) = (\sigma(\alpha))^{-1}\alpha, \tag{8.6}$$

$$\bar{\gamma}(\alpha) = \alpha(\sigma(\alpha))^{-1}. \tag{8.7}$$

Any element α of $\tilde{\mathfrak{L}}_n^2$ can be written uniquely as products of an element of L_n^1 and an element of $\tilde{\mathfrak{N}}_n^2$ by

$$\alpha = \sigma(\alpha)\gamma(\alpha) = \bar{\gamma}(\alpha)\sigma(\alpha). \tag{8.8}$$

Then, putting

$$b_i^j(\alpha) = a_i^j((\sigma(\alpha))^{-1})$$

---

10) We do not demand that $|p_i^j| \neq 0$, and so any α has not always its inverse α^{-1}.

on $\tilde{\mathfrak{L}}^2_n$, we have

$$(b^j_i,\ 0,\ b^j_i)\ (a^j_i,\ a^j_{ih},\ p^j_i)$$
$$=(\delta^j_i,\ b^j_k a^k_{ih},\ b^j_k p^k_i)$$

and

$$(a^j_i,\ a^j_{ih},\ p^j_i)\ (b^j_i,\ 0,\ b^j_i)$$
$$=(\delta^j_i,\ a^j_{kl} b^k_i b^l_h,\ p^j_k b^k_i),$$

hence

$$\eta(\alpha) = (\delta^j_i,\ b^j_k(\alpha) a^k_{ih}(\alpha),\ b^j_k(\alpha) p^k_i(\alpha)), \tag{8.9}$$
$$\bar{\eta}(\alpha) = (\delta^j_i,\ a^j_{kl}(\alpha) b^k_i(\alpha) b^l_h(\alpha),\ p^j_k(\alpha) b^k_i(\alpha)). \tag{8.10}$$

Now, we shall make some formulas on $\tilde{\mathfrak{L}}^2_n$ in order to utilize in the following theory.

For any two $\alpha, \beta \in \widetilde{\mathfrak{N}}^2_n$, we get from (8.2)

$$a^j_{ih}(\alpha\beta) = a^j_{ih}(\alpha) p^l_i(\beta) + a^j_{ih}(\beta). \tag{8.11}$$

For any $\alpha \in \mathfrak{L}^2_n$, we have

$$a^j_k(\alpha^{-1})\, a^k_{ih}(\alpha) + a^j_{kl}(\alpha^{-1})\, a^k_i(\alpha)\, a^l_h(\alpha) = 0,$$

hence

$$a^j_{ih}(\alpha^{-1}) = -\, b^j_k(\alpha)\, a^k_{lm}(\alpha)\, b^l_i(\alpha)\, b^m_h(\alpha). \tag{8.12}$$

For any $\alpha \in \mathfrak{L}^2_n$, $\beta \in \widetilde{\mathfrak{N}}^2_n$, we have

$$a^j_i(\alpha^{-1}\beta) = a^j_i(\alpha^{-1}),$$
$$a^j_{ih}(\alpha^{-1}\beta) = a^j_k(\alpha^{-1})\, a^k_{ih}(\beta) + a^j_{kh}(\alpha^{-1})\, p^k_i(\beta),$$
$$p^j_i(\alpha^{-1}\beta) = a^j_k(\alpha^{-1})\, p^k_i(\beta)$$

and furthermore

$$a^j_i(\alpha^{-1}\beta\alpha) = a^j_k(\alpha^{-1}\beta) a^k_i(\alpha) = \delta^j_i,$$
$$a^j_{ih}(\alpha^{-1}\beta\alpha) = a^j_k(\alpha^{-1}\beta) a^k_{ih}(\alpha) + a^j_{kl}(\alpha^{-1}\beta) a^k_i(\alpha) a^l_h(\alpha),$$
$$= a^j_k(\alpha^{-1}) a^k_{ih}(\alpha) + \{a^j_m(\alpha^{-1}) a^m_{kl}(\beta) + a^j_{ml}(\alpha^{-1}) p^m_k(\beta)\} a^k_i(\alpha) a^l_h(\alpha)$$
$$= b^j_k(\alpha) a^k_{ih}(\alpha) + b^j_k(\alpha) a^k_{ml}(\beta) a^m_i(\alpha) a^l_h(\alpha)$$
$$- b^j_k(\alpha) a^k_{mh}(\alpha) b^m_t(\alpha) p^t_l(\beta) a^l_i(\alpha),$$
$$p^j_i(\alpha^{-1}\beta\alpha) = p^j_k(\alpha^{-1}\beta)\, p^k_i(\alpha) = b^j_k(\alpha) p^k_l(\beta) a^l_i(\alpha),$$

that is, for any $\alpha \in \mathfrak{L}^2_n$, $\beta \in \widetilde{\mathfrak{N}}^2_n$, we have

$$a^j_i(\alpha^{-1}\beta\alpha) = \delta^j_i, \tag{8.15}$$

$$\left.\begin{aligned} a^j_{ih}(\alpha^{-1}\beta\alpha) &= b^j_k(\alpha) a^k_{ml}(\beta) a^m_i(\alpha) a^l_h(\alpha) \\ &\quad + b^j_k(\alpha) a^k_{mh}(\alpha) \{\delta^m_i - b^m_t(\alpha) p^t_l(\beta)\ a^l_i(\alpha)\} \\ &= b^j_k(\alpha) a^k_{ml}(\beta) a^m_i(\alpha) a^l_h(\alpha) \\ &\quad + a^j_{ml}(\alpha^{-1}) \{p^m_k(\beta) - \delta^m_k\}\ a^k_i(\alpha) a^l_h(\alpha), \end{aligned}\right\} \tag{8.14}$$

$$p^j_i(\alpha^{-1}\beta\alpha) = b^j_k(\alpha)p^k_l(\beta)a^l_i(\alpha). \tag{8.15}$$

Accordingly, for any $\alpha \in L^1_n$, $\beta \in \widetilde{\mathfrak{N}}^2_n$, we have

$$a^j_{ih}(\alpha^{-1}\beta\alpha) = b^j_k(\alpha)a^k_{ml}(\beta)a^m_i(\alpha)a^l_h(\alpha), \tag{8.16}$$

since $a^j_{ih}(\alpha) = 0$.

Lastly, for any two $\alpha, \beta \in \widetilde{\mathfrak{L}}^2_n$, we have

$$\begin{aligned}\gamma(\alpha\beta) &= \sigma(\alpha\beta)^{-1}\alpha\beta = \sigma(\beta)^{-1}\sigma(\alpha)^{-1}\alpha\beta\\ &= \sigma(\beta)^{-1}\gamma(\alpha)\sigma(\beta)\gamma(\beta).\end{aligned}$$

Since $\gamma(\alpha), \gamma(\beta) \in \widetilde{\mathfrak{N}}^2_n$, we get from (8.11)

$$a^j_{ih}(\gamma(\alpha\beta)) = a^j_{lh}(\sigma(\beta)^{-1}\gamma(\alpha)\sigma(\beta))\,p^l_i(\gamma(\beta)) + a^j_{ih}(\gamma(\beta)).$$

By means of (8.9) and (8.16), the first term of the right hand side of the above equation can be written as

$$\begin{aligned}&b^j_k(\beta)a^k_{mt}(\gamma(\alpha))a^m_l(\beta)a^t_h(\beta)p^l_i(\gamma(\beta))\\ &= b^j_k(\beta)a^k_{ml}(\gamma(\alpha))p^m_i(\beta)a^l_h(\beta).\end{aligned}$$

Analogously, we have

$$\begin{aligned}\bar{\gamma}(\alpha\beta) &= \alpha\beta\sigma(\alpha\beta)^{-1} = \alpha\beta\sigma(\beta)^{-1}\sigma(\alpha)^{-1}\\ &= \bar{\gamma}(\alpha)\sigma(\alpha)\bar{\gamma}(\beta)\sigma(\alpha)^{-1},\end{aligned}$$

hence

$$\begin{aligned}a^j_{ih}(\bar{\gamma}(\alpha\beta)) &= a^j_{kh}(\bar{\gamma}(\alpha)\,p^k_i(\sigma(\alpha)\bar{\gamma}(\beta)\sigma(\alpha)^{-1})\\ &\quad + a^j_{ih}(\sigma(\alpha)\bar{\gamma}(\beta)\sigma(\alpha)^{-1}).\end{aligned}$$

Thus, for any two $\alpha, \beta \in \widetilde{\mathfrak{L}}^2_n$, we have

$$a^j_{ih}(\gamma(\alpha\beta)) = b^j_k(\beta)a^k_{ml}(\gamma(\alpha))p^m_i(\beta)a^l_h(\beta) + a^j_{ih}(\gamma(\beta)) \tag{8.17}$$

and

$$\begin{aligned}a^j_{ih}(\bar{\gamma}(\alpha\beta)) &= a^j_{kh}(\bar{\gamma}(\alpha))a^k_l(\alpha)p^l_m(\bar{\gamma}(\beta))b^m_i(\alpha)\\ &\quad + a^j_k(\alpha)a^k_{lm}(\bar{\gamma}(\beta))b^l_i(\alpha)b^m_h(\alpha).\end{aligned} \tag{8.18}$$

We get also easily

$$p^j_i(\gamma(\alpha\beta)) = b^j_k(\beta)p^k_h(\gamma(\alpha))p^h_i(\beta), \tag{8.19}$$

$$p^j_i(\bar{\gamma}(\alpha\beta)) = p^j_k(\alpha)p^k_h(\bar{\gamma}(\beta))b^h_i(\alpha). \tag{8.20}$$

§ 9. General frames and general principal bundles of the second order.

Let $\mathfrak{X}$ be any n-dimensional differentiable manifold. At any point $x \in \mathfrak{X}$, we call a set $\{e_i, e_{ih}\}$ of $n+n^2$ tangent vectors of the second order at x such that

$$e_i = \partial u_j a^j_i(\alpha), \tag{9.1}$$

$$e_{ih} = \partial u_j a^j_{ih}(\alpha) + \partial^2 u_{jk} p^j_i(\alpha)a^k_h(\alpha), \tag{9.2}$$

where we take any $\alpha \in \tilde{\mathfrak{L}}^2_n$, *a general frame of the second order* at x. For any two coordinate neighborhood (U, u^i), (V, v^i), $U \cap V \ni x$, we get from (1. 7) and (1. 8) the equations

$$
\begin{aligned}
e_i &= \partial v_k \frac{\partial v^k}{\partial u^j} a^j_i(\alpha) = \partial v_j a^j_i(\beta),\\
e_{ih} &= \partial v_k \frac{\partial v^k}{\partial u^j} a^j_{ih}(\alpha) +\\
&+\left(\partial v_l \frac{\partial^2 v^l}{\partial u^k \partial u^j} + \partial^2 v_{lm} \frac{\partial v^l}{\partial v^j}\frac{\partial v^m}{\partial n^k}\right) p^j_i(\alpha) a^k_h(\alpha)\\
&= \partial v_k \left\{\frac{\partial v^k}{\partial u^j} a^j_{ih}(\alpha) + \frac{\partial^2 v^k}{\partial u^l \partial u^i} p^j_i(\alpha) a^l_h(\alpha)\right\}\\
&+ \partial^2 v_{lm} \frac{\partial v^l}{\partial u^j} p^j_i(\alpha) \frac{\partial v^m}{\partial u^k} a^k_h(\alpha)\\
&= \partial v_k a^k_{ih}(\beta) + \partial^2 v_{kl} p^k_i(\beta) a^l_h(\beta).
\end{aligned}
$$

Hence, putting

$$
g_{VU}:\ U \cap V \to L^2_n \subset \tilde{\mathfrak{L}}^2_n,
$$

$$
\begin{cases}
a^j_i(g_{VU}) = p^j_i(g_{VU}) = \dfrac{\partial v^j}{\partial u^i},\\
a^j_{ih}(g_{VU}) = \dfrac{\partial^2 v^j}{\partial u^h \partial u^i},
\end{cases} \tag{9. 3}
$$

we have

$$
\beta = g_{VU}\, \alpha. \tag{9. 4}
$$

The above equation shows that the general frame of the second order is defined without depending on the choice of local coordinates.

The set of all general frames of order 2 at the point x may be considered to be equivalent to $\tilde{\mathfrak{L}}^2_n$ · By means of (9. 4), the set of all general frames of order 2 of $\mathfrak{X}$ is the bundle space of an associated fibre bundle of $\mathfrak{L}^2(\mathfrak{X})$, which we denote by $(\tilde{\mathfrak{B}}^2(\mathfrak{X}), \mathfrak{X}, \tilde{\pi}_2)$ or simply $\tilde{\mathfrak{B}}^2(\mathfrak{X})$, which we call *the general principal bundle of the second order of* $\mathfrak{X}$.

By means of its definition, $\mathfrak{B}^2(\mathfrak{X})$ is clearly a subbundle of $\tilde{\mathfrak{B}}^2(\mathfrak{X})$. For any point $b \in \tilde{\mathfrak{B}}^2(\mathfrak{X})$, we denote the corresponding general frame by $\{e_i(b), e_{ih}(b)\}$. $\{e_i(b)\}$ is clearly a frame of order 1 that is a point of $\mathfrak{B}(\mathfrak{X})$ over $\tilde{\pi}_2(b)$. We define a mapping

$$
\tilde{\sigma}:\ \tilde{\mathfrak{B}}^2(\mathfrak{X}) \to \mathfrak{B}(\mathfrak{X})
$$

by

$$
\tilde{\sigma}(\{e_i(b),\ e_{ih}(b)\}) = \{e_i(b)\}, \tag{9. 5}
$$

which is a natural extension of $\sigma:\ \mathfrak{B}^2(\mathfrak{X}) \to \mathfrak{B}(\mathfrak{X})$ given by (1. 16).

Since $\tilde{\mathfrak{B}}^2(\mathfrak{X})$ is the space of all general frames, any element of $\tilde{\mathfrak{L}}_n^2$ operates on it as a right translation which in general, a homomrphism of this bundle covering the identity mapping of $\mathfrak{X}$ and an isomorphism when it belongs to $\mathfrak{L}_n^2$. That is, for any $b \in \tilde{\mathfrak{B}}^2(\mathfrak{X})$, $\alpha \in \tilde{\mathfrak{L}}_n^2$, $r(\alpha)(b) = b\alpha$ is defined by

$$\begin{cases} e_i(b\alpha) = e_j(b)a_i^j(\alpha), \\ e_{ih}(b\alpha) = e_j(b)a_{ih}^j(\alpha) + e_{jk}(b)p_i^j(\alpha)a_h^k(\alpha). \end{cases} \tag{9.6}$$

Using local coordinates (u^i), we put

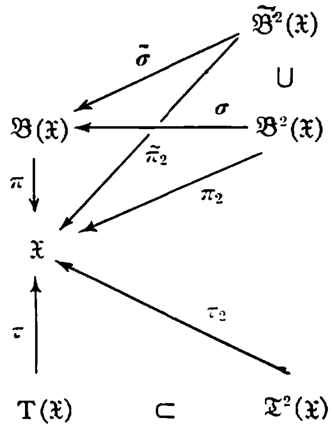
$$e_i(b) = \partial u_j a_i^j(\beta),$$
$$e_{ih}(b) = \partial u_j a_{ih}^j(\beta) + \partial^2 u_{jk} p_i^j(\beta) a_h^k(\beta)^{11)},$$

then

$$e_i(b\alpha) = \partial u_j a_i^j(\beta\alpha)$$
$$\begin{aligned} e_{ih}(b\alpha) &= \partial u_j a_k^j(\beta) a_{ih}^k(\alpha) \\ &\quad + \{\partial u_l a_{jk}^l(\beta) + \partial^2 u_{lm} p_j^l(\beta) a_k^m(\beta)\} p_i^j(\alpha) a_h^k(\alpha) \\ &= \partial u_j \{a_k^j(\beta) a_{ih}^k(\alpha) + a_{lk}^j(\beta) p_i^l(\alpha) a_h^k(\alpha)\} \\ &\quad + \partial^2 u_{lm} p_j^l(\beta) p_i^j(\alpha) a_k^m(\beta) a_h^k(\alpha) \\ &= \partial u_j a_{ih}^j(\beta\alpha) + \partial^2 u_{lm} p_i^l(\beta\alpha) a_h^m(\beta\alpha), \end{aligned}$$

which shows that $\{e_i(b\alpha), e_{ih}(b\alpha)\}$ is a general frame of order 2 of $\mathfrak{X}$ at $\tilde{\pi}_2(b)$ in the above mentioned sense.

It is to be remarked that properties, which principal bundles have, are not always maintained in $\tilde{\mathfrak{B}}^2(\mathfrak{X})$, because it is not a principal bundle.



Diag. 1

11) We denote also the point b by (u^i, β) in the following.

Any $b \in \widetilde{\mathfrak{B}}^2(\mathfrak{X})$ defines a homomorphism

$$b : \widetilde{\mathfrak{L}}^2_n \longrightarrow \tilde{\pi}_2^{-1}(\tilde{\pi}_2(b))$$

by

$$b(\alpha) = \mathrm{r}(\alpha)(b) = b\alpha, \tag{9.7}$$

which is called an admissible mapping for a principal bundle and we call it also so in this case.

Using local coordinates u^i, we put $b=(u^i, \beta)$, $\beta \in \widetilde{\mathfrak{L}}^2_n$, then the set of the last coordinates of all elements of $b(\widetilde{\mathfrak{L}}^2_n)$ is $\beta\widetilde{\mathfrak{L}}^2_n$. Accordingly, in order that

$$b(\widetilde{\mathfrak{L}}^2_n) = \tilde{\pi}_2^{-1}(\tilde{\pi}_2(b)),$$

it is necessary and sufficient that

$$\beta\widetilde{\mathfrak{L}}^2_n = \widetilde{\mathfrak{L}}^2_n.$$

Therefore, β must have its inverse and conversely if β has its inverse then the above equation holds good. This property does not depend on the choice of local coordinates.

Now, we put

$$\overline{\mathfrak{L}}^2_n = \{\alpha \mid |p^j_i(\alpha)| \neq 0,\ \alpha \in \widetilde{\mathfrak{L}}^2_n\}$$

then this is a group and the set of all elements which have inverses. It is evident that

$$\mathfrak{L}^2_n \subset \overline{\mathfrak{L}}^2_n.$$

We denote the associated principal bundle of $\mathfrak{T}^2(\mathfrak{X})$, regarding its structure group as $\overline{\mathfrak{L}}^2_n$, by $\{\overline{\mathfrak{B}}^2(\mathfrak{X}), \mathfrak{X}, \bar{\pi}_2\}$, then it may be regarded as a subbundle of $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$, which consists of all point b of $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$ such that the last coordinate β of $b=(u^i, \beta)$ belongs to $\overline{\mathfrak{L}}^2_n$. Thus we can say the above mentioned fact as follows.

