ON RIEMANNIAN MANIFOLDS ADMITTING A CONCIRCULAR TRANSFORMATION

SHIGERU ISHIHARA and YOSHIHIRO TASHIRO

In 1940 to 42, K. Yano[1] introduced the concept of concircular transformation of Riemannian manifolds, developed the theory of concircular geometry and obtained many suggestive theorems. A concircular transformation of a Riemannian manifold M to a Riemannian one $'M$ is by definition a conformal transformation of M to $'M$, which carries geodesic circles in M to geodesic circles in $'M$ [CG, I] ; a geodesic circle in M with metric tensor $g_{\mu\lambda}$ is a curve $x^\kappa = x^\kappa(s)$ satisfying the differential equation

$$\frac{\delta^3 x^\kappa}{ds^3} + g_{\mu\lambda}\frac{\delta^2 x^\mu}{ds^2}\frac{\delta^2 x^\lambda}{ds^2}\frac{dx^\kappa}{ds} = 0,$$

s being the arc length of the curve and δ/ds denoting the covariant differentiation along the curve in M.

The purpose of this paper is to study the structure, topological and differential-geometrical, of compact or complete Riemannian manifolds admitting a concircular transformation. In §1 we shall recall the arguments developed by K. Yano [CG, I, II] as preliminaries. In §2, we shall discuss the local structure of the manifolds in a neighborhood of an isolated stationary point of a concircular transformation. §3 will be devoted to the study of compact manifolds and §4 to the study of complete manifolds of constant scalar curvature, which admit a concircular transformation. The principal results are Theorem 2 in §3 and Theorem 4 in §4. In §5, the holonomy group of such manifolds will be discussed. Speaking in short, Theorem 2 states that a compact manifold admitting a non-homothetic concircular transformation is conformal to a sphere, and Theorem 4 states that a complete Riemannian manifold of constant scalar curvature admitting a non-homothetic concircular transformation onto itself is a sphere.

1) K. Yano, Concircular geometry,
- I. Concircular transformations, Proc. Imp. Acad. Tokyo, vol. 16 (1940), pp. 195—200.
- II. Integrability conditions of $\rho_{\mu\nu} = \phi g_{\mu\nu}$, ibid., vol. 16 (1940), pp. 354—360.
- III. Theory of curves, ibid., vol. 16 (1940), pp. 442—448.
- IV. Theory of subspaces, ibid,. vol. 16 (1940), pp. 505—511.
- V. Einstein spaces, ibid., vol. 18 (1942), pp. 446—451.

These papers will be referred to as CG.

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§ 1. Concircular transformation.

Let M and $'M$ be n-dimensional Riemannian manifolds[2]. We denote by $\{^{\kappa}_{\mu\lambda}\}$, $K_{\nu\mu\lambda}{}^{\kappa}$, $K_{\mu\lambda}$ and k the Christoffel symbol, the curvature tensor, the Ricci tensor and the scalar curvature ($k=K/n(n-1)$, $K=K_{\mu\lambda}g^{\mu\lambda}$) of M respectively, and by preceding primes the corresponding quantities of $'M$.[3]

A conformal transformation

$$(1.1)\qquad 'g_{\mu\lambda} = \rho^2 g_{\mu\lambda}$$

of M into $'M$ is concircular if and only if the equation

$$(1.2)\qquad \rho_{\mu\lambda} = \phi\, g_{\mu\lambda}$$

holds for a certain function ϕ [CG, I], where ρ is a positive valued scalar function on M and

$$(1.3)\qquad \begin{aligned} \rho_{\mu\lambda} &= \nabla_\mu\rho_\lambda - \rho_\mu\rho_\lambda + \tfrac{1}{2} g_{\mu\lambda}\rho_\kappa\rho^\kappa, \\ \rho_\lambda &= \partial_\lambda \log\rho, \end{aligned}$$

∇ denoting the covariant differentiation with respect to $\{^{\kappa}_{\mu\lambda}\}$. For a concircular transformation, the following formulas are known [CG, I] :

$$(1.4)\qquad '\{^{\kappa}_{\mu\lambda}\} = \{^{\kappa}_{\mu\lambda}\} + \delta^\kappa_\mu\rho_\lambda + \delta^\kappa_\lambda\rho_\mu - g_{\mu\lambda}\rho^\kappa,$$

$$(1.5)\qquad 'K_{\nu\mu\lambda}{}^{\kappa} = K_{\nu\mu\lambda}{}^{\kappa} - 2\phi(\delta^\kappa_\nu g_{\mu\lambda} - \delta^\kappa_\mu g_{\nu\lambda}),$$

$$(1.6)\qquad 'K_{\mu\lambda} = K_{\mu\lambda} - 2(n-1)\phi\, g_{\mu\lambda},$$

$$(1.7)\qquad 'k = \frac{1}{\rho^2}(k-2\phi).$$

If the function ρ is a constant, the conformal transformation is a homothety, and a homothety is a concircular transformation. However, throughout this paper, we shall be concerned only with non-homothetic concircular transformations, and the term "concircular" will always denote "non-homothetic concircular".

The equation (1.2) is equivalent to

$$(1.8)\qquad \nabla_\mu\rho_\lambda - \rho_\mu\rho_\lambda = \psi\, g_{\mu\lambda},$$

2) Throughout this paper we shall suppose that manifolds are connected and of dimension greater than 2, and that the differentiability of manifolds, transformations and quantities is of class C^∞.

3) Greek indices run from 1 to n and Latin indices from 1 to $n-1$, unless otherwise is stated.

where we have put

(1.9) $$\psi = \phi - \tfrac{1}{2} g^{\mu\lambda} \rho_\mu \rho_\lambda.$$

If we put $\tau = 1/\rho$, then the equation (1.8) is reduced to

(1.10) $$\nabla_\mu \nabla_\lambda \tau + \psi \tau g_{\mu\lambda} = 0.$$

Differentiating covariantly the both sides of (1.8) and taking account of the Ricci formula, we have

$$K_{\nu\mu\lambda}{}^{\kappa} \rho_\kappa = (\psi \rho_\nu - \partial_\nu \psi) g_{\mu\lambda} - (\psi \rho_\mu - \partial_\mu \psi) g_{\nu\lambda},$$

and, transvecting this equation with ρ^λ, we see

(1.11) $$\partial_\nu \psi = \alpha \rho_\nu,$$

where α is a proportional factor. Putting

(1.12) $$\gamma = \psi - \alpha,$$

we obtain

(1.13) $$K_{\nu\mu\lambda}{}^{\kappa} \rho_\kappa = \gamma (\rho_\nu g_{\mu\lambda} - \rho_\mu g_{\nu\lambda}).$$

A point of M is called a *stationary* point or an *ordinary* one of a concircular transformation if the gradient vector field ρ_λ vanishes at the point or not. In a neighborhood of an ordinary point we consider the integral curves of the vector field ρ^κ. By means of (1.8), we can easily see that such an integral curve is a geodesic arc. A geodesic is called a *ρ-curve* if it contains such an arc. [CG, II].

Let P be an ordinary point in M and U a coordinate neighborhood of P which contains no stationary point. Then we can define in U a family of hypersurfaces by the equation $\rho =$ const. The hypersufaces will be called *ρ-hypersurfaces in U* [CG, II]. Given a point Q in U, there exists in the family one and only one ρ-hypersurface $V(Q)$ passing through Q. The ρ-curves form the normal congruence to the family of the ρ-hypersurfaces in U.

Let i^κ be the unit vector field of the vector field ρ^κ, which is definable in M except at the stationary points. i^κ is given by

$$i^\kappa = \frac{1}{\sigma} \rho^\kappa, \qquad \sigma = \sqrt{\rho_\lambda \rho^\lambda}.$$

From this equation and (1.8), we have

$$(\partial_\mu \sigma) i_\lambda + \sigma \nabla_\mu i_\lambda = \psi g_{\mu\lambda} + \sigma^2 i_\mu i_\lambda,$$

and, by transvection of this equation with i^λ,

(1.14) $$\partial_\mu \sigma = (\psi + \sigma^2) i_\mu.$$

The two above equations imply

$$\nabla_\mu i_\lambda = \frac{\psi}{\sigma}(g_{\mu\lambda} - i_\mu i_\lambda). \tag{1.15}$$

