SPECTRAL THEORY OF OPERATOR ALGEBRAS I.

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Introduction.

In this paper we shall deal with non-commutative extension problems of some well-known and primary procedures in the classical integration and ideal theories of abelian C^*-algebras.

I. *Absolute variation of linear functional on C^*-algebra.* (*§1 in Chap. 1*)

II. *Quotient space of C^*-algebra* A *devided by left ideal. The regular representation of* A *in the quotient space, and its Q^*-topology.* (*§1 in Chap. 3.*)

III. *Non-commutative extension of regularity of measure and the extension of the Lusin's Theorem.* (*§2, 3, 4 in Chap. 2*)

IV. *Banach space of vector fields, and a non-commutative extension of the Gelfand-Naimark Theory.* (*§2 in Chap. 3*)

V. *Non-commutative extension of the concept of the spectrum.* (*§2, 4 in Chap. 3*)

The existence of the absolute variation of linear functional founds the classical integration theory. It shall be extended in general C^*-algebra using an easy and elementary calculation, and offer some interesting problem in the future development of the "non-commutative extension of absolute integration"[1)] and of the weak convergence theory of linear functionals.

The quotient space of a C^*-algebra devided by a left ideal is a Banach space. The algebra is represented as an operator algebra on the quotient space by the regular reprsentation. The quotient-strong topology (or abreviately a Q^*-topology) is the self-adjoint strong topology of a suitable extended $*$-algebra on this quotient space. It has an intermediate strength between the uniform topology and the strong topology in the ordinary sence. The uniform and the strong topologies of operator algebra in the ordinary sence is a special form of this Q^*-topology. (Prop. 1. 2, 1. 3 in Chapter 3.). And an extended v. Neumann-Kaplansky Density Theorem holds (Theorem 8, 9, 10). In composing and decomposing operator algebras, the Q^*-topology and the extended concept of the regularity seem to relate in essence to their known and unknown structure problems.

1). Segal. (11).

A special roll of the regularity problem of projections in the v. Neumann-Mautner Reduction Theory shall be seen in its pecuilar measure theoretical aspect. The existence problem of exceptional null sets is changed to the problem that a projection in the strong closure of the algebra is not generally regular, but is approximated by regular projections. (Theorem 5, 6, 7 ; Prop. 4. 1 in Chap. 2).

A vector field is a mapping of a state space into the dual space of the algebra which satisfies a suitable norm condition. (§ 2. Chap. 2). If the algebra is abelian, a continuous field on its spectrum is a multiplied form of a continuous function and the coordinate field. The Gelfand-Naimark Representation Theorem is paraphrased to the isometric representativity of the algebra on the totality of continuous fields on the spectrum.

Even if the algebra **A** is non-abelian, the quotient space of **A** devided by any abelian projection is represented as a Banach space of continuous vector fields in a compact space of states. (Theorem 14).

It leads further a sort of non-commutative extension of the corresponding theory between ideals and closed sub-spaces of the spectrum in abelian algebra (Theorem 15).

But unlike in abelian case, even in a compact space of pure states, the corresponding quotient space of non-commutative algebra **A** may not generally be represented as the totality of continuous fields on that space. Hence a compact space of pure states may be called a sub-spectrum if the corresponding quotient space of **A** is justly represented on the totality of continuous fields in the space; in a word, a sub-spectrum is a compact space of pure states in which an extended Gelfand-Naimark Representation Theorem holds.

The structure theory of Q^*-algebra on the quotient space devided by abelian projection, which is the same thing with the structure theory of Q^*-algebra on a Banch space of vector fields, includes necessarily the v. Neumann's Reduction Theory. The commutor theory in Q^*-algebra (which is an extension of the v. Neumann's Density Theorem) relates to the problem of the continuity of absolute variations of continuous fields, and results an improvement of the v. Neumann's measurability theorem of the commutor family of measurable family of algebras in his Reduction Theory.

The last section is a research of a compact space of pure traces. Such a space becomes a pre-spectrum, and an extended Gelfand-Naimark Theorem, as similar as that in a sub-spectrum, holds.

Chapter 1. Preliminary state theory.

Terms and notations (1). A *C^*-algebra* is a uniformly closed self-adjoint algebra of operators in a Hilbert space $\mathfrak{H}$. In this paper we assume moreover that every C^*-algebra contains the identity. A *W^*-algebra* is a weakly closed C^*-algebra. Let $\mathfrak{M}$ be a sub-set of a Hilbert space $\mathfrak{H}$. Then $[\mathfrak{M}]$ denotes the smallest closed linear set which contains $\mathfrak{M}$. If A is a C^*-algebra on $\mathfrak{H}$, $\mathrm{A}x$ denotes the set $(Ax,\ A \in \mathrm{A})$, and $E_x^{\mathrm{A}}x$ denotes the projection of $\mathfrak{H}$ in the space $[\mathrm{A}x]$. $x \in \mathfrak{H}$ is said to be a *cyclic element of a sub-space* $\mathfrak{M}$ (relative to A) if $\mathfrak{M} = [\mathrm{A}x]$ holds.

A $*$-algebraic homomorphism of a C^*-algebra A in another $*$-algebra on a Hilbert space is said to be a *representation* of A. A representation is denoted by $A \to A_s$, and the represented algebra is by A_s, where the suffix s characterizes this representation, and shall be taken as a Hilbert space, a projection, a positive functional, a set of states, and so on, We omitt the suffix s when the consiedered representation is certified even if we do so. The *commutor* of a C^*-algbra A is denoted by A', then the *bicommutor* A'' of A is the weak closure of A, and is a W^*-algebra.

§ 1. Canonical form and absolute variation of a linear functional.

We consider a fixed C^*-algebra A, and its dual Banach space $\overline{\mathrm{A}}$. If $f \in \overline{\mathrm{A}}$ and $A \in \mathrm{A}$, we denote by Af, $A_l f$ and f^* such functionals in A as $Af(B) = f(BA)$, $A_l f(B) = f(AB)$ and $f^*(B) = \overline{f(B^*)}$.

$A \in \mathrm{A}$ is represented as an operator $f \in \overline{\mathrm{A}} \to Af$ in $\overline{\mathrm{A}}$, we call it the *regular right representation*, or merely the *representation*, of A in $\overline{\mathrm{A}}$. The operator A_l : $f \in \overline{\mathrm{A}} \to A_l f$ is said to be the regular left representation of A in $\overline{\mathrm{A}}$. f^* is the *adjoint functional* of f. A functional $p \in \overline{\mathrm{A}}$ with $p(A^*A) \geqq 0$ is *positive*. A positive functional p with $p(I) = 1$ is a *state*.

Definition 1.1. If p is a positive functional, and f is a functional in A, then $\|f\|_p = \sup_{\substack{p(A^*A) \leqq 1, \\ A \in \mathrm{A}}} |f(A^*)|$ (where $0 \leqq \|f\|_p \leqq \infty$). A functional $f \in \overline{\mathrm{A}}$ with $\|f\|_p < \infty$ is said to be *observable* in the state p. The set $(f \in \overline{\mathrm{A}} : \|f\|_p < \infty)$ is denoted by $L^2(p)$.

Lemma 1.1. *$L^2(p)$ is a Hilbert space with the norm $\|f\|_p$. Let $(x, y)_p$ denote the inner-product in $L^2(p)$, then every $f \in L^2(p)$ satisfies $f(A) = (Af, p)_p$. $L^2(p)$ is an invariant sub-space of $\overline{\mathrm{A}}$ (by the regular right representation of* A) *and* A *is represented in a C^*-algebra A_p in $L^2(p)$.*

p is a cyclic element of $L^2(p)$. We denote by E_p the projection of $L^2(p)$ in $[\mathrm{A}'p]$, then $E_p \in \mathrm{A}'' = \mathrm{A}_p''$.

Definition 1.2. If p is a positive functional and X is an operator in $L^2(p)$, $p(X)$ denotes $p(X) = (Xp, p)_p$. If X is an operator in A_p'' and if

$f \in L^2(p)$, $X_l f$ and Xf denote such functionals $\in \bar{\mathrm{A}}$ that $Xf(A)=f(AX) = (AXf,\ p)_p$ and $X_l f(A) = f(XA) = (XAf,\ p)_p$.

Proof of Lemma 1. 1. If p is a positive functional, it satisfies $|p(B^*A)|^2 \leqq p(B^*B)p(A^*A)$. Then it follows immediately that $\|Ap\|_p^2 = p(A^*A)$, $(Ap,\ Bp)_p = p(B^*A)$, and $f(A^*) = (Af,\ p)_p$ hold for every $f = Ap$ $(A \in \mathrm{A})$, and every $f \in [\mathrm{A}p] \subseteqq L^2(p)$. The Lemma is completed by proving $L^2(p) = [\mathrm{A}p]$. If $f \in L^2(p)$, then $|f(A^*)| \leqq \|f\|_p p(A^*A)^{\frac{1}{2}} = \|f\|_p \|Ap\|_p$, and we can choose $x \in [\mathrm{A}p]$ with $f(A^*) = (x,\ Ap)_p$. We see $f(A) = x(A)$ for every $A \in \mathrm{A}$; and consequently $f \in [\mathrm{A}p]$.

Lemma 1. 2. *$\bar{\mathrm{A}}$ is the linear spann of the totality of positive functionals.*

Proof. The totality $\mathcal{S}_0$ of positive functionals p with $p(I)=1$ is a weakly compact sub-set of $\bar{\mathrm{A}}$. $\mathcal{W}_0 = (p_1 - p_2 + ip_3 - ip_4 :\ p_i \in \mathcal{S}_0)$ is a regularly convex sub-set of $\bar{\mathrm{A}}$, and contains the unit ball of $\bar{\mathrm{A}}$, because, if $\mathfrak{H}$ is the underlying Hilbert space of A, $\mathcal{W}_0$ contains every $y_x \in \bar{\mathrm{A}}$ with $\|x\|$, $\|y\| \leqq 1$, where y_x denotes such a functional in A that $y_x(A) = (Ay,\ x) = \frac{1}{4}((A(x+y),\ x+y) - (A(x-y),\ x-y) + i(A(x+iy),\ x+iy) - i(A(x-iy),\ x-iy)$.

Theorem 1. *Every $f \in \bar{\mathrm{A}}$ is written for $f = Up$, where p is a positive functional and U is a partially isometric operator in A_p'' with $U^*f = p$ and $U^*U = E_p$.*

Proof. Consider a fixed $f \in \bar{\mathrm{A}}$. There is at least one positive p with $|f(A^*)|^2 \leqq p(A^*A)$ (Lemma 1. 2). Chosen a sufficiently large number γ, the totality of positive p with $|f^*(A)|^2 \leqq p(A^*A)$ and $p(I) \leqq \gamma$ consists of a non-empty compact sub-set of $\bar{\mathrm{A}}$, and contains at least a functional r with the least norm value $r(I)$. If r is such a functional, f belongs to $L^2(r)$ and satisfies $\|f\| \leqq 1$. Let α be any number $\neq 0$ and E a projection in A_r'. Then $f(A) = (Af,\ r)_r = (A(Ef + \alpha(I-E)f),\ Er + \alpha^{-1}(I-E)r)_r$. The positive functional $t(A) = (A(Er + \alpha^{-1}(I-E)r),\ Er + \alpha^{-1}(I-E)r)_r / \|Ef + \alpha(I-E)f\|^2$ satisfies $f(A^*)^2 \leqq t(A^*A)$ and $t(I) \geqq r(I) \geqq r(I)\|f\|^2 = (\|Er\|^2 + \|(I-E)r\|^2)\ (\|Ef\|^2 + \|(I-E)f\|^2)$. When α tends to $\frac{\|Er\|\ \|(I-E)r\|}{\|Ef\|\ \|(I-E)f\|}$, $t(I)$ tends to $(\|Er\|\,\|Ef\| + \|(I-E)r\|\,\|(I-E)f\|)\ (\|Er\|\ \|(I-E)f\| + \|(I-E)r\|\ \|Ef\|)$: Then $\frac{\|Er\|}{\|Ef\|} = \frac{\|(I-E)r\|}{\|(I-E)f\|} = \frac{\|r\|}{\|f\|} = k$.

If K is an Hermitian operator in A_r', K has a spectral resolution $K = \int \lambda\ d\ E(\lambda)$, where $E(\lambda)$ are projections in A_r', and $(Kr,\ r)_r = \int \lambda\ \mathrm{d}\|E(\lambda)r\|^2$

$= k^2\int \lambda \, d\|E(\lambda)f\|^2 = k^2(Kf, f)_r$. Now every $A \in A_r'$ satisfies $(A^*Ar, r)_r = k^2(A^*Af, f)_r$ and $\|Ar\| = k\|Af\|$. The transform $x = Ar \rightarrow Ux = kAf$ is extended to an isometric transform $x \in [Ar] \rightarrow Ux \in [Af]$ between $[Ar]$ and $[Af]$. U is extended to a partially isometric operator in $L^2(r)$ which vanishes in the orthogonal complement of $[Ar]$ in $L^2(r)$. U commutes to every element of A_r', and belongs to A_r''. $p = k^{-1}r$ is the desired state with $L^2(p) = L^2(r)$ and $f = k^{-1}Ur = Up$.

Definition 1.2. If a functional f in A is written for $f = Up$ by a partially isometric operator U and a positive functional p as in Theorem 3, then we call it the *canonical form* of f, and p is denoted by f^v, and is called the *absolute variation* of p.

Corollary of Theorem 1. *Given $f \in \bar{A}$, its absolute variation, f^v and the canonical form $f = Uf^v$ are uniquely determined.*

The following two are both the n. & s. conditions for a positive functional p to be the absolute variation of a given $f \in \bar{A}$.

(1. 1). *Two sequences U_n, $V_n (n = 1, 2, \dots)$ in A with norms $\leqq 1$ exist and $|U_n f - p| \rightarrow 0$, $|V_n p - f| \rightarrow 0$.*

(1. 2). *p satisfies $p(I) = |f|$ and $|f(A^*)|^2 \leqq p(A^*A)p(I)$.*

Proof. Let $f = Up$ be one of the canonical form of f, then p satisfies (1. 1) because U and U^* are in the unit ball of A_p'' and has two sequences U_n, $V_n (n = 1, 2, \dots)$ in the unit ball of A with $\|U_n p - Up\|_p \rightarrow 0$ and $\|V_n f - U^*f\|_p \rightarrow 0$ by the Kaplansky's Density Theorem.

Next, if a positive functional p satisfies (1. 1), then we have $\|U_n p\|_p \leqq \|p\|_p$ and $\|V_n f\|_p \leqq \|f\|_p$. $U_n p$ and $V_n f$ converge weakly to f and p respectively in the space $L^2(p)$, and satisfy $\|f\|_p \leqq \|p\|_p = p(I)^{\frac{1}{2}}$ and the condition (1. 2).

Now to conclude the Corollary, it is sufficient to see that two positive p_0 and p_1 which satisfy (1. 1) and (1. 2) respectively agree with each other. From $|V_n f - p_0| \rightarrow 0$ and $|V_n| \leqq 1$, it follows that $\|Ap\|_p \|f\|_p \geqq \|Ap\|_p \|V_n f\|_p \geqq |(V_n f, Ap)_p| = |V_n f(A^*)| \rightarrow |p_0(A^*)|$. Then $\|p_0\|_p \leqq \|f\|_p$ and $p_0 \in L^2(p)$. By (1. 2), $p_0(I) = |f| = p(I)$ and $(p_0, p_0)_p \leqq (p_0, p)_p = (p, p)_p = p(I) = p_0(I)$. Using the Schwaltz's inequality, we have $p = p_0$. Q. E. D.

A functional $f \in \bar{A}$ with $f = f^*$ is said to be *self-adjoint*. The existence of the absolute variation of a self-adjoint functional has been already asserted by Grothendiek[1].

Theorem 2. *Let $f = Uf^v$ denote the canonical form of a self-adjoint functional f, and put $p = f^v$. Then $U = E - (E_p - E)$, where E*

1) Grothendiek (3).

is a projection in A_p'' *with* $EE_p = E_pE = E$, $f(EA) = f(AE)$ *and* $p(EA) = p(AE)$.

Proof. $p = f^v$ and $f = Up$ are both self-adjoint, and the relations $U^*Up = p$, $p(AU) = p(U^*A)$, $p(A) = p(AU^*U)$ are preserved for $A \in A_p''$. Then $p(AU^*) = \overline{p(UA^*)} = \overline{p(UA^*U^*U)} = p(U^*UAU^*) = p(UAU^*U) = p(UA)$. $W = i(U - U^*)/2$ vanishes since $W = W^*$, $p(W^*W) = p(WW) = -p(WW) = 0$, $p(AW) = p(WA) = 0$ and $Wp = 0$. Then $Up = U^*p$, $UKp = U^*Kp(K \in A_p')$, $U = UE_p = U^*E_p$ and $V^* = E_pU = U$ holds. $E = (E_p + U)/2$ is a desired projection in A_p''.