Lemma 9.1. *In order that for a point $b \in \widetilde{\mathfrak{B}}^2(\mathfrak{X})$, the corresponding admissible mapping $b : \widetilde{\mathfrak{L}}^2_n \rightarrow \tilde{\pi}_2^{-1}(\tilde{\pi}_2(b))$ is an isomorphism onto, it is necessary and sufficient $b \in \overline{\mathfrak{B}}^2(\mathfrak{X})$.*

§ 10. Canonical mappings for general connections.

Now, we shall show in this section that we can define the canonical mapping for any general connection.

Let Γ be any general connection of $\mathfrak{X}$ and φ_Γ be the corresponding mapping defined in § 2. Utilizing $\varphi = \varphi_\Gamma$, we calculate some equations as follows. From

$$\partial u_i = \frac{\partial v^j}{\partial u^i}\partial v_j,$$

we get

$$\varphi\partial u_i = \frac{\partial v^j}{\partial u^i}\varphi\partial v_j,$$

herce

$$d(\varphi\partial u_i) = \frac{\partial^2 v^j}{\partial u^h \partial u^i}\varphi\partial v_j \otimes du^h + \frac{\partial v^j}{\partial u^i} d(\varphi\partial v_j). \tag{10. 1}$$

From

$$\partial^2 u_{ih} = \frac{\partial^2 v^j}{\partial u^h \partial u^i}\partial v_j + \frac{\partial v^j}{\partial u^i}\frac{\partial v^k}{\partial u^h}\partial^2 v_{jk},$$

we get

$$\varphi\partial^2 u_{ih} = \frac{\partial^2 v^j}{\partial u^h \partial u^i}\varphi\partial v_j + \frac{\partial v^j}{\partial u^i}\frac{\partial v^k}{\partial u^h}\varphi\partial^2 v_{jk},$$

hence

$$\varphi\partial^2 u_{ih}\otimes du^h = \frac{\partial^2 v^j}{\partial u^h \partial u^i}\varphi\partial v_j \otimes du^h + \frac{\partial v^j}{\partial u^i}\varphi\partial^2 v_{jk}\otimes dv^k. \tag{10. 2}$$

Subtracting (10. 2) from (10. 1), we get the following vectorial equations

$$d(\varphi\partial u_i) - \varphi\partial^2 u_{ih}\otimes du^h = \frac{\partial v^j}{\partial u^i}\{d(\varphi\partial v_j) - \varphi\partial^2 v_{jk}\otimes dv^k\}. \tag{10. 3}$$

These equations yield immediately

$$\frac{\partial d(\varphi\partial u_i)}{\partial(du^h)} - \varphi\partial^2 u_{ih} = \left\{\frac{d(\varphi\partial v_j)}{\partial(dv^k)} - \varphi\partial^2 v_{jk}\right\}\frac{\partial v^j}{\partial u^i}\frac{\partial v^k}{\partial u^h}, \tag{10. 4}$$

where we denote by $\frac{\partial d(\varphi\partial u_i)}{\partial(du^h)}$ the coefficient of du^h of $d(\varphi\partial u_i)\in\Psi(T(U)\otimes T^*(U))$. In fact, in the coordinate neighborhood (U, u^i), we have

$$\Gamma = \partial u_j \otimes (P^j_i d^2 u^i + \Gamma^j_{ih} du^i \otimes du^h),$$

$$\varphi\partial u_j = P^j_i \partial u_j,\ \varphi\partial^2 u_{ih} = \Gamma^j_{ih}\partial u_j,$$

hence we obtain from these equations

$$\begin{aligned} d(\varphi\partial u_i) - \varphi\partial^2 u_{ih}\otimes du^h &= d(P^j_i\partial u_j) - \Gamma^j_{ih}\partial u_j\otimes du^h \\ &= P^j_i\partial^2 u_{jh}\otimes du^h + \frac{\partial P^j_i}{\partial u^h}\partial u_j\otimes du^h - \Gamma^j_{ih}\partial u_j\otimes du^h \\ &= \left\{P^j_i\partial^2 u_{jh} - \left(\Gamma^j_{ih} - \frac{\partial P^j_i}{\partial u^h}\right)\partial u_j\right\}\otimes du^h. \end{aligned}$$

Therefore, we define for the coordinate neighborhood (U, u^i) a local cross-section h_U of the fibre bundle $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$ by

$$\begin{cases} e_i \cdot h_U = \partial u_i, \\ e_{ih} \cdot h_U = \partial^2 u_{jh} P^j_i - \partial u_j \Lambda^j_{ih}. \end{cases} \tag{10. 5}$$

(10. 3) shows that for any two coordinate neighborhoods (U, u^i), (V, v^i), $U \cap V \neq \emptyset$, we have

$$h_U(x) = h_V(x)\sigma(g_{VU}(x))$$

by means of (9. 6). For any $\alpha \in L^1_n$, we have

$$h_U(x)\alpha = h_V(x)\sigma(g_{VU}(x))\alpha = h_V(x)\beta,$$

hence

$$\beta = \sigma(g_{VU}(x)\alpha. \tag{10. 6}$$

Now, for any $b \in \pi^{-1}(U)$, $\mathrm{e}_i(b) = \partial u_j a^j_i(\alpha)$, if we put

$$\rho(b) = h_U(x)\alpha, \tag{10. 7}$$

then (10. 6) shows that this definition of ρ does not depend on the choice of local coordinates. Hence, by (10. 7) we can define

$$\rho = \rho_\Gamma : \mathfrak{B}(\mathfrak{X}) \to \widetilde{\mathfrak{B}}^2(\mathfrak{X}), \tag{10. 8}$$

such that

$$\tilde{\sigma} \cdot \rho = 1. \tag{10. 9}$$

Since we can regard $\{\widetilde{\mathfrak{B}}^2(\mathfrak{X}), \mathfrak{B}(\mathfrak{X}), \tilde{\sigma}\}$ as fibre bundle with fibre $\widetilde{\mathfrak{N}}^2_n$, $\rho = \rho_\Gamma$ is a cross-section of this fibre bundle. From (10. 7) we have the equation

$$\rho \cdot \mathrm{r}(\alpha) = \mathrm{r}(\alpha) \cdot \rho, \quad \alpha \in L^1_n, \tag{10. 10}$$

that is ρ_Γ commutes with any right translation $\mathrm{r}(\alpha)$, $\alpha \in L^1_n$. We call the mapping ρ_Γ *the canonical mapping* for the general connection Γ.

Theorem 10. 1. *For any general connection Γ, there exists a mapping $\rho_\Gamma : \mathfrak{B}(\mathfrak{X}) \to \widetilde{\mathfrak{B}}^2(\mathfrak{X})$ with the properties (10. 9) and (10. 10). Conversely, for any mapping ρ with such properties, there exists a general connection Γ such that $\rho = \rho_\Gamma$.*

Proof. By the above exposition, the first part is clear. We shall prove here the second part.

Let us suppose that a mapping $\rho : \mathfrak{B}(\mathfrak{X}) \to \widetilde{\mathfrak{B}}^2(\mathfrak{X})$ with the properties (10. 9) and (10. 10) is given. For any $b \in \pi^{-1}(U)$, $\bar{b} = \rho(b)$, we put

$$\mathrm{e}_i(b) = \partial u_j a^j_i(\beta)$$

and

$$\mathrm{e}_i(\bar{b}) = \partial u_j a^j_i(\bar{\beta}),$$
$$\mathrm{e}_{ih}(\bar{b}) = \partial u_j a^j_{ih}(\bar{\beta}) + \partial^2 u_{jk} p^j_i(\bar{\beta}) a^k_h(\bar{\beta}),$$

where $\beta \in L^1_n$, $\bar{\beta} \in \widetilde{\mathfrak{L}}^2_n$.

From (10. 9), we have

$$\rho(\bar{\beta}) = \beta.$$

From (10. 10), we get for any $\alpha \in L^1_n$

$$\rho(b\alpha)=\bar{b}\,\alpha,$$

of which local coordinates are $(u^j, \bar{\beta}\alpha)$. By the definition (8.7) of the mapping $\bar{\gamma}$, we have

$$\bar{\gamma}(\bar{\beta}\alpha)=\bar{\beta}\alpha(\sigma(\bar{\beta}\alpha))^{-1}=\bar{\beta}\alpha(\sigma(\bar{\beta})\sigma(\alpha))^{-1}$$
$$=\bar{\beta}\alpha(\beta\alpha)^{-1}=\bar{\beta}\beta^{-1}=\bar{\gamma}(\bar{\beta}).$$

Hence, if we put

$$p_i^j(\bar{\gamma}(\bar{\beta}))=P_i^j,\quad a_{ih}^j(\bar{\gamma}(\bar{\beta}))=-\Lambda_{ih}^j, \tag{10.11}$$

these functions P_i^j, Λ_{ih}^j depend only on the coordinate neighborhood (U, u^i) including the point $x=\pi(b)$, that is, they are functions defined on U.

If we take another coordinate neighborhood (V, v^i), $U\cap V\neq\emptyset$, then the corresponding last coordinates of b and $\bar{b}$ are

$$\sigma(g_{VU}(x))\beta \text{ and } g_{VU}(x)\bar{\beta}$$

respectively. By means of (8.20), we get

$$p_i^j(\bar{\gamma}(g_{VU}(x)\bar{\beta}))=p_k^j(g_{VU}(x))p_h^k(\bar{\gamma}(\bar{\beta}))b_i^h(g_{VU}(x))$$
$$=a_k^j(g_{VU}(x))P_h^k(u)b_i^h(g_{VU}(x)),$$

hence

$$P_i^j(v)=\frac{\partial v^j}{\partial u^k}P_h^k(u)\frac{\partial u^h}{\partial v^i}, \tag{10.12}$$

where $P_h^k(u)$ and $P_i^j(v)$ denote the corresponding functions defined on U and V by the first equations of (10.11) respectively.

Analogously, by means of (8.18), we get

$$a_{ih}^j(\bar{\gamma}(g_{VU}(x)\bar{\beta}))$$
$$=a_{kh}^j(\bar{\gamma}(g_{VU}(x)))a_i^k(g_{VU}(x))p_m^l(\bar{\gamma}(\bar{\beta}))b_i^m(g_{VU}(x))$$
$$+a_k^j(g_{VU}(x))a_{lm}^k(\bar{\gamma}(\bar{\beta}))b_i^l(g_{VU}(x))b_h^m(g_{VU}(x))$$
$$=a_{kh}^j(\bar{\gamma}(g_{VU}(x)))\frac{\partial v^k}{\partial u^l}P_m^l(u)\frac{\partial u^m}{\partial v^i}$$
$$-\frac{\partial v^j}{\partial u^k}\Lambda_{lm}^k(u)\frac{\partial u^l}{\partial v^i}\frac{\partial u^m}{\partial v^h}.$$

By means of (8.10) and (9.3), the first term of the right hand side can be written as

$$\frac{\partial^2 v^j}{\partial u^t\partial u^s}\frac{\partial u^s}{\partial v^k}\frac{\partial u^t}{\partial v^h}\frac{\partial v^k}{\partial u^l}P_m^l(u)\frac{\partial u^m}{\partial v^i}=\frac{\partial^2 v^j}{\partial u^t\partial u^l}\frac{\partial u^t}{\partial v^h}P_m^l(u)\frac{\partial u^m}{\partial v^i},$$

hence the equation goes over into

$$\Lambda_{ih}^j(v)=-a_{ih}^j(\bar{\gamma}(g_{VU}(x)\bar{\beta}))$$
$$=\frac{\partial v^j}{\partial u^k}\Lambda_{lm}^k(u)\frac{\partial u^l}{\partial v^i}\frac{\partial u^m}{\partial v^h}-\frac{\partial^2 v^j}{\partial u^m\partial u^k}P_l^k(u)\frac{\partial u^l}{\partial v^i}\frac{\partial u^m}{\partial v^h}. \tag{10.13}$$

By virtue of (2.26) and (2.28), the equations show that P_i^j, $\Gamma_{ih}^j=\Lambda_{ih}^j+$

$\frac{\partial P_i^j}{\partial u^h}$ are the components of a general connection Γ with respect to local coordinates u^i.

For this connection Γ, we make h_U for each coordinate neighborhood (U, u^i). Then, for the point $b=(\partial u_i)$, we have $\beta=1$. Accordingly putting $\bar{b}=\rho(b)=(u^i, \bar{\beta})$, we have $\bar{\gamma}(\bar{\beta})=\bar{\beta}$. Hence, by (10. 11), we get

$$e_i(\bar{b}) = \partial u_i$$
$$e_{ih}(\bar{b}) = \partial u_j a^j_{ih}(\bar{\beta}) + \partial^2 u_{jk} p^j_i(\bar{\beta}) a^k_h(\bar{\beta})$$
$$= \partial^2 u_{jk} P^j_i \delta^k_h - \partial u_j \Lambda^j_{ih}.$$

Accordingly we get

$$\rho(b) = h_U(x), \quad x = \pi(b).$$

Since ρ and ρ_Γ satisfy (10. 10), it must be

$$\rho = \rho_\Gamma$$

The proof is finished.

§ 11. Some mappings on $\mathfrak{B}(\mathfrak{X})$ and $\tilde{\mathfrak{B}}^2(\mathfrak{X})$.

For any $P \in \Psi(T(\mathfrak{X}) \otimes T^*(\mathfrak{X}))$, we define a mapping

$$\tilde{P}: \tilde{\mathfrak{B}}^2(\mathfrak{X}) \to R_n = L(R^n, R^n),$$

where R_n denotes the space of endomorphisms of R^n, as follows. For any $b \in \tilde{\mathfrak{B}}^2(\mathfrak{X})$,

$$e_i(b) = \partial u_j a^j_i(\beta), \quad e_{ih}(b) = \partial u_j a^j_{ih}(\beta) + \partial^2 u_{jk} p^j_i(\beta) a^k_h(\beta),$$

we put

$$\tilde{P}(b) = ((b^\lambda_i(b) P^i_j(u) a^j_\mu(\beta))) = ((\tilde{P}^\lambda_\mu(b))). \tag{11. 1}$$

Now ι be the imbedding of R_n into $\tilde{\mathfrak{L}}^2_n$ defined by

$$\iota(p^j_i) = (\delta^j_i, 0, p^j_i). \tag{11. 2}$$

Combining $\tilde{P}$ and ι, we define

$$P_\iota = \iota \cdot \tilde{P}: \tilde{\mathfrak{B}}^2(\mathfrak{X}) \to \tilde{\mathfrak{L}}^2_n \tag{11. 3}$$

and furthermore a mapping $\bar{P}: \tilde{\mathfrak{B}}^2(\mathfrak{X}) \to \tilde{\mathfrak{B}}^2(\mathfrak{X})$ by

$$\bar{P}(b) = \mathrm{r}(P_\iota(b))(b) = b P_\iota(b). \tag{11. 4}$$

Clearly, the mapping $\bar{P}$ maps each fibre of $\tilde{\mathfrak{B}}^2(\mathfrak{X})$ into itself, that is a mapping of $\tilde{\mathfrak{B}}^2(\mathfrak{X})$ covering the identity mapping of $\mathfrak{X}$.

Theorem 11. 1. *$\bar{P}$ is not a bundle homomorphism of $\tilde{\mathfrak{B}}^2(\mathfrak{X})$, but it is commutative with the right translation $\mathrm{r}(\alpha)$ of $\tilde{\mathfrak{B}}^2(\mathfrak{X})$ for any $\alpha \in L^1_n$. $\bar{P}$ is commutative with $\mathrm{r}(\alpha)$, for any $\alpha \in \mathfrak{L}^2_n$, if and only if $P = I$.*

Proof. For any $\alpha \in \mathfrak{L}^2_m$, $b=(u^i, \beta)$, $\beta \in \tilde{\mathfrak{L}}^2_n$, we have

$$\bar{P}(b\alpha) = b\alpha\, P_\iota(b\alpha),$$
$$P_\iota(b\alpha) = (\delta^\lambda_\mu,\ 0,\ \tilde{P}^\lambda_\mu(b\alpha)),$$
$$\tilde{P}^\lambda_\mu(b\alpha) = b^\lambda_i(\beta\alpha) P^i_j a^j_\mu(\beta\alpha) = b^\lambda_\rho(\alpha)\, \tilde{P}^\rho_\sigma(b)\, a^\sigma_\mu(\alpha),$$

and hence

$$\alpha P_\iota(b\alpha) = (a^\lambda_\mu(\alpha),\ a^\lambda_{\mu\nu}(\alpha),\ p^\lambda_\mu(\alpha))(\delta^\lambda_\mu,\ 0,\ b^\lambda_\rho(\alpha)\tilde{P}^\rho_\sigma(b)\, a^\sigma_\mu(\alpha))$$
$$= (a^\lambda_\mu(\alpha),\ a^\lambda_{\tau\nu}(\alpha) b^\tau_\rho(\alpha)\tilde{P}^\rho_\sigma(b)\, a^\sigma_\mu(\alpha),\ p^\lambda_\tau(\alpha) b^\tau_\rho(\alpha)\tilde{P}^\rho_\sigma(b)\, a^\sigma_\mu(\alpha)).$$

On the other hand, we have

$$\bar{P}(b)\alpha = b P_\iota(b)\alpha$$
$$P_\iota(b)\alpha = (\delta^\lambda_\mu,\ 0,\ \tilde{P}^\lambda_\mu(b))(a^\lambda_\mu(\alpha),\ a^\lambda_{\mu\nu}(\alpha),\ p^\lambda_\mu(\alpha))$$
$$= (a^\lambda_\mu(\alpha),\ a^\lambda_{\mu\nu}(\alpha),\ \tilde{P}^\lambda_\rho(b)\, p^\rho_\mu(\alpha)).$$

Hence, in general, we have

$$\alpha P_\iota(b\alpha) \neq P_\iota(b)\alpha$$

that is

$$\bar{P}\cdot r(\alpha) \neq r(\alpha)\cdot \bar{P}. \tag{11.5}$$

This shows that $\bar{P}$ is not a bundle homomorphism of $\tilde{\mathfrak{B}}^2(\mathfrak{X})$.
Especially, if $\alpha \in L^1_n$, then we get

$$\alpha P_\iota(b\alpha) = P_\iota(b)\alpha,$$

since $a^\lambda_{\mu\nu}(\alpha)=0$, $p^\lambda_\mu(\alpha)=a^\lambda_\mu(\alpha)$. Hence

$$\bar{P}\cdot \mathrm{r}(\alpha) = \mathrm{r}(\alpha)\cdot \bar{P},\quad \alpha \in L^1_n. \tag{11.6}$$

Nextly, if $\alpha \in \mathfrak{L}^2_n$, we have

$$\alpha P_\iota(b\alpha) = (a^\lambda_\mu(\alpha),\ a^\lambda_{\tau\nu}(\alpha) b^\tau_\rho(\alpha)\tilde{P}^\rho_\sigma(b)\, a^\sigma_\mu(\alpha),\ \tilde{P}^\lambda_\rho(b)\, a^\rho_\mu(\alpha)),$$
$$P_\iota(b)\alpha = (a^\lambda_\mu(\alpha),\ a^\lambda_{\mu\nu}(\alpha),\ \tilde{P}^\lambda_\rho(b)\, a^\rho_\mu(\alpha)).$$

Hence, in order that $\alpha P_\iota(b\alpha) = P_\iota(b)\alpha$, it is necessary and sufficient $\tilde{P}^\lambda_\mu(b) = \delta^\lambda_\mu$. The proof is finished.