Let Q be a point in U and $V(Q)$ the ρ-hypersurface in U passing through the point Q. Then the vector i^κ is the normal unit vector of $V(Q)$ at any point of $V(Q)$. We choose a system of local coordinates u^h in $V(Q)$ and suppose that $V(Q)$ is expressed by parametric equations

$$x^\kappa = x^\kappa(u^h)$$

in U. The second fundamental tensor h_{ji} of the ρ-hypersurface $V(Q)$ is given by

$$h_{ji} = B_j^\mu B_i^\lambda \nabla_\mu i_\lambda, \tag{1.16}$$

where $B_i^\kappa = \partial_i x^\kappa$. Denoting by $\bar{g}_{ij}$ the induced metric tensor $B_j^\mu B_i^\lambda g_{\mu\lambda}$ of $V(Q)$, we see on account of (1.15)

$$h_{ji} = h\,\bar{g}_{ji}, \qquad h = \frac{\psi}{\sigma}, \tag{1.17}$$

because of $B_i^\lambda i_\lambda = 0$. Therefore the ρ-hypersurface $V(Q)$ is totally umbilical. Moreover, in virtue of the Weingarten equation, we obtain

$$\partial_j B_i^\kappa + \left\{ {\kappa \atop \mu\lambda} \right\} B_j^\mu B_i^\lambda - \overline{\left\{ {h \atop ji} \right\}} B_h^\kappa = h\,\bar{g}_{ji}\, i^\kappa, \tag{1.18}$$

where $\overline{\left\{ {h \atop ji} \right\}}$ denotes the Christoffel symbol constructed from $\bar{g}_{ji}$ in $V(Q)$. From the Codazzi equation it follows that

$$\bar{\nabla}_k h_{ji} - \bar{\nabla}_j h_{ki} = B_k^\nu B_j^\mu B_i^\lambda K_{\nu\mu\lambda}{}^\kappa i_\kappa,$$

$\bar{\nabla}$ denoting the covariant differentiation with respect to $\overline{\left\{ {h \atop ji} \right\}}$ in $V(Q)$. Taking account of (1.13) and (1.17), we have hence

$$(\partial_k h)\,\bar{g}_{ji} - (\partial_j h)\,\bar{g}_{ki} = 0.$$

Provided $n > 2$, this implies $\partial_j h = 0$, which means that the function h is constant on $V(Q)$. On the other hand, (1.14) implies $\partial_j \sigma = B_j^\mu \partial_\mu \sigma = 0$, that is, σ is constant on $V(Q)$, and consequently so is ψ, because of (1.17). This means that h, σ and ψ are functions of ρ in U.

Now we can choose a system of coordinates u^κ in U such that the hypersurfaces defined by u^n = const. are the ρ-hypersurfaces in U and the curves defined by the equations u^h = const. are the ρ-curves in U. The ρ-curves being normal to the ρ-hypersurfaces u^n = const., we have at first

$$g_{ni} = g_{in} = 0.$$

Since the ρ-curves are geodesics, we have

$$\frac{d}{du^n}\frac{du^\kappa}{du^n} + \left\{ {\kappa \atop \mu\lambda} \right\} \frac{du^\mu}{du^n}\frac{du^\lambda}{du^n} = \beta \frac{du^\kappa}{du^n}, \qquad \frac{du^\kappa}{du^n} = \delta^\kappa_n,$$

β being a function of u^n, and hence

$$\left\{ {\kappa \atop nn} \right\} = \beta\, \delta^\kappa_n.$$

In particular, we obtain

$$\left\{ {h \atop n\, n} \right\} = 0 \quad \text{or} \quad \frac{1}{2} g^{h\lambda}\left(\frac{\partial g_{\lambda n}}{\partial u^n} + \frac{\partial g_{\lambda n}}{\partial u^n} - \frac{\partial g_{nn}}{\partial u^\lambda}\right) = 0.$$

Taking account of $g_{ni} = 0$ and $g^{nh} = 0$, we have

$$\frac{\partial g_{nn}}{\partial u^i} = 0,$$

from which it follows that g_{nn} depends only on u^n. Hence, by a suitable transformation of the n-th coordinate, we may suppose from now that g_{nn} is always equal to 1 in U. Then we have

$$\left\{ {\kappa \atop nn} \right\} = 0, \tag{1.19}$$

and the variable u^n is the arc length of ρ-curves in U. Therefore the arcs of ρ-curves cut off by two ρ-hypersurfaces $u^n = s_1$ and $u^n = s_2$ have a constant length $s_2 - s_1$, that is, ρ-hypersurfaces in U are geodesically parallel to each other.

If we take the variables u^h as local coordinates in each ρ-hypersurface in U, then we have

$$B^\kappa_i = \partial_i u^\kappa = \delta^\kappa_i \quad \text{and} \quad \overline{g}_{ji} = g_{ji},$$

on each ρ-hypersurface. Therefore the equation (1. 18) is reduced to

$$\left\{ {\kappa \atop j\, i} \right\} - \overline{\left\{ {h \atop j\, i} \right\}}\, \delta^\kappa_h = h\, g_{ji}\, \delta^\kappa_n,$$

because the unit vector i^κ has δ^κ_n as components with respect to the local coordinates u^κ. For $\kappa = n$, the above equation is reduced to

$$\left\{ {n \atop j\, i} \right\} = \frac{1}{2} g^{n\lambda}\left(\frac{\partial g_{\lambda i}}{\partial u^j} + \frac{\partial g_{\lambda j}}{\partial u^i} - \frac{\partial g_{ji}}{\partial u^\lambda}\right) = h g_{ji}.$$

On account of $g^{nh} = g_{ni} = 0$ and $g_{nn} = g^{nn} = 1$, we have

$$\frac{1}{2}\frac{\partial g_{ji}}{\partial u^n} = h g_{ji}. \tag{1.20}$$

Since the function h is constant on each ρ-hypersurface in U, h is a function dependent only of the variable u^n in U. Thus, integrating the equation (1.20), we obtain

(1.21) $$g_{ji} = \lambda(u^n)^2 f_{ji}(u^h),$$

where $f_{ji}(u^h)$ are certain functions of the $n-1$ variables u^h such that the matrix (f_{ji}) is positive definite, and $\lambda(u^n)$ is a positive-valued function of the variable u^n. Consequently the line element of the Riemannian manifold M is written in the form

(1.22) $$ds^2 = g_{\mu\lambda}\, du^\mu du^\lambda = \lambda(u^n)^2 f_{ji}(u^h)\, du^j du^i + (du^n)^2$$

with respect to the system of coordinates u^κ in U.

In the following, primes on the right shoulder will indicate the derivatives with respect to the n-th coordinate u^n in U, i.e., with respect to the arc length of the ρ-curves. By means of (1.22), we have easily

(1.23) $$\left\{ {n \atop j\ i} \right\} = -\lambda\lambda' f_{ji}.$$

Since ρ is constant on each ρ-hypersurface in U, the function ρ depends only on the variable u^n in U. Hence, putting $\lambda = i$, $\mu = j$ in (1.8), we have

$$-\left\{ {n \atop j\ i} \right\} \frac{\rho'}{\rho} = \psi\, g_{ji}.$$

Comparing this equation with (1.23), we have

(1.24) $$\psi = \frac{\lambda'\rho'}{\lambda\rho}.$$

On the other hand, putting $\lambda = \mu = n$ in (1.10) and taking account of (1.19), we obtain

(1.25) $$\psi = -\frac{\tau''}{\tau}.$$

From (1.24) and (1.25), it follows that

$$\frac{\lambda'}{\lambda} = \frac{\tau''}{\tau'}$$

and consequently

(1.26) $$\lambda = c\,\tau',$$

c being a non-zero constant. Writing f_{ji} instead of $c^2 f_{ji}$ from the biginning, we have from (1.22)

(1.27) $$ds^2 = (\tau')^2 f_{ji}(u^h)\, du^j du^i + (du^n)^2.$$

Summarizing results, we have the following theorem [Cf. CG, II] :

Theorem 1. *If a Riemannian manifold M admits a concircular transformation into a Riemannian manifold 'M, then, for any ordinary point of the transformation, there exists a coordinate neighborhood U of the point, where we can choose a system of coordinates* u^κ *having the following properties : The function* ρ *depends only on the n-th variable* u^n *in U. The line element of M is given by* (1. 27) *in U. The hypersurfaces defined by the equation* $u^n = const.$ *are the* ρ*-hypersurfaces and the curves defined by the equations* $u^h = const.$ *are the* ρ*-curves and* u^n *indicates the arc length of the* ρ*-curves.*

A systsm of coordinates u^κ having the above properties will be called a system of *adapted coordinates*, and a coordinate neighborhood U of an ordinary point, where we can introduce a system of adapted coordinates, will be called a regular one.