Definition 1.4. Those positive functionals $f^+ = Ef = Ef^v$, and $f^- = -(I-E)f = (I-E)f^v$ in Theorem 2 is said to be the *positive* and *negative variations* of f, respectively.

Corollary of Theorem 2. *Every self-adjoint functional* f *is a difference* $f = f^+ - f^-$ *of two positive functionals, where* $f^v = f^+ + f^-$ *holds.*

§ 2. Induced functionals.

Consider a representation of A in a Hilbert space $\mathfrak{H}$. If x and y are two elements of $\mathfrak{H}$, the functional $y_x \in \bar{A}$ is defined by $y_x(A) = (Ay, x)$ and is said to be the *functional representation*, or merely the *functional*, of y *induced by* x. x_x is said to be the *self-induced* (positive) *functional* of x.

Proposition 2.1. *Consider a representation of* A *in a Hilbert space* $\mathfrak{H}$, *an element* x *of* $\mathfrak{H}$, *and its self-induced functional* $p = x_x$. *Then* $L^2(p) = (y_x : y \in \mathfrak{H})$. *Let* E *denote the projection* E_x^A *in* $[Ax]$, *then* $\| Ey \| = \| y_x \|_p$ *for every* $y \in \mathfrak{H}$. *Especially,* $y \to y_x$ *is an isometry between* $[Ax]$ *and* $L^2(p)$.

Definition 2.1. The mapping $y \in \mathfrak{H} \to y_x$, which is determined by the functional representation of elements of $\mathfrak{H}$ induced by a fixed element x of $\mathfrak{H}$, is said to be the induction of $\mathfrak{H}$ by x.

We now observe a relation between the above induction and the classical elementary operation[1] of algebras.

Terms and notatins (2). If E is a projection and A is an operator in $\mathfrak{H}$, the operator EAE, regarded as an operator in $\mathfrak{M} = $ *Range* E, is said to be the ***reduced operator*** of A in E. If A is a C^*-algebra and E is a projection, the smallest C^*-algebra on the *Range* E which contains $(EAE : A \in A)$ is said to be the reduced algebra of A in E (and in $\mathfrak{M}$), and is denoted by EAE. If E is a projection in the commutor A′ to A, then $A \to AE$ is a representation of A as a C^*-algebra on $\mathfrak{M} = $ *Range* E. We call it the ***induction*** of A by E, (or by $\mathfrak{M}$), and the representative algebra is said to be the ***induced algebra*** of A and denoted

1) Dixmier [1].

by A_E. If A is a W^*-algebra on $\mathfrak{H}$ and F is a projection in A′, then $(A_E)' = EA'E$ holds.

If K is a bounded or unbounded self-adjoint operator in $\mathfrak{H}$ which commutes to every element of A′, we denote by $K\eta$A.

Definition 2. 2. Consider a representation of A in a Hilbert space $\mathfrak{H}$, and an induction $y \in \mathfrak{H} \to y_x$ of $\mathfrak{H}$ by a fixed element x in $\mathfrak{H}$. If T is any bounded operator in $\mathfrak{H}$, a bounded operator T_x in the space $L^2(x_x)$ is so determined as $T_x y_x = (Ty)_x$ for every $y \in \mathfrak{H}$. T_x is said to be the *reduced operator* of T in $L^2(x_x)$. If T commutes with the projection $E = E_x^{A}$ on $[Ax]$, then T_x is said to be the *induced operator* of T in $L^2(x_x)$. If B is a C^*-algebra on $\mathfrak{H}$, then the smallest C^*-algebra which contains $(B_x : B \in \mathrm{B})$ is denoted by (B_x) and is said to be the *reduced algebra* of B by x. If the algebra B on $\mathfrak{H}$ commutes with E, the reduced-algebra is said to be the *induced algebra*, and is denoted by B_x.

Proposition 2. 2. *Consider a representation of* A *in a Hilbert space* $\mathfrak{H}$, *an element* x *of* $\mathfrak{H}$, *its self-induced functional* $p=x_x$, *and the projection* $E=E_x^{A}$ *on* $[Ax]$. *Then the isometry* $y \longleftrightarrow y_x$ *between* $[Ax]$ *and* $L^2(p)$ *determines a spatial isomorphism between* $(A_E, (A'')_E, EA'E)$ *and* (A_p, A_p'', A_p'). *Especially, we have* $A_x = (A_E)_x = A_p$, $(A'')_x = A_p''$ *and* $((A')_x) = (EA'E)_x = A_p'$.

Lemma 2. 1. *Let* p *and* q *be two positve functionals so that* $\gamma^2 p - q$ *is positive for a suitable number* γ^2. *Then* $q = Kp$, *where* K *is a definite Hermitian in* A_p'.

Proof. By Definition 1. 1 and Lemma 1. 1, we have $\|f\|_p \leqq \gamma \|f\|_q$ $(f \in \bar{A})$ and $\|Aq\|_q \leqq \gamma \|Ap\|_p$. Then $\|Aq\|_p \leqq \gamma \|Aq\|_q \leqq \gamma^2 \|Ap\|_p$. The map $Ap \to Aq$ is extended to a bounded operator K in $L^2(p)$ which commutes to every $A \in$ A. K is the desired definite Hermitian because $(KAp, Ap)_p = (Aq, Ap)_p = q(A^*A) \geqq 0$.

Lemma 2. 2. *Every positive functional* q *in* $L^2(p)$ (p *is positive*) *is written for* $q = Kp$, *where* K *is a* (*bounded or unbounded*) *self-adjoint operator* η A_p'.

Proof. q and $t = p + q$ belongs to $L^2(p)$. Consider a definite Hermitian $T \in A_t'$ with $T^2 t = p$ and $(I - T^2) = q$. $x \in L^2(t) \to Tx$ is the induction in $L^2(p)$ by Tt, because $x_{Tt}(A) = (Ax, Tt)_t = Tx(A)$. Now $L^2(p) = Range$ T contains $(At : A \in \mathrm{A})$ which is dense everywhere in $L^2(t)$, and the induction is one-to-one isometric between $L^2(t)$ and $L^2(p)$. Since the induced operator of an operator X in $L^2(t)$ is TXT^{-1}, the induced operator of T is T itself and belongs to A_p'. $K = T^{-2}(I - T^2)$ is therefore a definite self-adjoint operator with the desired properties in $L^2(p)$.

Lemma 2.3. *Let p be a positive functional and f be a linear functional in $L^2(p)$. Then $f^v \in L^2(p)$, and f has the canonical form $f = Uf^v$ by a suitable partially isometric operator U in A_p''.*

Proof. f^v is a uniform limit $f^v = \lim U_n f$, where U_n are of norms $\leqq 1$, and $\| U_n f\|_p \leqq \|f\|_p$. $U_n f$ converges to f^v by the weak topology of $L^2(p)$, then $f^v \in L^2(p)$. Let $f^v = K^2 p = q$ be a representation of f^v by a definite self-adjoint operator $K \gamma$ A_p', $f = Uq$ the canonical form of f, and U a partially isometric operator in A_q''. Then q is self-induced by K_p and the induction $X \to X_k$ is an isometry between $L^2(p)$ and $L^2(q)$. U is an induced operator of a suitable partially isometric operator V in A_p''. Then $UKx = KVx$ holds in the domain of the operator K. Especially, $f^v = Vf$ and $V^* f = f^v$ hold.

Lemma 2.4. *Let x and y be two functionals in A, then $|(x+y)^v(A^*)^2| \leqq (|x| + |y|)\,(x^v(A^*A) + y^v(A^*A))$.*

Proof. Let $x = Ux^v$ and $y = Vy^v$ denote the canonical forms of x and y. Let $t = x^v + y^v$, and K a definite Hermitian in A_t' with $|K| \leqq 1$, $x^v = Kt$ and $y^v = (I-K)t$. Then U and V are still a partially isometric operator in $L^2(t)$. Now $x(A) + y(A) = (A(UK + V(I-K))t,\ t)_t$, and $UK + V(I-K)$ is an operator in $L^2(t)$ with norm $\leqq 1$. In fact $UK = KU$ and $VK = KV$, then $\|(UK + V(I-K))z\|_t^2 = (U^*UKz,\ Kz)_t + ((V^*U + U^*V)Kz,\ (I-K)z)_t + (V^*V(I-K)z,\ (I-K)z)_t \leqq \|z\|_t^2$. Especially, $\|x+y\|_t = \|(UK + V(I-K))t\|_t \leqq \|t\|_t$ and $\|(x+y)^v\|_t \leqq \|t\|_t$. This is equivalent to the desired inequality.

Lemma 2.5. *If x_n is a sequence of functionals in A with $|x_n - x| \to 0$, then x_n^v converges to x^v by the point weak topology of $\overline{A}$.*

Proof. If $|x_n - x| \to 0$, then $|x_n^v| = |x_n| \to |x|$, and $|x_n^v|$ is bounded. Let y be any accumulating point of x_n^v by the functional weak topology of $\overline{A}$. From $x_n^v(I) = |x_n|$ and $|x_n|\, x_n^v(A^*A) \geqq |x_n(A^*)|^2$, it follows $y(I) = |x|$, $y(I)y(A^*A) \geqq |x(A^*)|^2$, and $y = x^v$. Then x_n^v converges to x^v by the functional weak topology. Choose a sub-sequence $z_n = x_{m_n}$ such that $|z_n - x| \leqq 2^{-n}$, and put $t_n = (z_{n+1} - z_n)^v$ ($n = 0, 1, 2,$ *where* $z_0 = o$). $t = \sum_{n=0}^{\infty} t_n$ converges uniformly, and $\| z_{n+1} - z_n\|_t \leqq \|t_n\|_t \leqq 2^{-n+1} (n \geqq 1)$. Then $x = \sum_{n=0}^{\infty}(z_{n+1} - z_n)$ converges in $L^2(t)$, and x^v belongs to $L^2(t)$. Now x_n^v converges to x weakly in $L^2(t)$ (since they are bounded in $L^2(t)$), and x^v is contained in the uniform convex span of x_n^v in $L^2(t)$. The functional norm is swallowed by the norm in $L^2(t)$; and the uniform convex span of x_n^v in $\overline{\mathrm{A}}$ contains x^v. Then x_n^v converges to x^v by the point weak topology of $\overline{\mathrm{A}}$.

Lemma 2. 6. *Let p, and q be two states with $q \in L^2(p)$, then $(p-q)^+$ belongs to $L^2(p)$, and $p+(p-q)^+$ is cyclic in $L^2(p)$.*

Proof. $(p-q)^+$ belongs to $L^2(p)$, because $(p-q)$ and $(p-q)^v$ belong to $L^2(p)$. Choose a definite self-adjoint operator K η A_p' with $(p-q)^+=Kp$; let E denote the projection in A_p'' with $(p-q)^+(A) = (p-q)\,(AE) = (p-q)\,(EA)$, and put $r=(p-q)^+$, then $p(EAE) - r(A) = q(EAE)$, $(KAEp,\ AEp)_p = (AEp,\ AEp)_p$, and K is of norm $\leqq 1$ on the space $[\mathrm{A}Ep]$. Put $s=p+r=p+(p-q)^+$, then $s=(I+K)p$, and $(I+K)$ is a bounded regular operator in $[\mathrm{A}Ep]$. The range $(I+K)$ contains $[\mathrm{A}Ep]$ and especially Ep, Then, Ep belongs to $[\mathrm{A}Es] = [(I+K)\mathrm{A}Ep]$. On the other hand $(I-E)s = (I-E)p$. Hence $p \in [\mathrm{A}s]$ and s is cyclic in $L^2(p)$.

Proposition 2. 3. *Consider a representation of* A *on a Hilbert space* $\mathfrak{H}$. *Let E be a projection in* A″, *$\Phi(E)$ the smallest uniformly closed linear sub-set of* A *which contains all those induced functionals* $((Ex)_y:\ y,\ x \in \mathfrak{H})$, *and $\mathcal{S}(E)$ the totality of those states* $p(A) = \sum_1^\infty (Ax_i,\ x_i)$ $(x_i \in \mathfrak{H},\ Ex_i = x_i$ *and* $\sum_1^\infty \|x_i\|^2 = 1)$. *Then the adjoint f^* of every $f \in \Phi(E)$ with $|f|=1$, has a canonical form $f^* = Uq$, (i. e., $f(A) = \sum_1^\infty (Ax_i,\ Ux_i)$), where U is a partially isometric operator in* A″, *and q is a state in $\mathcal{S}(E)$.*

Proof. We first assume that $f^* \in L^2(p)$ for a suitable $p \in \mathcal{S}(E)$. Consider the product space $\overline{\overline{\mathfrak{H}}} = \sum \oplus \mathfrak{H}_i$ of Hilbert spaces $\mathfrak{H} = \mathfrak{H}_i (i = 1, 2, \ldots)$. If A is an operator in $\mathfrak{H}$, $\overline{\overline{A}}$ denotes the operator in $\overline{\overline{\mathfrak{H}}}$ so that $\overline{\overline{A}}(\sum \oplus y_i) = \sum \oplus (Ay_i)$. $\overline{\overline{\mathrm{A}}} = (\overline{\overline{A}} : A \in \mathrm{A})$ is a representation of A on $\overline{\overline{\mathfrak{H}}}$, and the strong closure $\overline{\overline{\mathrm{A}}}''$ of $\overline{\overline{\mathrm{A}}}$ is $(\overline{\overline{A}} : A \in \mathrm{A}'')$. By the assumption $p(A) = \sum (Ax_i,\ x_i)$, $x_i \in \mathfrak{H}$, $Ex_i = x_i$ and $\sum \|x_i\|^2 = 1$ hold. Then p is a self-induced state of $x = \sum \oplus x_i$. Since $f^* \in L^2(p)$ is assumed, f^* has a canonical form $f^* = Uq = UK^2p$, where U is partially isometric in A_p'', and K is definite, self-adjoint and η A_p'. Apply Proposition 2. 1 to the induction $y \in \overline{\overline{\mathfrak{H}}} \rightarrow y_x \in L^2(p)$, then U is the induced operator $U = \overline{\overline{V}}_x$ of a suitable partially isometric operator $\overline{\overline{V}} \in \overline{\overline{\mathrm{A}}}''$ (then V is a partially isometric operator in A″), and K is the induced operator $K = T_x$ of a suitable definite seif-adjoint operator $T\gamma\overline{\overline{\mathrm{A}}}'$ in $\overline{\overline{\mathfrak{H}}}$. Let $Tx = \sum_1^\infty \oplus z_i$, then $f^*(A) = UK^2p(A) = (AVT^2x, x) = \sum_1^\infty (AVz_i, z_i)$. $q(A) = \sum_1^\infty (Az_i, z_i)$ belongs to $\mathcal{S}(E)$ since $\sum_1^\infty \|z_i\|^2 = q(I) = |f^*| = 1$, and $Ez_i = z_i$ follows from $Ex = x$ and $ETx = TEx = Tx$.

To complete the Proposition it is sufficient to see that for each $f \in \Phi(E)$ we can choose a $p \in \mathcal{S}(E)$ with $f \in L^2(p)$. If $f \in \Phi(E)$, a sequence $\{g_n\}$ in $\overline{\mathrm{A}}$ can be so chosen that each g_n is a sum of finite number of functionals y_{Ex} $(x, y \in \mathfrak{H})$ and satisfies $|g_n - f| < 2^{-n-1}$. Put $g_0 = 0$ and $h_n = g_n - g_{n-1}$ $(n \geqq 1)$, then $f = \sum_1^\infty h_n$ and $|h_n| < 2^{-n}$ $(n \geqq 2)$. Each h_n is contained in a suitable $L^2(q)$ with $q \in \mathcal{S}(E)$, and its absolute variation $s_n = \alpha_n p_n$ satisfies $p_n \in \mathcal{S}(E)$, $\alpha_n = s_n(I) = |h_n|$ $(\leqq 2^{-n}$ for $n \geqq 2)$, and $|h_n(A^*)|^2 \leqq s_n(I)s_n(A^*A) < \alpha_n^2 p_n(A^*A)$. Then $f = \sum_1^\infty h_n$ satisfies $|f(A)|^2 \leqq (|\sum h_n^*(A)|)^2 \leqq (\sum \alpha_n p_n(A^*A)^{\frac{1}{2}})^2 < (2\alpha_1^2 + \sum 2^{-n})(\sum 2^{-n}p_n(A^*A))$. $p = \sum 2^{-n}p_n$ belongs to $\mathcal{S}(E)$, and f^* belongs to $L^2(p)$. Q. E. D.