Now, for another $R \in \Psi(T(\mathfrak{X})\otimes T^*(\mathfrak{X}))$, we make $\tilde{R}$, R_ι and $\bar{R}$ as P. By means of (11.1), (11.2), (11.3), we have

$$\bar{P}(b) = b P_\iota(b),$$
$$\bar{R}(\bar{P}(b)) = \bar{R}(bP_\iota(b)) = bP_\iota(b) R_\iota(bP_\iota(b)),$$
$$P_\iota(b) R_\iota(bP_\iota(b)) = (\delta^\lambda_\mu,\ 0,\ \tilde{P}^\lambda_\rho(b)\tilde{R}^\rho_\mu(bP_\iota(\mathrm{b}))),$$

and

$$\tilde{R}^\lambda_\rho(b)\tilde{R}^\rho_\mu(bP_\iota(\mathrm{b}))$$
$$= \tilde{R}^\lambda_\rho(b)\, b^\rho_\sigma(P_\iota(b))\tilde{R}^\sigma_\tau(b)\, a^\tau_\mu(P_\iota(b))$$
$$= \tilde{P}^\lambda_\rho(b)\,\delta^\rho_\sigma\, \tilde{R}^\sigma_\tau(b)\,\delta^\tau_\mu = \tilde{P}^\lambda_\rho(b)\tilde{R}^\rho_\mu(b) = \widetilde{PR}^\lambda_\mu(b).$$

Hence, we have

$$\bar{R}(\bar{P}(b)) = b(PR)_{\iota}(b) = \overline{PR}(b).$$

Thus we have proved

Theorem 11.2. *For any P, $R \in \Psi(T(\mathfrak{X}) \otimes T^*(\mathfrak{X}))$, the isomorphic equation*

$$\bar{R} \cdot \bar{P} = \overline{PR} \tag{11.7}$$

holds good. If P is an isomorphism of $T(\mathfrak{X})$, then

$$\overline{P^{-1}} = \bar{P}^{-1}.$$

Now, let $\hat{P}: \mathfrak{B}(\mathfrak{X}) \to R_n$ be the mapping defined by

$$\hat{P}(b) = (b_i^{\lambda}(\beta) P_j^i(u) a_{\mu}^j(\beta)), \tag{11.8}$$

where $b = (\partial u_i a_j^i(\beta))$. Then

$$\tilde{P} = \tilde{\sigma}^* \hat{P} = \hat{P} \cdot \tilde{\sigma}. \tag{11.9}$$

If P is an isomorphism of $T(\mathfrak{X})$, we can define a mapping P_r : $\mathfrak{B}(\mathfrak{X}) \to \mathfrak{B}(\mathfrak{X})$ by

$$P_r(b) = \mathrm{r}(\hat{P}(b))(b) = b\,\hat{P}(b). \tag{11.10}$$

Lemma 11. 3. *For any $\alpha \in L_n^1$, we have*

$$\hat{P} \cdot \mathrm{r}(\alpha) = \mathrm{adj}(\alpha^{-1}) \cdot \hat{P} \tag{11.11}$$

Proof. From the definition (11.8) of $\hat{P}$, we get

$$\hat{P}(b\alpha) = (b_{\rho}^{\lambda}(\alpha)\, \hat{P}_{\nu}^{\rho}(b)\, a_{\mu}^{\nu}(\alpha))$$
$$= \alpha^{-1} \hat{P}(b)\alpha.$$

Therem 11.4. *P_r for any $P \in \Psi(T(\mathfrak{X}) \otimes T^*(\mathfrak{X}))$, which is an isomorphism of $T(\mathfrak{X})$, is a bundle mapping of $\mathfrak{B}(\mathfrak{X})$ and P_r is the identity mapping, when $P = I$.*

Proof. For any $\alpha \in L_n^1$, by (11.11) we get

$$\begin{aligned} P_r(\mathrm{r}(\alpha)(b)) &= P_r(b\alpha) \\ &= b\alpha\, \hat{P}(b\alpha) = b\, \hat{P}(b)\alpha \\ &= P_r(b)\alpha = \mathrm{r}(\alpha)(P_r(\mathrm{b})), \end{aligned}$$

that is

$$P_r \cdot \mathrm{r}(\alpha) = \mathrm{r}(\alpha) \cdot P_r. \tag{11.12}$$

Theorem 11.5. *For any P, $R \in \Psi(T(\mathfrak{X}) \otimes T^*(\mathfrak{X}))$, which are isomorphisms of $T(\mathfrak{X})$, the isomorphic equation*

$$R_r \cdot P_r = (RP)_r \tag{11.13}$$

holds good.

Proof. By (11.13), we get

$$R_r(P_r(b)) \doteq R_r(b\,\hat{P}(b))$$
$$\doteq R_r(b)\,\hat{P}(b) \doteq b\,\hat{R}(b)\,\hat{P}(b)$$
$$= b\,\widehat{(RP)}(b),$$

since we can easily prove that

$$\hat{R}(b)\,\hat{P}(b) = \widehat{(RP)}(b). \tag{11.14}$$

Corollary 11.6. *For any isomorphism P of $T(\mathfrak{X})$, we have*

$$(P_r)^{-1} = Q_r \tag{11.15}$$

where $Q = P^{-1}$.

Lastly, for any isomorphism P of $T(\mathfrak{X})$, we define a mapping $\overline{P}_r$: $\tilde{\mathfrak{B}}^2(\mathfrak{X}) \to \tilde{\mathfrak{B}}^2(\mathfrak{X})$ by

$$\overline{P}_r(b) = b\widetilde{P}(b), \tag{11.16}$$

where we regard as

$$\widetilde{P}(b) \in L_n^1 \subset \mathfrak{L}_n^2 \subset \tilde{\mathfrak{L}}_n^2.$$

Lemma 11.7. *For any isomorphism P of $T(\mathfrak{X})$ and any $\alpha \in \widetilde{L}_n^2$, we have*

$$\widetilde{P} \cdot \mathrm{r}(\alpha) \doteq \mathrm{adj}(\sigma(\alpha)^{-1}) \cdot \widetilde{P}. \tag{11.17}$$

Proof. It is evident from the equation

$$\widetilde{P}(b\alpha) = (b_\rho^\lambda(\alpha)\,\widetilde{P}_\nu^\rho(b)\,a_\mu^\nu(\alpha)) = \sigma(\alpha)^{-1}\,\widetilde{P}(b)\,\sigma(\alpha).$$

Theorem 11.8. *For any isomorphism P of $T(\mathfrak{X})$ and any $\alpha \in \tilde{\mathfrak{L}}_n^2$, the following equation*

$$\overline{P}_r \cdot \mathrm{r}(\alpha) = \mathrm{r}(\sigma(\alpha)) \cdot \overline{P}_r \cdot \mathrm{r}(\bar{\eta}(\alpha))$$

holds good.

Proof. By means of (11.16) and (11.17), we have

$$\overline{P}_r(b\alpha) = b\alpha\,\widetilde{P}(b\alpha) = b\alpha\sigma(\alpha)^{-1}\,\widetilde{P}(b)\,\sigma(\alpha)$$
$$= b\bar{\eta}(\alpha)\,\widetilde{P}(b)\,\sigma(\alpha).$$

On the other hand, since

$$\sigma(\bar{\eta}(\alpha)) = 1,$$

we have always

$$\widetilde{P}(b) = \widetilde{P}(b\bar{\eta}(\alpha)).$$

Accordingly, we have

$$\overline{P}_r(b\alpha) = b\bar{\eta}(\alpha)\,\widetilde{P}(b\bar{\eta}(\alpha))\,\sigma(\alpha)$$
$$= \overline{P}_r(b\bar{\eta}(\alpha))\,\bar{\sigma}(\alpha).$$

Corollary 11.9. *$\overline{P}_r$ is commutative with $r(\alpha)$, if and only if $\alpha \in L^{\mathrm{II}}_m$.*

Theorem 11.10. *For any two isomorphisms P and R of $T(\mathfrak{X})$, the isomorphic equation*

$$\overline{R}_r \cdot \overline{P}_r = (\overline{RP})_r \tag{11.18}$$

holds good.

Proof. By means of (11.16) and Corollary 11.9, we have

$$\overline{R}_r(\overline{P}_r(b)) = \overline{R}_r(b\widetilde{P}(b)) = \overline{R}_r(b)\widetilde{P}(b)$$
$$= b\widetilde{R}(b)\widetilde{P}(b).$$

Since we have easily

$$(\widetilde{RP}) = \widetilde{R}\ \widetilde{P}. \tag{11.19}$$

Therefore, we get

$$\overline{R}_r(\overline{P}_r(b)) = b\widetilde{RP}(b) = (\overline{RP})_r(b).$$

Corollary 11.11. *For any isomorphism P of $T(\mathfrak{X})$, we have*

$$(\overline{P}_r)^{-1} = \overline{Q}_r,$$

where $Q = P^{-1}$.

§ 12. The relation between the canonical mapping for a regular general connection and the ones for its contravariant part and covariant part .

Let Γ be a regular general connection, then $P = \lambda(\Gamma)$ is an isomorphism of $T(\mathfrak{X})$. By virtue of Theorem 11.2, the mapping P on $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$ is a homeomorphism covering the identity mapping of $\mathfrak{X}$ and

$$\overline{P}^{-1} = \overline{Q}, \tag{12.1}$$

where

$$Q = P^{-1}. \tag{12.2}$$

Theorem 12.1. *Let $\rho = \rho_\Gamma : \mathfrak{B}(\mathfrak{X}) \to \widetilde{\mathfrak{B}}^2(\mathfrak{X})$ be the canonical mapping for a regular general connection Γ defined in § 10 and $\rho'' = \rho_{''\Gamma} : \mathfrak{B}(\mathfrak{X}) \to \mathfrak{B}^2(\mathfrak{X}) \subset \widetilde{\mathfrak{B}}^2(\mathfrak{X})$ be the one for the covariant part $''\Gamma$ of Γ. Then.*

$$\rho = \overline{P} \cdot \rho''. \tag{12.3}$$

Proof. For any $b \in \mathfrak{B}(\mathfrak{X})$, $b = (u^i, \beta)$, $\beta \in L^1_m$, we put

$$\rho(b) = \bar{b} = (u^i, \bar{\beta}),\ \bar{\beta} \in \widetilde{\mathfrak{L}}^2_m.$$

Then we have

$$\bar{\beta} = \bar{\gamma}(\bar{\beta})\sigma(\bar{\beta}) = \bar{\gamma}(\bar{\beta})\beta,$$

since $\sigma(\bar{\beta}) = \beta$. By (10.11), we have

$$\bar{\gamma}(\bar{\beta}) = (\delta^j_i,\ -A^j_{ih},\ P^j_i).$$

Now, we have

$$\overline{Q}(\rho(b)) = \rho(b)\, Q_\iota(\rho(b)),$$

$$Q_\iota(\rho(b)) = (\delta^\lambda_\mu,\ 0,\ \widetilde{Q}^\lambda_\mu(\rho(b)),$$

$$\widetilde{Q}^\lambda_\mu(\rho(b)) = b^\lambda_j(\beta)\, Q^j_i\, a^i_\mu(\beta),$$

and

$$\begin{aligned}\beta\, Q_\iota(\rho(b)) &= (a^\lambda_\mu(\beta),\ 0,\ a^\lambda_\mu(\beta))\,(\delta^\lambda_\mu,\ 0,\ b^\lambda_j(\beta)\, Q^j_i\, a^i_\mu(\beta)) \\ &= (a^\lambda_\mu(\beta),\ 0,\ Q^\lambda_i\, a^i_\mu(\beta)) \\ &= (\delta^j_i,\ 0,\ Q^j_i)\,\beta.\end{aligned}$$

Hence, the last coordinate of $\overline{Q}(\rho(b))$ can be written as

$$\begin{aligned}\bar{\beta}\, Q_\iota(\rho(b)) &= \bar{\gamma}(\bar{\beta})(\delta^j_i,\ 0,\ Q^j_i)\,\beta \\ &= (\delta^j_i,\ -A^j_{ih},\ P^j_i)\,(\delta^j_i,\ 0,\ Q^j_i)\,\beta \\ &= (\delta^j_i,\ -A^j_{ih}\, Q^h_k,\ \delta^j_i)\,\beta,\end{aligned}$$

which is

$$= (\delta^j_i,\ -{}''\Gamma^j_{ik},\ \delta^j_i)\,\beta$$

by (3. 3) and so the last coordinate of $\rho''(b)$. Therefore, we obtain

$$\overline{Q}(\rho(b)) = \rho''(b).$$

By means of (12. 1), we obtain the equation

$$\rho = \overline{P} \cdot \rho''.$$

Theorem 12. 2. *For any regular general connection Γ, we have*

$$\rho \cdot P_r = \overline{P}_r \cdot \rho \tag{12. 4}$$

where $\rho = \rho_\Gamma$, $P = \lambda(\Gamma)$.

Proof. For any $b \in \mathfrak{B}(\mathfrak{X})$, $b = (u^i, \beta)$, $\beta \in L^1_n$, we put

$$\bar{b} = \rho(b) = (u^i, \bar{\beta}),\ \bar{\beta} \in \widetilde{\mathfrak{L}}^2_n.$$

By (11. 10), we have

$$P_r(b) = b\, \hat{P}(b).$$

Since $\hat{P}(b) \in L^1_n$, by (10. 10), we get

$$\rho(P_r(b)) = \rho(b\,\hat{P}(b)) = \rho(b)\,\hat{P}(b).$$

Since $\sigma(\bar{\beta}) = \beta$, it must be

$$\widetilde{P}(\bar{b}) = \hat{P}(b).$$

Hence we have

$$\rho(P_r(b)) = \rho(b)\,\widetilde{P}(\rho(b)) = \overline{P}_r(\rho(b)).$$

Now, $'\Gamma$ be the contravariant part of a regular general connection Γ. $'\Gamma$ is a classical affine connection, hence for any $b\in\widetilde{\mathfrak{B}}^2(\mathfrak{X})$, $b=(u^i,\beta)$, $\beta\in\tilde{\mathfrak{L}}_n^2$, we have the equation

$$\widetilde{P}^\lambda_{\mu;\nu}(b)=b^\lambda_j(\beta)P^j_{i;h}a^i_\mu(\beta)a^h_\nu(\beta), \tag{12.5}$$

where the notation " ; " denotes the covariant derivative with respect to $'\Gamma$ in the right hand side and the one with respect to the induced connection $\tilde{\pi}_2^\sharp\,'\Gamma$ of the induced vector bundle $\tilde{\pi}_2^\sharp T(\mathfrak{X})$, $\tilde{\pi}_2$ being the projecton of $\{\widetilde{\mathfrak{B}}^2(\mathfrak{X}),\mathfrak{X},\tilde{\pi}_2\}$. By means of (3.14), (12.5) can be written as

$$\widetilde{P}^\lambda_{\mu;\nu}(b)=b^\lambda_j(\beta)Q^j_k\delta^k_{i.h}a^i_\mu(\beta)a^h_\nu(\beta)\,. \tag{12.6}$$

Let P_ι': $\widetilde{\mathfrak{B}}^2(\mathfrak{X})\to\tilde{\mathfrak{L}}_n^2$ be the mapping defined by

$$P_\iota'(b)=(\delta^\lambda_\mu,\ \widetilde{P}^\lambda_{\mu;\nu}(b),\ \widetilde{P}^\lambda_\mu(b)) \tag{12.7}$$

and $\bar{P}_1$: $\widetilde{\mathfrak{B}}^2(\mathfrak{X})\to\widetilde{\mathfrak{B}}^2(\mathfrak{X})$ be the mapping defined by

$$\bar{P}_1(b)=b\,P_\iota'(b) \tag{12.8}$$

which are analogous to P_ι and $\bar{P}$. By (12.6), we get easily

$$P_\iota'(b)=(\widetilde{Q}^\lambda_\mu(b),\ 0,\ \widetilde{Q}^\lambda_\mu(b))(\widetilde{P}^\lambda_\mu(b),\ \tilde{\delta}^\lambda_{\mu,\nu}(b),\ \widetilde{M}^\lambda_\mu(b)),$$

where $M=P^2$ and we put

$$\tilde{\delta}^\lambda_{\mu,\nu}(b)=b^\lambda_j(\beta)\delta^j_{i,h}a^i_\mu(\beta)a^h_\nu(\beta). \tag{12.9}$$

Let the mapping

$$\tilde{J}=\tilde{J}_\Gamma:\ \widetilde{\mathfrak{B}}^2(\mathfrak{X})\to\widetilde{\mathfrak{L}}_n^2$$

be defined by

$$\tilde{J}(b)=(\widetilde{P}^\lambda_\mu(b),\ \tilde{\delta}^\lambda_{\mu,\nu}(b),\ \widetilde{M}^\lambda_\mu(b)), \tag{12.10}$$

which is defined by using only Γ and D, that is even though Γ is not regular. Then, the above equation can be written as

$$P_\iota'(b)=\widetilde{Q}(b)\tilde{J}(b). \tag{12.11}$$

Lemma 12.3. *The mapping* $\bar{P}_1$ *has the inverse and the equation*

$$\bar{P}_1^{-1}(b)=b\,P_\iota'(b)^{-1} \tag{12.12}$$

holds good, when Γ *is regular.*

Proof. For any $b\in\widetilde{\mathfrak{B}}^2(\mathfrak{X})$, we put

$$\bar{P}_1^{-1}(b)=b_1=b\alpha.$$

We have

$$b=\bar{P}_1(b\alpha)=b\alpha P_\iota'(b\alpha),$$

hence

$$1 = \alpha P_\iota'(b\alpha)$$
$$= (a^\lambda_\mu(\alpha), a^\lambda_{\mu\nu}(\alpha), p^\lambda_\mu(\alpha))\,(\delta^\lambda_\mu, \widetilde{P}^\lambda_{\mu;\nu}(b\alpha), \widetilde{P}^\lambda_\mu(b\alpha)),$$

that is

$$\delta^\lambda_\mu = a^\lambda_\mu(\alpha),$$
$$0 = a^\lambda_\rho(\alpha)\,\widetilde{P}^\rho_{\mu;\nu}(b\alpha) + a^\lambda_{\rho\nu}(\alpha)\,\widetilde{P}^\rho_\mu(b\alpha),$$
$$\delta^\lambda_\mu = p^\lambda_\rho(\alpha)\,\widetilde{P}^\rho_\mu(b\alpha).$$

Using the first equation, we have

$$\widetilde{P}^\lambda_\mu(b\alpha) = \widetilde{P}^\lambda_\mu(b),\ \widetilde{P}^\lambda_{\mu;\nu}(b\alpha) = \widetilde{P}^\lambda_{\mu;\nu}(b).$$

Hence the second equation of the above yields

$$0 = \widetilde{P}^\lambda_{\mu;\nu}(b) + a^\lambda_{\rho\nu}(\alpha)\,P^\rho_\mu(b),$$

that is

$$a^\lambda_{\mu\nu}(\alpha) = -\,\widetilde{P}^\lambda_{\rho;\nu}(b)\,\widetilde{Q}^\rho_\mu(b)$$
$$= -\,\widetilde{Q}^\lambda_\rho(b)\,{}''\nabla_\nu\,\widetilde{P}^\lambda_\rho(b)$$

by means of (3. 14). From the last equation we get

$$\delta^\lambda_\mu = p^\lambda_\rho(\alpha)\,\widetilde{P}^\rho_\mu(b),$$

hence

$$p^\lambda_\mu(\alpha) = \widetilde{Q}^\lambda_\mu(b).$$

Thus we obtain

$$\alpha = (\delta^\lambda_\mu, -\,\widetilde{P}^\lambda_{\rho;\nu}(b)\,\widetilde{Q}^\rho_\mu(b), \widetilde{Q}^\lambda_\mu(b)) \qquad (12.\,13)$$
$$= (P_\iota'(b))^{-1}.$$

Theorem 12. 4. *For the canonical mapping* $\rho = \rho_\Gamma$ *of a regular general connection* Γ *and the one* $\rho' = \rho_{'\Gamma}: \mathfrak{B}(\mathfrak{X}) \to \mathfrak{B}^2(\mathfrak{X})$ *of the contravariant part* $'\Gamma$ of Γ, *the equation*

$$\rho = \overline{P}_1 \cdot \rho' \qquad (12.\,14)$$

holds good.