With respect to a system of adapted coordinates, the Christoffel symbol of the line element (1. 27) has the components

$$\begin{aligned} &\left\{ {h \atop j\,i} \right\} = \overline{\left\{ {h \atop j\,i} \right\}}, \qquad \left\{ {n \atop j\,i} \right\} = \left\{ {n \atop j\,i} \right\} = -\tau'\tau'' f_{ji}, \\ (1.28) \qquad &\left\{ {h \atop n\,i} \right\} = \left\{ {h \atop i\,n} \right\} = \frac{\tau''}{\tau'}\delta_i^h, \qquad \left\{ {n \atop n\,i} \right\} = \left\{ {n \atop i\,n} \right\} = 0, \\ &\left\{ {h \atop n\,n} \right\} = 0, \qquad \left\{ {n \atop n\,n} \right\} = 0, \end{aligned}$$

where $\overline{\left\{ {h \atop j\,i} \right\}}$ denotes the Christoffel symbol constructed from f_{ji}. Moreover we can verify that the curvature tensor $K_{\nu\mu\lambda}{}^\kappa$ of M is given by

$$\begin{aligned} &K_{kji}{}^h = \overline{K}_{kji}{}^h - (\tau'')^2 (\delta_k^h f_{ji} - \delta_j^h f_{ki}), \\ (1.29) \qquad &K_{nji}{}^n = -K_{jni}{}^n = -\tau'\tau''' f_{ji}, \\ &K_{njn}{}^h = -K_{jnn}{}^h = \left(\frac{\tau'''}{\tau'}\right) \delta_j^h, \end{aligned}$$

the other components being zero, and the Ricci tensor $K_{\mu\lambda}$ of M by

$$\begin{aligned} &K_{ji} = \overline{K}_{ji} - [(n-2)(\tau'')^2 + \tau'\tau'''] f_{ji}, \\ (1.30) \qquad &K_{ni} = K_{in} = 0, \\ &K_{nn} = -(n-1)\frac{\tau'''}{\tau'}, \end{aligned}$$

and the scalar curvature k by

$$(1.31) \qquad k = \frac{1}{n(\tau')^2}[(n-2)\overline{k} - (n-2)(\tau'')^2 - 2\tau'\tau'''],$$

where $\overline{K}_{kji}{}^h$, $\overline{K}_{ji}$ and $\overline{k}$ are the curvature tensor, the Ricci tensor and the scalar curvature of the $(n-1)$-dimensional metric f_{ji}; $\overline{K}_{ji} = \overline{K}_{hji}{}^{h}$ and $\bar{k} = \overline{K}/(n-1)(n-2)$, $\overline{K} = \overline{K}_{ji} f^{ji}$.

Since the function ρ depends only on u^n in U, the gradient vector field ρ_λ has the components $(\rho'/\rho)\delta^n_\lambda$, and its length is given by

$$\sigma^2 = \rho_\lambda \rho^\lambda = \left(\frac{\rho'}{\rho}\right)^2, \tag{1. 32}$$

with respect to a system of adapted coordinates.

§ 2. Concircular transformation with isolated stationary points.

In this paragraph, we suppose that a concircular transformation of a Riemannian manifold M has an isolated stationary point O in M. We take a sufficiently small spherical neighborhood W of O, such that it contains no stationary point except O and, for any point P in W, there exists a unique geodesic arc joining O to P. As it is noticed in § 1, the function ψ in (1. 4) is a function of ρ in $W-O$. However, since the stationary point O is isolated, ψ is a function of ρ in the whole neighborhood W, because of the continuity of ψ and ρ. Along any geodesic curve, the equation (1. 4) is reduced to the ordinary differential equation

$$\frac{d^2 \log \rho}{ds^2} - \left(\frac{d \log \rho}{ds}\right)^2 = \psi(\rho), \tag{2. 1}$$

s being the arc length of the geodesic. In particular, we consider the solutions of (2. 1) along the geodesics issuing from O, and make the value $s=0$ correspond to O. Since the vector field ρ_λ vanishes at O, we have $(d\rho/ds)_0 = 0$ along every geodesic issuing from O. In virtue of the uniqueness of solution of an ordinary differential equation, the solutions of (2. 1) along all geodesics issuing from O, with initial conditions $\rho(0) = \rho_0$ and $(d\rho/ds)_0 = 0$, are given in W by a same function $\rho = \rho(s)$ of the arc length s. Moreover, since such a geodesic curve in W contains no stationary point except O, the function $\rho(s)$ is monotone and is a univalent function of the arc length s along any geodesic arc issuing from O. Therefore it follows that every Riemannian hypersphere of radius s with center O in W is a ρ-hypersurface and every geodesic curve issuing from O is a ρ-curve. Conversely, since through a point $P \neq O$ in W there exist only one ρ-hypersurface and only one ρ-curve, we can state the following

Lemma 1. *If a concircular transformation has an isolated stationary point O, then, in a sufficiently small spherical neighborhood W of O, the ρ-hypersurface $V(P)$ through a point $P \neq O$ in W is a Riemannian hypersphere with O as center and the ρ-curve through P*

coincides with the geodesic arc joining O to P in W.

We may also introduce in W a system of normal coordinates y^κ with center O. The metric tensor $g_{\mu\lambda}$ has components $\delta_{\mu\lambda}$ at O with respect to the system of normal coordinates y^κ. The coordinates y^κ of a point P in W are expressed in the form $y^\kappa = st^\kappa$, where t^κ satisfy the equation $\sum_{\kappa=1}^{n} t^\kappa t^\kappa = 1$ and s is the geodesic distance of P from O. Consequently, the ρ-hypersurface $V(P)$ through P in W is the Riemannian hypersphere defined by the equation

$$\sum_{\kappa=1}^{n} (y^\kappa)^2 = s^2 \tag{2.2}$$

with respect to the system of normal coordinates.

On the other hand, we take a regular neighborhood U contained in W and denote by u^κ adapted coordinates in U. The transformation

$$u^\kappa = u^\kappa(y^1, \ldots, y^n) \tag{2.3}$$

from the adapted coordinates u^κ to the normal coordinates y^κ has the following properties : The functions $u^h(y^\kappa)$ are homogeneous of degree zero in y^κ, and

$$u^n = \{\sum_{\kappa=1}^{n} (y^\kappa)^2\}^{\frac{1}{2}} \tag{2.4}$$

in U. Hence the derivatives $\partial u^h/\partial y^\lambda$ are homogeneous of degree -1 in y^κ and $\partial u^n/\partial y^\lambda$ are homogeneous of degree zero in y^κ. Accordingly we can easily see that the derivatives $\partial y^\kappa/\partial u^\iota$ are homogeneous of degree one in y^κ.

Now we consider a parallel vector field $v(s)$ along a ρ-curve l, s being the arc length of l, and denote by $v^\kappa(s)$ and $\xi^\kappa(s)$ the componenets of the vector field $v(s)$ with respect to the adapted coordinates in U and to the normal coordinates in W respectively. If the ρ-curve l is given by $u^h = c^h$ in U, then, by taking account of (1.25), the components $v^\kappa(s)$ satisfy the equations

$$\frac{dv^h(s)}{ds} + \frac{\tau''}{\tau'} v^h(s) = 0, \quad \frac{dv^n(s)}{ds} = 0.$$

Integrating these equations, we have the following

Lemma 2. *With respect to a system of adapted coordinates, the components $v^\kappa(s)$ of a parallel vector field $v(s)$ along a ρ-curve l are functions of the form*

$$v^h(s) = \frac{1}{\tau'(s)} v^h, \quad v^n(s) = v^n, \tag{2.5}$$

where v^κ are constants and s being the arc length of l.

In particular, if the parallel vector field $v(s)$ is tangent to one of the ρ-hypersurfaces in U, then the vector field $v(s)$ is always tangent to the ρ-hypersurfaces, and its components are given by

$$v^h(s) = \frac{1}{\tau'(s)} v^h, \quad v^n(s) = 0 \tag{2.6}$$

in U.

If a parallel vector field $v(s)$ along a ρ-curve l is tangent to the ρ-hypersurfaces in U and the ρ-curve l is expressed by the equation $y^\kappa = st^\kappa$ with respect to a system of normal coordinates in W with center O, then, under the coordinate transformation (2.3) in U, the components $\xi^\kappa(s)$ of $v(s)$ with respect to the system of normal coordinates are given by

$$\begin{aligned}\xi^\kappa(s) &= \left(\frac{\partial y^\kappa}{\partial u^i} v^i(s)\right)_{y^\kappa = st^\kappa} \\ &= \frac{1}{\tau'(s)}\left(\frac{\partial y^\kappa}{\partial u^i}\right)_{y^\kappa = st^\kappa} v^i = \frac{s}{\tau'(s)}\left(\frac{\partial y^\kappa}{\partial u^i}\right)_{y^\kappa = t^\kappa} v^i\end{aligned} \tag{2.7}$$

in U, because $\partial y^\kappa/\partial u^i$ are homogeneous of degree one in y^κ. If we put

$$\nu(s) = \frac{s}{\tau'(s)}, \qquad \xi^\kappa = \left(\frac{\partial y^\kappa}{\partial u^i}\right)_{y^\kappa = t^\kappa} v^i, \tag{2.8}$$

then we have

$$\xi^\kappa(s) = \nu(s)\,\xi^\kappa. \tag{2.9}$$

We notice here that ξ^κ are constants and the function $\nu(s)$ does not depend on the choices of ρ-curve l and of parallel vertor field $v(s)$ along l. Since the vector field $v(s)$ is parallel along l, the limiting values $\lim_{s\to 0} \xi^\kappa(s)$ have to exist and these limiting values, say $\xi^\kappa(0)$, define the vector of the field $v(s)$ at the stationary point O. Since ξ^κ are constants, we see that the limiting value $\lim_{s\to 0}\nu(s) = \lim_{s\to 0} s/\tau'(s)$ should exist and the value, say $\nu(0)$, is non-zero finite. Summarizing the results, we say

Lemma 3. *Under the same assumption as that in Lemma* 1, *we consider a parallel vector field $v(s)$ along a ρ-curve l in W and tangent to the ρ-hypersurfaces in W. Then, with respect to a system of normal coordinates in W, the components $\xi^\kappa(s)$ of the vector field $v(s)$ are given by*

$$\xi^\kappa(s) = \nu(s)\,\xi^\kappa,$$

where ξ^κ are constants and s is the arc length of l such that $s = 0$ corresponds to the stationary point O.