Corollary. *Let p be a state, and $\mathfrak{M}$ the range of E_p. Then $\mathcal{S}(E_p)$ is the set $(x_x :\ x \in \mathfrak{M}$ and $\|x\|_p = 1)$. Every $f \in \Phi(E_p)$ is written for* $f(A) = (Ax, Ux)_p$, where V is *a suitable partially isometric operator* in A_p'', *and* x is an element in $\mathfrak{M}$.

Proof. $\mathcal{S}(E_p)$ is the uniform convex hull of the totality of self-induced states $(Kp)_{Kp} = K^*Kp$ with $K \in \mathrm{A}_p'$ and $K^*Kp(I) = 1$. Consider a fixed $q \in \mathcal{S}(E_p)$ and choose a sequence of definite Hermitians $K_n \in \mathrm{A}_p'$ so that $|q - K_np| \leqq 2^{-n-1}$ and $K_np(I) = 1$. Put $r_0 = p$, $r_1 = K_1p$, $r_n =$ the absolute variation of $(K_n - K_{n-1})p$ for $n \geqq 2$ and $t = \sum r_n$. Then $r_n \in L^2(p)$, $r_n(I) = 2^{-n}$, $t(I) < \infty$ and $r_0 = p = K^2t = (Kt)_{Kt}$, where K is a suitable definite Hermitian $\in \mathrm{A}_t'$. We consider the induction in $L^2(t)$: $x \in L^2(t) \to x_{Kt} = Kx \in L^2(p)$. Then *Range* K $(= L^2(p))$ contains r_n, its uniform closure contains $t = \sum r_n$ and all $L^2(t)$, and the induction $x \to Kx$ is an isometry between $L^2(t)$ and $L^2(p)$. Now $q = K_1p + \sum(K_n - K_{n-1})p \in L^2(t)$ follows from $K_1p = r_1$, $|(K_n - K_{n-1})p(A)|^2 \leqq r_n(I)r_n(A^*A)$ $(n \geqq 2)$ and $|q(A)|^2 \leqq (\sum(r_n(I)r_n(A^*A))^{\frac{1}{2}})^2 \leqq (\sum r_n(I))(\sum r_n(A^*A)) \leqq t(I)t(A^*A)$. q is positive in $L^2(t)$, and written for $q = St$, where S is a definite self-adjoint operator $\gamma \mathrm{A}_t''$. The induced operator $T = S_{Kt}(= KSK^{-1})$ is defined in $L^2(p)$. $L^2(p) =$ *Range* K contains $Kt = K^{-1}p$; $K^{-1}p$ and $TK^{-1}p$ belong to $\mathfrak{M} = (Kp\colon K \in \mathrm{A}_p')$ and satisfy $q(A) = (ASt, St)_t = (ATK^{-1}p, TK^{-1}p)_p = (Ax, x)_p$; then $\|x\|_p^2 = q(I) = 1$.

Finally, if $f \in \Phi(E_p)$, and $|f| = 1$, f^{*v} belongs to $\mathcal{S}(E_p)$ and is a self-induction x_x of an $x \in \mathfrak{M}$. Then the canonical form of f^* is $f^*(A) = (AUx, x)_p$, where U is a partially isometric operator in $\mathbf{A}_p'$ with $U^*Ux = x$. Hence $f(A) = (Ax, U^*x)_p$.

§ 3. The intersection operator.

Consider a fixed state p. We denote by $\mathfrak{F}(p)$ the totality of such functionals f in $L^2(p)$ that f^* also belongs to $L^2(p)$, and by $[\mathfrak{F}(p)]$ its uniform closure. If K is any bounded operator in $\mathbf{A}_p'$, $x = Kp$ belongs to $\mathfrak{F}(p)$. We call such an x a *bounded element* of $\mathfrak{F}(p)$. If x is bounded in $\mathfrak{F}(p)$, the operator $K \in \mathbf{A}_p'$ with $x = Kp$ is uniquely determined. If $x = Xp$ and $y = Yp$ are two bounded elements with $X, Y \in \mathbf{A}_p'$, then we put $xy = XYp$. We denote by $\mathfrak{S}(p)$ the totality of self-adjoint elements of $L^2(p)$.

Lemma 3. 1. *$\mathfrak{S}(p)$ is a real Hilbert sub-space of $L^2(p)$. And every $f \in \mathfrak{F}(p)$ is a sum $f = f_1 + if_2$ ($f_1, f_2 \in \mathfrak{S}(p)$).*

Lemma 3. 2. $[\mathfrak{F}(p)] = [\mathbf{A}_p'p]$. *$E_p$ is the projection on* $[\mathfrak{F}(p)]$.

Definition 4. 1. We denote by E the projection E_p, and by $\mathbf{A}_E''$ the reduced algebra $(EAE : A \in \mathbf{A}_p'')$ of $\mathbf{A}_p''$ on the space $[\mathfrak{F}(p)]$.

Then $\mathbf{A}_E''$ and $\mathbf{A}_p{}'_E$ are a commutor-pair on the space $[\mathfrak{F}(p)]$.

Every $x \in [\mathfrak{F}(p)]$ is regarded as a linear functional $x(K) = (Kx, p)_p$ on $\mathbf{A}_p'$. If $x \in [\mathfrak{F}(p)]$ is a state on $\mathbf{A}_p'$, it is said to be a *dual state.* If x is positive on $\mathbf{A}_p'$, it is said to be *dually positive.* If x is self-adjoint (bounded) on $\mathbf{A}_p'$, it is said to be *dually self-adjoint* (*dually bounded*).

We denote by $\mathfrak{F}^*(p)$ the totality of $x \in [\mathfrak{F}(p)]$ whose dual adjoint belongs to $[\mathfrak{F}(p)]$, and by $\mathfrak{S}^*(p)$ the totality of dually self-adjoint element of $[\mathfrak{F}(p)]$. As a dual of Lemma 3. 1 and Lemma 3. 2, we have following

Lemma 3.3. *$\mathfrak{S}^*(p)$ is a real Hilbert sub-space of $L^2(p)$, and every $f \in \mathfrak{F}^*(p)$ is written for $f = f_1 + if_2$ with $f_1, f_2 \in \mathfrak{S}^*(p)$. If $f \in \mathfrak{S}(p)$ and $x \in \mathfrak{S}^*(p)$, then (f, x) is real.*

The last assertion follows from the fact that by the next Lemma 3. 4 bounded self-adjoint (and dually bounded dually self-adjoint) elements are dense everywhere in $\mathfrak{S}(p)$ and $\mathfrak{S}^*(p)$, respectively.

Lemma 3. 4. *Given any* $x \in \mathfrak{F}(p)$, we *can choose a sequence of bounded elements x_n with $\|x_n - x\|_p \to 0$ and $\|x_n^* - x_n^*\|_p \to 0$.*

Let $x = y + iz$ ($y, z \in \mathfrak{S}(p)$). If y and z are approximated by bounded self-adjoint elements y_n, z_n in $\mathfrak{S}(p)$, then $x_n = y_n + iz_n$ satisfies $\|x_n - x\|_p \to 0$ and $\|x_n^* - x^*\|_p \to 0$. $y \in \mathfrak{S}(p)$ is written for $y = y^+ - y^-$, where y^+, y^- are positive and negative variations of y, and they are approximated by bounded definite Hermitians in $\mathfrak{S}(p)$. Then the Lemma is concluded.

Theorem 3. *Consider a fixed state p in* A. *Then the mapping $x \to x^*$ is a closed operator in $\mathfrak{C}(p)$, and there exists an unbounded definite self-adjoint operator J in $\mathfrak{C}(p)$ whose domain $\mathfrak{D}(J)$ is dense everywhere in $\mathfrak{C}(p)$ and satisfies $(Jx, y)_p = (y^*, x^*)_p$. $\mathfrak{D}(J^{\frac{1}{2}})$ contains $\mathfrak{C}(p)$ and satisfies $\|J^{\frac{1}{2}}x\|_p = \|x^*\|_p$.*

Proof. The closedness of $x \to x^*$ in $\mathfrak{C}(p)$ is obvious, since $\|x_n - x\|_p \to 0$ and $\|x_n^* - y\|_p = 0$ imply $y = x^*$. The existence of J is shown by the well-known method. Introduce in $\mathfrak{C}(p)$ a new inner-product $<x, y> = (x, y)_p + (y^*, x^*)_p$; then $\mathfrak{C}(p)$ is a Hilbert space by this inner-product. Let T denote the definite Hermitian in $\mathfrak{C}(p)$ such that $(x, y)_p = <Tx, y> = <T^{\frac{1}{2}}x, T^{\frac{1}{2}}y>$. Then $\|x\|_p^2 = \||T^{\frac{1}{2}}x\||^2 = <T^{\frac{1}{2}}x, T^{\frac{1}{2}}x>$ and $<x, x> \neq 0$ imply $\|x\|_p \neq 0$. *Range* T is therefore dense everywhere in $\mathfrak{C}(p)$, and $T^{-1}(I - T)$ exists. This self-adjoint operator in $\mathfrak{C}(p)$ is maped on the self-adjoint operator on $[\mathfrak{C}(p)]$ by the isometry $\|x\|_p = \||T^{\frac{1}{2}}x\||$.

Definition 3.1. The operator J in Theorem 3 is said to be the *intersection operator* in $L^2(p)$.

Proposition 3.1. *Let x and y be two elements in $\mathfrak{C}(p)$ so that Jx, Jy exist and are bounded in $\mathfrak{C}(p)$. Then $(Jx)(Jy) = J(xy)$.*

Proof. If z is a bounded element of $\mathfrak{C}(p)$, then $(xy, Jz) = (z^*, y^*x^*) = (yz^*, x^*) = (Jx, zy^*) = (z^*(Jx), y^*) = (Jy, (Jx)^*z) = ((Jx)(Jy), z)$. Hence $J(xy) = (Jx)(Jy)$.

Lemma 3.5. *Every $x \in \mathfrak{C}(p)$ is written for $x = UKp$, where U is a partially isometric operator in A_p' and K is a definite (not necessarily bounded) self-adjoint operator which $\eta A_p''$.*

Proof. Regard x as a functional on A_p' in the sence of Definition 3.1. Then its canonical form is the above. (See Lemma 2.2, 2.3.)

We call the representation form $x = Uy = UKp$ in Lemma 3.5 the *dual canonical form*, and $y = Kp$ the *dual absolute variation* of x.

Lemma 3.6. *Let x be any bounded element in $\mathfrak{C}(p)$, and $y = ux = Kp$ be the dual absolute variation of x, where u is partially isometric in $\mathfrak{C}(p)$, and $K \eta A_p''$ is definite self-adjoint. Then we have $J(y^*) = y$.*

Proof. $(y^*, Jz) = (z^*, Kp) = (Kp, z) = (y, z)$, (*for* $z \in \mathfrak{C}(p)$). Hence y^* belongs to the domain $\mathfrak{D}(J)$ of J, and $J(y^*) = y$ holds.

Lemma 3.7. *The domain $\mathfrak{D}(J^{-1})$ of J^{-1} is $(x^* : x \in \mathfrak{D}(J))$, and satisfies $J^{-1}x = (Jx^*)^*$. Similarly, $\mathfrak{D}(J^{-1}) = (x^* : x \in \mathfrak{D}(J^{\frac{1}{2}})) = \mathfrak{C}(p)$,* and $J^{-\frac{1}{2}}x = (J^{\frac{1}{2}}x^*)^*$.

Proof. Apply the notation in the proof of Theorem 3. By the inner-product $<x, y>$, $\mathfrak{E}(p)$ becomes a Hilbert space, and satisfies $<x, y> = <y^*, x^*>$ and $(Tx^*)^* = I-T$, where T is a definite Hermitian operator so that $(x, y)_p = <Tx, y>$. $J = T^{-1}(I-T)$ satisfies $(Jx^*)^* = J^{-1}x$ and $(J^{\frac{1}{2}}x^*)^* = J^{-\frac{1}{2}}x$, applying the theory of unitary equivalence of Hermitian operators.

Theorem 4. *$J^{\frac{1}{2}}$ determines an isometry between $\mathfrak{S}^*(p)$ and $\mathfrak{S}(p)$.*

Proof. Let $x \in \mathfrak{S}(p)$; then $(J^{\frac{1}{2}}x, y) = (y^*, J^{\frac{1}{2}}x)$ (where $y \in \mathfrak{E}(p)$) and hence $J^{\frac{1}{2}}x \in \mathfrak{S}^*(p)$. Conversely, let x be any bounded element in $\mathfrak{E}(p)$ and in $\mathfrak{S}^*(p)$ commonly. By Lemma 3. 6, $J(x^*) = x$ and $y = J^{-\frac{1}{2}}(x^*)$ exist. We have $y = y^*$, and $\|x\|_p = \|y\|_p$.

Given any dual positive functional x_0 in $\mathfrak{E}(p)$, we can choose a sequence $\{z_n\}$ of bounded elements in $\mathfrak{E}(p)$ with $\|z_n - x_0\|_p \to 0$. Let x_n denote the dual absolute variation of z_n, and y_n an element of $\mathfrak{S}(p)$ with $J^{\frac{1}{2}}y_n = x_n$. By Lemma 2. 5, x_n converges weakly to x_0 in $L^2(p)$. $y \in \mathfrak{S}(p) \to J^{\frac{1}{2}}y$ is an isometry, and y_n converges weakly to a $y_0 \in \mathfrak{S}(p)$. Then $x_0 = J^{\frac{1}{2}}(y_0) \in$ *Range* $J^{\frac{1}{2}}$. $J^{\frac{1}{2}}$ is an isometry between $\mathfrak{S}(p)$ and $\mathfrak{S}^*(p)$.

Corollary 1. $J^{\frac{1}{2}}(\mathfrak{E}(p)) = \mathfrak{E}^*(p)$.

Corollary 2. *J^{-1} is such an operator that $(J^{-1}x, y)_p = (y^s, x^s)_p$, where $x, y \in \mathfrak{E}^*(p)$, and x^s is the dual adjoint of x. Hence $x^s = Jx^*$ holds in $\mathfrak{E}(p)$.*

Proof. If $x = x_1 + ix_2$ $(x_1, x_2 \in \mathfrak{S}(p))$ is any element of $\mathfrak{E}(p)$, $J^{\frac{1}{2}}x = (J^{\frac{1}{2}}x_1) + i(J^{\frac{1}{2}}x_2)$ belongs to $\mathfrak{E}^*(p)$, and the Corollary 1 follows. Now J^{-1} is the intersection operator in $\mathfrak{E}^*(p)$ relative to the algebra A_E'', then Corollary 2 follows.

Remark. There are many unsolved questions relative to the general properties of intersection operators.

(1). *Does $J^{\frac{1}{2}}$ satisfy $J^{\frac{1}{2}}(xy) = (J^{\frac{1}{2}}x)(J^{\frac{1}{2}}y)$?*

(2). *Can we construct directly (without use the dimension theory) the trace theory developing our theory?*

(3). *The concept of the intersection operator is defined in the generalized standard algebra.*

Let M be a W^*-algebra. A functional t, which is defined in the totality of definite Hermitians in M, is called an essentially bounded positive functional, if $0 \leq t(A) \leq \infty$, $t(\alpha A) + = \alpha t(A)$ for $\alpha \geq 0$, $t(A+B) = t(A) + t(B)$ and $t(A) = \sup_{\substack{0 \leq B \leq A \\ t(B) < \infty}} t(B)$ is satisfied.

t is said to be *faithful* if $t(A) \neq 0$ for every definite Hermitian $A \neq 0$. t is said to be *normal*[1] if $t(A) = \sup_{B \in \mathfrak{F}} t(B)$ holds whenever A is the supremum of an assending filter $\mathfrak{F}$ of definite Hermitians in M. Every W^*-algebra has at least one normal faithful essentially bounded positive functional.

1) Dixmier [1].