Proof. For any $b \in \mathfrak{B}(\mathfrak{X})$, $b = (u^i, \beta)$, we put

$$\bar{b} = \rho(b) = (u^i, \bar{\beta}).$$

We have

$$\bar{\gamma}(\bar{\beta}) = \bar{\beta}\sigma(\bar{\beta})^{-1} = \bar{\beta}\,\beta^{-1} = (\delta^j_i, -\,A^j_{ih}, P^j_i),$$

and by the above lemma

$$\overline{P}_1^{-1}(\bar{b}) = \bar{b}\,P_\iota'(\bar{b})^{-1} = (u^j, \bar{\beta}P_\iota'(b)^{-1}).$$

Therefore, we have

$$\sigma(\bar{\beta}\,P_\iota'(\bar{b})^{-1}) = \beta\sigma(P_\iota'(b)^{-1}) = \beta$$

by (12. 13). Hence we have

$$\bar{\eta}(\bar{\beta}P_\iota'(\bar{b})^{-1}) = \bar{\beta}P_\iota'(\bar{b})^{-1}\beta^{-1}$$
$$= \bar{\eta}(\bar{\beta})\beta P_\iota'(\bar{b})\beta^{-1},$$

and

$$\beta P_\iota'(\bar{b})^{-1}\beta^{-1} = \beta(\delta^\lambda_\mu, -\widetilde{P}^\lambda_{\rho;\nu}\widetilde{Q}^\rho_\mu(b), \widetilde{Q}^\lambda_\mu(b))\beta^{-1}$$
$$= (\delta^j_i, -P^j_{l,h}Q^l_i, Q^j_i).$$

Accordingly, we get

$$\bar{\eta}(\bar{\beta}P_\iota'(\bar{b})^{-1}) = (\delta^j_i, -\Lambda^j_{ih}, P^j_i)(\delta^j_i, -P^j_{l;h}Q^l_i, Q^j_i)$$
$$= (\delta^j_i, -P^j_{l;h}Q^l_i - \Lambda^j_{lh}Q^l_i, \delta^j_i).$$

Since we have

$$P^j_{l;h}Q^l_i + \Lambda^j_{lh}Q^l_i = (P^j_{l;h} + \Lambda^j_{lh})Q^l_i$$
$$= \left\{\frac{\partial P^j_l}{\partial u^h} + {}'\Gamma^j_{mh}P^m_l - P^j_m{}'\Gamma^m_{lh} + \Gamma^j_{lh} - \frac{\partial P^j_l}{\partial u^h}\right\}Q^l_i$$
$$= {}'\Gamma^j_{ih}.$$

Thus, we obtain

$$\bar{\eta}(\bar{\beta}P_\iota'(\bar{b})^{-1}) = (\delta^j_i, -{}'\Gamma^j_{ih}, \delta^j_i), \tag{12. 15}$$

which is the last coordinate of $\rho'(b)$. Hence we have

$$\bar{P}_1^{-1}(\rho(b)) = \rho'(b).$$

Now, from Theorem 12. 1 and Theorem 12. 4, we get

$$\rho = \bar{P}\cdot\rho'' = \bar{P}_1\cdot\rho',$$

hence

$$\rho'' = \bar{P}^{-1}\cdot\bar{P}_1\cdot\rho'. \tag{12. 16}$$

For any point $b \in \widetilde{\mathfrak{B}}^2(\mathfrak{X})$, we have

$$\bar{P}^{-1}(\bar{P}_1(\mathrm{b})) = \bar{P}^{-1}(bP_\iota'(b))$$
$$= \bar{Q}(bP_\iota'(b))$$
$$= bP_\iota'(b)Q_\iota(bP_\iota'(b)).$$

On the other hand, since we have

$$\sigma(P_\iota'(b)) = 1,$$

hence

$$Q_\iota(bP_\iota'(b)) = Q_\iota(b).$$

Accordingly, we have

$$P_\iota'(b)Q_\iota(b) = (\delta^\lambda_\mu, \widetilde{P}^\lambda_{\mu;\nu}(b), \widetilde{P}^\lambda_\mu(b))(\delta^\lambda_\mu, 0, \widetilde{Q}^\lambda_\mu(b))$$
$$= (\delta^\lambda_\mu, \widetilde{P}^\lambda_{\rho;\nu}(b)\widetilde{Q}^\rho_\mu(b), \delta^\lambda_\mu), \tag{12. 17}$$

that is

$$P_\iota' Q_\iota : \widetilde{\mathfrak{B}}^2(\mathfrak{X}) \to \sigma^{-1}(1) \cap \mathfrak{L}_n^2 = \mathfrak{N}_n^2. \tag{12.18}$$

Theorem 12. 5. *Let $'\Gamma$ and $''\Gamma$ be the contravariant part and the covariant part of a regular general connection Γ. Then, the canonical mappings ρ' and ρ'' of $'\Gamma$ and $''\Gamma$ satisfy the equation*

$$\rho'' = \overline{P}^{-1} \cdot \overline{P}_1 \cdot \rho'.$$

The mapping $\overline{P}^{-1} \cdot \overline{P}_1$ on $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$ is written as

$$\overline{P}^{-1}(\overline{P}_1(b)) = b P_\iota'(b) Q_\iota(b)$$

and the mapping $P_\iota' Q_\iota : \widetilde{\mathfrak{B}}^2(\mathfrak{X}) \to \mathfrak{N}_n^2$ is written as (12. 17).

§ 13. The universal general connection.

In [11], the author defined an affine connection for the induced vector bundle $\pi_2^\sharp(T(\mathfrak{X}))$ over $\mathfrak{B}^2(\mathfrak{X})$ which was said the universal affine connection of $\mathfrak{X}$, because the differential forms on $\mathfrak{B}(\mathfrak{X})$ for any classical affine connection of $\mathfrak{X}$ can be induced from the ones of this connection by the canonical mapping of the former defined in § 10. But, for general connections, it has the geometrical meaning through their contravariant parts or covariant parts which are classical affine connections of $\mathfrak{X}$, if and only if they are regular.

In this section, the author will show that this concept can be extended so that for general connections, even though they are not regular, the differential forms on $\mathfrak{B}(\mathfrak{X})$ for them can be induced from the ones on $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$ of a connection for the induced vector bundle $\tilde{\pi}_2^\sharp(T(\mathfrak{X}))$ over $\widetilde{\mathfrak{B}}_2(\mathfrak{X})$ by their canonical mappings which were defined for these connections too.

We call anew the induced vector bundle

$$\mathfrak{U}(\mathfrak{X}) = \tilde{\pi}_2^\sharp(T(\mathfrak{X})) \tag{13.1}$$

over $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$, which is induced from the tangent bundle $T(\mathfrak{X})$ by the projection $\tilde{\pi}_2$ of the general principal bundle of order 2 of $\mathfrak{X}$, $\{\widetilde{\mathfrak{B}}^2(\mathfrak{X}), \mathfrak{X}, \tilde{\pi}_2\}$, *the universal vector bundle of* $\mathfrak{X}$.

We may consider as

$$\mathfrak{U}(\mathfrak{X}) \subset \tilde{\pi}_2^\sharp(\mathfrak{T}^2(\mathfrak{X})) \tag{13.2}$$

since $T(\mathfrak{X}) \subset \mathfrak{T}^2(\mathfrak{X})$.

Let

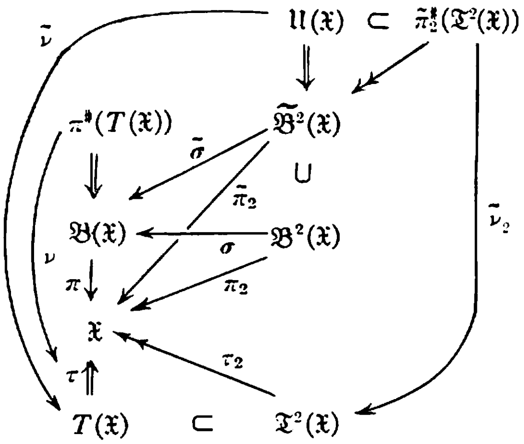
$$\tilde{\nu} : \mathfrak{U}(\mathfrak{X}) \to T(\mathfrak{X}),$$
$$\tilde{\iota}_2 : \tilde{\pi}^\sharp \mathfrak{T}^2(\mathfrak{X})) \to \mathfrak{T}^2(\mathfrak{X})$$

be the induced bundle mappings which are determined by $\tilde{\pi}_2$. It is clear

$$\tilde{\nu}_2 \mid \mathfrak{U}(\mathfrak{X}) = \tilde{\nu}.$$

The vector bunble $\tilde{\pi}_2^{\#}(\mathfrak{T}^2(\mathfrak{X}))$ over $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$ has $n+n^2$ natural cross-sections

$$\tilde{e}_\lambda,\ \tilde{e}_{\lambda\mu}:\ \widetilde{\mathfrak{B}}^2(\mathfrak{X}) \to \tilde{\pi}_2^{\#}(\mathfrak{T}^2(\mathfrak{X}))$$



Diag. 2

such that

$$\tilde{\nu}_2 \cdot \tilde{e}_\lambda(b) = e_\lambda(b), \qquad \tilde{\nu}_2 \cdot \tilde{e}_{\lambda\mu}(b) = e_{\lambda\mu}(b), \tag{13.3}$$

where we may regard as

$$\tilde{e}_\lambda:\ \widetilde{\mathfrak{B}}^2(\mathfrak{X}) \to \mathfrak{U}(\mathfrak{X}). \tag{13.4}$$

Since $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$ is not a principal bundle, for a point b, $b=(u^i, \beta)$, $\beta \in \tilde{\mathfrak{L}}_n^2$, $\tilde{e}_{\lambda\mu}(b)$ out of $\{\tilde{e}_\lambda(b), \tilde{e}_{\lambda\mu}(b)\}$ are mutually linearly dependent mod $\{\tilde{e}_\lambda(b)\}$, if $\beta \in \tilde{\mathfrak{L}}_n^2 - \mathfrak{L}_n^2$.

Now, we denote the natural cross-sections of the vector bundle $\pi^{\#}(T(\mathfrak{X}))$ over $\mathfrak{B}(\mathfrak{X})$ induced from the tangent bundle $T(\mathfrak{X})$ by the projection π of the associated principal bundle of order 1 of $\mathfrak{X}$, $\{\mathfrak{B}(\mathfrak{X}), \mathfrak{X}, \pi\}$, by

$$e_\lambda:\ \mathfrak{B}(\mathfrak{X}) \to \pi^{\#}(T(\mathfrak{X}))$$

and the induced bundle mapping by

$$\nu:\ \pi^{\#}(T(\mathfrak{X})) \to T(\mathfrak{X}).$$

Then, we have

$$\nu \cdot e_\lambda(\mathrm{b}) = e_\lambda(b). \tag{13.5}$$

In § 6, we introduced the connection forms on $\mathfrak{B}(\mathfrak{X})$ for any general connection Γ, which are written in terms of local coordinates (u^j, β), $\beta \in L_n^1$, as

$$\theta^{\mu}_{\lambda} = b^{\mu}_{j}(\beta)\{P^{j}_{i}(u)da^{i}_{\lambda}(\beta) + \Gamma^{j}_{ih}(u)a^{i}_{\lambda}(\beta)du^{h}\}$$
$$= b^{\mu}_{j}(b)\{P^{j}_{i}(u)da^{i}_{\lambda}(\beta) + A^{j}_{ih}(u)a^{i}_{\lambda}(\beta)du^{h} + dP^{j}_{i}(u)a^{i}_{\lambda}(\beta)\}$$
$$= b^{\mu}_{j}(\beta)\{d(P^{j}_{i}(u)a^{i}_{\lambda}(\beta)) + A^{j}_{ih}(u)a^{i}_{\lambda}(\beta)du^{h}\}.$$

Let $\rho = \rho_{\Gamma} : \mathfrak{B}(\mathfrak{X}) \rightarrow \widetilde{\mathfrak{B}}^{2}(\mathfrak{X})$ be the canonical mapping of Γ and put

$$\bar{b} = \rho(b) = (u^{j}, \bar{\beta}), \quad \bar{\beta} \in \tilde{\mathfrak{L}}^{2}_{n}.$$

By means of (8. 10) and (10. 11), the right hand side of the above equation can be written as

$$\theta^{\mu}_{\lambda} = b^{\mu}_{j}(\bar{\beta})\{d(p^{j}_{i}(\bar{\gamma}(\bar{\beta}))a^{i}_{\lambda}(\bar{\beta})) - a^{j}_{ih}(\bar{\gamma}(\bar{\beta}))a^{i}_{\lambda}(\bar{\beta})du^{h}\}$$
$$= b^{\mu}_{j}(\bar{\beta})\{d(p^{j}_{\lambda}(\bar{\beta})) - a^{j}_{\lambda\nu}(\bar{\beta})b^{\nu}_{h}(\bar{\beta})du^{h}\}.$$

Hence, θ^{μ}_{λ} may be considered as they are induced from the n^{2} differential forms on $\widetilde{\mathfrak{B}}^{2}(\mathfrak{X})$:

$$\tilde{\theta}^{\mu}_{\lambda} = b^{\mu}_{j}(dp^{j}_{\lambda} - a^{j}_{\lambda\nu}\theta^{\nu}) \tag{13. 6}$$

by the dual mapping ρ^{*} of ρ, where θ^{ν} are the differential forms on $\mathfrak{B}(\mathfrak{X})$ defined by (6. 8) and we regard them as differential forms on $\widetilde{\mathfrak{B}}^{2}(\mathfrak{X})$ by transforming them under $\tilde{\sigma}^{*}$.

Lemma 13. 1. *$\tilde{\theta}^{\mu}_{\lambda}$ are defined on the whole space $\widetilde{\mathfrak{B}}^{2}(\mathfrak{X})$ and do not depend on the choice of local coordinates.*

Proof. Let (U, u^{i}) and (V, v^{i}) be any two coordinate neighborhoods of $\mathfrak{X}$, such that $U \cap V \neq \emptyset$. Let (u^{i}, β) and (v^{i}, γ) be the corresponding local coordinates of $\widetilde{\mathfrak{B}}^{2}(\mathfrak{X})$ on $\tilde{\pi}_{2}^{-1}(U)$ and $\tilde{\pi}_{2}^{-1}(V)$. In $\tilde{\pi}_{2}^{-1}(U \cap V)$, we have

$$\gamma = g_{VU}\beta$$

by (9. 4). Accodingly we have

$$b^{\mu}_{j}(\gamma)\{d(p^{j}_{\lambda}(\gamma)) - a^{j}_{\lambda\mu}(\gamma)b^{\mu}_{k}(\gamma)dv^{k}\}$$
$$= b^{\mu}_{j}(g_{VU}\beta)[dp^{j}_{\lambda}(g_{VU}\beta) - a^{j}_{\lambda\mu}(g_{VU}\beta)b^{\mu}_{k}(g_{VU}\beta)dv^{k}]$$
$$= b^{\mu}_{l}(\beta)a^{l}_{j}(g_{UV})[d(a^{j}_{i}(g_{VU})p^{i}_{\lambda}(\beta)) -$$
$$- \{a^{j}_{i}(g_{VU})a^{i}_{\lambda\mu}(\beta) + a^{j}_{ih}(g_{VU})p^{i}_{\lambda}(\beta)a^{h}_{\mu}(\beta)\}b^{\mu}_{m}(\beta)a^{m}_{k}(g_{UV})dv^{k}]$$
$$= b^{\mu}_{l}(\beta)a^{l}_{j}(g_{UV})[a^{j}_{i}(g_{VU})d(p^{i}_{\lambda}(\beta)) + a^{j}_{ih}(g_{VU})p^{i}_{\lambda}(\beta)du^{h}$$
$$- \{a^{j}_{i}(g_{VU})a^{i}_{\lambda\mu}(\beta) + a^{j}_{ih}(g_{VU})p^{i}_{\lambda}(\beta)a^{h}_{\mu}(\beta)\}b^{\mu}_{m}(\beta)du^{m}]$$
$$= b^{\mu}_{i}(\beta)\{d(p^{i}_{\lambda}(\beta)) - a^{i}_{\lambda\mu}(\beta)b^{\mu}_{m}(\beta)du^{m}\},$$

which shows that the forms $\tilde{\theta}^{\mu}_{\lambda}$ are determined on $\widetilde{\mathfrak{B}}^{2}(\mathfrak{X})$ without depending on the choice of local coordinates.

Analogously, corresponding to the differential forms

$$\pi^{\mu}_{\lambda} = \theta^{\mu}_{\lambda} - d\widetilde{P}^{\mu}_{\lambda}$$

on $\mathfrak{B}(\mathfrak{X})$, we shall determine other n differential forms on $\widetilde{\mathfrak{B}}^{2}(\mathfrak{X})$. Putting

$\mathfrak{B}(\mathfrak{X}) \ni b = (u^j, \beta)$, $\beta \in L^1_{n}$, and $\bar{b} = \rho(b) = (u^j, \bar{\beta})$, $\bar{\beta} \in \tilde{\mathfrak{L}}^2_n$, we have

$$\begin{aligned} d\tilde{P}^\mu_\lambda &= d(b^\mu_j(\beta) P^j_i(u) a^i_\lambda(\beta)) \\ &= d(b^\mu_j(\beta) p^j_i(\bar{\gamma}(\bar{\beta})) a^i_\lambda(\beta)) \\ &= d(b^\mu_j(\bar{\beta}) p^j_\lambda(\bar{\beta})). \end{aligned}$$

Hence, if we introduce n differential forms on $\tilde{\mathfrak{B}}^2(\mathfrak{X})$ as follows:

$$\begin{aligned} \tilde{\pi}^\mu_\lambda &= \tilde{\theta}^\mu_\lambda - d(b^\mu_j p^j_\lambda) \\ &= b^\mu_j(dp^j_\lambda - a^j_{\lambda\nu}\theta^\nu) - d(b^\mu_j p^j_\lambda) \\ &= -(db^\mu_j p^j_\lambda + b^\mu_j a^j_{\lambda\nu}\theta^\nu), \end{aligned}$$

that is

$$\tilde{\pi}^\mu_\lambda = -(db^\mu_j p^j_\lambda + b^\mu_j a^j_{\lambda\nu}\theta^\nu), \tag{13. 7}$$

then the differential forms π^μ_λ for the connection Γ are induced from $\tilde{\pi}^\mu_\lambda$ by the canonical mapping ρ.