The function $\nu(s) = s/\tau'(s)$ is independent of the choices of ρ-curve l and of parallel vector field along l, and the limiting value $\nu(0) =$

$\lim_{s \to 0} \nu(s)$ *exists and is non-zero finite.*

In the spherical neighborhood W of a stationary point O, we consider a transformation Φ defined by

$$'y^\kappa = a_\lambda{}^\kappa y^\lambda \tag{2.10}$$

with respect to the normal coordinates with center O, where the constant matrix $(a_\lambda{}^\kappa)$ is an arbitrary orthogonal one. The transformation Φ leaves the stationary point O invariant, and preserves also any ρ-hypersurface in W, because a ρ-hypersurfaces in W is a Riemannian hypersphere with center O expressed by $\sum_{\kappa=1}^{n} (y^\kappa)^2 = s^2$. The group G of all transformations such as defined above is isomorphic to the group $O(n)$ of all orthogonal transformations of the tangent space of M at O. Thus we have $\dim G = n(n-1)/2$. The group G may be considered as a group of transformations of a ρ-hypersurface in W. Now we shall prove the following

Lemma 4. *Under the same assumption as that in Lemma* 1, *the group G is a group of isometries of a ρ-hypersurface in W.*

Proof. Let V be a ρ-hypersurface in W, P a point of V, and l the ρ-curve joining O to P. Let l be expressed by $y^\kappa = st^\kappa$ with respect to a system of normal coordinates with center O, and suppose that $s = 0$ and $s = s_1$ correspond to O and P respectively. We take two tangent vectors v and w to V at P, and construct from v and w the two parallel vector fields $v(s)$ and $w(s)$ along the ρ-curve l: $v(s_1) = v$ and $w(s_1) = w$. We denote by $\xi^\kappa(s)$ and $\eta^\kappa(s)$ the components of $v(s)$ and $w(s)$ with respect to the system of normal coordinates respectively. By means of Lemma 3, we have

$$\xi^\kappa(s) = \nu(s)\,\xi^\kappa, \qquad \eta^\kappa(s) = \nu(s)\,\eta^\kappa, \tag{2.11}$$

where ξ^κ and η^κ are constants. Since the inner product of two vectors is invariant under a parallel displacement, the inner product (v, w) of the two vectors v, w at P is equal to

$$(v, w) = (v(0), w(0)) = \nu(0)^2 \sum_{\kappa=1}^{n} \xi^\kappa \eta^\kappa. \tag{2.12}$$

Let Φ be an element of G. Putting $'P = \Phi(P)$, we see that the point $'P$ lies in V and the curve $'l = \Phi(l)$ is the ρ-curve joining O to $'P$. We denote by $'v(s)$ and $'w(s)$ the images $d\Phi(v(s))$ and $d\Phi(w(s))$ by the differential mapping $d\Phi$ of the transformation Φ. In virtue of the linearity of the transformation Φ, we see from (2.11) that the components of $'v(s)$ and $'w(s)$ are given by

$$(2.13)\qquad \begin{aligned} {}'\xi^\kappa(s) &= a_\lambda{}^\kappa\, \xi^\lambda(s) = \nu(s)\, {}'\xi^\kappa, \\ {}'\eta^\kappa(s) &= a_\lambda{}^\kappa\, \eta^\lambda(s) = \nu(s)\, {}'\eta^\kappa \end{aligned}$$

respectively, where we have put

$$(2.14)\qquad {}'\xi^\kappa = a_\lambda{}^\kappa\, \xi^\kappa, \quad {}'\eta^\kappa = a_\lambda{}^\kappa\, \eta^\lambda.$$

Hence the vector fields $'v(s)$ and $'w(s)$ are parallel along the ρ-curve $'l$ and tangent to the ρ-hypersurfaces. Therefore, the inner product of the images $'v = d\Phi(v)$ and $'w = d\Phi(w)$ at $'P$ is equal to

$$(2.15)\qquad ('v, 'w) = \nu(0)^2 \sum_{\kappa=1}^{n} {}'\xi^\kappa\, {}'\eta^\kappa .$$

Since the matrix $(a_\lambda{}^\kappa)$ is orthogonal, we have

$$(2.16)\qquad \sum_{\kappa=1}^{n} \xi^\kappa \eta^\kappa = \sum_{\kappa=1}^{n} {}'\xi^\kappa\, {}'\eta^\kappa,$$

and, from (2.12) and (2.15),

$$(v, w) = ('v, 'w).$$

This means that the transformation Φ preserves the inner product of any two tangent vectors of the ρ-hypersurface V, that is, Φ is an isometry of V. Thus the proof of the lemma is completed.

As a direct consequence of Lemma 4, we can prove the following

Lemma 5. *Uuder the same assumption as that in Lemma* 1, *any ρ-hypersurface V in W is isometrically homeomorphic to an $(n-1)$-dimensional spherical space S_{n-1}, that is, a hypershere S_{n-1} of an n-dimensional Euclidean space, which is endowed with the naturally induced Riemannian metric of positive constant sectional curvature.*

Proof. By Lemma 4, the ρ-hypersurface V admits a group G of isometries, and G is of dimension $n(n-1)/2$. Hence V is a Riemannian manifold of constant sectional curvature. On the other hand, V is homeomorphic to an $(n-1)$-dimensional sphere S_{n-1}. Combining these facts, we obtain the lemma.

§ 3. Compact manifold.

In this paragraph we shall confine ourselves to a compact Riemannian manifold admitting a concircular transformation. Let P be an ordinary point of the concircular transformation. We consider the hypersurface defined by $\rho = \rho(P)$ in M, and denote by $V(P)$ the connected component of the hypersurface containing the point P. M being compact, the hypersurface $V(P)$ is also compact. If U is a regular neighborhood of an ordinary point of $V(P)$, then $V(P) \cap U$ is a ρ-hypersurface in U. As is proved in § 1, the length σ of the vector field ρ_λ is constant on

$V(P) \cap U$. Therefore σ is constant on the hypersurface $V(P)$ and consequently any point of $V(P)$ is ordinary. We call $V(P)$ the ρ-hypersurface passing through the point P.

Since $V(P)$ is compact, it follows from Theorem 1 that there exists a positive number ε such that any point of the ε-neighborhood W_ε of $V(P)$ is ordinary and the ρ-hypersurface $V(Q)$ through any point Q of W_ε is contained in W_ε. Moreover the ε-neighborhood W_ε has the following property : R is a point of $V(P)$ and l the ρ-curve through R, then each connected component of the set $l \cap W_\varepsilon$ has one and only one point in common with $V(Q)$, because the function ρ is monotone along a connected arc of $l \cap W_\varepsilon$. Denote by $'R$ the point of intersection of $V(Q)$ with the connected arc of $l \cap W_\varepsilon$ containing the point R. The correspondence $R \rightarrow {'R}$ defines a homothetic homeomorphism of $V(P)$ onto $V(Q)$.

Let P be an ordinary point and l the ρ-curve passing through P. We consider the set of all ordinary points lying on l, and denote by $L(P)$ the connected component of the set containing P. Now we put

$$M^\circ = \bigcup_{Q \in L(P)} V(Q).$$

The set M° is open and connected, and any point of M° is ordinary. From the above arguments, it is easily seen that, starting from another point of M°, we obtain the same set M°. Moreover the set M° is homeomorphic to the product $V(P) \times L(P)$.

Since the manifold M is compact, there exists at least one stationary point of the concircular transformation. Then the set M° is not closed. In fact, if M° were closed, M° would coincide with the whole manifold M, because M is connected and M° is open. There exists hence a stationary point O belonging to the boundary of the open submanifold M°. Therefore there is in M° a sequence $\{P_m\}$ $(m = 1, 2, \dots)$ of points which converges to the stationary point O. We denote by σ_m the values of the function $(g^{\mu\lambda} \partial_\mu \tau \, \partial_\lambda \tau)^{\frac{1}{2}}$ at P_m, where $\tau = 1/\rho$. Then the sequence $\{\sigma_m\}$ tends to zero.

If we denote by d_m the diameter of the compact ρ-hypersurface $V(P_m)$, then, in virtue of Theorem 1, we obtain

$$d_m : \sigma_m = d_1 : \sigma_1$$

for any integer m. Hence the sequence $\{d_m\}$ of the diameters tends to zero. Since the sequence $\{P_m\}$ converges to O, the sequence $\{V(P_m)\}$ of the ρ-hypersurfaces converges to the stationary point O. Consequently, for any point P of M°, the connected arc $L(P)$ of the ρ-curve l passing through

P has the stationary point O as its boundary point and hence the ρ-curve l contains the stationary point O.