Let t be a fixed normal faithful essentially bounded positive functional on a W^*-algebra M. The totality $\mathfrak{H}_0$ of elements A in M with $t(A^*A) < \infty$ is a pre-Hilbert space with an inner-product $(A, B) = t(B^*A)$ and a norm $\|A\| = t(A^*A)^{\frac{1}{2}}$. The completed Hilbert space of $\mathfrak{H}_0$ is denoted by $\mathfrak{H}$. If $B \in \mathfrak{H}_0$, then $t_B : t_B(A) = t(B^*AB)$ is a normal positive functional $\in \overline{\mathrm{M}}$ in the Dixmier's sence. The representation of M as an operator algebra on $\mathfrak{H}$ is a normal isomorphism, and the representative algebra is a W^*-algebra on $\mathfrak{H}$. Those $x \in \mathfrak{H}$, whose representative functional $x(A) = (x, A^*)$ on M is bounded, consists of a dense linear sub-set of $\mathfrak{H}$. An $x \in \mathfrak{H}$ is said to be *bounded* if $\sup_{t(A^*A)\leqq 1,\ t(B^*B)\leqq 1} (Ax, B) < \infty$ holds. A bounded element x of $\mathfrak{H}$ determines an operator $L_x \in \mathrm{M}'$ with $L_xA = Ax$ (where $A \in \mathfrak{H}_0$). Those L_x are strongly dense everywhere in the commutor M′ in $\mathfrak{H}$. When $x^* \in \mathfrak{H}$ with $(L_x)^* = L_{x^*}$ exists, x^* is said to be the adjoint of $x \in \mathfrak{H}$. $x \rightarrow x^*$ is a closed operator whose domain is dense everywhere in $\mathfrak{H}$, and the intersection operator J is so defined as a closed operator in $\mathfrak{H}$ with $(Jx, y) = (y^*, x^*)$: Concerning these operators, analogous procedures in this section may be established, however the detail shall be omitted here.

§ 4. Algebraic properties of positive functionals.

A propertiy P of W^*-algebra is said to be an *algebraic property* if it is invariant by $*$-algebraic isomorphisms and anti-isomorphisms between algebras.

Lemma 4.1. *The following properties are algebraic properties of W^*-algebras.*

(1). *An algebra* R *is of one-dimensional* R $= (\alpha I)$.

(2). *An algebra* R *is abelian.*

(3). *An algebra is a factor.*

(4). *An algebra is of discrete type, continuous type, finite type, semi-finite type, etc.*

Terms and notations 3. A *factor* is a W^*-algebra whose center is of one-dimensional. A projection E in a W^*-algebra M is said to be *abelian* in M if the reduced algebra EME is abelian. A *discrete* algebra is a W^*-algebra which contains at least an abelian projection E so that there is non-zero projection Z in the center of the algebra with $ZE = 0$. An algebra is said to be of *continuous type* if it does not contain any non-zero abelian projection. An algebra is *of finite type* if every partially isometric operator U in it with $U^*U = I$ is unitary. An algebra R is of *semi-finite type* if every non-zero projection $E \in \mathrm{M}$ has at least an non-zero projection $F \leqq E$ so that the reduced algebra EME is of finite type.

Definition 4.1. Let p be a positive functional and P be an algebraic property of W^*-algebras. p is said to *have the property* P if the algebra A_p' has this property. Especially,

(1). p is said to be of *one-dimensional* if A_p' is of one-dimensional.

(2). p is said to be *abelian* if A_p' is abelian.

(3). p is said to be a *factor*, *of discrete type*, *of continuous type*, *of finite type*, etc., if A_p' is so.

Definition 4.2. Let P be an algebraic property of W^*-algebras. Then a projection E in a W^*-algebra R is said to *have the property* P if the reduced algebra $E\mathrm{R}E$ has this property.

Lemma 4.2. *Let P be an algebraic property of W^*-algebras, and p a positive functional of semi-finite type. Then, p has the property P if and only if the projection E_p has the same property in A_p''.*

(1). *A positive functinal is of one-dimensional if and only if E_p is of one-dimensional.*

(2). *p is abelian if and only if E_p is abelian in A_p''.*

(3). *p is a factor, of disctete type, of continuous type, of finite type, etc., if and only if E_p is so in A_p''.*

The proof follows from the next well-known sub-lemma.

Sub-lemma.[1] *Let R be a W^*-algebra of semi-finite type in a Hilbert space $\mathfrak{H}$ which contains at least one g with $\mathfrak{H} = [\mathrm{R}g] = [\mathrm{R}'g]$*[2]*, Then $\mathfrak{H}$ is regarded as a completed Hilbert space of a suitable Hilbert algebra, whose right and left associated algebras are R and R' respectively. Hence there is an $*$-algebraic anti-isomorphism between R and R'.*

Let p be positive functional of semi-finite type, and put $E = E_p$, then the induction $K \in \mathrm{A}_p' \rightarrow KE \in (\mathrm{A}_p')_E$ is an algebraic isomorphism between A_p' and $(\mathrm{A}_p')_E$. These induced and reduced algebras $\mathrm{R} = E(\mathrm{A}_p'')E$ and $\mathrm{R}' = (\mathrm{A}_p')_E$ are coupled W^*-algebras of semi-finite types on the Hilbert space $\mathfrak{H} = [\mathrm{A}_p'p] = Range\ E$, and it satisfies $\mathfrak{H} = [\mathrm{R}p] = [\mathrm{R}'p]$. A_p', R' and R are $*$-algebraically anti-isomorphic. Then p has the property P if and only if $E = E_p$ has this property.

(b) **Pure and relatively pure states.**

Definition 4.3. A one-dimensional state p is said to be a *pure state.* If p is a state, we denote by $\mathrm{N}(p)$ the left ideal $\mathrm{N}(p) = (A \in \mathrm{A}:\ p(A^*A) = 0)$.

Lemma 4.4. $\mathrm{N}(p) = \mathrm{N}(E_p) = (A \in \mathrm{A}:\ AE_p = 0)$. *A state p vanishes in a left ideal N of A if and only if* $\mathrm{N}(p) \supseteqq \mathrm{N}$.

By the Kadison's irreducibility Theorem (cf. Proposition 2.1 in

1) Dixmier [1] 2) g is a *totalizateur* and *separateur* in the Dixmier's sence.

Chap. 2), a state p is pure if and only if $\mathrm{N}(p)$ is a maximal left ideal of A. Conversely, if N is a maximal left ideal of A, a pure state p with $\mathrm{N}(p) = \mathrm{N}$ is uniquely determined.

Definition 4.4. A state p is said to be *pure relative to a given element A of* A, if $p(A) = q(A)$ holds whenever q be a state vanishing on $\mathrm{N}(p)$.

Lemma 4.5. *A state p is pure if and only if it is pure relative to every element of* A.

Lemma 4.6. *If p is a pure state relative to a fixed $A \in \mathrm{A}$, then there exists at least a pure state q with $p(A) = q(A)$ and $\mathrm{N}(p) \subseteqq \mathrm{N}(q)$.*

(c). **Abelian projection.**

Terms and notations 4. An abelian C^*-algebra C is isomorphic to the C^*-algebra $\mathrm{C}(\Omega)$ of the totality of continuous functions on its *spectrum* Ω. Ω is the totality of pure states on C. A state p on C is pure if and only if $f \to p(f)$ is a homomorphism of C in the complex number field.

A regular measure μ on Ω with the total mass 1 is said to be a *distribution.* A positive functional p on C is an *integral* $p(f) = \int f\,d\mu$ by a regular measure μ on Ω. If p is a state, the measure μ is a distribution.

Consider a set $\mathcal{W}$ of regular measures on Ω, and a function f in $\mathrm{C}(\Omega)$. The *primitive function* J_f of f in $\mathcal{W}$ is a function $J_f(p) = \int f(\omega)\,dp(\omega)$ on $\mathcal{W}$. The mapping $f \in \mathrm{C}(\Omega) \to J_f$ is said to be the *integration* in $\mathrm{C}(\Omega)$. The primitive function $J_f(x) = \int_a^x f(t)\,dt$ just falls to the definition regarding the variable x as the indicator of the Lebesgue mesure in the interval $[a, x]$.

Definition 4.5. Consider a representation of A in a Hilbert space $\mathfrak{H}$. A projection E in $\mathfrak{H}$ is said to be *abelian relative to* A if the reduced algebra $E\mathrm{A}E$ is abelian. If E is a projection in $\mathfrak{H}$, the set $\mathfrak{H}_0 = [\mathrm{A}E\mathfrak{H}] = [AEx : A \in \mathrm{A} \text{ and } x \in \mathfrak{H}]$ is said to be the sub-space of $\mathfrak{H}$ *generated by E*, and E a *generative projection* in $\mathfrak{H}_0$.

Lemma 4.7. *Let* A *be a C^*-algebra in $\mathfrak{H}$, E an abelian projection in $\mathfrak{H}$ relative to* A, *$\mathfrak{H}_0$ the generated space $\mathfrak{H}_0 = [\mathrm{A}E\mathfrak{H}]$, and A an operator in* A. *Then a bounded linear operator J_A in $\mathfrak{H}_0$ is so determined as $J_A BE = BEAE$.*

Definition 4.6. The operator J_A in Lemma 4.7 is said to be the *primitive operator* of A reduced by E. The smallest C^*-algebra C_E which contains the totality of primitive operators of elements of A is said to be the *carrier algebra* of E. The spectrum of C_E (C_E is an abelian algebra) is said to be the *carrier* of E. The mapping $A \in \mathrm{A} \to J_A \in \mathrm{C}_E$ is said to be the *integration*. The smallest C^*-algebra $\mathrm{K}_E = \mathrm{A} \cup \mathrm{C}_E$ which contains C_E and the induced algebra of A in $\mathfrak{H}_0$ is said to be the *diagona-*

lizable extension, or *diagonalizor,* of A.

Lemma 4.8. *The carrier algebra* C_E *of an abelian projection* E *relative to* A *is an abelian algebra in* $\mathfrak{H}_0 = [AE\mathfrak{H}]$, *and commutes to* E *and* A.

To prove these two Lemmas, we recall the next two sub-lemmas in the theory of v. Neumann's elementary operations.

Sub-lemma 1. *Let* R *be a* W^**-algebra in a Hilbert space* $\mathfrak{H}$, *and* E *be a generative projection in* R. *Then the induction* $A \in R' \rightarrow AE \in R'_E$ *of* R' *in* R'_E *is an isomorphism.*

Proof. If $A \in R'$ and $AE = 0$, then $Ax = 0$ for every $x = AEy$ $(A \in A,\ y \in \mathfrak{H})$ and every $x \in [AE\mathfrak{H}] = \mathfrak{H}$. Hence A vanishes, and the induction becomes an isomorphism.

Sub-lemma 2. *Let the projection* E *in Sub-lemma* 1 *be a generative abelian projection in* $\mathfrak{H}$. *Then the reduced algebra* ERE *of* R *in* E *is the induced algebra of the center* Z *of* R, *and the induction of* Z *in* E *is an isomorphism.*

Proof. ERE is abelian, and becomes the center of the commuter $(ERE)' = R'_E$. Since the induction of R'' in E is an isomorphism, ERE is the induced algebra of the center Z of R', which is simultaneously the center of R.

Proof of Lemma 4.7 *and* 4.8. We can assume without loss of generality that E is generative in $\mathfrak{H}$ relative to A (because we can replace the original representation of A in $\mathfrak{H}$ by the induced representation of A in $\mathfrak{H}_0$). Consider the W^*-algebra $R = (A \cup E)''$. $ERE = E(A \cup E)E'' = EAE''$ is abelian, and E is a generative abelian projection in R. Then ERE is the induced algebra of the center Z of R. EAE is a sub-algebra of ERE, and is an induced algebra of a suitable C^*-sub-algebra C_E of Z. Every induced operator EAE of $A \in A$ is an induced operator $J_A E = EAE$ of a suitable J_A in C_E, and satisfies $J_A BE = BJ_A E = BEAE$ for every $B \in A$. Operator J_A is uniquely determined, because E is generative relative to A.

Definition 4.7. Consider a representation of A in a Hilbert space $\mathfrak{H}$, and an abelian W^*-sub-algebra M of the commutor A'. An element g of $\mathfrak{H}$ is said to be *compoundly cyclic* relative to A and M if g is cyclic in $\mathfrak{H}$ relative to A$\cup$M, and in $[Mg]$ relative to the reduced algebra of A in $[Mg]$ (, ie., $\mathfrak{H} = [(A \cup M)g]$ and $[Mg] = [EAEg]$ holds, where E is the projection in $[Mg]$).

Lemma 4.9. *Consider a representation of* A *in a Hilbert space* $\mathfrak{H}$ *and an abelian generative projection* E *in* $\mathfrak{H}$ *relative to* A. *If* g *is an element of Range* E *and is cyclic relative to* $K_E = A \cup C_E$, *then the*

W^*-closure C_E'' is an abelian W^*-sub-algebra of the commutor A', and g is compoundly cyclic in $\mathfrak{H}$ relative to A and C_E''.

Proof. $\mathfrak{H} = [K_E g] = [(A \cup C_E'')g]$ and *Range* $E = [C_E g] = [EAEg]$ holds. Then g is compoundly cyclic relative to A and C_E''.

Lemma 4.10. *Consider a representation of A in a Hilbert space $\mathfrak{H}$, an abelian W^*-sub-algebra M of A' and a compoundly cyclic element g of $\mathfrak{H}$ relative to A and M. Then the projection E on $[Mg]$ is generative and abelian, M is the W^*-closure of the carrier algebra C_E of E, and g is cyclic in $\mathfrak{H}$ relative to K_E.*

Proof. E is clearly generative and abelian, Then it is sufficient to show $C_E'' = M$. The induction $A \rightarrow AE$ of M in $[Mg] =$ *Range* E is an isomorphism. Then $C_E'' = M$ holds if and only if the induced algebra M_E is the W^*-closure of the induced algebra $(C_E)_E = EAE$ of C_E, where EAE is the reduced algebra of A in $[Mg]$. g is cyclic in $[Mg]$ relative to the abelian algebra EAE. Then $(EAE)'' = (EAE)'$. Hence $(EAE)' \supseteqq M_E \supseteqq EAE$ implies $M_E = (EAE)''$ and $M = C_E''$.

Lemma 4.11. *Consider a representation of A in a Hilbert space $\mathfrak{H}$ with a cyclic element g, and an abelian W^*-sub-algebra M of A'. Then g is compoundly cyclic relative to A and M.*

Proof. Let E denote the projection in $[Mg]$, then $\mathfrak{H} = [Ag] = [(A \cup M)g]$ and $[Mg] =$ *Range* $E = [EAEg : A \in A] = [Xg : X \in EAE]$ holds. Hence g is compoundly cyclic relative to A and M.

Chapter 2. Reduction Theory and regularity problem in Operator algebra.

Terms and notations 5. Consider a compact space Ω, a distribution μ on it, a normed space $\mathfrak{E}$ and its dual space $\bar{\mathfrak{E}}$. A vector-valued function x, whose range is contained in $\mathfrak{N}$, is said to be *regularly weakly measurable* (*or measurable in the Lusin's sence*) if it becomes weakly continuous removing any small mass from Ω. The value of a vector valued function x at $\omega \in \Omega$ is denoted by x_ω. The *carrier* of μ is the smallest compact sub-set of Ω whose total mass is 1. The totality of continuous functions on the carrier of μ is denoted by $C\mu$ and said to be the *carrier algebra* of μ. The totality of bounded measurable functions on Ω is denoted by $\mathbf{M}(\mu)$.

An open sub-set $\mathfrak{D}$ of Ω is said to be *maximally open* if there is not any open set which contains $\mathfrak{D}$ properly and has same the mass with $\mathfrak{D}$. For any open sub-set $\mathfrak{D}$ of Ω, there is at least one maximally open set which contains $\mathfrak{D}$ and has the same mass with $\mathfrak{D}$. We notice further,

Lemma. *Let x be an $\bar{\mathfrak{E}}$-valued regularly weakly meaurable function on Ω. If $x_\omega(A)$ vanishes almost everywhere for each fixed $A \in \mathfrak{E}$, then x_ω vanishes almost everywhere.*

Proof. Given any $\varepsilon > 0$, we can choose a maximally open set $\mathfrak{D}$ whose mass

$<\varepsilon$, and x becomes weakly continuous in $\Omega - \mathfrak{D}$. x_ω vanishes on $\Omega - \mathfrak{D}$, and consequently x vanishes on Ω, almost everywhere.

§ 1. Hilbert space of vector fields. (A measure theory in a state space).

(a). Hilbert space of vector fields.

We denote by $\mathcal{S}$ the totality of states on A, $\mathcal{S}$ is a weakly compact sub-set of $\bar{\mathrm{A}}$. In what follows the topology of $\mathcal{S}$ and its sub-space shall be always meant by the weak topology. Let $\mathcal{W}$ be a compact sub-set of $\mathcal{S}$. We denote by C the totality of continuous functions on $\mathcal{S}$, and by $\mathrm{C}_{\mathcal{W}}$ the totality of continuous functions on $\mathcal{W}$. If A is an element of A, the function J_A on $\mathcal{S}$ so that $J_A(\omega) = \omega(A)\,(\omega \in \mathcal{S})$ is said to be the *primitive function* of A. The mapping $A \in \mathrm{A} \rightarrow J_A$ is said to be the *integration*. The primitive function and the integration (restricted) in $\mathcal{W}$ is similarly defined.

Lemma 1.1. *Let $\mathcal{W}$ be a compact sub-set of $\mathcal{S}$. Then $\mathrm{C}_{\mathcal{W}}$ is the smallest C^*-algebra of continuous functions in $\mathcal{W}$, which contains the totality of primitive functions of elements in* A.