Lemma 13. 2. *The functions*

$$\varphi^\mu_\lambda = b^\mu_j(\bar{\beta}) p^j_\lambda(\bar{\beta}) = p^\mu_\lambda(\gamma(\bar{\beta})) \tag{13. 8}$$

are defined on the whole space $\tilde{\mathfrak{B}}^2(\mathfrak{X})$ without depending on the choice of local coordinates $(u^j, \bar{\beta})$ and they are the components of a homomorphism of the universal vector bundle $\mathfrak{U}(\mathfrak{X})$.

Proof. Using the notations in the proof of Lemma 13. 1, we have

$$\begin{aligned} b^\mu_j(\gamma) p^j_\lambda(\gamma) &= b^\mu_j(g_{VU}\beta) p^j_\lambda(g_{VU}\beta) \\ &= b^\mu_k(\beta)\frac{\partial u^k}{\partial v^j}\frac{\partial v^j}{\partial u^h} p^h_\lambda(\beta) = b^\mu_k(\beta) p^k_\lambda(\beta). \end{aligned}$$

By means of Lemma 13. 2, $\tilde{\pi}^\mu_\lambda$ can be written as

$$\tilde{\pi}^\mu_\lambda = \tilde{\theta}^\mu_\lambda - d\varphi^\mu_\lambda. \tag{13. 9}$$

Now, we define a natural differential operator

$$d: \Psi(\tilde{\pi}^\sharp_2(T(\mathfrak{X})) \to \Psi(\tilde{\pi}^\sharp_2(T^2(\mathfrak{X})) \otimes T^*(\tilde{\mathfrak{B}}^2(\mathfrak{X})))$$

by

$$d\tilde{e}_\lambda = \tilde{e}_\mu \otimes \tilde{\theta}^\mu_\lambda + \tilde{e}_{\lambda\mu} \otimes \theta^\mu. \tag{13. 10}$$

On the other hand, let $\tilde{e}^\mu$ be the natural cross-sections of the dual vector bundle

$$\mathfrak{U}^*(\mathfrak{X}) = \tilde{\pi}^\sharp_2(T^*(\mathfrak{X}))$$

of the universal vector bundle $\mathfrak{U}(\mathfrak{X}) = \tilde{\pi}^\sharp_2(T(\mathfrak{X}))$. Let $\tilde{\nu}_1: \mathfrak{U}^*(\mathfrak{X}) \to T^*(\mathfrak{X})$ be the induced bundle mapping, then

$$\tilde{\nu}_1(\tilde{e}^\mu(b)) = e^\mu(b), \tag{13. 11}$$

where we put

$$\mathrm{e}^{\mu}(b) = b^{\mu}_{j}(\beta)\, du^{j}(x), \tag{13. 12}$$

and $\widetilde{\mathfrak{B}}^2(\mathfrak{X}) \ni b = (u^j, \beta)$, $\beta \in \tilde{\mathfrak{L}}^2_n$, $x = \tilde{\pi}_2(b)$.

Now, we define a general connection of the universal vector bundle $\mathfrak{U}(\mathfrak{X})$ of $\mathfrak{X}$ by the equations

$$D\tilde{\mathrm{e}}_{\lambda} = \tilde{\mathrm{e}}_{\mu} \otimes \tilde{\theta}^{\mu}_{\lambda}, \tag{13. 13}$$

$$D\tilde{\mathrm{e}}^{\mu} = -\tilde{\mathrm{e}}^{\lambda} \otimes \tilde{\pi}^{\mu}_{\lambda}. \tag{13. 14}$$

We call this connection of $\mathfrak{U}(\mathfrak{X})$ *the universal general connection of* $\mathfrak{X}$, by virtue of the following

Theorem 13. 3. *The differential forms* θ^{μ}_{λ}, π^{μ}_{λ} *defined on* $\mathfrak{B}(\mathfrak{X})$ *by* (6. 1) *and* (6. 19) *for a general connection* Γ *of* $\mathfrak{X}$ *are induced from the differential forms* $\tilde{\theta}^{\mu}_{\lambda}$, $\tilde{\pi}^{\mu}_{\lambda}$ *on* $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$ *defined by* (13. 6) *and* (13. 9) *for the universal general connection of* $\mathfrak{X}$ *by the canonical mapping* $\rho = \rho_{\Gamma}$: $\mathfrak{B}(\mathfrak{X}) \to \widetilde{\mathfrak{B}}^2(\mathfrak{X})$ *of this connection* Γ *defined in* § 10.

This theorem is evident from the above mentioned facts.

Lemma 13. 4. *For any right translation* $\mathrm{r}(\alpha)$, $\alpha \in \tilde{\mathfrak{L}}^2_n$, *of* $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$, *the following equations hold good* :

$$\mathrm{r}(\alpha)^* \theta^{\mu} = b^{\mu}_{\lambda}(\alpha)\theta^{\lambda}, \tag{13. 15}$$

$$\begin{aligned} \mathrm{r}(\alpha)^* \tilde{\theta}^{\mu}_{\lambda} &= b^{\mu}_{\rho}(\alpha)\tilde{\theta}^{\rho}_{\sigma} p^{\sigma}_{\lambda}(\alpha) - a^{\mu}_{\lambda\nu}(\gamma(\alpha))\, \mathrm{r}(\alpha)^* \theta^{\nu} \\ &= b^{\mu}_{\rho}(\alpha)\{\tilde{\theta}^{\rho}_{\sigma} p^{\sigma}_{\lambda}(\alpha) - a^{\rho}_{\sigma\nu}(\gamma(\alpha))\theta^{\nu} a^{\sigma}_{\lambda}(\alpha)\}, \end{aligned} \tag{13. 16}$$

$$\mathrm{r}(\alpha)^* \varphi^{\mu}_{\lambda} = b^{\mu}_{\rho}(\alpha)\varphi^{\rho}_{\sigma} p^{\sigma}_{\lambda}(\alpha) \tag{13. 17}$$

and

$$\mathrm{r}(\alpha)^* \tilde{\pi}^{\mu}_{\lambda} = b^{\mu}_{\rho}(\alpha)\tilde{\pi}^{\rho}_{\sigma} p^{\sigma}_{\lambda}(\alpha) - a^{\mu}_{\lambda\nu}(\gamma(\alpha))\, \mathrm{r}(\alpha)^* \theta^{\nu}. \tag{13. 18}$$

Proof. (13. 15) and (13. 17) are evident from their definitions (6. 8) and (13. 8). By (13. 6), we have

$$\begin{aligned} \mathrm{r}(\alpha)^* \tilde{\theta}^{\mu}_{\lambda} &= b^{\mu}_{j}(\beta\alpha)\,[d(p^{j}_{\lambda}(\beta\alpha)) - a^{j}_{\lambda\nu}(\beta\alpha)\, \mathrm{r}(\alpha)^* \theta^{\nu}] \\ &= b^{\mu}_{\rho}(\alpha) b^{\rho}_{j}(\beta)\,[d(p^{j}_{\sigma}(\beta)) p^{\sigma}_{\lambda}(\alpha) \\ &\quad - \{a^{j}_{\sigma}(\beta) a^{\sigma}_{\lambda\nu}(\alpha) + a^{j}_{\sigma\tau}(\beta) p^{\sigma}_{\nu}(\alpha) a^{\tau}_{\nu}(\alpha)\}\, \mathrm{r}(\alpha)^* \theta^{\nu}] \\ &= b^{\mu}_{\rho}(\alpha) b^{\rho}_{j}(\beta)\,[d(p^{j}_{\sigma}(\beta)) - a^{j}_{\sigma\tau}(\beta)\theta^{\tau}]\, p^{\sigma}_{\lambda}(\alpha) \\ &\quad - b^{\mu}_{\rho}(\alpha) a^{\rho}_{\lambda\nu}(\alpha)\, \mathrm{r}(\alpha)^* \theta^{\nu} \\ &= b^{\mu}_{\rho}(\alpha)\widetilde{\theta^{\rho}_{\sigma}} p^{\sigma}_{\lambda}(\alpha) - a^{\mu}_{\lambda\nu}(\gamma(\alpha))\, \mathrm{r}(\alpha)^* \theta^{\nu}. \end{aligned}$$

(13. 18) follows immediately from (13. 9), (13. 16) and (13. 17).

In the last place, we define mappings on $\mathfrak{U}(\mathfrak{X})$ and $\mathfrak{U}^*(\mathfrak{X})$ such that they cover the right translation $\mathrm{r}(\alpha)$, $\alpha \in \tilde{\mathfrak{L}}^2_n$, on $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$, by

$$\tilde{\mathrm{r}}(\alpha)(\tilde{\mathrm{e}}_{\lambda}(b)) = \tilde{\mathrm{e}}_{\mu}(b\alpha) b^{\mu}_{\lambda}(\alpha), \tag{13. 19}$$

$$\tilde{\mathrm{r}}(\alpha)(\tilde{\mathrm{e}}^{\mu}(b)) = \tilde{\mathrm{e}}^{\lambda}(b\alpha) a^{\mu}_{\lambda}(\alpha). \tag{13. 20}$$

Lemma 13.5. *For the mapping* $\tilde{\mathfrak{r}}(\alpha)$, $\alpha\in\mathfrak{L}^{\tilde{n}}_{n}$, *we have*

$$\tilde{\nu}\cdot\tilde{\mathfrak{r}}(\alpha)=\tilde{\nu} \quad\text{and}\quad \tilde{\nu}_1\cdot\tilde{\mathfrak{r}}(\alpha)=\tilde{\nu}_1. \tag{13.21}$$

Proof. By means of (13.3), we have

$$\begin{aligned}\tilde{\nu}(\tilde{\mathfrak{r}}(\alpha)(\tilde{\mathfrak{e}}_\lambda(b)))&=\tilde{\nu}(\tilde{\mathfrak{e}}_\mu(b\alpha))b^\mu_\lambda(\alpha)\\&=\mathfrak{e}_\mu(b\alpha)b^\mu_\lambda(\alpha)=\mathfrak{e}_\lambda(b)\\&=\tilde{\nu}(\tilde{\mathfrak{e}}_\lambda(b))\end{aligned}$$

and

$$\begin{aligned}\tilde{\nu}_1(\tilde{\mathfrak{r}}(\alpha)\tilde{\mathfrak{e}}^\mu(b))&=\tilde{\nu}_1(\tilde{\mathfrak{e}}^\lambda(b\alpha))a^\mu_\lambda(\alpha)\\&=\mathfrak{e}^\lambda(b\alpha)a^\mu_\lambda(\alpha)=\mathfrak{e}^\mu(b),\end{aligned}$$

since from (13.12) we have easily

$$\mathfrak{e}^\mu(b\alpha)=b^\mu_\lambda(\alpha)\mathfrak{e}^\lambda(b). \tag{13.22}$$

§ 14. The covariant differentiation with respect to the universal connection.

Making use of φ^μ_λ defined on $\mathfrak{U}(\mathfrak{X})$ by (13.8), we define a homomorphism of the vector bundles $\mathfrak{U}(\mathfrak{X})$ and $\mathfrak{U}^*(\mathfrak{X})$ by

$$\tilde{\varphi}\,\tilde{\mathfrak{e}}_\lambda(b)=\tilde{\mathfrak{e}}_\mu(b)\varphi^\mu_\lambda(b), \tag{14.1}$$

$$\tilde{\varphi}\,\tilde{\mathfrak{e}}^\mu(b)=\varphi^\mu_\lambda(b)\,\tilde{\mathfrak{e}}^\lambda(b). \tag{14.2}$$

Extending this homomorphism to any tensor product bundle of $\mathfrak{U}(\mathfrak{X})$ and $\mathfrak{U}^*(\mathfrak{X})$ by the well known method, we define the covariant differential operator for tensor fields of $\mathfrak{U}(\mathfrak{X})$ as follows.

Let

$$\mathfrak{U}^{\otimes(p,q)}=\mathfrak{U}^{\otimes p}\otimes(\mathfrak{U}^*)^{\otimes q}, \tag{14.3}$$
$$\mathfrak{U}=\mathfrak{U}(\mathfrak{X}),\ \mathfrak{U}^*=\mathfrak{U}^*(\mathfrak{X})$$

be the vector bundle over $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$ which is the tensor product bundle of p copies $\mathfrak{U}(\mathfrak{X})$ and q copies of $\mathfrak{U}^*(\mathfrak{X})$. The covariant differential operator

$$D:\ \Psi(\mathfrak{U}^{\otimes(p,q)})\to\Psi(\mathfrak{U}^{\otimes(p,q)}\otimes T^*(\widetilde{\mathfrak{B}}^2(\mathfrak{X})))$$

is given, by means of (13.13), (13.14), (14.1) and (14.2), for

$$V=V^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\tilde{\mathfrak{e}}_{\lambda_1}\otimes\cdots\otimes\tilde{\mathfrak{e}}_{\lambda_p}\otimes\tilde{\mathfrak{e}}^{\mu_1}\otimes\cdots\otimes\tilde{\mathfrak{e}}^{\mu_q},$$

by the equations

$$\begin{aligned}DV=&\ \tilde{\varphi}(\tilde{\mathfrak{e}}_{\lambda_1}\otimes\cdots\otimes\tilde{\mathfrak{e}}_{\lambda_p}\otimes\tilde{\mathfrak{e}}^{\mu_1}\otimes\cdots\otimes\tilde{\mathfrak{e}}^{\mu_q})\otimes dV^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\\&+V^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\sum_{s=1}^{p}\tilde{\varphi}(\tilde{\mathfrak{e}}_{\lambda_1}\otimes\cdots\otimes\tilde{\mathfrak{e}}_{\lambda_{s-1}})\otimes\tilde{\mathfrak{e}}_\rho\otimes\tilde{\varphi}(\tilde{\mathfrak{e}}_{\lambda_{s+1}}\otimes\cdots\otimes\tilde{\mathfrak{e}}_{\lambda_p}\\&\qquad\otimes\tilde{\mathfrak{e}}^{\mu_1}\otimes\cdots\otimes\tilde{\mathfrak{e}}^{\mu_q})\otimes\tilde{\theta}^\rho_{\lambda_s}\\&-V^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\sum_{t=1}^{q}\tilde{\varphi}(\tilde{\mathfrak{e}}_{\lambda_1}\otimes\cdots\otimes\tilde{\mathfrak{e}}_{\lambda_p}\otimes\tilde{\mathfrak{e}}^{\mu_1}\otimes\cdots\otimes\tilde{\mathfrak{e}}^{\mu_{t-1}})\otimes\tilde{\mathfrak{e}}^\rho\otimes\end{aligned}$$

$$\otimes\tilde{\varphi}(\tilde{e}^{\mu_{t+1}}\otimes\cdots\otimes\tilde{e}^{\mu_q})\otimes\tilde{\pi}^{\mu_t}_{\rho}, \tag{14.4}$$

which can be written as, putting

$$DV=\tilde{e}_{\sigma_1}\otimes\cdots\otimes\tilde{e}_{\sigma_p}\otimes\tilde{e}^{\tau_1}\otimes\cdots\otimes\tilde{e}^{\tau_q}\otimes DV^{\sigma_1\cdots\sigma_p}_{\tau_1\cdots\tau_q}, \tag{14.5}$$

$$\begin{aligned} DV^{\sigma_1\cdots\sigma_p}_{\tau_1\cdots\tau_q}&=\varphi^{\sigma_1}_{\lambda_1}\cdots\varphi^{\sigma_p}_{\lambda_p}dV^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\varphi^{\mu_1}_{\tau_1}\cdots\varphi^{\mu_q}_{\tau_q}\\ &+\sum_{s=1}^{p}\varphi^{\sigma_1}_{\lambda_1}\cdots\varphi^{\sigma_{s-1}}_{\lambda_{s-1}}\theta^{\sigma_s}_{\lambda_s}\varphi^{\sigma_{s+1}}_{\lambda_{s+1}}\cdots\varphi^{\sigma_p}_{\lambda_p}V^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\varphi^{\mu_1}_{\tau_1}\cdots\varphi^{\mu_q}_{\tau_q}\\ &-\sum_{t=1}^{q}\varphi^{\sigma_1}_{\lambda_1}\cdots\varphi^{\sigma_p}_{\lambda_p}V^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\varphi^{\mu_1}_{\tau_1}\cdots\varphi^{\mu_{t-1}}_{\tau_{t-1}}\tilde{\pi}^{\mu_t}_{\tau_t}\varphi^{\mu_{t+1}}_{\tau_{t+1}}\cdots\varphi^{\mu_q}_{\tau_q}. \end{aligned} \tag{14.6}$$

Furthermore, we extend naturally the covariant differential operator D so that

$$\begin{aligned} D:\ &\Psi(U^{\otimes(p,q)}\otimes\Lambda^r(T^*(\widetilde{\mathfrak{B}}^2(\mathfrak{X}))))\to\\ &\Psi(U^{\otimes(p,q)}\otimes\Lambda^{r+1}(T^*(\widetilde{\mathfrak{B}}^2(\mathfrak{X})))). \end{aligned}$$

For

$$W=\tilde{e}_{\lambda_1}\otimes\cdots\otimes\tilde{e}_{\lambda_p}\otimes\tilde{e}^{\mu_1}\otimes\cdots\otimes\tilde{e}^{\mu_q}\otimes\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q},$$

where $\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}$ are exterior differential forms on $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$ of order r, DW is defined by

$$\begin{aligned} DW&=\tilde{\varphi}(\tilde{e}_{\lambda_1}\otimes\cdots\otimes\tilde{e}_{\lambda_p}\otimes\tilde{e}^{\mu_1}\otimes\cdots\otimes\tilde{e}^{\mu_q})\otimes d\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\lambda_q}\\ &+\sum_{s=1}^{p}\tilde{\varphi}(\tilde{e}_{\lambda_1}\otimes\cdots\otimes\tilde{e}_{\lambda_{s-1}})\otimes\tilde{e}_{\rho}\otimes\tilde{\varphi}(\tilde{e}_{\lambda_{s+1}}\otimes\cdots\otimes\tilde{e}_{\lambda_p}\otimes\\ &\qquad\otimes\tilde{e}^{\mu_1}\otimes\cdots\otimes\tilde{e}^{\mu_q})\otimes(\theta^{\rho}_{\lambda_s}\wedge\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q})\\ &-\sum_{t=1}^{q}\tilde{\varphi}(\tilde{e}_{\lambda_1}\otimes\cdots\otimes\tilde{e}_{\lambda_p}\otimes\tilde{e}^{\mu_1}\otimes\cdots\otimes\tilde{e}^{\mu_{t-1}})\otimes\tilde{e}^{\rho}\otimes\\ &\qquad\otimes\tilde{\varphi}(\tilde{e}^{\mu_{t+1}}\otimes\cdots\otimes\tilde{e}^{\mu_q})\otimes(\tilde{\pi}^{\mu_t}_{\rho}\wedge\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q})\\ &=\tilde{e}_{\sigma_1}\otimes\cdots\otimes\tilde{e}_{\sigma_p}\otimes\tilde{e}^{\tau_1}\otimes\cdots\otimes\tilde{e}^{\tau_q}\otimes D\omega^{\sigma_1\cdots\sigma_p}_{\tau_1\cdots\tau_q}. \end{aligned} \tag{14.7}$$