Let Q be a point of the ρ-hypersurface $V(P)$ and l the ρ-curve passing through Q. The ρ-curve l passes also through the stationary point O, and let us denote by $e(Q)$ the unit tangent vector of l at O. The correspondence $Q \to e(Q)$ defines in a natural way a continuous mapping of $V(P)$ into the unit hypersphere S_{n-1} of the tangent space of M at O. The mapping is obviously one-to-one. Since the ρ-hypersurface $V(P)$ is compact, the mapping is therefore a homeomorphism of $V(P)$ onto S_{n-1}. Thus the set M° is homeomorphic to the product $S_{n-1} \times L(P)$.

Since the sequence $\{V(P_m)\}$ of ρ-hypersurfaces, which are homeomorphic to an $(n-1)$-dimensional sphere S_{n-1}, converges to O, the stationary point O is an isolated stationary one, that is, there is a neighborhood of O whose points except O belong to M°. This implies that any boundary point of M° is an interior point of the closure $\overline{M}^{\circ}$ of the set M°. Hence the closure $\overline{M}^{\circ}$ is open in M. By the connectedness of M, the set $\overline{M}^{\circ}$ have to coincide with the whole manifold M. Since any stationary point is isolated and M is compact, the manifold M is the union of the set M° and a finite number of stationary points.

It is easily seen that, if two geodesic curves issuing from a stationary point O have in common a point O' different from O, the point O' is also a stationary point. If there were in M only one stationary point O, then, by means of the above arguments, there would exist no conjugate point of O on any geodesic curve issuing from O, and consequently the manifold M would not be compact. It is a contradiction to the compactness of M. Therefore there exist at least two stationary points in M.

As is mentioned above, for any point P of M°, the connected geodesic arc $L(P)$ possesses any stationary point as its boundary point. It is however obvious that the arc $L(P)$ has at most two boundary points. Hence there must exist exactly two stationary point O and O' in M. Since M° is homeomorphic to the product $S_{n-1} \times L(P)$ of an $(n-1)$-dimensional sphere S_{n-1} and an open interval $L(P)$, the manifold M, which is the union of M° and the two stationary points O and O', is homeomorphic to an n-dimensional sphere S_n. The homeomorphism θ of M onto S_n can be defined in a natural and differentiable way. Summarizing the results in this paragraph, we have the following

Lemma 6 *If a compact Riemannian manifold M admits a concircular transformation, then the manifold M is differentiably homeomorphic to an n-dimensional sphere S_n, and there exist exactly two stationary points O and O' in M. When S_n is represented by the unit* *hypersphere*

$$(x^1)^2 + (x^2)^2 + \dots + (x^{n+1})^2 = 1$$

in an $(n+1)$*-dimensional Euclidean space* E_{n+1}, $(x^1, x^2, \dots, x^{n+1})$ *being rectangular coordinates in* E_{n+1}, *the homeomorphism* θ *of* M *onto* S_n *maps a* ρ*-hypersurface on a sphere*

$$(x^1)^2+(x^2)^2 + \dots + (x^{n+1})^2 = 1, \quad x^{n+1} = c, \quad -1 < c < 1,$$

and a ρ*-curve on a great circle passing through the antipodal points* $(0, \dots, 0, 1)$ *and* $(0, \dots, 0, -1)$, *which are the images of the two stationary points.*

We have seen that, in a compact manifold M, the ρ-hypersurfaces are homothetically related to each other. On the other hand, since the stationary points in M are isolated, it follows from Lemma 5 that a ρ-hypersurface is isometrically homeomorphic to an $(n-1)$-dimensional spherical space, if it lies sufficiently near to a stationary point. Therefore any ρ-hypersurface of the compact manifold M is homothetically homeomorphic to a unit hypersphere S_{n-1} of E_n. Accordingly we may now assume that the line element $f_{ji}(u^h)du^j du^i$ appearing in (1. 24) has constant sectional curvature 1 : $\bar{k} = \bar{K}/(n-1)(n-2) = 1$.

As is mentioned in § 1, the arcs of ρ-curves cut off by two ρ-hypersurfaces have the same length if they contain no stationary point. This implies that any ρ-curve has a constant arc length, say s_1, between the two stationary points O and O'. We denote by s the arc length of a ρ-curve joining O to O' such as $s = 0$ at O and $s = s_1$ at O'. s is the arc length in common with the ρ-curves joining O to O'. Along each of such arcs, we define a parameter t by

(3. 1) $$t = 2 \tan^{-1} \chi(s), \qquad 0 \leqq s \leqq s_1,$$

where we have put

(3. 2) $$\chi(s) = \exp \int_{\frac{s_1}{2}}^{s} \frac{ds}{\tau'(s)}.$$

When s varies from 0 to s_1, it is obvious that the parameter t is a monotone function of s, and, by use of Lemma 3, it is verified that t runs over the range $0 \leqq t \leqq \pi$. We obtain

(3. 3) $$\frac{ds}{dt} = \frac{\tau'(s)}{\sin t}.$$

From this equation and Theorem 1, it follows that, in a regular neighborhood U, the line element of M is written in the form

(3.4) $$ds^2 = \tau'(u^n)^2 f_{ji}(u^h)du^j du^i + \left(\frac{\tau'(u^n)}{\sin t}\right)^2 (dt)^2.$$

We shall now define a function ω on the manifold M as follows: For an ordinary point P at distance s from O, we put

(3.5) $$\omega(P) = \frac{\sin t}{\tau'(s)},$$

and, for the stationary points O and O',

(3.6) $$\omega(P) = \begin{cases} \lim\limits_{s\to 0} \dfrac{\sin t}{\tau'(s)}, & \text{if } P = O, \\ \lim\limits_{s\to s_1} \dfrac{\sin t}{\tau'(s)}, & \text{if } P = O', \end{cases}$$

where t is the parameter defined by (3.1).

By use of the function ω, we effect a conformal change

(3.7) $$\bar{g}_{\mu\lambda} = \omega^2 g_{\mu\lambda}$$

of the metric on the manifold M. In a regular neighborhood U, the new line element $d\bar{s}^2 = \bar{g}_{\mu\lambda}\, dx^\mu dx^\lambda$ takes the form

(3.8) $$d\bar{s}^2 = (\sin t)^2 f_{ji}(u^h)du^j du^i + (dt)^2$$

with respect to the coodinates u^h, t, where u^h are parts of the adapted coordinates in U and t is defined by (3.1). Since the line element $f_{ji}(u^h)du^j du^i$ is of constant sectional curvature 1, we see, from the similar equations to (1.29), that the new Riemannian metric (3.7) is also of constant sectional curvature 1 except at O and O'. However, by the continuity, the exception for the points O and O' are removed. Therefore the compact manifold M with Riemannian metric $\bar{g}_{\mu\lambda}$ is isometrically homeomorphic to a sphereical space of curvature 1. Thus we have established the following

Theorem 2. *If a compact Riemannian manifold M admits a concircular transformation, then it is conformally homeomorphic to an n-dimensional spherical space of curvature* 1. *The homeomorphism of M onto the unit hypershere S_n in an $(n+1)$-dimensional Euclidean space E_{n+1} is given by the mapping θ in Lemma* 6. *The ratio of the metric tensor at a point P of M to that at the corresponding point of S_n by θ is constant when P moves in a ρ-hypersurface of M.*

Conversely, if a compact Riemannian manifold M is conformally homeomorphic to S_n in such a way, the manifold M admits a concircular transformation.

§4. Complete manifolds of constant scalar curvature.

We shall determine a complete Riemannian manifold M of constant scalar curvature k, which admits a concircular transformation into a Riemannian manifold $'M$ of constant scalar curvature $'k$.

From the equation (1. 7), we have

(4. 1) $$'k = (k-2\phi)\tau^2,$$

or

(4. 2) $$2\phi = k - \frac{'k}{\tau^2} = k - 'k\rho^2.$$

Putting

(4. 3) $$\sigma^2 = \rho_\lambda \rho^\lambda = \frac{1}{\tau^2} g^{\mu\lambda}(\partial_\mu \tau)(\partial_\lambda \tau),$$

we have, from (1. 9) and (4. 2),

(4. 4) $$\begin{aligned} 2\psi &= 2\phi - \sigma^2 \\ &= k - 'k\rho^2 - \sigma^2. \end{aligned}$$

Since k and $'k$ are constants, we have from (4. 1)

$$\nabla_\nu \phi = \frac{1}{\tau}(k - 2\phi)\nabla_\nu \tau$$

or, taking account of (1. 8), (4. 2), (4. 3), (4. 4),

$$\nabla_\nu \psi = \frac{k-\psi}{\tau}\nabla_\nu \tau.$$

Integrating this equation, we have

(4. 5) $$\psi = \frac{1}{\tau}(k\tau - a) = k - a\rho,$$

where a is an arbitrary constant. Therefore the equation (1. 10) is written in the form

(4. 6) $$\nabla_\mu \nabla_\lambda \tau + (k\tau - a)g_{\mu\lambda} = 0.$$

Let l : $x^\kappa = x^\kappa(s)$ be an arbitrary geodesic curve in M, s being the arc length of l. Then, along the curve l, the equation (4. 6) is reduced to the ordinary differential equation

(4. 7) $$\frac{d^2\tau}{ds^2} + k\tau - a = 0.$$

According to the sign of the constant scalar curvature k, we put

(4. 8) $$k = \begin{cases} \text{I)} & 0, \\ \text{II)} & c^2, \\ \text{III)} & -c^2, \end{cases}$$

c being a positive constant. Then, by choosing suitably the arc length s of l, a solution of (4.7) is given by

$$(4.9)\qquad \tau = \begin{cases} \text{I)} & \frac{1}{2}as^2 + A, \quad \text{if } a \neq 0, \\ \text{I}') & As, \quad \text{if } a = 0, \\ \text{II)} & A \cos cs + a/c^2, \\ \text{III)} & A \cosh cs - a/c^2, \\ \text{III}') & A \sinh cs - a/c^2 \end{cases}$$

in the respective case, where A is an arbitrary constant.