The Lemma follows immediately from the reducibility of any two points in $\mathcal{W}$ by the primitive function of an element of A.

Definition 1.1. Let $\mathcal{W}$ be a compact seb-set of $\mathcal{S}$. An $\bar{\mathrm{A}}$-valued function x, whose each value x_ω at ω is observable (i.e. $x \in L^2(\omega)$) in that state ω, is said to be a *vector field*, or merely a *field*, on $\mathcal{W}$.

A special field ω, whose value at $\omega \in \mathcal{S}$ is ω, is said to be the *coordinate field.*

An operator valued function X on $\mathcal{W}$, whose value X_ω at $\omega \in \mathcal{W}$ is a bounded operator in $L^2(\omega)$ and satisfies $|X|_{\mathcal{W}} = \sup_{\omega \in \mathcal{W}} |X_\omega| < \infty$, is said to be an operator-field on $\mathcal{W}$. A projection field P on $\mathcal{S}$ so that each $P_\omega(\omega \in \mathcal{S})$ is the projection of $L^2(\omega)$ in the one-dimensional space $(\alpha\omega)$, (i. e., $P_\omega x_\omega = (x_\omega, \omega)_\omega \omega$ for every $x \in L^2(\omega)$) is said to be the *coordinate projection field.*

If X is an operator field and x is a vector field, then Xx denotes the vector field so that $(Xx)_\omega = X_\omega x_\omega$,

The totality T of operator fields on $\mathcal{W}$ is a C^*-algebra, and the norm of $X \in \mathrm{T}$ is $|X|_{\mathcal{W}}$. (T is an operator algebra on the product space $\sum_{\omega \in \mathcal{W}} \oplus L^2(\omega)$.)

Every element A of A is regarded as an operator field by its regular right representation in the dual space $\bar{\mathrm{A}}$, and every bounded numerical function f on $\mathcal{W}$ is regarded as an operator field on $\mathcal{W}$ whose value at

$\omega \in \mathcal{W}$ is $f(\omega)$ I. Then C and $\mathrm{C}_{\mathcal{W}}$ are C^*-algebras of operator fields on $\mathcal{S}$ and $\mathcal{W}$. The smallest C^*-algebra $\mathrm{K}_{\mathcal{W}}$ of operator fields on $\mathcal{W}$ which contains $\mathrm{C}_{\mathcal{W}}$ and $\mathrm{A}_{\mathcal{W}}$ is said to be the *diagonalizable extension*, or *diagonalizor*, of A in $\mathcal{W}$, The *diagonalizor* of A in $\mathcal{S}$ is denoted by K, and said to be the *diagonalizor* of A.

Definition 1.2. Let $\mathcal{W}$ and $\mathcal{U}$ be compact sub-sets of $\mathcal{S}$ with $\mathcal{U} \subseteqq \mathcal{W}$, and $\mathfrak{M}$ and X be sets of vector fields and operator fields in $\mathcal{W}$, respectively. Then the totality of restricted fields of elements of the respective sets in the space $\mathcal{U}$ are denoted by $\mathfrak{M}_{\mathcal{U}}$ and $\mathrm{X}_{\mathcal{U}}$. Especially, if ω is a point in $\mathcal{W}$, then $\mathfrak{M}_\omega$ and X_ω denotes the sets $(x_\omega : x \in \mathfrak{M})$ and $(X_\omega : X \in \mathrm{X})$, respectively.

Lemma 1.2. *Every value X_ω of an operator field X in* K *satisfies $X_\omega \in \mathrm{A}$. (More exactly, X agrees to a regular representation of a suitable element of* A *in $L^2(\omega)$). Every $\omega \in \mathcal{S}$ is regarded as a state on* K *so that $\omega(X) = \omega(X_\omega)$. The extended state ω satisfies $\omega(fX) = f(\omega)\omega(X)$ (where $f \in \mathrm{C}$, $X \in \mathrm{K}$). Every functional f in* A*, observable in a state $\omega \in \mathcal{S}$, is extended to a fuuctional in* K *with $f(X) = f(X_\omega)$. And even if we regard $L^2(\omega)$ as a set of functionals in* K*, it becomes the totality of functionals $\in \overline{\mathrm{K}}$ observable in the state ω being in* K*. The representative algebras of* A *and* K *in $L^2(\omega)$ agree to each other.*

Lemma 1.3. *Let μ be a distribution on $\mathcal{S}$, and x be a field regularly weakly measurable*[1] *by μ, Then $\|x_\omega\|_\omega$ is a measurable function on $\mathcal{S}$.*

Proof. $\|x_\omega\|_\omega = \sup_{A \in \mathrm{A}} |x_\omega(A^*)|/\omega(A^*A)^{\frac{1}{2}}$ (where $0/0 = 0$) becomes lower-semi-continuous removing any small open mass from $\mathcal{S}$.

Definition 1.3. Let μ be a distribution on $\mathcal{S}$. A vector field x on $\mathcal{S}$ is said to be *square summable*, if it is regularly weakly measurable and has a square summable norm function $\|x_\omega\|_\omega$.

we denote by $L^2(\mu)$ the totality of square summable vector fields on $\mathcal{S}$.

Proposition 1.1. *$L^2(\mu)$ is a Hilbert space, The norm and the inner-product of elements in $L^2(\mu)$ is so determined as $\|x\|_\mu = \left(\int \|x_\omega\|_\omega^2 \, d\mu(\omega)\right)^{\frac{1}{2}}$ and $(x, y)_\mu = \int (x_\omega, y_\omega)_\omega \, d\mu(\omega)$. $(x_\omega, y_\omega)_\omega$ is measurable and summable whenever $x, y \in L^2(\mu)$.*

Proof. $(x_\omega, y_\omega)_\omega$ is summable as a linear sum of four summable measurable functions $\|x_\omega \pm y_\omega\|_\omega^2$, $\|x_\omega \pm iy_\omega\|_\omega^2$. Only the metrical completeness of $L^2(\mu)$ needs the proof. Let $\{x_n\}$ be a sequence in $L^2(\mu)$ with $\|x_n - x_{n-1}\|_\mu \leqq 2^{-n}$. $t_n(\omega) = \|x_n)_\omega - (x_{n-1})_\omega\|_\omega$, (where $t_1(\omega) = \|x_{1\omega}\|_\omega$) satisfies $\int t_n(\omega)^2 \, d\mu(\omega) \leqq 4^{-n+1} (n \leqq 2)$, and removing any small open mass

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from $\mathcal{S}$, $\sum_{n=1}^{\infty} t_n(\omega)$ converges uniformly. Now $\|x_{n\omega} - x_{m\omega}\|_\omega \leqq \sum_{m+1}^{n} t_k(\omega) \to 0$ uniformly when n, $m \to \infty$. The limit field x exists and belongs to $L^2(\mu)$, and hence $L^2(\mu)$ is a Hilbert space.

From the proof of the Proposition 1. 1 it follows immediately that

Lemma 1.4. *Assume that for each $\omega \in \mathcal{S}$ a closed linear sub-set $\mathfrak{M}_\omega$ of $L^2(\omega)$ is determined. Then the set $(x \in L^2(\mu)$: $x_\omega \in \mathfrak{M}_\omega)$ is a closed linear sub-set of $L^2(\mu)$.*

Proposition 1.2. *The coordinate field ω is cyclic in $L^2(\mu)$ relative to the algebra* K. *Namely, $L^2(\mu) = [X\omega : X \in \mathrm{K}]$.*

Proof. Let a vector field x be orthogonal to $(X\omega : X \in \mathrm{K})$. Then $\int x_\omega(A)f(\omega)\, d\mu(\omega) = (x, \bar{f}A^*\omega)_\mu = 0$ for every $A \in \mathrm{A}$ and $f \in \mathrm{C}$. $x_\omega(A)$ vanishes almost everywhere for each fixed $A \in \mathrm{A}$. Since X_ω is regularly weakly measurable, x_ω vanishes almost everywhere.

Lemma 1.5. *If x, $y \in L^2(\mu)$, then the $\bar{\mathrm{A}}$-valued function $(x_\omega)_{y_\omega}$ (where $(x_\omega)_{y_\omega}(A) = (Ax_\omega, y_\omega)_\omega$) is regularly weakly measurable.*

Proof. We can choose x_n, $y_n \in L^2(\mu)$ so that $\|x - x_n\|_\mu < 2^{-n}$, $\|y - y_n\|_\mu < 2^{-n}$, $x_n = X_n\omega$, $y_n = Y_n\omega$ and X_n, $Y_n \in \mathrm{K}$. $(Ax_{n\omega}, y_{n\omega})_\omega = (x_{n\omega})_{y_{n\omega}}(A)$ are weakly continuous $\bar{\mathrm{A}}$-valued functions on $\mathcal{S}$, and converge to $(Ax_\omega, y_\omega)_\omega$ uniformly, removing any small open mass from $\mathcal{S}$.
Then $(x_\omega)_{y_\omega}$ is regularly weakly measurable.

Definition 1.4. An operator field X on the carrier $\mathcal{D}$ of μ, which maps every $x \in L^2(\mu)$ to an element of $L^2(\mu)$, is said to be a *measurable operator field.*

The restricted algebras $\mathrm{A}_{\mathcal{D}}$, $\mathrm{C}_{\mathcal{D}}$, $\mathrm{K}_{\mathcal{D}}$ are C^*-algebras of measurable operator fields. We denote these by A_μ, C_μ, and K_μ, respectively. *Then $|X|_{\mathcal{D}} = \sup_{\omega \in \mathcal{D}} |X_\omega|$ becomes the operator-norm of $X \in \mathrm{K}_\mu$ as an operator in $L^2(\mu)$.*

Proposition 1. 3. *Every bounded operator X in the commutor K_μ' of K_μ in the Hilbert space $L^2(\mu)$ is a measurable operator field. Each value X_ω (ω is a state in the carrier of μ) is essentially uniquely determined and belongs to A_ω'.*

Proof. We can assume without loss of generality that X is a definite Hermitian in the unit ball of K_μ'. Consider the field $x = X\omega \in L^2(\mu)$. Then $(\omega, K^*K\omega)_\mu \geqq (x, K^*K\omega)_\mu \geqq 0$ for every $K = fA$ with $f \in \mathrm{C}$ and $A \in \mathrm{A}$. Hence

$$\int |f(\omega)|^2\, \omega(A^*A)\, d\mu \geqq \int |f(\omega)|^2 x_\omega(A^*A)\, d\mu \geqq 0$$

(where $f \in \mathrm{C}$ and $A \in \mathrm{A}$), and $\omega(A^*A) \geqq x_\omega(A^*A) \geqq 0$ almost everywhere for each fixed $A \in \mathrm{A}$. Removing any small maximally open mass from $\mathcal{S}$, x becomes a weakly continuous field, whose each value x_ω is a positive functional together with $\omega - x_\omega$, and written for $x_\omega = X_\omega \omega$ by a suitable definite Hermitian X_ω in the unit ball of A_ω'.

Now $(Xz)_\omega = X_\omega z_\omega$ holds for every $z \in (K\omega : K \in \mathrm{K})$. Then X is a measurable operator field in $L^2(\mu)$, whose value at ω is X_ω.

Corollary. *Every bounded measurable function φ on $\mathcal{S}$ is a measurable operator field in $L^2(\mu)$. And the totality $\mathrm{M}(\mu)$ of bounded measurable functions on $\mathcal{S}$ is a W^*-sub-algebra of K_μ'.*

Proposition 1.4. *Let μ be a distribution in $\mathcal{S}$. Then the coordinate field is compoundly cyclic relative to A_μ and $\mathrm{M}(\mu)$. The coordinate projection field P, as the projection of $L^2(\mu)$ in $[\mathrm{M}(\mu)\omega]$, is generative and abelian relative to A_μ. The primitive operator of $A \in \mathrm{A}$ is the primitive function J_A* of A, *and C_μ is the carrier algebra of P.*

Proof. Every $x \in L^2(\mu)$ satisfies $(Px)_\omega = x_\omega(I)\omega$ and $Px \in [\mathrm{M}(\mu)\omega]$. Then P is measurable in $L^2(\mu)$ and is the projection in $\mathfrak{M} = [\mathrm{M}(\mu)\omega]$. The reduced operator of $A \in \mathrm{A}$ satisfies $P_\omega A P_\omega = (A\omega, \omega)_\omega P_\omega = J_A(\omega) P_\omega$ and $PAP = J_A P$. Hence P is abelian and generative in $L^2(\mu)$ relative to A, because $[\mathrm{A}PL^2(\mu)]$ contains $(Af\omega : A \in \mathrm{A}$ and $f \in \mathrm{C})$. The carrier algebra of P is C_μ because C_μ is the smallest C^*-algebra which contains the totality of primitive operators J_A of elements of A reduced by P. By Lemma 4. 9, ω is compoundly cyclic relative to A and $\mathrm{C}_\mu'' = \mathrm{M}(\mu)$.

Proposition 1. 5. *Consider a representation of* A *in a Hilbert space $\mathfrak{H}$, with an abelian W^*-sub-algebra* M *of the commutor* A$'$ *and a compoundly cyclic element g with $\|g\| = 1$ relative to* A *and* M. *Then a distribution μ on $\mathcal{S}$ is so uniquely determined that $\mathfrak{H}$ is isometrially maped on $L^2(\mu)$, where g is maped to the coordinate field, and the isometry determines a spatial isomorphism between algebras* $(\mathrm{A}_{\mathfrak{H}}, \mathrm{M})$ *and* $(\mathrm{A}_\mu, \mathrm{M}(\mu))$.

Proof. Let E be the projection in $[\mathrm{M}g]$, C_E the carrier algebra of E, K_E the diagonalizor of A, and $\mathcal{W}$ the spectrum of C_E. C_E is represented on the totality of continuous functions on $\mathcal{W}$. For each $\omega \in \mathcal{W}$ let ω_E denote the linear functional in A with $\omega_E(A) = J_A(\omega)$, where J_A is the primitive operator of $A \in \mathrm{A}$. Every ω_E is a state since $J_{A^*A}E = EA^*AE$ and J_{A^*A} are definite Hermitians. Any two points in $\mathcal{W}$ are reduced by the representative function of a suitable J_A. Then $\omega \longleftrightarrow \omega_E$ is a homeomorphism, and its range is a compact sub-set of $\mathcal{S}$. We regard $\mathcal{W}$ as a compact sub-set of $\mathcal{S}$ identified each $\omega \in \mathcal{W}$ to ω_E. A distribution μ on $\mathcal{W}$ is so determined as $(Fg, g) = \int F(\omega)\, d\mu(\omega)$ for every $F \in \mathrm{C}_E$. Consider an $x = \sum_1^n F_i A_i g$ in $\mathfrak{H}$ with $F_i \in \mathrm{C}_E$ and $A_i \in \mathrm{A}$, and for each $\omega \in \mathcal{W}$ let x_ω denote an element of $L^2(\omega)$ with $x_\omega = \sum F_i(\omega) A_i \omega$. Then $\|x\|^2 = \int \|x_\omega\|_\omega^2\, d\mu(\omega)$ $= \sum_{i,j} \int F_i(\omega)\overline{F_j(\omega)}\,\omega(A_j{}^* A_i)\, d\mu(\omega)$. The representation of x as a field in $L^2(\mu)$, whose value x_ω at $\omega \in \mathcal{W}$ is defined as above, is extended to an isometry between $\mathfrak{H}$ and $L^2(\mu)$, because $[Kg] = [(\mathrm{A} \cup \mathrm{C}_E) g] = [(\mathrm{A} \cup \mathrm{M}) g]$ $= \mathfrak{H}$ and $[\mathrm{K}\omega] = L^2(\mu)$ hold. This isometry satisfies clearly the desired conditions in the Proposition.

(b). Fourier induction and pre-spectral distribution.

Definition 1. 5. Let μ be a distribution on $\mathcal{S}$. The weak integration $m_\mu = \int \omega d(\omega)$ of the coordinate field ω by μ is said to be the *mean* of μ.

Proposition 1. 6. *The mean $m_\mu = \int \omega d\mu$ of a distribution μ in $\mathcal{S}$ is a self-induced state $m_\mu(A) = \int \omega(A) d\mu = (A\omega, \omega)_\mu$ of the coordinate field ω in $L^2(\mu)$.*

The weak integration $\int x_\omega d\mu(\omega)$ of a field x in $L^2(\mu)$ is the induced functional $\int x_\omega(A)\, d\mu = (Ax, \omega)_\mu$ of x by the coordinate field ω in $L^2(\mu)$.

Definition 1. 6. The induction $x \in L^2(\mu) \to \int x_\omega\, d\mu \in L^2(m_\mu)$ is said to be the *Fourier induction* of $L^2(\mu)$.