Now, for any two

$$\begin{aligned} V&\in\Psi(\mathfrak{U}^{\otimes(p,q)}\otimes\Lambda^r(T^*(\widetilde{\mathfrak{B}}^2(\mathfrak{X})))),\\ W&\in\Psi(\mathfrak{U}^{\otimes(s,t)}\otimes\Lambda^w(T^*(\widetilde{\mathfrak{B}}^2(\mathfrak{X})))), \end{aligned}$$

we define a product of V and W

$$V\odot W\in\Psi(\mathfrak{U}^{\otimes(p+s,q+t)}\otimes\Lambda^{r+w}(T^*(\widetilde{\mathfrak{B}}^2(\mathfrak{X})))),$$

by

$$\begin{aligned} &(\tilde{e}_{\lambda_1}\otimes\cdots\otimes\tilde{e}_{\lambda_p}\otimes\tilde{e}^{\mu_1}\otimes\cdots\otimes\tilde{e}^{\mu_q}\otimes\omega_r)\odot\\ &\qquad(\tilde{e}_{\sigma_1}\otimes\cdots\otimes\tilde{e}_{\sigma_s}\otimes\tilde{e}^{\tau_1}\otimes\cdots\otimes\tilde{e}^{\tau_t}\otimes\theta_w)\\ &=\tilde{e}_{\lambda_1}\otimes\cdots\otimes\tilde{e}_{\lambda_p}\otimes\tilde{e}_{\sigma_1}\otimes\cdots\otimes\tilde{e}_{\sigma_s}\otimes\tilde{e}^{\mu_1}\otimes\cdots\otimes\tilde{e}^{\mu_q}\otimes\\ &\qquad\otimes\tilde{e}^{\tau_1}\otimes\cdots\otimes\tilde{e}^{\tau_t}\otimes(\omega_r\wedge\theta_w), \end{aligned}$$

where ω_r and θ_w are exterior differential forms of order r and w. The product $\odot$ goes over into the tensor product $\otimes$ and the exterior product $\wedge$ when $r=w=0$ and $p=q=s=t=0$ respectively. By virtue of (14.7) and analogous calculations to the ones done to obtain the formula (2.19) in § 2, we can prove the formula

$$D(V\odot W)=DV\odot\tilde{\varphi}W+(-1)^r\tilde{\varphi}V\odot DW, \tag{14.8}$$

where we regard as $\tilde{\varphi}$ operates only on the parts of the tensor product bundles of $\mathfrak{U}(\mathfrak{X})$ and $\mathfrak{U}^*(\mathfrak{X})$.

For any fidre bundle $\{\mathfrak{B}, \mathfrak{X}, \psi\}$, a differential form ω on $\mathfrak{B}$ of order $r>0$ is said to be *primitive*, if for any r-dimensional tangent hyperplane H at any point $b\in\mathfrak{B}$ such that dim $d\psi(H)<r$, the restriction ω on H vanishes always, that is ω can be written locally in terms of differentials of functions on $\mathfrak{X}$ only ψ through. A tensor field W of $\mathfrak{U}$ of type $(p, q; r)$, $r>0$, is said to be primitive if its components $\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}$ are all primitive, as differential forms on $\tilde{\mathfrak{B}}^2(\mathfrak{X})$.

§ 15. Relations between $\tilde{\varphi}$, $\tilde{\mathrm{r}}(\alpha)$ and the covariant differentiation.

Here, we discribe some notations on bundle homomorphisms of vector bundles. Let $\mathfrak{F}=\{\mathfrak{Z}, \mathfrak{X}, \pi, \mathfrak{Y}\}$ and $\mathfrak{F}'=\{\mathfrak{Z}', \mathfrak{X}', \pi', \mathfrak{Y}'\}$ be two vector bundles and $h: \mathfrak{Z}'\to\mathfrak{Z}$ be a bundle homomorphism covering a given mapping ψ: $\mathfrak{X}'\to\mathfrak{X}$. We denote the dual vector bundle of $\mathfrak{F}$ by $\mathfrak{F}^*$. Then, we can define naturally a transformation of $\Gamma(\mathfrak{F}^*)$ into $\Gamma((\mathfrak{F}')^*)$ denoted by $h^{\circledcirc}$ as follows: For any $\xi\in\Gamma(\mathfrak{F}^*)$, $z'\in\mathfrak{Z}'$, we put

$$\langle z', h^{\circledcirc}\xi\rangle=\langle hz', \xi\rangle. \tag{15.1}$$

When $\mathfrak{F}=T(\mathfrak{X})$, $\mathfrak{F}'=T(\mathfrak{X})$ and h is the differential mapping $d\psi$ of ψ, $h^{\circledcirc}=(d\psi)^{\circledcirc}$ is ψ^* in the ordinary sense. Furthermore, when dim $\mathfrak{Y}$ = dim $\mathfrak{Y}'$ and h is a bundle mapping, that is an isomorphism on each fibre of $\mathfrak{F}'$, we define naturally a transformation of $\Gamma(\mathfrak{F})$ into $\Gamma(\mathfrak{F}')$ denoted by $h\ominus$ as follows: For any $\xi\in\Gamma(\mathfrak{F})$, we put

$$h\ominus\xi(z')=(h\mid\mathfrak{Y}_{x'})^{-1}(\xi(\psi(x'))). \tag{15.2}$$

When h is the induced bundle mapping of ψ, we denote $h\ominus$ by ψ°. [12)]

Now, for the bundle mapping $\tilde{\mathrm{r}}(\alpha)$, $\alpha\in\tilde{\mathfrak{L}}^2_n$, by means of (13.19) and (13.20), we have

$$\begin{aligned}(\tilde{\mathrm{r}}(\alpha)\ominus\tilde{\mathrm{e}}_\lambda)(b)&=(\tilde{\mathrm{r}}(\alpha)\mid\mathfrak{U}_b)^{-1}(\tilde{\mathrm{e}}_\lambda(b\alpha))\\&=\tilde{\mathrm{e}}_\mu(b)a^\mu_\lambda(\alpha),\end{aligned}$$

where $\mathfrak{U}_b$ is the fibre of $\mathfrak{U}=\mathfrak{U}(\mathfrak{X})$ over $b\in\tilde{\mathfrak{B}}^2(\mathfrak{X})$ which is an n dimensional vector space. Analogously, we have

12) See [11], § 8.

$$< (\tilde{\mathfrak{r}}(\alpha)\odot \tilde{\mathfrak{e}}^{\mu})(b),\ \tilde{\mathfrak{e}}_{\lambda}(b) >$$
$$= < \tilde{\mathfrak{e}}^{\mu}(b\alpha),\ \tilde{\mathfrak{r}}(\alpha)(\tilde{\mathfrak{e}}_{\lambda}(b)) >$$
$$= < \tilde{\mathfrak{e}}^{\mu}(b\alpha),\ \tilde{\mathfrak{e}}_{\rho}(b\alpha) b^{\rho}_{\lambda}(\alpha) > = b^{\mu}_{\lambda}(\alpha).$$

Hence, we obtain the formulas

$$\tilde{\mathfrak{r}}(\alpha)\ominus \tilde{\mathfrak{e}}_{\lambda} = \tilde{\mathfrak{e}}_{\mu} a^{\mu}_{\lambda}(\alpha), \tag{15. 3}$$
$$\tilde{\mathfrak{r}}(\alpha)\odot\tilde{\mathfrak{e}}^{\mu} = b^{\mu}_{\lambda}(\alpha)\tilde{\mathfrak{e}}^{\lambda}. \tag{15. 4}$$

(15. 4) shows that $\tilde{\mathfrak{e}}^{\lambda}$ may be regarded as θ^{λ} by means of (13. 15) but they are different with each other because they are considered on different spaces.

Lemma 15. 1. *The tensor field* $\tilde{\mathfrak{e}}_{\lambda}\otimes\tilde{\mathfrak{e}}^{\lambda}$ *of* $\mathfrak{U}(\mathfrak{X})$, *which represents the identity isomorphism of* $\mathfrak{U}(\mathfrak{X})$, *is invariant under* $\tilde{\mathfrak{r}}(\alpha)$, *for any* $\alpha\in\tilde{\mathfrak{L}}^2_n$.

Proof. By means of (15. 3) and (15. 4), we have

$$\begin{aligned}\tilde{\mathfrak{r}}(\alpha)(\tilde{\mathfrak{e}}_{\lambda}\otimes\tilde{\mathfrak{e}}^{\lambda}) &= (\tilde{\mathfrak{r}}(\alpha)\ominus\otimes\mathfrak{r}(\alpha)\odot)(\tilde{\mathfrak{e}}_{\lambda}\otimes\tilde{\mathfrak{e}}^{\lambda})\\ &= \tilde{\mathfrak{e}}_{\rho} a^{\rho}_{\lambda}(\alpha)\otimes b^{\lambda}_{\sigma}(\alpha)\tilde{\mathfrak{e}}^{\sigma}\\ &= \tilde{\mathfrak{e}}_{\rho}\otimes\delta^{\rho}_{\sigma}\tilde{\mathfrak{e}}^{\sigma} = \tilde{\mathfrak{e}}_{\rho}\otimes\tilde{\mathfrak{e}}^{\rho}.\end{aligned}$$

Theorem 15. 2. *In order that a tensor field* $\widetilde{W}$ *of* $\mathfrak{U}(\mathfrak{X})$ *of type* $(p, q\,;\,r)$:

$$\widetilde{W} = \tilde{\mathfrak{e}}_{\lambda_1}\otimes\cdots\tilde{\mathfrak{e}}_{\lambda_p}\otimes\tilde{\mathfrak{e}}^{\mu_1}\otimes\cdots\otimes\tilde{\mathfrak{e}}^{\mu_q}\otimes\tilde{\omega}^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}, \tag{16. 5}$$

and

$\tilde{\omega}^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_p}$ *are exterior differential forms of order r on* $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$,

is induced from a tensor field of $T(\mathfrak{X})$ *of the same type by the bundle mapping* $\tilde{\nu}$: $\mathfrak{U}(\mathfrak{X})\to T(\mathfrak{X})$, *it is necessary and sufficient that it is invariant under* $\tilde{\mathfrak{r}}(\alpha)$, *any* $\alpha\in\tilde{\mathfrak{L}}^2_n$ *and primitive when* $r>0$.

Proof. Sufficiency. By means of (15. 3), (15. 4) and (15. 5), we have

$$\begin{aligned}\tilde{\mathfrak{r}}(\alpha)\widetilde{W} &= (\tilde{\mathfrak{r}}(\alpha)\ominus\otimes\tilde{\mathfrak{r}}(\alpha)\odot\otimes\tilde{\mathfrak{r}}(\alpha)^{*})\widetilde{W}\\ &= \tilde{\mathfrak{e}}_{\sigma_1}\otimes\cdots\otimes\tilde{\mathfrak{e}}_{\sigma_p}\otimes\tilde{\mathfrak{e}}^{\tau_1}\otimes\cdots\otimes\tilde{\mathfrak{e}}^{\tau_q}\otimes\\ &\otimes\{a^{\sigma_1}_{\lambda_1}(\alpha)\cdots a^{\sigma_p}_{\lambda_p}(\alpha)\mathfrak{r}(\alpha)^{*}\tilde{\omega}^{\mu_1\cdots\lambda_p}_{\lambda_1\cdots\mu_q}\ b^{\mu_1}_{\tau_1}(\alpha)\cdots b^{\mu_p}_{\tau_q}(\alpha)\}.\end{aligned} \tag{15. 6}$$

Let us suppose that $\tilde{\mathfrak{r}}(\alpha)\widetilde{W}=\widetilde{W}$, which is equivalent to

$$\tilde{\omega}^{\sigma_1\cdots\sigma_p}_{\tau_1\cdots\tau_q} = a^{\sigma_1}_{\lambda_1}(\alpha)\cdots a^{\sigma_p}_{\lambda_p}(\alpha)\,\mathfrak{r}(\alpha)^{*}\,\tilde{\omega}^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\ b^{\mu_1}_{\tau_1}(\alpha)\cdots b^{\mu_q}_{\tau_q}(\alpha),$$

that is

$$\mathfrak{r}(\alpha)^{*}\tilde{\omega}^{\mu_1\cdots\lambda_p}_{\lambda_1\cdots\mu_q} = b^{\lambda_1}_{\sigma_1}(\alpha)\cdots b^{\lambda_p}_{\sigma_p}(\alpha)\tilde{\omega}^{\sigma_1\cdots\sigma_p}_{\tau_1\cdots\tau_q}\ a^{\tau_1}_{\mu_1}(\alpha)\cdots a^{\tau_q}_{\mu_q}(\alpha). \tag{15. 7}$$

Let $(u^j,\ \beta)$, $\beta\in\tilde{\mathfrak{L}}^2_n$, be the local coordinates of $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$ and put

$$\omega_{j_1\cdots j_q}^{i_1\cdots i_p} = a_{\lambda_1}^{i_1}(\beta)\cdots a_{\lambda_p}^{i_p}(\beta)\,\tilde{\omega}_{\mu_1\cdots\mu_q}^{\lambda_1\cdots\lambda_p}\, b_{j_1}^{\mu_1}(\beta)\cdots b_{j_q}^{\mu_q}(\beta). \tag{15.8}$$

For any $\alpha\in\widetilde{\mathfrak{L}}_n^2$, we have

$$\begin{aligned}
\mathrm{r}(\alpha)^*\omega_{j_1\cdots j_q}^{i_1\cdots i_p} &= a_{\lambda_1}^{i_1}(\beta\alpha)\cdots a_{\lambda_p}^{i_p}(\beta\alpha)\,\mathrm{r}(\alpha)^*\tilde{\omega}_{\mu_1\cdots\mu_q}^{\lambda_1\cdots\lambda_p} b_{j_1}^{\mu_1}(\beta\alpha)\cdots b_{j_q}^{\mu_q}(\beta\alpha)\\
&= a_{\lambda_1}^{i_1}(\beta\alpha)\cdots a_{\lambda_p}^{i_p}(\beta\alpha)\, b_{\sigma_1}^{\lambda_1}(\alpha)\cdots b_{\sigma_p}^{\lambda_p}(\alpha)\tilde{\omega}_{\tau_1\cdots\tau_q}^{\sigma_1\cdots\sigma_p}\, a_{\mu_1}^{\tau_1}(\alpha)\cdots a_{\mu_q}^{\tau_q}(\alpha)\times\\
&\qquad\times b_{j_1}^{\mu_1}(\beta\alpha)\cdots b_{j_q}^{\mu_q}(\beta\alpha)\\
&= a_{\sigma_1}^{i_1}(\beta)\cdots a_{\sigma_p}^{i_p}(\beta)\tilde{\omega}_{\tau_1\cdots\tau_q}^{\sigma_1\cdots\sigma_p}\, b_{j_1}^{\tau_1}(\beta)\cdots b_{j_q}^{\tau_q}(\beta)\\
&= \omega_{j_1\cdots j_q}^{i_1\cdots i_p},
\end{aligned}$$

which shows that $\omega_{j_1\cdots j_q}^{i_1\cdots i_p}$ is differential forms of order r on $\widetilde{\mathfrak{B}}^2(\mathfrak{X})$ depending only on the local coordinates u^i. This fact and (15.8) show that $\widetilde{\mathrm{W}}$ is induced from a tensor field of type $(p, q ; r)$ of $T(\mathfrak{X})$

$$W = \partial u_{i_1}\otimes\cdots\otimes\partial u_{i_p}\otimes du^{j_1}\otimes\cdots du^{j_q}\otimes\omega_{j_1\cdots j_q}^{i_1\cdots i_p}, \tag{15.9}$$

that is

$$\widetilde{W} = \tilde{\nu}\ W = (\tilde{\nu}^{\ominus}\otimes\tilde{\nu}^{\odot}\otimes\tilde{\pi}_2^*)W. \tag{15.10}$$

The necessity is easily proved, because we can obtain (15.7) from

$$\tilde{\omega}_{\mu_1\cdots\mu_q}^{\lambda_1\cdots\lambda_p} = b_{i_1}^{\lambda_1}(\beta)\cdots b_{i_p}^{\lambda_p}(\beta)\,\omega_{j_1\cdots j_q}^{i_1\cdots i_p}\, a_{\mu_1}^{j_1}(\beta)\cdots a_{\mu_q}^{j_q}(\beta).$$

Lemma 15.3. *$d\mathfrak{p} = \mathfrak{e}_\mu\otimes\theta^\mu$ is invariant under $\tilde{\mathrm{r}}(\alpha)$, $\alpha\in\widetilde{\mathfrak{L}}_n^2$.*

The proof of this lemma is evident from (13.15) and (15.3).