If the geodesic curve l is a ρ-curve, the length σ of the vector field ρ_λ is given by

$$\sigma = \left| \rho_\lambda \frac{dx^\lambda}{ds} \right| = \left| \frac{d \log \rho}{ds} \right| = \left| \frac{1}{\tau} \frac{d\tau}{ds} \right|$$

along l, i.e., by

$$(4.10)\qquad \sigma = \begin{cases} \text{I)} & \left| \frac{a}{\tau} s \right|, \\ \text{I}') & \left| \frac{A}{\tau} \right|, \\ \text{II)} & \left| \frac{Ac}{\tau} \sin cs \right|, \\ \text{III)} & \left| \frac{Ac}{\tau} \sinh cs \right|, \\ \text{III}') & \left| \frac{Ac}{\tau} \cosh cs \right|. \end{cases}$$

The functions τ and σ are given respectively by (4.9) and (4.10) with respect to a system of adapted coordinates, if we put $u^n = s$.

Comparing (4.4) and (4.5), we have

$$2(k - a\rho) = k - {}'k\rho^2 - \sigma^2,$$

and, substituting (4.10) and (4.11) in this equation, the constant A is equal to

$$(4.11)\qquad A = \begin{cases} \text{I)} & {}'k/2a, \\ \text{I}') & \pm \sqrt{-{}'k}, \\ \text{II)} & \pm \sqrt{a^2 - c^2{}'k}/c^2, \\ \text{III)} & \pm \sqrt{a^2 + c^2{}'k}/c^2, \\ \text{III}') & \pm \sqrt{-(a^2 + c^2{}'k)}/c^2 \end{cases}$$

for any ρ-curve. For the concircular transformation to be real, the following inequalities should hold :

$$
\text{(4.12)} \qquad \begin{array}{ll} \text{I}') & 'k < 0, \\ \text{II}) & a^2 > c^2\,'k, \\ \text{III}) & a^2 > -\,c^2\,'k, \\ \text{III}') & a^2 < -\,c^2\,'k. \end{array}
$$

(4.11) emphasizes that the constant A is independent of the choice of ρ-curves. Accordingly, along any ρ-curve, the function τ is given by

$$
\text{(4.13)} \qquad \tau = \begin{cases} \text{I}) & \dfrac{a}{2}(s^2 + 'k), \\ \text{I}') & \sqrt{-\,'k}\,s, \\ \text{II}) & \dfrac{1}{c^2}(\pm\sqrt{a^2 - c^2\,'k}\cos cs + a), \\ \text{III}) & \dfrac{1}{c^2}(\pm\sqrt{a^2 + c^2\,'k}\cosh cs - a), \\ \text{III}') & \dfrac{1}{c^2}(\pm\sqrt{-(a^2 + c^2\,'k)}\sinh cs - a). \end{cases}
$$

In the following, we shall always assume that the manifold M is complete. We shall call the point, where τ vanishes, a *singular* point of the concircular transformation. In order that a concirculrar transformation be defined on the whole manifold M, it is necessary that there exist no singular point in M.

From (4.13) it is seen that, in Case I′) or Case III′), there exists a singular point on a ρ-curve. Hence Cases I′) and III′) do not occur for a complete manifold. In Case I), if $'k \leqq 0$, then there is also a singular point on a ρ-curve. Hence the constant scalar curvature $'k$ of $'M$ should be positive. Moreover, since τ is positive valued, the constant a should be positive. Therefore we have the following

Lemma 7. *Let M be a complete Riemannian manifold of constant scalar curvature k, and assume that M admits a concircular transformation in a Riemannian manifold $'M$ of constant scalar curvature $'k$. Then the function τ is given by*

$$
\text{(4.14)} \qquad \tau = \begin{cases} \text{I}) & \dfrac{a}{2}(s^2 + 'k), \quad \text{if } k = 0, \\ \text{II}) & \dfrac{1}{c^2}(\pm\sqrt{a^2 - c^2\,'k}\cos cs + a), \quad \text{if } k = c^2, \\ \text{III}) & \dfrac{1}{c^2}(\pm\sqrt{a^2 + c^2\,'k}\cosh cs - a), \quad \text{if } k = -c^2, \end{cases}
$$

along a ρ-curve in M, where s is a suitably chosen arc-length of the ρ-curve.

In Case I), *the constant scalar curvature $'k$ of $'M$ and the constant* *a should be positive.*

By means of (4. 10) and (4. 11), the length σ of the vector field ρ_λ is given by

$$\text{(4. 15)} \qquad \sigma = \begin{cases} \text{I)} & \dfrac{a}{\tau}|s|, \\ \text{II)} & \dfrac{\sqrt{a^2 - c^2{}'k}}{c\tau} \,|\sin cs|, \\ \text{III)} & \dfrac{\sqrt{a^2 + c^2{}'k}}{c\tau} \,|\sinh cs|, \end{cases}$$

along a ρ-curve l in M, where τ is given by (4. 14) in the respective case. In Case I) or Case III), there exists a point corresponding to $s = 0$, where σ vanishes. That is, the point is stationary, and the other points on l are ordinary. While, in Case II), there are two points corresponding to $s = 0$ and $s = \pi/c$ respectively on l. These two points are distinct, because the function τ given by (4. 14, II) has different values for $s = 0$ and $s = \pi/c$. Since σ vanishes at these points, they are stationary points. Thus we have the following

Lemma 8. *Under the same assumptions as those in Lemma* 7,

I) *if* $k = 0$, *there exists one and only one stationary point on a* ρ*-curve,*

II) *if* $k = c^2 > 0$, *there exist at least two stationary points on a* ρ*-curve, and*

III) *if* $k = -c^2 < 0$, *there exists one and only one stationary point on a* ρ*-curve.*

Let O be a stationary point, l an arbitrary geodesic issuing from O, and s the arc length of l such that $s = 0$ at O. Then the function τ along the geodesic l is the solution of the differential equation (4. 7) with initial conditions $\tau(0) = \tau_0$ and $(d\tau/ds)_{s=0} = 0$, where τ_0 is a non-zero constant :

$$\tau_0 = \begin{cases} \text{I)} & a'k/2, \\ \text{II)} & (\pm\sqrt{a^2 - c^2{}'k} + a)/c^2, \\ \text{III)} & (\pm\sqrt{a^2 + c^2{}'k} - a)/c^2. \end{cases}$$

Solving (4. 7), we have

$$\text{(4. 16)} \qquad \tau(s) = \begin{cases} \text{I)} & a(s^2 + {}'k)/2, \\ \text{II)} & A\cos cs + a/c^2, \qquad A = \tau_0 - a/c^2, \\ \text{III)} & A\cosh cs - a/c^2, \qquad A = \tau_0 + a/c^2, \end{cases}$$

along the geodesic l. From (4. 16), we see that the function $\tau = 1/\rho$ is constant on any Riemannian hypersphere with center O. Therefore a

Riemannian hypersphere with center O is a ρ-hypersurface, if it lies sufficiently near to the stationary point O. Hence the point O is an isolated stationary one, and, in a spherical neighborhood W with center O, any geodesic curve issuing from O is a ρ-curve. Combining these facts with Lemma 5, we have the following

Lemma 9. *We keep the assumptions in Lemma 7. In either Case* I), II) *or* III), *a stationary point O is isolated and there is a spherical neighborhood W with center O such that a Riemannian hypersphere in W is a ρ-hypersurface and is isometrically homeomorphic to an $(n-1)$-dimensional spherical space.*

First we deal with Case I), $k=0$. Let O be a stationary point and N the union of all geodesics issuing from O. Then the set N is an open submanifold of M and may be regarded as a Riemannian manifold with the restriction of the metric of M. Moreover, in virtue of Lemma 8, N contains no stationary point except O. If a point P of M lies sufficiently near to O, then the ρ-hypersurface $V(P)$ through P is contained in N and $V(P)$ is isometrically homeomorphic to a hypersphere S_{n-1} of a Euclidean space E_n. However, from the definition of N, it follows in more general that, for any ordinary point P of N, the ρ-hypersurface $V(P)$ through P is contained in N and is isometrically homeomorphic to a hypersphere S_{n-1}. Therefore, the set $N-O$ is homeomorphic to the product $S_{n-1}\times L$ of an $(n-1)$-dimensional sphere S_{n-1} and a straight line L. Consequently the set N is homeomorphic to an n-dimensional Euclidean space E_n. The homeomorphism is obviously differentiable.