Lemma 1. 7. *Let μ be a distribution in $\mathcal{S}$, and m its mean.*

Every $x \in L^2(\mu)$ is a sum $x = x_1 + x_2$ of two mutually orthogonal fields x_1 and x_2 in $L^2(\mu)$ so that $\int x_2\, d\mu = 0$ and $\| \int x d\mu \|_m = \|x_1\|_\mu$.

The reduced operator of a $\varphi \in \mathrm{M}(\mu)$ by the Fourier induction is so uniquely determined as an operator K_φ in A_m' with $K_\varphi m = \int \varphi(\omega)\omega\, d\mu(\omega)$.

Proof. By Proposition 2. 1 in Chapter 1, $x_1 = E_\omega^{\mathrm{A}} x$ and $x_2 = x - x_1$ satisfies the desired properties.

Let K_φ be any operator in A_m' with $K_\varphi m = \int \varphi(\omega)\omega\, d\mu(\omega)$.

Then every $x = Am$ with $A \in \mathrm{A}$ satisfies $K_\varphi(\int x_\omega d\mu) = K_\varphi Am =$ $AK_\varphi m = \int \varphi(\omega) x_\omega\, d\mu(\omega)$. Hence K_φ is the reduced operator of φ by the Fourier induction.

Definition 1. 7. A distribution μ on $\mathcal{S}$ is said to be *pre-spectral* if the Fourier induction $x \in L^2(\mu) \to \int x_\omega d\mu(\omega) \in L^2(m_\mu)$ is an isometry between $L^2(\mu)$ and $L^2(m_\mu)$.

Then a distribution μ is pre-spectral if and only if the coordinate field

ω is cyclic in $L^2(\mu)$ relative to A.

Proposition 1. 7. *If μ is a pre-spectral distribution, the Fourier induction determines a spatial isomorphism between algebras* (A_μ, C_μ, K_μ, $M(\mu)$) *and its induced algebras* A_m, C_m, K_m, M_m *in* $L^2(m)$ *respectively, where m is the mean of* μ.

Proposition 1. 8. *Let p be a state and* M_p *be any abelian W*-subalgebra of* A_p'. *Then there exists a pre-spectral distribution μ in $\mathcal{S}$ whose mean is p and whose Fourier induction determines spatial isomorphisms between algebras* (A_μ, $M(\mu)$) *and* (A_p, M_p). *The distribution μ is uniquely determined.*

Proof. p is cyclic in $L^2(p)$, then it is compoundly cyclic relative to A_p and M_p. Consider the distribution μ in Propsition 1. 5 which characterizes the above compoundly cyclic representation. The isometry between $L^2(\mu)$ and $L^2(p)$ becomes clearly the Fourier induction, and then μ is pre-spectral.

Lemma 1. 8. *A distribution μ on $\mathcal{S}$ is pre-spectral if and only if for every $f \in$ C we can choose a sequence $\{A_n\}$ in A with* $\int \|f(\omega)\omega - A_n\omega\|_\omega^2 \, d\mu = \|A_n\omega - f\omega\|_\mu^2 \to 0$.

Proof. If μ is a pre-spectral distribution, ω is cyclic in $L^2(\mu)$ relative to A, and A contains such a sequence $\{A_n\}$. Conversely, if μ is a distribution in $\mathcal{S}$ so that $(f\omega : f\in C) \subsetneqq [A\omega]$, , then $[A\omega] \supseteqq [fA\omega : f\in C, A\in A]$ $= [K\omega] = L^2(\mu)$, and μ is pre-spectral.

Lemma 1. 9. *Let μ be a distribution on $\mathcal{S}$, and K_f the reduced operator of $f \in$ C by the Fourier induction. If $K_{fg} = K_f K_g$ is satisfied for every $f \in$ C, then μ is pre-spectral.*

Proof. Let m denote the mean of μ. Then by the assumption of the Lemma, every $f \in$ C satisfies $\|K_f m\|_m^2 = ((K_f^* K_f)m, m)_m = (K_{|f|^2} m, m)_m = \int |f(\omega)|^2 d\mu = \|f\omega\|_\mu^2$. By Lemma 1. 7, $f\omega$ belongs to $[A\omega]$, then by Lemma 1. 8 μ is a pre-spectral distribution.

Lemma 1. 10. *Regard $\mathcal{S}$ as a compact set of states on* K *as in Lemma 1. 1, Then every $\bar{A}$-valued vector field on $\mathcal{S}$ becomes an $\bar{K}$-valued vector field. Every distribution on $\mathcal{S}$ becomes pre-spectral relative to the algebra* K.

Proof. Let x be a $\bar{K}$-valued vector field on a compact sub-set $\mathcal{W}$ of $\mathcal{S}$. By Lemma 1. 1, each value x_ω at $\omega \in \mathcal{W}$ is determined as a functional in $L^2(\omega)$ and contained in $\bar{A}$. The totality $L^2(\mu)$ of $\bar{A}$-valued square summable fields on $\mathcal{S}$ is therefore the totality of $\bar{K}$-valued square summable fields, and by Proposition 1. 2, μ is pre-spectral relative to the algebra K.

Lemma 4. 11. *Let μ be a distribution on $\mathcal{S}$. $M(\mu) = A_\mu'$ holds if and only if the coordinate projection field P in $L^2(\mu)$ belongs to A_μ''.*

Proof. If $M(\mu) = A_\mu'$, then P is the projection of $L^2(\mu)$ in the space $[M(\mu)\omega] = [A_\mu'\omega]$, and belongs to A_μ''. Conversely, assume that $P \in A_\mu''$. Then P is a generative abelian projection in A_μ''. And $(PA_\mu''P)' = A_\mu'{}_P$ holds. Since ω is compoundly cyclic in $L^2(\mu)$, ω is cyclic in the space $\mathfrak{M} = [M(\mu)\omega]$ relative to the abelian W^*-algebra $PA_\mu''P$, and $PA_\mu''P = (PA_\mu''P)' = A_\mu'{}_P = M(\mu)_P$ holds. P is generative relative to A_μ'', and the induction $A \in A_\mu' \to AP$ is an isomorphism. Then we have $A_\mu' = M(\mu)$.

Lemma 4. 12. *Let μ be a distribution on $\mathcal{S}$. $M(\mu)$ becomes a maximal abelian sub-algebra of A_μ' if and only if the coordinate projection field P belongs to K_μ''.*

Proof. $M(\mu)$ becomes a maximal abelian sub-algebra of A_μ' if and only if $M(\mu) = K_\mu'$ holds. Regard $\mathcal{S}$ as a compact space of states on K, and $L^2(\mu)$ the Hilbert space of $\overline{K}$-valued field. Then Lemma follows immediately the Lemma 4. 11.

§ 2. **Regular projections and the Lusin's Theorem.**

We consider a representation of A in a C^*-algebra A_λ on a Hilbert space $\mathfrak{H}$, and a C^*-sub-algebra B of A. A projetion E in the strong closure A_λ'' of A_λ is said to be *regular relative to* B if $|AE| = |A/N(E)| = \inf_{B \in N(E)} |A - B|$ for every $A \in B$, where $|A|$ is the operator-norm of A, $N(E)$ is a left ideal $N(E) = (B \in A: BE = 0)$ of A, and $A/N(E)$ is the residue-class of A in the quotient Banach space $A/N(E)$. E is said to be *regular* if it is regular relative to A itself.

Kaplansky[1)] asserts that every projections in the center $A_\lambda'' \cap A_\lambda'$ of A_λ'' is regular. However, when A is non-commutative, projections in A_λ'' may not be necessarily regular.

Theorem 5. *Let A_λ be a representative algebra of* A. *A_λ'' its strong closure, E a projection in A_λ'', and* B *a separable *-sub-algebra of* A. *Then every strong neighbourhood of E contains at least one projection F, $E \geqq F \in A_\lambda''$* regular *relative to* B.

Corollary. If A *is separable, every projection E in A_λ'' is a strong limit of a sequence of regular projecton F_n with $E \geqq F_n \in A_\lambda''$.*

Theoem 5 follows from the next Lusin's Theorem.

1). Kaplansky (6)

Theorem 6. (*Non-commutative extension of the Lusin's Theorem*). *Let* A_λ *be a representative algebra of* A. *E be a projection in* A_λ'', *and A be an element of* A_λ''. *Then for any* $\varepsilon>0$ *and any strong neighbourhood* U *of E we can choose a* $B\in A$ *and a projection* $F\in U\cap A_\lambda''$ *with* $E\geqq F$, $AF=BF$ *and* $|B|\leqq|AF|(1+\varepsilon)$.

When A is abelian, and A_λ is the representation of A in $L^2(\mu)$ by a measure μ in the spectrum of A, then Theorem 6 implies that every bounded measurable function on a measurable set becomes conitnuous removing any small mass. To prove these Theorems we need the next Lemma.

Lemma 2. 1. *Let* $\mathfrak{H}$ *denote the underlying Hilbert space of the representative algebra* A_λ. *Given a projection* $E\in A_\lambda''$, *an* $A\in A_\lambda''$, *a number* $\varepsilon>0$ *and n-elements* $g_1, \ldots . g_n\in\mathfrak{H}$ *with* $Eg_i=g_i$ *and* $\|g_i\|=1$. *Then there exists a projection* $F\in A_\lambda''$ *and a* $B\in A$ *so that* $E\geqq F$, $|AF-BF|<\varepsilon$, $|AF|=|B|$ *and* $\|Eg_i-Fg_i\|<\varepsilon$ $(1\leqq i\leqq n)$.

Proof. Besides $g_1, \ldots, g_n$ we consider a $g_0\in\mathfrak{H}$ with $Eg_0=g_0$, $\|g_0\|=1$ and $\|AEg_0\|\geqq(1-\varepsilon)|AE|$. By the Kaplansky's Density Theorem the unit ball of A_λ is strongly dense everywhere in the unit ball of A_λ''. For $AE\in A_\lambda''$ we can choose a $B_0\in A$ with $|B_0|=|AE|$ and $\|Bg_i-AEg_i\|\leqq\varepsilon^4$ $(0\leqq i\leqq n)$. The definite Hermitian $G=E(A^*-B_0^*)(A-B_0)E$ has the spectral resolution $G=\int\lambda dF(\lambda)$. We put $F=F(\varepsilon^2)E$ and estimate

$$|AF-B_0F|^2=|GF(\varepsilon^2)|<\varepsilon^2,$$
$$\varepsilon^2\|(E-F)g_i\|^2\leqq\int\lambda d\|F(\lambda)g_i\|^2\leqq(Gg_i,\ g_i)=\|(A-B_0)Eg_i\|^2<\varepsilon^4$$

and

$$\|Eg_i-Fg_i\|<\varepsilon\ (0\leqq i\leqq n).$$

It means $|AF|\geqq\|AFg_0\|\geqq\|AEg_0\|-\|A(E-F)g_0\|\geqq(1-\varepsilon)|AE|$. Hence from $B=\alpha B_0$ and $\alpha=|AF|/|AE|$, the Lemma follows.

Proof of Theorem 6. Let ε be any prositive number and let g_0, g_1, $\ldots$ g_n be $(n+1)$-elements as in Lemma 2.1 and in its proof. To prove the Theorem 2 it is sufficient to see the existence of requiered B and F in the conditions of $F\in A_\lambda''$ and $\|Eg_i-Fg_i\|<\varepsilon$ $(1\leqq i\leqq n)$, instead of $F\in A_\lambda''\cap U$.

We choose a $B_1\in A$ and a projection $F_1\in A_\lambda''$ as $E\geqq F_1$, $|AF_1-BF_1|<\varepsilon/2$, $|AF_1|=|B_1|$ and $\|Eg_i-F_1g_i\|<\varepsilon/2$ $(0\leqq i\leqq n)$. By induction, $B_k\in A$ and a projection $F_k\in A_\lambda''$(for $k=2,\ 3, \ldots$) are so chosen as $E\geqq F_1\geqq F_2$ $\ldots\geqq F_k$, $|(A-\sum_1^{k-1}B_j)\,F_k-B_kF_k|<\varepsilon/2^k$, $|(A-\sum_1^{k-1}B_j)F_j|=|B_k|$ and $\|F_k g_i-F_{k+1}g_i\|<\varepsilon/2^k$ $(0\leqq i\leqq n)$. F_i converges strongly to a projection F in A_λ'', and $B=\sum_1^\infty B_i$ converges uniformly, belongs to A_λ and satisfies

$|B| \leqq \sum_1^\infty |B_i| < |AE| + \sum \varepsilon/2^i < |AE| + \varepsilon$. B and F are thus acquired, and satisfy $AF = BF$, $\|Eg_i - Fg_i\| < \varepsilon$ $(0 \leqq i \leqq n)$, $|AF| \geqq \|AFg_0\| \geqq \|AEg_0\| - \|AE(E-F)g_0\| \geqq (1-2\varepsilon)|AE| \geqq (1-3\varepsilon)|B|$. It concludes Theorem 2.

Proof of Theorem 5. If B is a separable sub-algebra of A_λ'', it contains a conuntable system $A_1, A_2, \ldots$ uniformly dense everywhere in B. Let $g_1, \ldots, g_n$ be n-elements in $\mathfrak{H}$ so that $Eg_i = g_i$ and $\|g_i\| = 1$. To prove the Theorem 5, it is sufficient to see the existence of such a projection $F \in A_\lambda''$ that $\|Eg_i - Fg_i\| < \varepsilon$ $(1 \leqq i \leqq n)$, and $|A_i F| = \inf(|B|: A_i F = BF, B \in A)$. We define inductively a triple pair of systems $g_j^k \in \mathfrak{H}$, $B_j^k \in A$ and projections $F_j^k (1 \leqq k \leqq \infty, 1 \leqq j \leqq k)$ in the following way.

(a). g_1^1 is so chosen as $Eg_1^1 = g_1^1$, $\|g_1^1\| = 1$ and $\|AEg_1^1\| \geqq |AE|(1-\varepsilon/2)$. F_1^1 and B_1^1 are so chosen as $E \geqq F_1^1$, $\|Eg_i - F_1^1 g_i\| < \varepsilon/2$, $\|g_1^1 - F_1^1 g_1^1\| < \varepsilon/2$, $AF_1^1 = B_1^1 F_1^1$ and $|AF_1^1| \geqq |B_1^1| - \varepsilon/2$.

(b) For convenience we put $F_k^k = F_0^{k+1}$, $B_k^k = B_0^{k+1}$ and $g_k^k = g_0^{k+1}$. For a pair of numbers k and $1 \leqq j \leqq k$ assume that every F_s^t, B_s^t and g_s^t (for every $t \leqq k-1$, $s \leqq t-1$ and $t \leqq k$, $s \leqq j-1$) has been defined and satisfies $E \geqq F_1^1 \geqq F_1^2 \geqq F_2^2 \geqq F_1^3 \geqq \ldots \geqq F_{j-1}^k$. Then g_j^k is so chosen as $F_{j-1}^k g_j^k = g_j^k$, $\|g_j^k\| = 1$ and $\|A_j F_{j-1}^k g_j^k\| \geqq |A_j F_{j-1}^k|(1-\varepsilon/2^k)$. F_j^k and B_j^k are so chosen as $F_{j-1}^k \geqq F_j^k \in A_\lambda''$, $\|(F_{j-1}^k - F_j^k) g_i\| < \varepsilon/2^{k+1} k$ $(1 \leqq i \leqq n)$, $\|(F_{j-1}^k - F_j^k) g_s^t\| \leqq /2^k k$, $A_j F_j^k = B_j^k F_j^k$ and $|A_j F_j^k| \geqq |B_j^k|(1-\varepsilon/2^k)$. As $k \to \infty$, F_1^k converges strongly to a projection $F \in A_\lambda''$. F satisfies $\|Eg_i - Fg_i\| < \varepsilon$ $(1 \leqq i \leqq n)$, $\|g_j^k - Fg_j^k\| \leqq \varepsilon/2^k$ and $A_j F = B_j^k F$, then

$$\begin{aligned} |A_j F| &\geqq \|A_j F g_j^k\| \geqq \|A_j g_j^k\| - \|A_j F_j^k (g_j^k - F g_j^k)\| \\ &\geqq |A_j F_j^k (1-\varepsilon/2^{k-1})| \geqq |B_j^k|(1-\varepsilon/2^{k-2}). \end{aligned}$$

It follows that $|FA| = \inf(|B|: B \in A$ and $BF = FA)$ for each $A = A_j$ $(j = 1, 2, \ldots)$, and consequently for every $A \in A$. Then Theorem 1 is completed.

Improvement of Theorem 5 and 6 may be an interesting problem because it relates some unsolved problems in ideal theory of C^*-algebra.