Now, we shall investigate the relation between $\tilde{\mathrm{r}}(\alpha)$ and the covariant differentiation of $\mathfrak{U}(\mathfrak{X})$. From (13.13), (15.3) and (13.16), we have, for any $\alpha\in\widetilde{\mathfrak{L}}_n^2$,

$$\begin{aligned}
\tilde{\mathrm{r}}(\alpha)D\tilde{\mathfrak{e}}_\lambda &= \tilde{\mathrm{r}}(\alpha)(\tilde{\mathfrak{e}}_\mu\otimes\tilde{\theta}_\lambda^\mu)\\
&= (\tilde{\mathrm{r}}(\alpha)^{\ominus}\otimes\mathrm{r}(\alpha)^*)(\tilde{\mathfrak{e}}_\mu\otimes\tilde{\theta}_\lambda^\mu)\\
&= \tilde{\mathrm{r}}(\alpha)^{\ominus}\,\tilde{\mathfrak{e}}_\mu\otimes\mathrm{r}(\alpha)^*\,\tilde{\theta}_\lambda^\mu,\\
&= \tilde{\mathfrak{e}}_\nu\, a_\mu^\nu(\alpha)\otimes\{b_\rho^\mu(\alpha)\,\tilde{\theta}_\sigma^\rho\, p_\lambda^\sigma(\alpha) - a_{\lambda\rho}^\mu(\gamma(\alpha))\,\mathrm{r}(\alpha)^*\theta^\rho\}\\
&= \tilde{\mathfrak{e}}_\nu\otimes\theta_\sigma^\nu\, p_\lambda^\sigma(\alpha) - \tilde{\mathfrak{e}}_\nu\otimes a_{\lambda\rho}^\nu(\alpha)\,\mathrm{r}(\alpha)^*\theta^\rho,
\end{aligned}$$

that is

$$\tilde{\mathrm{r}}(\alpha)D\tilde{\mathfrak{e}}_\lambda = (D\tilde{\mathfrak{e}}_\sigma)p_\lambda^\sigma(\alpha) - \tilde{\mathfrak{e}}_\nu\otimes a_{\lambda\rho}^\nu(\alpha)\,\mathrm{r}(\alpha)^*\theta^\rho. \tag{15.11}$$

Let W be a tensor field of $\mathfrak{U}(\mathfrak{X})$ of type $(p, q ; r)$ which is invariant under $\tilde{\mathrm{r}}(\alpha)$, $\alpha\in\widetilde{\mathfrak{L}}_n^2$, and is written as

$$W = \tilde{\mathfrak{e}}_{\lambda_1}\otimes\cdots\otimes\tilde{\mathfrak{e}}_{\lambda_p}\otimes\tilde{\mathfrak{e}}^{\mu_1}\otimes\cdots\otimes\tilde{\mathfrak{e}}^{\mu_q}\otimes\omega_{\mu_1\cdots\mu_q}^{\lambda_1\cdots\lambda_p}.$$

By means of the assumption, we get from Theorem 15.2

$$r(\alpha)^*\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q} = b^{\lambda_1}_{\sigma_1}(\alpha)\cdots b^{\lambda_p}_{\sigma_p}(\alpha)\omega^{\sigma_1\cdots\sigma_p}_{\tau_1\cdots\tau_q}\, a^{\tau_1}_{\mu_1}(\alpha)\cdots a^{\tau_q}_{\mu_q}(\alpha) \tag{15.12}$$

for the element α. By the definition (14.7) of D, we have

$$DW = \tilde{e}_{\sigma_1}\otimes\cdots\otimes\tilde{e}_{\sigma_p}\otimes\tilde{e}^{\tau_1}\otimes\cdots\otimes\tilde{e}^{\tau_q}\otimes D\omega^{\sigma_1\cdots\sigma_p}_{\tau_1\cdots\tau_q},$$

$$\begin{aligned} D\omega^{\sigma_1\cdots\sigma_p}_{\tau_1\cdots\tau_q} &= \varphi^{\sigma_1}_{\lambda_1}\cdots\varphi^{\sigma_p}_{\lambda_p}\, d\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\,\varphi^{\mu_1}_{\tau_1}\cdots\varphi^{\mu_q}_{\tau_q} \\ &+ \sum_{s=1}^{p}\varphi^{\sigma_1}_{\lambda_1}\cdots\varphi^{\sigma_{s-1}}_{\lambda_{s-1}}\bar{\omega}^{\sigma_s}_{\lambda_s}\varphi^{\sigma_{s+1}}_{\lambda_{s+1}}\cdots\varphi^{\sigma_p}_{\lambda_p}\wedge\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\varphi^{\mu_1}_{\tau_1}\cdots\varphi^{\mu_q}_{\tau_q} \\ &-(-1)^r\sum_{t=1}^{q}\varphi^{\sigma_1}_{\lambda_1}\cdots\varphi^{\sigma_p}_{\lambda_p}\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\wedge\varphi^{\mu_1}_{\tau_1}\cdots\varphi^{\mu_{t-1}}_{\tau_{t-1}}\tilde{\pi}^{\mu_t}_{\tau_t}\varphi^{\mu_{t+1}}_{\tau_{t+1}}\cdots\varphi^{\mu_q}_{\tau_q}. \end{aligned} \tag{15.13}$$

Making use of (13.17), (13.16), (13.18) and (15.12), we have

$$\begin{aligned} & r(\alpha)^* D\omega^{\sigma_1\cdots\sigma_p}_{\tau_1\cdots\tau_q} \\ &= b^{\sigma_1}_{\rho_1}(\alpha)\cdots b^{\sigma_p}_{\rho_p}(\alpha)\varphi^{\rho_1}_{\nu_1}\cdots\varphi^{\rho_p}_{\nu_p}p^{\nu_1}_{\lambda_1}(\bar{\gamma}(\alpha))\cdots p^{\nu_p}_{\lambda_p}(\bar{\gamma}(\alpha))\times \\ &\quad\times d\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\,\varphi^{\mu_1}_{\kappa_1}\cdots\varphi^{\mu_q}_{\kappa_q}p^{\kappa_1}_{\tau_1}(\alpha)\cdots p^{\kappa_q}_{\tau_q}(\alpha) \\ &+\sum_{s=1}^{p} b^{\sigma_1}_{\rho_1}(\alpha)\cdots b^{\sigma_p}_{\rho_p}(\alpha)\varphi^{\rho_1}_{\nu_1}\cdots\varphi^{\rho_{s-1}}_{\nu_{s-1}}\tilde{\omega}^{\rho_s}_{\nu_s}\varphi^{\rho_{s+1}}_{\nu_{s+1}}\cdots\varphi^{\rho_p}_{\nu_p}\wedge \\ &\quad p^{\nu_1}_{\lambda_1}(\bar{\gamma}(\alpha))\cdots p^{\nu_p}_{\lambda_p}(\bar{\gamma}(\alpha))\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\,\varphi^{\mu_1}_{\kappa_1}\cdots\varphi^{\mu_q}_{\kappa_q}p^{\kappa_1}_{\tau_1}(\alpha)\cdots p^{\kappa_q}_{\tau_q}(\alpha) \\ &-\sum_{s=1}^{p} b^{\sigma_1}_{\rho_1}(\alpha)\cdots b^{\sigma_p}_{\rho_p}(\alpha)\,\varphi^{\rho_1}_{\nu_1}\cdots\varphi^{\rho_{s-1}}_{\nu_{s-1}}\,a^{\rho_s}_{\lambda_s\gamma}(\bar{\gamma}(\alpha))\,\theta^{\gamma}\varphi^{\rho_{s+1}}_{\nu_{s+1}}\cdots\varphi^{\rho_p}_{\nu_p}\wedge \\ &\quad p^{\nu_1}_{\lambda_1}(\bar{\gamma}(\alpha))\cdots p^{\nu_{s-1}}_{\lambda_{s-1}}(\bar{\gamma}(\alpha))p^{\nu_{s+1}}_{\lambda_{s+1}}(\bar{\gamma}(\alpha))\cdots p^{\nu_p}_{\lambda_p}(\bar{\gamma}(\alpha))\,\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q} \\ &\qquad\times\varphi^{\mu_1}_{\kappa_1}\cdots\varphi^{\mu_q}_{\kappa_q}p^{\kappa_1}_{\tau_1}(\alpha)\cdots p^{\kappa_q}_{\tau_q}(\alpha) \\ &-(-1)^r\sum_{t=1}^{q} b^{\sigma_1}_{\rho_1}(\alpha)\cdots b^{\sigma_p}_{\rho_p}(\alpha)\varphi^{\rho_1}_{\nu_1}\cdots\varphi^{\rho_p}_{\nu_p}\,p^{\nu_1}_{\lambda_1}(\bar{\gamma}(\alpha))\cdots p^{\nu_p}_{\lambda_p}(\bar{\gamma}(\alpha))\times \\ &\quad\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\wedge\varphi^{\mu_1}_{\kappa_1}\cdots\varphi^{\mu_{t-1}}_{\kappa_{t-1}}\,\tilde{\pi}^{\mu_t}_{\kappa_t}\,\varphi^{\mu_{t+1}}_{\kappa_{t+1}}\cdots\varphi^{\mu_q}_{\kappa_q}\,p^{\kappa_1}_{\tau_1}(\alpha)\cdots p^{\kappa_q}_{\tau_q}(\alpha) \\ &+(-1)^r\sum_{t=1}^{q} b^{\sigma_1}_{\rho_1}(\alpha)\cdots b^{\sigma_p}_{\rho_p}(\alpha)\varphi^{\rho_1}_{\nu_1}\cdots\varphi^{\rho_p}_{\nu_p}\,p^{\nu_1}_{\lambda_1}(\bar{\gamma}(\alpha))\cdots p^{\nu_p}_{\lambda_p}(\bar{\gamma}(\alpha))\times \\ &\quad\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\wedge\varphi^{\mu_1}_{\kappa_1}\cdots\varphi^{\mu_{t-1}}_{\kappa_{t-1}}\,a^{\mu_t}_{\kappa_t\gamma}(\bar{\gamma}(\alpha))\,\theta^{\gamma}\,\varphi^{\mu_{t+1}}_{\kappa_{t+1}}\cdots\varphi^{\mu_q}_{\kappa_q}\times \\ &\qquad\times p^{\kappa_1}_{\tau_1}(\alpha)\cdots p^{\kappa_{t-1}}_{\tau_{t-1}}(\alpha)\,a^{\kappa_t}_{\tau_t}(\alpha)\,p^{\kappa_{t+1}}_{\tau_{t+1}}(\alpha)\cdots p^{\kappa_q}_{\tau_q}(\alpha). \end{aligned} \tag{15.14}$$

This equation shows that DW is not invariant, in general, under $\tilde{r}(\alpha)$, $\alpha\in\tilde{\mathfrak{L}}^2_n$, even if W is so.

If we take especially any $\alpha\in\mathfrak{L}^2_n$, then since we have

$$p^j_i(\alpha) = a^j_i(\alpha),\quad p^j_i(\bar{\gamma}(\alpha)) = \delta^j_i,$$

and so the above equation (15.14) goes over into

$$\begin{aligned} r(\alpha)^* D\omega^{\sigma_1\cdots\sigma_p}_{\tau_1\cdots\tau_q} &= b^{\sigma_1}_{\rho_1}(\alpha)\cdots b^{\sigma_p}_{\rho_p}(\alpha)\,[D\,\omega^{\rho_1\cdots\rho_p}_{\kappa_1\cdots\kappa_q} - \\ &-\sum_{s=1}^{p}\varphi^{\rho_1}_{\lambda_1}\cdots\varphi^{\rho_{s-1}}_{\lambda_{s-1}}\,a^{\rho_s}_{\lambda_s\gamma}(\bar{\gamma}(\alpha))\,\theta^{\gamma}\,\varphi^{\rho_{s+1}}_{\lambda_{s+1}}\cdots\varphi^{\rho_p}_{\lambda_p}\wedge\omega^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\,\varphi^{\mu_1}_{\kappa_1}\cdots\varphi^{\mu_q}_{\kappa_q} \end{aligned}$$

$$+(-1)^r \sum_{t=1}^{q} \varphi^{\rho_1}_{\lambda_1} \cdots \varphi^{\rho_p}_{\lambda_p} \omega^{\lambda_1 \cdots \lambda_p}_{\mu_1 \cdots \mu_q} \wedge \varphi^{\mu_1}_{\kappa_1} \cdots \varphi^{\mu_{t-1}}_{\kappa_{t-1}} a^{\mu_t}_{\kappa_t \gamma}(\bar{\gamma}(\alpha))\theta^{\gamma} \times$$
$$\times \varphi^{\mu_{t+1}}_{\kappa_{t+1}} \cdots \varphi^{\mu_q}_{\kappa_q}]\ a^{\kappa_1}_{\tau_1}(\alpha) \cdots a^{\kappa_q}_{\tau_q}(\alpha). \qquad (15.15)$$

Furthermore, if $\alpha \in \mathfrak{L}^2_n$ and $a^j_{ih}(\bar{\gamma}(\alpha)) = 0$, that is $\alpha \in L^1_n$, then (15. 15) goes over into

$$\mathfrak{r}(\alpha)\, D\omega^{\sigma_1 \cdots \sigma_p}_{\tau_1 \cdots \tau_q} = b^{\sigma_1}_{\rho_1}(\alpha) \cdots b^{\sigma_p}_{\rho_p}(\alpha)\, D\omega^{\rho_1 \cdots \rho_p}_{\kappa_1 \cdots \kappa_q}\, a^{\kappa_1}_{\tau_1}(\alpha) \cdots a^{\kappa_q}_{\tau_q}(\alpha). \qquad (15.16)$$

From the above consideration, we have obtained

Theorem 15. 4. *If a tensor field W of $\mathfrak{U}(\mathfrak{X})$ is invariant under $\bar{\mathfrak{r}}(\alpha)$, $\alpha \in \widetilde{\mathfrak{L}}^2_n$, then DW is transformed by $\bar{\mathfrak{r}}(\alpha)$ as (15. 14). Especially, if $\alpha \in \mathfrak{L}^2_n$ or L^1_n, then (15. 14) becomes (15. 15) or (15. 16) respectively.*

Theorem 15. 5. *If a tensor field W of $\mathfrak{U}(\mathfrak{X})$ is invariant under the group of isomorphisms $\bar{\mathfrak{r}}(\alpha)$, $\alpha \in L^1_n$, of $\mathfrak{U}(\mathfrak{X})$, then DW is also invariant under the group. This property of $\mathfrak{U}(\mathfrak{X})$ can not extend to the group $\mathfrak{L}^2_n \supset L^1_n$.*

§ 16. The torsion and curvature forms of the universal general connection.

In the first place, we shall make some formulas for the sake of the consideration in this section. By means of (14. 1) and (14. 6), we have

$$\begin{aligned}\bar{\varphi}\, D\bar{\mathfrak{e}}_\lambda &= \bar{\varphi}(\mathfrak{e}_\mu \otimes \bar{\theta}^\mu_\lambda) = \bar{\mathfrak{e}}_\rho \otimes \varphi^\rho_\mu \bar{\theta}^\mu_\lambda,\\ D\bar{\varphi}\mathfrak{e}_\lambda &= D(\bar{\mathfrak{e}}_\rho \varphi^\rho_\lambda) = D\bar{\mathfrak{e}}_\rho\, \varphi^\rho_\lambda + \bar{\varphi}\,\bar{\mathfrak{e}}_\rho \otimes d\varphi^\rho_\lambda\\ &= \bar{\mathfrak{e}}_\mu \otimes (\varphi^\mu_\rho\, d\varphi^\rho_\lambda + \bar{\theta}^\mu_\rho \varphi^\rho_\lambda),\end{aligned}$$

hence

$$\begin{aligned}(D\bar{\varphi} - \bar{\varphi}D)\bar{\mathfrak{e}}_\lambda &= \bar{\mathfrak{e}}_\mu \otimes (\bar{\theta}^\mu_\rho \varphi^\rho_\lambda - \varphi^\mu_\rho \bar{\theta}^\rho_\lambda + \varphi^\mu_\rho\, d\varphi^\rho_\lambda)\\ &= \bar{\mathfrak{e}}_\mu \otimes (\bar{\theta}^\mu_\rho \varphi^\rho_\lambda - \varphi^\mu_\rho \bar{\pi}^\rho_\lambda) = \bar{\mathfrak{e}}_\mu \otimes D\delta^\mu_\lambda.\end{aligned}$$

Analogously, we have

$$\begin{aligned}(D\bar{\varphi} - \bar{\varphi}D)\bar{\mathfrak{e}}^\mu &= D(\varphi^\mu_\rho \bar{\mathfrak{e}}^\rho) + \bar{\varphi}(\bar{\mathfrak{e}}^\rho \otimes \bar{\pi}^\mu_\rho)\\ &= \bar{\mathfrak{e}}^\lambda \otimes (d\varphi^\mu_\rho \varphi^\rho_\lambda - \varphi^\mu_\rho \bar{\pi}^\rho_\lambda) + \bar{\mathfrak{e}}^\lambda \otimes \bar{\pi}^\mu_\rho \varphi^\rho_\lambda\\ &= \bar{\mathfrak{e}}^\lambda \otimes (\bar{\theta}^\mu_\rho \varphi^\rho_\lambda - \varphi^\mu_\rho \bar{\pi}^\rho_\lambda) = \bar{\mathfrak{e}}^\lambda \otimes D\delta^\mu_\lambda.\end{aligned}$$

Thus we obtained the formulas :

$$(D\bar{\varphi} - \bar{\varphi}D)\bar{\mathfrak{e}}_\lambda = \bar{\mathfrak{e}}_\mu \otimes D\delta^\mu_\lambda, \qquad (16.1)$$
$$(D\bar{\varphi} - \bar{\varphi}D)\bar{\mathfrak{e}}^\mu = \bar{\mathfrak{e}}^\mu \otimes D\delta^\mu_\lambda, \qquad (16.2)$$

where $D\delta^\mu_\lambda$ is exactly obtained as follows :

$$D(\bar{\mathfrak{e}}_\lambda \otimes \bar{\mathfrak{e}}^\lambda) = \bar{\mathfrak{e}}_\mu \otimes \bar{\varphi}\,\bar{\mathfrak{e}}^\lambda \otimes \bar{\theta}^\rho_\lambda - \bar{\varphi}\bar{\mathfrak{e}}_\lambda \otimes \bar{\mathfrak{e}}^\rho \otimes \bar{\pi}^\lambda_\rho$$

$$= \tilde{\mathfrak{e}}_\mu \otimes \tilde{\mathfrak{e}}^\lambda \otimes (\tilde{\theta}^\mu_\rho \varphi^\rho_\lambda - \varphi^\mu_\rho \tilde{\pi}^\rho_\lambda) = \tilde{\mathfrak{e}}_\mu \otimes \tilde{\mathfrak{e}}^\lambda \otimes D\delta^\mu_\lambda,$$

that is

$$D(\tilde{\mathfrak{e}}_\lambda \otimes \tilde{\mathfrak{e}}^\lambda) = \tilde{\mathfrak{e}}_\mu \otimes \tilde{\mathfrak{e}}^\lambda \otimes D\delta^\mu_\lambda, \tag{16. 3}$$

$$D\delta^\mu_\lambda = \tilde{\theta}^\mu_\rho \varphi^\rho_\lambda - \varphi^\mu_\rho \tilde{\pi}^\rho_\lambda. \tag{16. 4}$$

By virtue of Lemma 15. 1, $\tilde{\mathfrak{e}}_\lambda \otimes \tilde{\mathfrak{e}}^\lambda$ is invariant under all $\tilde{\mathfrak{r}}(\alpha)$, $\alpha \in \tilde{\mathfrak{L}}^2_n$, hence we get, from Theorem 15. 5, for any $\alpha \in \tilde{\mathfrak{L}}^2_n$

$$\begin{aligned}
\mathfrak{r}(\alpha)^* D\delta^\sigma_\tau &= b^\sigma_\rho(\alpha)\tilde{\theta}^\rho_\nu p^\nu_\lambda(\bar{\eta}(\alpha))\varphi^\lambda_\kappa p^\kappa_\tau(\alpha) - \\
&\quad - b^\sigma_\rho(\alpha) a^\rho_{\lambda\gamma}(\bar{\eta}(\alpha))\theta^\gamma \varphi^\lambda_\kappa p^\kappa_\tau(\alpha) - b^\sigma_\rho(\alpha)\varphi^\rho_\nu p^\nu_\lambda(\bar{\eta}(\alpha))\tilde{\pi}^\lambda_\kappa p^\kappa_\tau(\alpha) \\
&\quad + b^\sigma_\rho(\alpha)\varphi^\rho_\nu p^\nu_\lambda(\bar{\eta}(\alpha)) a^\lambda_{\kappa\gamma}(\bar{\eta}(\alpha))\theta^\gamma a^\kappa_\tau(\alpha) \\
&= b^\sigma_\rho(\alpha)\{\tilde{\theta}^\rho_\nu p^\nu_\lambda(\bar{\eta}(\alpha))\varphi^\lambda_\kappa - \varphi^\rho_\nu p^\nu_\lambda(\bar{\eta}(\alpha))\tilde{\pi}^\lambda_\kappa\} p^\kappa_\tau(\alpha) \\
&\quad - b^\sigma_\rho(\alpha)\{a^\rho_{\nu\gamma}(\bar{\eta}(\alpha))\theta^\gamma \varphi^\nu_\lambda p^\lambda_\kappa(\bar{\eta}(\alpha)) \\
&\qquad - \varphi^\rho_\nu p^\nu_\lambda(\bar{\eta}(\alpha)) a^\lambda_{\kappa\gamma}(\bar{\eta}(\alpha))\theta^\gamma\} a^\kappa_\tau(\alpha),
\end{aligned}$$

hence

$$\begin{aligned}
\mathfrak{r}(\alpha)^* D\delta^\sigma_\tau &= b^\sigma_\rho(\alpha)\,[D(p^\rho_\nu(\bar{\eta}(\alpha)))p^\nu_\kappa(\bar{\eta}(\alpha)) \\
&\quad - \{a^\rho_{\nu\gamma}(\bar{\eta}(\alpha))\theta^\gamma(\varphi^\nu_\lambda p^\lambda_\kappa(\bar{\eta}(\alpha))) - (\varphi^\rho_\lambda p^\lambda_\nu(\bar{\eta}(\alpha))) a^\nu_{\kappa\gamma}(\bar{\eta}(\alpha))\theta^\gamma\}]\, a^\kappa_\tau(\alpha).
\end{aligned} \tag{16. 5}$$

Now, we covariantly differentiate $d\tilde{\mathfrak{p}} = \tilde{\mathfrak{e}}_\lambda \otimes \theta^\lambda$ which is invariant under all $\tilde{\mathfrak{r}}(\alpha)$, $\alpha \in \tilde{\mathfrak{L}}^2_n$. By means of (15. 13), we have

$$Dd\tilde{\mathfrak{p}} = \tilde{\mathfrak{e}}_\lambda \otimes D\theta^\lambda$$

$$D\theta^\lambda = \varphi^\lambda_\mu d\theta^\mu + \tilde{\theta}^\lambda_\mu \wedge \theta^\mu = \tilde{\Theta}^\lambda,$$

which we call *the torsion forms* of the universal connection of $\mathfrak{X}$.