Now consider a sequence of points of N converging to a point P of M. Then, by use of the projections of $N-O$ onto S_{n-1} and onto L, and by taking account of the infiniteness of length of geodesic rays issuing from O, we can easily see that N contains the limiting point P. That is to say, the set N is closed. Hence the manifold M has to coincide with N, and M is simply connected.

From (1. 27) and (4. 14), we see that, in a regular neighborhood of M, the line element of M is given by

$$ds^2 = (u^n)^2\bar{g}_{ji}(u^h)du^j du^i + (du^n)^2, \tag{4. 17}$$

where we have put

$$\bar{g}_{ji} = a^2 f_{ji}.$$

The metric $\bar{g}_{ji}$ is that of an $(n-1)$-dimensional spherical space and has a positive constant sectional curvature $\bar{k}$. Since $k=0$, we have $\bar{k}=1$ easily from (1. 31). Therefore the curvature tensor $\bar{K}_{kji}{}^h$ of the metric $\bar{g}_{ji}$ is equal to

(4.18) $$\bar{K}_{kji}{}^{h} = \delta_k^h \bar{g}_{ji} - \delta_j^h \bar{g}_{ki}.$$

From (1.29) and (4.18), we see that the manifold M is locally euclidean in any regular neighborhood. Since M is complete and simply connected, the manifold M is isometrically homeomorphic to an n-dimensional Euclidean space.

Next we consider Case II), $k = c^2$. By use of Lemmas 7, 8 and 9 and the same arguments just as the proof of Lemma 6, we can prove that there exist exactly two stationary points in M and the manifold M is differentiably homeomorphic to an n-dimensional sphere S_n. Moreover, from (1.27) and (4.14), we see that, in a regular neighborhood of the manifold M, the line element is given by

(4.19) $$ds^2 = (\sin cu^n)^2 \bar{g}_{ji}(u^h) du^j du^i + (du^n)^2,$$

where we have put

$$\bar{g}_{ji} = \frac{a^2 - c^{2\prime}k}{c^2} f_{ji}.$$

The metric $\bar{g}_{ji}$ is that of an $(n-1)$-dimensional spherical space and has a positive constant sectional curvature $\bar{k}$. Since $k = c^2$, we have easily $\bar{k} = c^2$ from (1.31). Therefore the curvature tensor $\bar{K}_{kji}{}^h$ of $\bar{g}_{ji}$ is equal to

(4.20) $$\bar{K}_{kji}{}^{h} = c^2(\delta_k^h \bar{g}_{ji} - \delta_j^h \bar{g}_{ki}).$$

Substituting (4.20) into (1.29), we see that the manifold M is of positive constant sectional curvature c^2. Since M is homeomorphic to S_n, the manifold M is isometrically homeomorphic to an n-dimensional spherical space of curvature c^2.

Finally we consider Case III), $k = -c^2$. By the same arguments as the first half of the arguments in Case I), we can also prove in this case that the complete manifold M is homeomorphic to an n-dimensional Euclidean space. Moreover, from (1.27) and (4.14), we see that, in a regular neighborhood of M, the line element is given by

(4.21) $$ds^2 = (\sinh cu^n)^2 \bar{g}_{ji}(u^h)\, du^j\, du^i + (du^n)^2,$$

where we have put

$$\bar{g}_{ji} = \frac{a^2 + c^{2\prime}k}{c^2} f_{ji}.$$

The metric $\bar{g}_{ji}$ is that of an $(n-1)$-dimensional spherical space and has a positive constant sectional curvature $\bar{k}$. Since $k = -c^2$, we have easily $\bar{k} = c^2$ from (1.31). Therefore the curvature tensor $\bar{K}_{kji}{}^h$ is also given by (4.20). By means of (1.29) and (4.20), we see that the manifold

M is of negative constant sectional curvature $-c^2$. Since M is complete and simply connected, the manifold M is isometrically homeomorphic to an n-dimensional hyperbolic space of curvature $-c^2$.

Thus we have established the following

Theorem 3. *We assume that a complete Riemanian manifold M of constant scalar curvature k admits a concircular transformation into a Riemannian manifold $'M$ of constant scalar curvature $'k$. Then the manifold M is isometrically homeomorphic*

I) *to an n-dimensional Euclidean space if* $k = 0$,

II) *to an n-dimensional spherical space if* $k > 0$, *or*

III) *to an n-dimensional hyperbolic space if* $k < 0$.

In addition to the assumptions of Theorem 3, we suppose now that $'M$ is also complete and the concircular transformation is a homeomorphism of M onto $'M$. If Case I) happened, then, in virtue of Lemma 7, the scalar curvature $'k$ of $'M$ should be positive and consequently, by the above theorem, the manifold $'M$ would be homeomorphic to a spherical space, which was compact. This contradicts to the existence of a homeomorphism of M onto $'M$. Therefore the constant scalar curvatures k and $'k$ are not equal to zero.

If one of the manifolds is of positive scalar curvature and the other is of negative scalar curvature, then the former is homeomorphic to a spherical space, which is compact, and the latter is homeomorphic to a hyperbolic space, which is non-compact. There cannot exist a homeomorphism between the manifolds.

Therefore, under our present assumptions, $'k$ should have the same sign as k. We put

$$(4.22) \qquad 'k = \begin{cases} \text{II)} & 'c^2, \\ \text{III)} & -'c^2, \end{cases}$$

$'c$ being a positive constant. Since we have supposed for τ to be positive, we can see the following facts from (4.14) : Along a ρ-curve in M,

in Case II), a should be positive, and, without loss of generality, A may be taken as positive : $A = \sqrt{a^2 - c^2{}'c^2}/c^2$, and

in Case III), A should be positive, $A = \sqrt{a^2 - c^2{}'c^2}/c^2$, and a should be negative.

Therefore the function τ is written in the form

$$(4.23) \qquad \tau = \begin{cases} \text{II)} & \dfrac{1}{c^2}(\sqrt{a^2 - c^2\,'c^2}\cos cs + a), \quad (a > 0), \\ \text{III)} & \dfrac{1}{c^2}(\sqrt{a^2 - c^2\,'c^2}\cosh cs - a), \quad (a < 0) \end{cases}$$

along a ρ-curve in M.

From the definitions of stationary points and ρ-curves, it is obvious that the image of a stationary point of a concircular transformation of M onto $'M$ is also a stationary point of the inverse concircular transformation and the image $'l$ of a ρ-curve l in M is also a ρ-curve in $'M$. Therefore the image $'l$ is a geodesic in $'M$ and has infinite length, because of the completeness of $'M$.

The change from the arc length s of a ρ-curve l in M to the arc length $'s$ of the image $'l$ in $'M$ is given by the equation

$$\frac{d's}{ds} = \frac{1}{\tau}, \tag{4.24}$$

where τ is given by (4.23) in the respective case. The solution of this equation with initial condition $'s = 0$ for $s = 0$ is

$$'s = \begin{cases} \text{II)} \ \dfrac{2}{c'} \tan^{-1} \dfrac{c\,'c}{a + \sqrt{a^2 - c^2\,'c^2}}\, t, & (a > 0), \\ \text{III)} \ \dfrac{1}{c'} \log \dfrac{\sqrt{-a + c\,'c}\; t + \sqrt{-a - c\,'c}}{\sqrt{-a - c\,'c}\; t + \sqrt{-a + c\,'c}}, & (a < 0), \end{cases} \tag{4.25}$$

where we have put

$$\text{II)} \quad t = \tan \frac{cs}{2}, \qquad \text{III)} \quad t = \exp cs \tag{4.26}$$

in the respective case.

In Case III), when the arc length s tends to the infinity, t tends monotonely to the infinity and we have

$$\lim_{s\to\infty} {}'s = \frac{1}{'c} \log \frac{\sqrt{-a + c\,'c}}{\sqrt{-a - c\,'c}}, \quad (a < 0). \tag{4.27}$$

This implies that, to a ρ-curve of infinite length in M, corresponds a ρ-curve of finite length in $'M$. This is a contradiction. Therefore Case III) does not happen.