Consider a C^*-algebra A and its represented Hilbert space $\mathfrak{H}$. Let E be a projection in A''. If E is regular, then the quotiont space $A/N(E)$ is represented as the space of operators $(AE: A \in A)$, whose norm is determined by $|A/N(E)| = |AE|$. Then even if E is not necessarily regular, we shall call the uniform clorure of the set of operators $(AE: A \in A)$ the quotiont space of A[1]. devided by the projection E. Then Theorem 1 implies that any quotient space A/E devided by a projection in A_λ'' is approxi

[1] cf. Section 1 in Capter 4.

mated *by* quotient spaces devided by left ideals of A in the following sense. Given any countable sub-algebra B of A, we can choose a sequence of left ideals N_1 N_2 .... so that $|A/E| = \sup |A/N_n|$ on B.

From Theorem 5 we obtain the following extended Kadison's Theorem.

Proposition 2.1. *Let* A *be a* C^**-algebra and* E *be a projection in* A'' *with a finite dimensional range, (or, more generally* E *be a finite sum* $E = \sum_1^n E_i$ *of mutually orthogonal minimal projections* E_i *in* A'',)

Then E *is regular.*

The quotient norm $|AE| = |A/N(E)|$ *is equivalent to a suitable Hilbert norm, and the quotient space* $A/N(E)$ *is isomorphic to a Hilbert space. If* E *is a minimal projection in* A'', *then* $N(E)$ *is a maximal left ideal of* A.

Proof. It is well-known that, if we choose a $g_i \in \mathfrak{H}$ ($i = 1, 2, .. n$) with $E_i g_i = g_j$ and $\|g_i\| = 1$, then $t(A) = \sum (Ag_i, g_i)$ is a trace of an operator A in $EA''E$. Then whenever F be a projection in A'' with $EF = FE = F$, $t(F) = \sum \|Fg_i\|^2$ is an integer. Hence by our Theorem 5, E is regular relative to any separable sub-algebra of A, what is equivalent to the regularity of E.

Notice that $|AE_i| = \|Ag_i\|$ holds, and the quotient norm $|AE|$ satisfies $\sum |AE_i| \geqq |AK| \geqq \max |AE_i|$. Then $|AE| = |A/N(E)|$ is equivalent to $\|A\|_t = (\sum \|Ag_i\|^2)^{\frac{1}{2}}$. The quotient space $A/N(E)$ is a Banach space, then $A/N(E)$ becomes a Hilbert space by the norm $\|A\|_t$.

Finally, if E is minimal and g is an element in $\mathfrak{H}$ with $Eg = g$ and $\|g\| = 1$, then $|AE| = \|Ag\|$. A is transitive in the Hilbert space $\mathfrak{M} = (Ag: A \in A)$, because $(Ah: A \in A) = \mathfrak{M}$ holds for any $h = Bg$ in $\mathfrak{M}$. Then $N(E)$ is a maximal left ideal of A.

The next Lemma shall be used in the latter section.

Lemma 2.2. *Let* A *be a representative algebra of* A *on a Hilbert space* $\mathfrak{H}$, B *a* C^**-sub-algebra of* A *with* $I \in B$, E *a projection in* A_λ'' *regular relative to* B, *and* p *a state on* A *with* $p(A) = 0$ *for* $A \in N(E) = (B: BE = 0)$. *Then a state* q *in the weak closure* $\overline{S(E)}$ of $S(E)$ *exists, and* $p(A) = q(A)$ *holds on* B.

Proof. Let $\mathfrak{U}(E)$ denote the unit ball of the space $\Phi(E)$, and $\mathfrak{U}(E)$ its weak closure. We see $|A/N(E)| = |AE| = \sup_{\|x\|, \|y\| = 1} |(AEx, y)| = \sup_{\varphi \in \mathfrak{U}(E)} |\varphi(A)|$ for $A \in B$, and

$$|p(A)| \leqq \inf_{B \in N(E)} |A - B| = |A/N(E)| = \sup_{\varphi \in \overline{\mathfrak{U}(E)}} |\varphi(A)|.$$

Then a $\varphi \in \mathfrak{U}(E)$ exists, and $p(A) = \varphi(A)$ holds on B[1]. The absolute variation q of φ^* belongs to $\overline{\mathcal{S}(E)}$. In fact, $\overline{\mathcal{S}(E)}$ is bounded and regularly convex, and each $\varphi \in \mathfrak{U}(E)$ has at least one $v_\varphi \in \mathcal{S}(E)$ with $|v_\varphi(A^*A)| \geqq |\varphi(A)|^2$ for $A \in$ A.[2] Then $\varphi \in \mathfrak{U}(E)$ has at least one $q \in \mathcal{S}(E)$ with $q(A^*A) \geqq |\varphi(A)|^2$, where $1 \geqq |\varphi| \geqq |\varphi(I)| = p(I) = 1 = q(I)$. Then q is the absolute variation of φ^*. We have now observed $p(I) = q(I) = 1$ and $|p(A)|^2 = |p^*(A)|^2 \leqq q(A^*A)$ for $A \in$ B. Then $p(A) = q(A)$ holds on B[3].

§ 3. Spectral distributions in a state space, and the Mautner Reduction Theory.

The next Theorem is a Mautner Reduction Theorem with an extended form.

Theorem 7. *Let μ be a distribution in $\mathcal{S}$. In order that* M(μ) *be a maximal abelian sub-algebra of* A_μ *in* $L^2(\mu)$, *it is necessary and sufficient that every state in the carrier* $\mathfrak{D}(\mu)$ *of* μ *becomes pure relative to each fixed* $A \in$ A, *almost everywhere.*

To prove the Theorem we need two Propositions.

Proposition 3.1. *Let μ be a distribution in $\mathcal{S}$ so that* M(μ) = A_μ' (; *then the coordinate projection field P belongs to* A_μ'').

Let A be a fixed element of A *which has at least one $B \in$* A *with $BP = J_A P = PAP$ (, where J_A is the primitive function of A). And assume that $P \in A_\mu''$ is regular relative to the smallest C^*-sub-algebra* B *of* A *which contains A, B and I. Then every state in the carrier $\mathfrak{D}(\mu)$ of μ is pure relative to A.*

Proof. We shall first prove two sub-lemmas.

Sub-lemma 1. *If a state r vanishes on* N(P), *then $r(A) = r(B)$.*

In fact, if r is such a state, a suitable state q in the weak closure $\overline{\mathcal{S}(P)}$ of $\mathcal{S}(P)$ can be so chosen as $r(X) = q(X)$ for every $X \in$ B. Then $r(A) = q(A)$ and $r(B) = q(B)$. $\mathcal{S}(P)$ is the weak closure of the totality of states t with $t(X) = (X f\omega, \omega)_\mu$, where $0 \leqq f \in$ M(μ). Since $BP = PAP$, we have $t(A) = t(B)$ and $q(A) = q(B)$. Hence $r(A) = r(B)$ holds.

Sub-lemma 2. *Every state ω in $\mathfrak{D}(\mu)$ satisfies* N(ω) $\supseteqq$ N(P) *and* $B - \omega(A)I \in$ N(ω).

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1). Notice that every $\varphi \in \bar{A}$ is reduced as a functional φ_B on the sub-algebra B. (φ_B: $\varphi \in \mathfrak{U}(E)$) is bounded and regularly convex in the dual space of B and it contains p_B because $|p_B| \leqq \sup_{|A| \leqq 1} (|\varphi_B(A)| : \varphi \in \mathfrak{U}(E))$.

2). v_φ may be chosen as the absolute variation of $\varphi^*/|\varphi|$.

3). q_B becomes the absolute variation of p_B on the sub-algebra B. But the absolute variation of a state is itself.

In fact, every $X \in \mathrm{N}(P)$ satisfies $XP = 0$ and $\|X\omega\|_\mu = \int \omega(X^*X) d_\mu(\omega) = 0$. Then $\omega(X^*X) = 0$ and $X \in \mathrm{N}(\omega)$ for every $\omega \in \mathcal{D}(\mu)$.

Next, for every $X \in \mathrm{A}$ and $\omega \in \mathrm{M}(\mu)$ we have

$\int \omega(XB) f(\omega) d\mu = (XB\mathrm{f}\omega, \omega)_\mu = (XJ_A f\omega, \omega)_\mu = \int \omega(X)\omega(A) f(\omega) d\mu(\omega)$.

Then $\omega(XB) = \omega(X)\omega(A)$ holds for every $X \in \mathrm{A}$, Now $\omega(B^*B) = \omega(B)\omega(A)$, and $\mathrm{N}(\omega) \supseteqq \mathrm{N}(P)$ imply $\omega(A) = \omega(B)$, $\omega((B-\omega(A)I)^*(B-\omega(A)I)) = 0$, and $B - \omega(A)I \in \mathrm{N}(\omega)$.

We shall now complete the Proposition. Let ω be any state in $\mathcal{D}(\mu)$, and q any state with $\mathrm{N}(q) \supseteqq \mathrm{N}(\omega)$. Then q vanishes on $\mathrm{N}(P)$ and satisfies $q(A) = q(B)$. On the other hand $B - \omega(A)I$ belongs to $\mathrm{N}(\omega)$ and $\mathrm{N}(q)$, then $q(B - \omega(A)) = 0$ and $q(A) = q(B) = \omega(A)$ hold. Hence every $\omega \in \mathcal{D}(\mu)$ is pure relative to A.

Proposition 3.2. *Let μ be a distribution on $\mathcal{S}$ with $\mathrm{M}(\mu) = \mathrm{A}_\mu'$, Then every state ω in $\mathcal{D}(\mu)$ is pure relative to each element of* A, *almost everywhere.*

Proof. Consider a fixed $A \in \mathrm{A}$ and $\varepsilon > 0$. By Theorem 5 and 6, a projection $E \in \mathrm{A}_\mu''$ and a $B \in \mathrm{A}$ can be so chosen that, $P \geqq E$, $J_A E = BE$, $\|(P-E)\omega\|_\mu < \varepsilon$, and E is regular relative to the smallest C^*-algebra B which contains A, B and I. P is an abelian projection in A_μ'', and then E is an induced operator $E = \varphi P$ of a suitable $\varphi \in \mathrm{M}(\mu) = \mathrm{A}_\mu'$, where φ is a characteristic function of a suitable measurable set $\mathfrak{X}$ in $\mathcal{S}$. The distribution $\nu(\mathfrak{Z}) = \mu(\mathfrak{X} \cap \mathfrak{Z})/\mu(\mathfrak{X})$ is a spectral distribution with $\mathrm{M}(\nu) = \mathrm{A}_\nu'$ and satisfies the condition in Propostion 3.1. every state in the carrier $\mathfrak{X}$ of ν is pure relative to A. Since $\mu(\mathfrak{Y} - \mathfrak{X}) = 0$ and $\mu(\mathcal{S} - \mathfrak{X}) < \varepsilon$, every state in the carrier of μ is pure relative to A, almost everywhere.

Proof of Theorem 7. (*Sufficiency*). Let μ be a distribution in $\mathcal{S}$ so that every state in the carrier $\mathcal{D}(\mu)$ of μ is pure relative to each fixed $A \in \mathrm{A}$, almost everywhere; and K be any definite Hermitian in the unit ball of K_μ'. By Proposition 1. 3, K is a measurable operator field, and each value K_ω is a definite Hermitian in the unit ball of A_ω'. $q = K\omega$ is a field in $L^2(\mu)$, whose each value q_ω is a positive functional, vanishing on $\mathrm{N}(\omega)$ and satisfying $q_\omega(A) = q_\omega(I)\omega(A)$ whenever ω is pure relative to $A \in \mathrm{A}$. q_ω is regularly weakly measurable, and $q_\omega - \mathrm{f}\omega$ (where $f(\omega) = q_\omega(I)$) vanishes for each $A \in \mathrm{A}$, almost everywhere. Then $K\omega = q_\omega = f\omega$ and $KX\omega = fX\omega$ hold for each $X \in \mathrm{K}$, almost everywhere. Hence $K = f$ belongs to $\mathrm{M}(\mu)$, and $\mathrm{M}(\mu)$ becomes a maximal abelian sub-algebra of A_μ'.

(*Necessity*). Conversely, let μ be a distribution on $\mathcal{S}$ so that $\mathrm{M}(\mu)$ is a maximal abelian $*$-sub-algebra of A_ω'. Then $\mathrm{K}_\omega' = (\mathrm{A}_\omega \cup \mathrm{M}(\mu))' = \mathrm{M}(\mu)$ holds. Regarded $\mathcal{S}$ as a compact space of states on K, and $L^2(\mu)$ as a

Hilbert space of $\overline{\mathrm{K}}$-valued fields, $\mathrm{K}_\mu' = \mathrm{M}(\mu)$ implies that every state in the carrier of μ becomes pure relative to each $X \in \mathrm{K}$, almost everywhere. Notice that a state ω in $\mathcal{S}$, regarded as being either of algebras A and K, is pure relative to an $A \in \mathrm{A}$ if and only if so, regarded ω as a state on the agreeing representative algebras $\mathrm{A}_\omega = \mathrm{K}_\omega$. Hence every state in the carrier of μ is pure relative to each $A \in \mathrm{A}$, almost everywhere.

Corollary 1. *Let* A *be a separable C^*-algebra, and μ a distribution on $\mathcal{S}$. Then* $\mathrm{M}(\mu)$ *is a maximal abelian sub-algebra of* A_μ' *if and only if every state in the carrier of μ is pure, almost everywhere.*

Proof. Let $\{A_i\}$ be a countable sub-set of A uniformly dense everywhere in it. Then a state p on A is pure if and only if it is pure relative to every A_i.

Definition 3.1. Let $\mathcal{P}$ denote the totality of pure states on A. The dual space $\overline{\mathrm{A}}$ is a locally convex linear topological space by its weak topology. Its uniform structure is determined by the totality of weakly continuous pre-norms of $\overline{\mathrm{A}}$, and is called the weak structure of $\overline{\mathrm{A}}$. A function f on $\mathcal{P}$ is said to be homogeneously weakly continuous if it is uniformly continuous by the weak structure of $\mathcal{P}$. The weak structure on $\mathcal{P}$ is totally bounded, and its completion is the weak closure $\overline{\mathcal{P}}$ of $\mathcal{P}$. A function f on $\mathcal{P}$ is homogeneously weakly continuous if and only if it is extensible to a continuous function on $\overline{\mathcal{P}}$.

A countablly additive measure μ on $\mathcal{P}$ is said to be *regular* (*relative to the weak structure of* $\mathcal{P}$) if every homogeneously weakly continuous function on $\mathcal{P}$ is measurable, and every measurable function on $\mathcal{P}$ becomes homogeneously weakly continuous removing any small open mass from it. A vector field x on $\mathcal{P}$ is said to be homogeneously continuous if x is extensible to a weakly continuous field on $\overline{\mathcal{P}}$ so that the norm function $\|x_\omega\|_\omega$ is simultaneously continuous on $\overline{\mathcal{P}}$. A vector field x on $\mathcal{P}$ is said to be *regularly weakly measurable* if x becomes homogeneously continuous removing any small open mass from it. Apply Theorem 12 and 14 which shall appear in Chapter 3, then the next Lemma shall follow immediately.

Lemma 3.1. *Let μ be a regular measure on $\mathcal{P}$ relative to its weak structure. Then the totality $L^2(\mu)$ of regularly weakly measurable and square summbale fields on $\mathcal{P}$ consists of a Hilbert space, in which the set* $\mathrm{K}\omega = (K\omega : K \in \mathrm{K})$ *is uniformly dense everywhere.*

To reform Theorem 7, for each $\omega \in \mathcal{S}$ we consider a pure state ω' which vanishes on $\mathrm{N}(\omega)$. Then $\omega \to \omega'$ is a mapping of $\mathcal{S}$ in $\mathcal{P}$. If μ is a distribution on $\mathcal{S}$, the induced measure ν of μ on $\mathcal{P}$ (by the mapping

$\omega \to \omega'$) is determined as follows: A sub-set X of $\mathcal{P}$ is said to be ν-measurable if its inverse image $X' = (\omega \in \mathcal{S}:\ \omega' \in X)$ is measurable by μ. The totality of ν-measurable subtsets of $\mathcal{P}$ consists of a countablly additive set-class. The induced measure ν on $\mathcal{P}$ is a countablly additive measure on the class $\mathfrak{X}$ of the totality of ν-measurables sub-sets of $\mathcal{P}$ so that $\nu(X) = \mu(X')$ for every $X \in \mathfrak{X}$. If f is a ν-measurable function on $\mathcal{P}$, we denote by f' a function on $\mathcal{P}$ so that $f'(\omega) = f(\omega')$. Then f is summable by ν if and only if f' is summable by μ, and satisfies $\int f d\nu = \int f' d\mu$.