For any $\alpha \in \tilde{\mathfrak{L}}^2_n$, we have by (13. 16), (13. 17)

$$\begin{aligned}
\mathfrak{r}(\alpha)^*\tilde{\Theta}^\lambda &= b^\lambda_\rho(\alpha)\varphi^\rho_\nu p^\nu_\mu(\bar{\eta}(\alpha))d\theta^\mu \\
&\quad + b^\lambda_\rho(\alpha)\{\tilde{\theta}^\rho_\nu \wedge p^\nu_\mu(\bar{\eta}(\alpha))\theta^\mu - a^\rho_{\nu\gamma}(\bar{\eta}(\alpha))\theta^\gamma \wedge \theta^\nu\} \\
&= b^\lambda_\rho(\alpha)\,[\{\varphi^\rho_\nu d(p^\nu_\mu(\bar{\eta}(\alpha)\theta^\mu) + \tilde{\theta}^\rho_\nu \wedge p^\nu_\mu(\bar{\eta}(\alpha))\theta^\mu\} \\
&\qquad + a^\rho_{\nu\gamma}(\bar{\eta}(\alpha))\theta^\nu \wedge \theta^\gamma],
\end{aligned}$$

that is

$$r(\alpha)^*\tilde{\Theta}^\lambda = b^\lambda_\mu(\alpha)\,[D(\sum_\nu p^\mu_\nu(\bar{\eta}(\alpha))\theta^\nu) + a^\mu_{\nu\tau}(\bar{\eta}(\alpha))\theta^\nu \wedge \theta^\tau]. \tag{16. 7}$$

We can obtain an identity as follows :

$$\begin{aligned}
D\tilde{\mathfrak{e}}_\lambda \odot \tilde{\mathfrak{e}}^\lambda &= \tilde{\mathfrak{e}}_\mu \otimes \tilde{\mathfrak{e}}^\lambda \otimes \tilde{\theta}^\mu_\lambda \\
&= \tilde{\mathfrak{e}}_\mu \otimes \tilde{\mathfrak{e}}^\lambda \otimes (\tilde{\pi}^\mu_\lambda + d\varphi^\mu_\lambda) \\
&= -\tilde{\mathfrak{e}}_\mu \odot D\tilde{\mathfrak{e}}^\mu + \tilde{\mathfrak{e}}_\mu \otimes \tilde{\mathfrak{e}}^\lambda \otimes d\varphi^\mu_\lambda,
\end{aligned}$$

that is

$$D\tilde{e}_\lambda \odot \tilde{e}^\lambda + \tilde{e}_\lambda \odot D\tilde{e}^\lambda - \tilde{e}_\mu \otimes \tilde{e}^\lambda \otimes d\varphi^\mu_\lambda = 0. \tag{16.8}$$

Covariantly differentiating this identity, we get

$$\begin{aligned} 0 &= D(D\tilde{e}_\lambda \odot \tilde{e}^\lambda) + D(\tilde{e}_\lambda \odot D\tilde{e}^\lambda) - D(\tilde{e}_\mu \otimes \tilde{e}^\lambda \otimes d\varphi^\mu_\lambda) \\ &= D(D\tilde{e}_\lambda \odot \tilde{e}^\lambda) + D(\tilde{e}_\lambda \odot D\tilde{e}^\lambda) - D\tilde{e}_\mu \odot \tilde{\varphi}(\tilde{e}^\lambda) \odot d\varphi^\mu_\lambda \\ &\qquad - \tilde{\varphi}(\tilde{e}_\mu) \odot D\tilde{e}^\lambda \odot d\varphi^\mu_\lambda, \end{aligned}$$

hence

$$\begin{aligned} &\{D(D\tilde{e}_\lambda \odot \tilde{e}^\lambda) - \tilde{\varphi}(\tilde{e}_\mu) \odot D\tilde{e}^\lambda \odot d\varphi^\mu_\lambda\} \\ &+ \{D(\tilde{e}_\lambda \odot D\tilde{e}^\lambda) - D\tilde{e}_\mu \odot \tilde{\varphi}(\tilde{e}^\lambda) \odot d\varphi^\mu_\lambda\} = 0. \end{aligned} \tag{16.9}$$

The quantity in the first parenthesis of the left hand side of the above equation can be written by (13. 13) and (16. 1), as

$$\begin{aligned} &D(D\tilde{e}_\lambda \odot \tilde{e}^\lambda) + \tilde{\varphi}(\tilde{e}_\mu) \odot (\tilde{\theta}^\mu_\lambda - \tilde{\pi}^\mu_\lambda) \odot D\tilde{e}^\lambda \\ &= D(D\tilde{e}_\lambda \odot \tilde{e}^\lambda) + \tilde{\varphi}(D\tilde{e}_\lambda) \odot D\tilde{e}^\lambda - \tilde{\varphi}(\tilde{e}_\mu) \odot \tilde{\pi}^\mu_\lambda \odot D\tilde{e}^\lambda \\ &= D(D\tilde{e}_\lambda \odot \tilde{e}^\lambda) + D(\tilde{\varphi}\,\tilde{e}_\lambda) \odot D\tilde{e}^\lambda - \tilde{e}_\mu \odot D\partial^\mu_\lambda \odot D\tilde{e}^\lambda - \tilde{\varphi}(\tilde{e}_\mu) \odot \tilde{\pi}^\mu_\lambda \odot D\tilde{e}^\lambda. \end{aligned}$$

On the other hand, since we have

$$\begin{aligned} \tilde{e}_\mu \otimes D\partial^\mu_\lambda + \tilde{\varphi}(\tilde{e}_\mu) \odot \tilde{\pi}^\mu_\lambda &= \tilde{e}_\mu \otimes \{\tilde{\theta}^\mu_\rho\, \varphi^\rho_\lambda - \varphi^\mu_\rho\, \tilde{\pi}^\rho_\lambda + \varphi^\mu_\rho\, \tilde{\pi}^\rho_\lambda\} \\ &= \tilde{e}_\mu \otimes \tilde{\theta}^\mu_\rho\, \varphi^\rho_\lambda = \varphi^\rho_\lambda\, D\tilde{e}_\rho, \end{aligned}$$

the above equation becomes

$$= D(D\tilde{e}_\lambda \odot \tilde{e}^\lambda) + D(\tilde{\varphi}\,\tilde{e}_\lambda) \odot D\tilde{e}^\lambda - \varphi^\mu_\lambda\, D\tilde{e}_\mu \odot D\tilde{e}^\lambda. \tag{16.10}$$

Analogously, the quantity in the second parenthesis of the left hand side of (16. 9) can be written as

$$\begin{aligned} &D(\tilde{e}_\lambda \odot D\tilde{e}^\lambda) - D\tilde{e}_\mu \odot \tilde{\varphi}(\tilde{e}^\lambda) \odot (\tilde{\partial}^\mu_\lambda - \tilde{\pi}^\mu_\lambda) \\ &= D(\tilde{e}_\lambda \odot D\tilde{e}^\lambda) - D\tilde{e}_\mu \odot \tilde{\varphi}(D\tilde{e}^\mu) - D\tilde{e}_\mu \odot \tilde{\varphi}(\tilde{e}^\lambda) \odot \tilde{\theta}^\mu_\lambda \\ &= D(\tilde{e}_\lambda \odot D\tilde{e}^\lambda) - D\tilde{e}_\mu \odot D\tilde{\varphi}\,\tilde{e}^\mu + D\tilde{e}_\mu \odot \tilde{e}^\lambda \odot (D\partial^\mu_\lambda - \tilde{\theta}^\mu_\rho\, \varphi^\rho_\lambda) \\ &= D(\tilde{e}_\lambda \odot D\tilde{e}^\lambda) - D\tilde{e}_\lambda \odot D\tilde{\varphi}\,\tilde{e}^\lambda - D\tilde{e}_\mu \odot \tilde{e}^\lambda \odot \varphi^\mu_\rho\, \tilde{\pi}^\rho_\lambda \\ &= D(\tilde{e}_\lambda \odot D\tilde{e}^\lambda) - D\tilde{e}_\lambda \odot D\tilde{\varphi}\,\tilde{e}^\lambda + \varphi^\mu_\lambda\, D\tilde{e}_\mu \odot D\tilde{e}^\lambda. \end{aligned} \tag{16.11}$$

Accordingly, the identity (16. 9) can be rewritten as

$$\begin{aligned} &\{D(D\tilde{e}_\lambda \odot \tilde{e}^\lambda) + D(\tilde{\varphi}\,\tilde{e}_\lambda) \odot D\tilde{e}^\lambda\} \\ &\quad + \{D(\tilde{e}_\lambda \odot D\tilde{e}^\lambda) - D\tilde{e}_\lambda \odot D(\tilde{\varphi}\,\tilde{e}^\lambda)\} = 0, \end{aligned} \tag{16.12}$$

of which the first and the second parts are symmetric with respect to the contravariant and covariant parts.

Now, we put

$$D(D\tilde{e}_\lambda \odot \tilde{e}^\lambda) + D(\tilde{\varphi}\,\tilde{e}_\lambda) \odot D\tilde{e}^\lambda = \tilde{e}_\mu \otimes \tilde{e}^\lambda \otimes \tilde{\Theta}^\mu_\lambda \tag{16.13}$$

and call $\tilde{\Theta}^\mu_\lambda$ *the curvature forms* of the universal general connection of $\mathfrak{X}$.

We shall write $\tilde{\Theta}^\mu_\lambda$ in terms of $\tilde{\theta}^\mu_\lambda$, $\tilde{\pi}^\mu_\lambda$, φ^μ_λ. Since we have

$$\begin{aligned}
&D(D\tilde{\mathfrak{e}}_\lambda \odot \tilde{\mathfrak{e}}^\lambda) + D(\tilde{\varphi}\,\tilde{\mathfrak{e}}_\lambda) \odot D\tilde{\mathfrak{e}}^\lambda \\
&= D^2\tilde{\mathfrak{e}}_\lambda \odot \tilde{\varphi}\,\tilde{\mathfrak{e}}^\lambda - \tilde{\varphi}(D\tilde{\mathfrak{e}}_\lambda) \odot D\tilde{\mathfrak{e}}^\lambda + D(\tilde{\varphi}\,\tilde{\mathfrak{e}}_\lambda) \odot D\tilde{\mathfrak{e}}^\lambda \\
&= D^2\tilde{\mathfrak{e}}_\lambda \odot \tilde{\varphi}\,\tilde{\mathfrak{e}}^\lambda + (D\tilde{\varphi} - \tilde{\varphi}D)\tilde{\mathfrak{e}}_\lambda \odot D\tilde{\mathfrak{e}}^\lambda \\
&= D(\tilde{\mathfrak{e}}_\mu \otimes \tilde{\theta}^\mu_\lambda) \odot \tilde{\varphi}\tilde{\mathfrak{e}}^\lambda + (\mathfrak{e}_\mu \otimes D\delta^\mu_\lambda) \odot D\tilde{\mathfrak{e}}^\lambda \\
&= (\tilde{\varphi}\,\tilde{\mathfrak{e}}_\mu \otimes d\tilde{\theta}^\mu_\lambda + D\tilde{\mathfrak{e}}_\mu \otimes \tilde{\theta}^\mu_\lambda) \odot \tilde{\varphi}\,\tilde{\mathfrak{e}}^\lambda + (\tilde{\mathfrak{e}}_\mu \otimes D\delta^\mu_\lambda) \odot D\tilde{\mathfrak{e}}^\lambda \\
&= (\tilde{\mathfrak{e}}_\mu \otimes \tilde{\mathfrak{e}}^\lambda) \otimes \{\varphi^\mu_\rho\, d\tilde{\theta}^\rho_\nu\, \varphi^\nu_\lambda + \tilde{\theta}^\mu_\rho \wedge \tilde{\theta}^\rho_\nu\, \varphi^\nu_\lambda - D\delta^\mu_\rho \wedge \tilde{\pi}^\rho_\lambda\},
\end{aligned}$$

it follows that

$$\tilde{\Theta}^\mu_\lambda = (\varphi^\mu_\rho\, d\tilde{\theta}^\rho_\nu + \tilde{\theta}^\mu_\rho \wedge \tilde{\theta}^\rho_\nu)\varphi^\nu_\lambda - D\delta^\mu_\rho \wedge \pi^\rho_\lambda. \qquad (16.14)$$

The formulas (16. 6), (16. 14) and Theorem 13. 3 follow immediately

Theorem 16. 1. *The torsion forms Θ^μ and the curvature forms Θ^μ_λ on $\mathfrak{B}(\mathfrak{X})$ of a given general connection Γ of $\mathfrak{X}$ are induced from the torsion forms $\tilde{\Theta}^\mu$ and the curvature forms $\tilde{\Theta}^\mu_\lambda$ on $\tilde{\mathfrak{B}}^2(\mathfrak{X})$ of the universal general connection by the canonical mapping $\rho_\Gamma : \mathfrak{B}(\mathfrak{X}) \to \tilde{\mathfrak{B}}^2(\mathfrak{X})$ of Γ.*

REFERENCES

[1] W. AMBROSE and I. SINGER: A theorem on holonomy, Trans. American Math. Soc., Vol. 75 (1953), 428—443.

[2] S. S. CHERN: Lecture note on differential geometry, Chicago University (1950).

[3] G. EHRESMANN: Les connexions infinitésimales dans un espace fibré différentiable, Colloque de Topologie (Espaces fibrés), (1950), 29—55.

[4] ————: Les prolongements d'une variété différentiable I. Calcul des jets, prolongement principal, C. R. Acad. Sci., Paris, 233 (1951), 598—600.

[5] A. NIJENHUIS: On the holonomy groups of linear connections, IA, IB & II, Indagationes Math., Vol. 15 (1953), 233—240, 241—249 & Vol. 16 (1954), 17—25.

[6] K. NOMIZU: Lie groups and differential geometry, Pub. Math. Soc. Japan (1956).

[7] T. ŌTSUKI: Geometries of connections (in Japanese), Kyōritsu Shuppan Co. (1957).

[8] ————: Theory of affine connections of the space of tangent directions of a differentiable manifold I, II, III, Math. J. Okayama Univ., Vol. 7 (1957), 1—49, 50—74, 95—122.

[9] ————: Note on homotopies of some curves in tangent bundles, Math. J. Okayama Univ., Vol. 7 (1957), 191—194.

[10] ————: On tangent bundles of order 2 and affine connections, Proc. Japan Acad., Vol. 34 (1958), 325—330.

[11] ————: Tangnt bundles of order 2 and general connections, Math. J. Okayama Univ., Vol. 8 (1958), 143—179.

[12] J. A. SCHOUTEN: Ricci-Calculus, Springer (1954).

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.

(*Received January 10, 1960*)

ERRATA: ON GENERAL CONNECTIONS I

(This Journal Vol. 9, 99—164)

TOMINOSUKE ŌTSUKI,

Page	Line	for	read
100	38	$\mathfrak{L}_n^2$ onto L_n^1	$\mathfrak{M}_n^2$ onto M_n^1 (the algebra of all $n\times n$ — matrices)
104	9	$\mathfrak{L}_n^2$	$\mathfrak{M}_n^2$
〃	14	$\mathfrak{L}_n^2$ onto L_n^1	$\mathfrak{M}_n^2$ onto M_n^1
〃	23	$\mu(\partial u_{jk})$	$\mu(\partial^2 u_{jk})$
108	11	$\dfrac{\partial u^j}{\partial u^k}\Gamma_{ih}^k$	$\dfrac{\partial v^j}{\partial u^k}\Gamma_{ih}^k$
109	1	$\eta(f_U)g_{VU}$	$\eta(f_V)g_{VU}$
113	28	*uniquely affine*	*uniquely within affine*
121	14	$U\rightarrow L_n^1$	$\pi^{-1}(U)\rightarrow L_n^1$
122	11	$U\rightarrow L_n^1$	$\pi^{-1}(U)\rightarrow L_n^1$
131	11	$\tilde{\mathfrak{L}}_n^2\rightarrow \mathrm{GL}(n+n^2, R)$	$\tilde{\mathfrak{L}}_n^2\rightarrow M_{n+n^2}^1$
138	34	$\rho(\bar{\beta})$	$\sigma(\bar{\beta})$

An addendum below (1.2) of p. 100: And we denote by $\mathfrak{M}_n^2$ the semi-group defined by the same formulas but $|a_i^j|\neq 0$.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.

(*Received December 27, 1960*)