Theorem 7′. *Let μ be a distribution on $\mathcal{P}$ so that* $\mathrm{M}(\mu)$ *is a maximally abelian *-sub-algebra of A_μ'. Consider a mapping $\omega \to \omega'$ of $\mathcal{S}$ in $\mathcal{P}$ with* $\mathrm{N}(\omega') \supseteq \mathrm{N}(\omega)$, *and the induced measure ν of μ on $\mathcal{P}$ by the mapping. Then ν is a regular measure on $\mathcal{P}$ relative to the weak structure of $\mathcal{P}$. The carrier $\mathcal{D}(\mu)$ of the measure μ is contained in the weak closure $\overline{\mathcal{P}}$ of $\mathcal{P}$. If f is any homogeneously weakly continuous function on $\mathcal{S}$, whose extension in $\mathcal{P}$ preserving the contininuty is expressed by the same f, Then f and f' agree essentially with each other on the carrier $\mathcal{D}(\mu)$ relative to the distribution μ. Hence $f \in \mathrm{M}(\nu) \longleftrightarrow f' \in \mathrm{M}(\mu)$ determines an isomorphism between these function algebras. Let X be any field in $L^2(\nu)$, then a field x' in $L^2(\mu)$ can be so chosen that $x_\omega(A) - (x')_\omega(A)$ vanishes for each $A \in \mathrm{A}$, almost everywhere relative to the distribution μ. $x \longleftrightarrow x'$ is an isometry between $L^2(\nu)$ and $L^2(\mu)$, where the coordinate field are leaved invariant, and the isometry determienes a spatial isomorphism between algebras* $(\mathrm{A}_\nu, \mathrm{M}(\nu))$ *and* $(\mathrm{A}_\mu, \mathrm{M}(\mu))$.

Proof. Consider the primitive function J_A on $\mathcal{P}$ and its induced function $J_A'(\omega) = J_A(\omega')$. Then J_A' is measurable by μ, and $J_A = J_A'$ holds on $\mathcal{P}$, almost everywhere relative to μ. Hence J_A is measurable by ν. Notice that the totality of homogeneously weakly continuous functions on $\mathcal{P}$ is the smallest C^*-algebra of functions on $\mathcal{P}$ which contains the totality of primitive functions of elements of A defined on $\mathcal{P}$. Then the assertion of the Theorem 7′ is concluded.

Definition 7.2. A pre-spectral distribution μ on $\mathcal{S}$ is said to be *spectral* if $\mathrm{M}(\mu)$ is a maximal abelian *-sub-algebra of A_μ'.

Theorem 8. *Let p be any state on* A, *and* M *be any maximal abelian *-sub-algebra of* A_p', *then a spectral distribution μ on $\mathcal{S}$ is so uniquely determined that p is its mean and* M *is the induced algebra of* $\mathrm{M}(\mu)$ *by the Fourier induction.*

If μ is a spectral distribution on $\mathcal{S}$, every state ω in the carrier of μ is pure relative to each $A \in \mathrm{A}$, almost everywhere.

Conversely, let μ be a distribution on S so that, (1). *every state in the carrier of μ as pure relative to each fixed $A \in \mathrm{A}$, almost everywhere, and* (2). *for every continuous function f on S we can choose a sequence A_n in A with $\int \|A_n\omega - f(\omega)\omega\|_\omega^2 \, d\mu(\omega) \to 0$.*

Then μ is a spectral distribution.

Corollary of Theorem 8. *Assume that A is separable,. A distribution μ on S is spectral if and only if it is pre-spectral and every state in the carrier of μ is pure, almost everywhere.*

§ 4. Density Theorems and the regularity of projections

The Kaplansky's Density Theorm follows the Helley's Theorem which asserts the weak denseness of the unit ball of any normed space in the unit ball of its bi-dual space. It is not hard to see that the bi-commutor of any representative algebra of A is contained in the bi-dual space of A. A dual concept of this Density Theorem is also dealt by Kaplansky. If A_λ is a representation of A (on a Hilbert space $\mathfrak{H}$), the norm $|A_\lambda|$ is the quotient norm $|A/\mathrm{N}|$ by a suitable two-sided ideal N of A. This Kaplansky's Theorem is obviously equivalent to the following statement : If $\mathfrak{M}$ is any uniformly closed two-sided invariant sub-space of the dual space $\bar{\mathrm{A}}$ of A, the unit ball of $\mathfrak{M}$ is weakly dense everywhere in the unit ball of the weak closure of $\mathfrak{M}$.

Then is the unit ball of a uniformly closed left invariant sub-space $\mathfrak{M}$ of $\bar{\mathrm{A}}$ weakly dense everywhere in the unit ball of the weak closure $\bar{\mathfrak{M}}$ of $\mathfrak{M}$? This question relates essentially to the regularity of the projection.

If A is represented as an operator algebra A_λ on a Hilbert space $\mathfrak{H}$ and E is a projection in A_λ'', then $\Phi(E)$ is a left invariant uniformly closed linear sub-space of $\bar{\mathrm{A}}$, Conversely, any uniformly closed left invariant sub-space of $\bar{\mathrm{A}}$ agrees to a $\Phi(E)$ of a suitable projection E in the W^*-algebra $\bar{\bar{\mathrm{A}}}$ of the bidual space of A($\bar{\bar{\mathrm{A}}}$ is a W^*-algebra. See Takeda (11).) The weak closure $\Phi(E)$ of $\Phi(E)$ is the totality of functionals which vanishes on $\mathrm{N}(E)$. Then E *is regular if and only if the unit ball of $\Phi(E)$ is dense everywhere in the unit ball of $\overline{\Phi(E)}$.*

Lemma 4. 1. *Let $\mathfrak{M}$ be a uniformly closed left invariant sub-space of $\bar{\mathrm{A}}$. $S(\mathfrak{M})$ the totality of states in $\mathfrak{M}$, and f an element of $\mathfrak{M}$ with $|f|=1$. Then $(f^*)^v \in S(\mathfrak{M})$. If p and q are two states so that $p \in S(\mathfrak{M})$ and $q \in L^2(p)$, then $q \in S(\mathfrak{M})$.*

Proof. Let $f \in \mathfrak{M}$ and $|f| = 1$. Then f^* has the canonical form $f^* = Up$ and $p = U_l f$, where U is partially isometric in A_p''. By Corollary

1 of Theorem 1, $p=(f^*)^v$ belongs to $\mathfrak{M}$ and $\mathcal{S}(\mathfrak{M})$. $\mathcal{S}(\mathfrak{M})$ is clearly uniformly closed and convex. Assume that $p\in\mathcal{S}(\mathfrak{M})$ and $q\in L^2(p)$, then we can choose a sequence $A_n\in\mathbf{A}$ with $|A_n p-q|\leqq A_n p-q_p\to 0$ and $|A_n^*p-q|\to 0$. Then $q\in\mathcal{S}(\mathfrak{M})$.

A left invariant uniformly closed sub-set of $\bar{\mathbf{A}}$, whose unit ball is weakly compact in $\bar{\mathbf{A}}$ is said to be a left-semi-ideal of $\mathbf{A}$. The unit ball of a left-semi-ideal is regularly convex, and the *Gelfand's extremity principle* is peserved.

Lemma 4. 2. *If $\mathfrak{M}$ is a left semi-ideal of $\mathbf{A}$, then $\mathcal{S}(\mathfrak{M})$ is regularly convex, and every extremal element of $\mathcal{S}(\mathfrak{M})$ is a pure state. The totality of finite dimensional states in $\mathcal{S}(\mathfrak{M})$ are dense everywhere in $\mathcal{S}(\mathfrak{M})$.*

Proof. Let p be an extremal element of $\mathcal{S}(\mathfrak{M})$, and $p=\alpha q+(1-\alpha)r$ be any decomposition of p to a sum of two positive functionals αq and $(1-\alpha)r$ (where q and r are states). Then q and r belong to $L^2(p)$, and belong to $\mathcal{S}(\mathfrak{M})$. This means $p=q=r$. Hence p is a pure state.

Lemma 4. 3. *Let $\mathcal{B}$ be any uniformly convex set of states on $\mathbf{A}$ so that $p\in\mathcal{B}$ and $q\in L^2(p)$ imply $q\in\mathcal{B}$. Then there exists a uniformly closed left invariant sub-space $\mathfrak{M}$ of $\bar{\mathbf{A}}$ with $\mathcal{S}(\mathfrak{M})=\mathcal{B}$. The correspondence $\mathfrak{M}\longleftrightarrow\mathcal{S}(\mathfrak{M})$ is one-to-one.*

Proof. We shall show that $\mathfrak{M}=(f\in\bar{\mathbf{A}}:\ p\in\mathcal{B}$ and $f^*\in L^2(p))$ is a closed left invariant linear sub-set of $\bar{\mathbf{A}}$. $f,\ g\in\mathfrak{M}$ imply $\alpha f+\beta g\in\mathfrak{M}$ (the linearity of $\mathfrak{M}$), because we can choose $f,\ g\in\mathcal{B}$ with $f^*\in L^2(p)$, $g^*\in L^2(q)$ and $\alpha f+\beta g\in L^2(\frac{1}{2}(p+q))$. If a sequence $\{f_n\}$ in $\mathfrak{M}$ converges uniformly to a $f\neq 0$, we have $f\in\mathfrak{M}$(the uniform closedness of $\mathfrak{M}$), because by Lemma 2. 5 of Chapter 1, $f^{*v}/|f|$ is contained in the uniform convex span of the sub-set $(f_n{}^*{}_v/|f_n|)$ of $\mathcal{B}$. $f\in\mathfrak{M}$ and $A\in\mathbf{A}$ implies $A_l f\in\mathfrak{M}$, (the left-invariance of $\mathfrak{M}$), because we can choose $p\in\mathcal{B}$ so that f^* and $(A_l f)^*=A^*f^*$ are contained in $L^2(p)$. Now $\mathcal{B}=\mathcal{S}(\mathfrak{M})$, and one-to-one between $\mathfrak{M}\longleftrightarrow\mathcal{S}(\mathfrak{M})$ follows immediately.

Lemma 4.4. *Assume that $\mathcal{S}(\mathfrak{M})$ is regulary convex, then $\mathfrak{M}$ is a left semi-ideal of $\mathbf{A}$.*

Proof. For every f in the unit ball $\mathfrak{U}(\mathfrak{M})$ of $\mathfrak{M}$ we can choose a $p\in\mathcal{S}(\mathfrak{M})$ with $|f(A)|^2\leqq p(A^*A)$. Then for any f in the weak closure $\overline{\mathfrak{U}(\mathfrak{M})}$ of $\mathfrak{U}(\mathfrak{M})$, we can choose such a p, and we have $\overline{\mathfrak{U}(\mathfrak{M})}=\mathfrak{U}(\mathfrak{M})$.

To study conditions for the carrier of a distribution on $\mathcal{S}$ to consists of pure states, we define the semi-regularity of projections.

Lemma 4. 5. *Consider a representation of* A *in a Hilbert space* $\mathfrak{H}$, *a projection* E *in* A'', *the weak closure* $\overline{\mathcal{S}(E)}$ *of* $\mathcal{S}(E)$, *and a norm* $|f|_E = \sup_{|AE|\leq 1} |f(A)|$ *of functionals* $f \in \bar{\mathrm{A}}$. *Then the following four conditions are mutually equivalent.*

(1). *Every state* p *with* $|AE| \geqq \|Ap\|_p$ $(A \in \mathrm{A})$ *satisfies* $|AE| \geqq |AE_p|$.

(2), *If* p *and* q *are states so that* $p \in \mathcal{S}(E)$ *and* $q \in L^2(p)$, *then* $q \in \mathcal{S}(E)$.

(3). $f \in \bar{\mathrm{A}}$ *and* $|f|_E < \infty$ *imply* $|f| = |f|_E$.

(4). *The weak closure of the unit ball of* $\Phi(E)$ *is the unit ball of a suitable left semi-ideal of* A.

Proof. Equivalences between (1), (2) and between (3), (4) are obvious. The equivalence between (2)--(4) follows immediately from what the weak closure of the unit ball of $\Phi(E)$ contains every $f \in \bar{\mathrm{A}}$ so that the absolute variation of f^* belongs to $\mathcal{S}(E)$.

Definition 4. 1. A projection E in A'' is said to be *semi-regular* if it satisfies one of the above four conditions.

Proposition 4. 1. *In order that every state in the carrier* $\mathfrak{D}(\mu)$ *of a distribution* μ *in* $\mathcal{S}$ *be pure, it is necessary and sufficient that, the coordinate projection field* P *belongs to* K_μ'', *and is semi-regular relative to* K.

Proof. Regard $\mathcal{S}$ as a compact space of states on K. Then $\omega \in \mathcal{S}$ is pure on A if and only if so, on K. If P belongs to K_μ'', $\Phi(P)$ and $\mathcal{S}(P)$ are defined relative to the algebra K, respetively.

(*Necessity*). Assume that every state in $\mathfrak{D}(\mu)$ is pure. Then $\mathrm{M}(\mu)$ is a maximal abelian $*$-subalgebra of A_μ', and P belongs to K_μ''. To see the semi-regularity of P, it is sufficient to show that, if q is a state on K with $q \in L^2(p)$ and $p \in \mathcal{S}(P)$, then $q \in \mathcal{S}(P)$. $\mathcal{S}(P)$ is the uniform closure of the set $(\int f(\omega)\, \omega\, d\mu : 0 \leqq f \in \mathrm{C}$ and $\int f d\mu = 1)$ of states on K. Then its weak closure $\overline{\mathcal{S}(P)}$ is the totality of means $\int \omega d\nu$ of distributions ν on $\mathcal{S}$ whose carrier is contained in $\mathfrak{D}(\mu)$. $\mathrm{M}(\nu)$ is a maximal abelian $*$-sub-algebra of A_ν' (ie., $\mathcal{S}(\cdot\cdot) = \mathrm{K}_\nu'$). Now let p and q be states on K so that $p = \int \omega d\nu \in \mathcal{S}(P)$ and $p \in L^2(p)$, where the carrier $\mathfrak{D}(\cdot)$ of ν is contained in $\mathfrak{D}(\mu)$. Then $q = \int f(\omega)\, \omega\, d\nu(\omega)$ holds, where f is a non-negative summable measurable function on $\mathcal{S}$. Hence q is contained in $\mathcal{S}(E)$.

(*Sufficiency*). Conversely, assume that P belongs to K_ν'' and is semi-regular. Then $\overline{\mathcal{S}(P)} = \mathcal{S}(\mathfrak{M})$ holds for a suitable left-semi-ideal $\mathfrak{M}$ of K. By Lemma 4. 2, $\mathcal{S}(P)$ satisfies the Gelfand's extremity principle, and every extremal element of $\mathcal{S}(P)$ is pure on K(and on A). The sufficiency shall be concluded by proving that $\mathcal{D}(\mu)$ is the totality of extremal elements of $\mathcal{S}(P)$.

By Lemma 4. 12, $\mathrm{M}(\mu)$ is a maximal abelian $*$-sub-algebra of A_μ', and $\mathrm{M}(\mu) = \mathrm{K}_\mu'$ holds. Then $\mathcal{S}(P)$ is the uniform closure of a set $(\int f(\omega)\, \omega\, d\mu : 0 \leqq f \in \mathrm{C},\ \int f\, d\mu = 1)$ of states on K, and $\mathcal{S}(P)$ is the totality of means $m_\nu = \int \omega\, d\nu$ of distributions on $\mathcal{S}$ whose carrier is contained in $\mathcal{D}(\mu)$. Remark that $m_\nu = m_\tau$ implies $m_\nu(f) = m_\tau(f) = \int f\, d\nu = \int f d\tau$ for every $f \in \mathrm{C} \subseteqq \mathrm{K}$ and $\nu = \tau$. Then $m_\nu \longleftrightarrow \nu$ is an weak homeomrphism between $\mathcal{S}(P)$ and the totality $\mathcal{S}$ of distributions on $\mathcal{S}$ whose carrier is contained in $\mathcal{D}(\mu)$. $m_\nu = \int \omega d\nu(\omega)$ is extremal in $\mathcal{S}(P)$ if and only if the distribution ν is extremal in $\mathcal{S}$; in other word, ν is a point mass distribution δ_ω, which distributes its total mass 1 at a point ω in $\mathcal{D}(\mu)$. Then $\mathcal{D}(\mu)$ is the totality of extremal elements of $\mathcal{S}(P)$, and its every element is a pure state in K and in A.

Regularity and semi-regularity are properties of projections which relates essentially to the natural representation $A \rightarrow AE$ of algebra in the quotient space devided by projections and left ideals. In the next chapter we shall study more systematically the quotient space of algebra and the the regular represention on it.

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