From (4.25, II), we obtain

$$\tan \frac{'c\,'s}{2} = \frac{c\,'c}{a + \sqrt{a^2 - c^2\,'c^2}} \tan \frac{cs}{2}. \tag{4.28}$$

By means of this equation, we can illustrate the concircular transformation as follows: We realize M and $'M$ on hyperspheres of radius $1/c$ and $1/'c$ respectively in an $(n+1)$-dimensional Euclidean space E_{n+1}, which are tangent to each other at the common south pole O. Let T be the hyperplane tangent to the hyperspheres at O. Let O' and $'O'$ be the north poles of M and $'M$ respectively. Denote by π and $'\pi$ the stereographic mappings of M from O' and of $'M$ from $'O'$ onto T respectively, and by ζ_a the similarity of manification $c^2/(a+\sqrt{a^2-c^2\,'c^2})$ on T with center O. Then the product $'\pi^{-1}\circ\zeta_a\circ\pi$ is the concircular transformation of M onto $'M$, for which the poles O and O' are the stationary points in M, the longitudes are the ρ-curves and the function τ is given by (4. 25, II) along any longitude in M.

Thus we have established the following

Theorem 4. *Let M and $'M$ be complete Riemannian manifolds of constant scalar curvature. If there exists a non-homothetic concircular transformation of M onto $'M$, then the scalar curvatures are positive, and both M and $'M$ are isometrically homeomorphic to spherical spaces, and conversely.*

Corlloary 1. *If a complete Riemannian manifold of constant scalar curvature admits a concircular transformation onto itself, then the manifold is a spherical space.*

Corollary 2. *If a homogeneous Riemannian manifold admits a concircular trnasformation onto itself, then the manifold is a spherical space.*

If an Einstein manifold M admits a concircular transformation, then we have from (1. 6)

$$'K_{\mu\lambda} = (n-1)(k-2\phi)\, g_{\mu\lambda}$$
$$= (n-1)(k-2\phi)\,\tau^2\, 'g_{\mu\lambda},$$

because of $K_{\mu\lambda}=(n-1)kg_{\mu\lambda}$. By means of this equation, K. Yano [CG, V] proved that, under a concircular transformation, an Einstein manifold is transformed to an Einstein one. Then the scalar curvature $'k$ of $'M$ is also constant. Hence we can apply the results of this paragraph on complete Einstein manifolds admitting a concircular transformation. In particular, we have

Corollary 3. *If a complete Einstein manifold admits a concircular transformation onto itself, then the manifold is a spherical space.*

A Riemannian manifold is said to have the parallel Ricci tensor if the covariant derivative of the Ricci tensor vanishes identically :

$$\nabla_\nu K_{\mu\lambda} = 0.$$

In such a manifold, the scalar curvature k is constant. Hence we have the following

Corollary 4. *If a complete Riemannian manifold with parallel*

Ricci tensor admits a concircular transformation onto itself, then it is a spherical space.

§ 5. Holonomy groups.

In matrix notation, we put

$$E_{\nu\mu} = (\delta_{\mu\lambda}\,\delta_{\nu\kappa} - \delta_{\nu\lambda}\,\delta_{\mu\kappa}) \tag{5.1}$$

and, at a point P of the manifold M,

$$R_{\nu\mu}(P) = (K_{\nu\mu\lambda\kappa}(P)) \tag{5.2}$$

for any pair of indices μ and ν. Then we recall the following theorem due to A. Nijenhuis[4]: The local homogeneous holonomy algebra at a point P is spanned by the matrices arising from the matrices $R_{\nu\mu}(Q)$ at the points Q in a suitable neighborhood of P by a suitable parallel transport from Q to P.

Now let P be an ordinary point, U a regular neighborhood of P and u^κ adapted coordinates in U. We may take a system of local coordinates in the ρ-hypersurface $V(P)$ such that $f_{ji}(P) = \delta_{ji}$. First we suppose that τ''' does not identically vanish in U. Then, from (1.26) we have

$$R_{nj}(P) = CE_{nj}, \tag{5.3}$$

and hence the bracket product of $R_{nk}(P)$ and $R_{nj}(P)$ is

$$[R_{nk}(P),\ R_{nj}(P)] = C^2[E_{nk},\ E_{nj}] = C^2 E_{kj}, \tag{5.4}$$

where $C = -\tau'(P)\,\tau'''(P) \neq 0$. Since the matrices $E_{\nu\mu}$ span the Lie algebra of the orthogonal group $O(n)$, the local homogeneous holonomy group of the manifold at an ordinary point is the special orthogonal group $SO(n)$, in virtue of the Nijenhuis' theorem. We have thus the following

Lemma 10. *If a Riemannian manifold M admits a concircular transformation and, in a regular neighborhood of an ordinary point P, τ''' does not vanish, then the local homogeneous holonomy group at P is the special orthogonal group $SO(n)$.*

If τ''' vanishes identically in a regular neighborhood U, then by a suitable choice of u^n, we have

$$\tau = \frac{1}{2}(a(u^n)^2 + b),$$

a and b being arbitrary constants. For a non-homothetic concircular transformation the constant a does not vanish. From (1.10) it follows

4) A. Nijenhuis, On the holonomy groups of linear connections, IA. Proc. Kon. Ned. Akad. Amsterdum, 56 = Indag. Math., Vol. 15 (1953), pp. 233—240.

$\psi = -a/\tau$. If we substitute this into (1. 10), we have in U

(5. 5) $$\nabla_\mu \nabla_\lambda \tau = a\, g_{\mu\lambda}.$$

The last equation shows that the vector field $\nabla_\lambda \tau$ is a concurrent one.

Now we assume that a complete Riemannian manifold M admits a concircular transformations such that $\tau''' = 0$ at any ordinary point. Then the set F of all stationary points contains no open set. In fact, if F contains an open set, denoting by F° the maximum open subset of F, we see that $\nabla_\lambda \tau = 0$ holds in F°. Then the function τ satisfies

(5. 6) $$\nabla_\mu \nabla_\lambda \tau = 0$$

in F°. Therefore, by means of continuity, we see that both of the equations (5. 5) and (5. 6) must hold in any boundary point of F. This contradicts the fact that the constant a does not equal to zero. Thus, the set F contains no open subset. Hence, the equation (5. 5) holds throughout the manifold M. That is to say, the vector field $\nabla_\lambda \tau$ is a concurrent one in M.

It is, however, well known that, if a complete Riemannian manifold admits a concurrent vector field, then it is flat.[5] Consequently, if a complete Riemannian manifold admits a concircular transformation such that $\tau''' = 0$ holds at any ordinary point, then it is flat. Thus, taking account of Lemma 10, we have the following

Theorem 5. *If a complete, non-flat Riemannian manifold admits a concircular transformation, then its local homogeneous holonomy group at any point is the special orthogonal group $SO(n)$.*

We shall next consider a conformally flat Riemannian manifold M admitting a concircular transformation. The conformal curvature tensor $C_{\nu\mu\lambda}{}^\kappa$ is given by

$$C_{\nu\mu\lambda}{}^\kappa = K_{\nu\mu\lambda}{}^\kappa - \frac{1}{n-2}(\delta^\kappa_\nu K_{\mu\lambda} - \delta^\kappa_\mu K_{\nu\lambda} + K_\nu^\kappa g_{\mu\lambda} - K_\mu^\kappa g_{\nu\lambda})$$
$$+ \frac{nk}{n-2}(\delta^\kappa_\nu g_{\mu\lambda} - \delta^\kappa_\mu g_{\nu\lambda}).$$

From (1. 29), (1. 30) and (1. 31) the tensor $C_{\nu\mu\lambda}{}^\kappa$ has the following components with respect to adapted coordinates u^κ :

$$C_{kji}{}^h = \overline{K}_{kji}{}^h - \frac{1}{n-2}(\delta^h_k \overline{K}_{ji} - \delta^h_j \overline{K}_{ki} + \overline{K}^h_k f_{ji} - \overline{K}^h_j f_{ki}) + \bar{k}(\delta^h_k f_{ji} - \delta^h_j f_{ki}),$$

(5. 7) $$C_{nji}{}^n = -C_{jni}{}^n = -\frac{1}{n-2}\overline{K}_{ji} + \bar{k} f_{ji},$$

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5) S. Sasaki and M. Gotō, Some theorems on holonomy groups of Riemannian manifold, Trans. Amer. Math. Soc., vol. 80 (1955), pp. 148—158.

$$C_{njn}{}^{h} = - C_{jnn}{}^{h} = \frac{1}{(\tau')^2}\left(\frac{1}{n-2}\bar{K}_j{}^h - \bar{k}\,\delta_j^h\right)$$

and the other components vanish identically. From these equations we can see that

$$\bar{K}_{kji}{}^{h} = \bar{k}\,(\delta_k^h f_{ji} - \delta_j^h f_{ki})$$

holds if and only if the tensor $C_{\nu\mu\lambda}{}^{\kappa}$ vanishes identically. Thus, we have the following

Lemma 11. *A Riemannian manifold admitting a concircular transformation is conformally flat, if and only if any ρ-hypersurface has constant sectional curvature.*

We now suppose that a conformally flat Riemannian manifold M admits a concircular transformation such that $\tau''' = 0$ holds at any ordinary point. By virtue of Lemma 11, we have

$$\bar{K}_{kji}{}^{h} = \bar{k}\,(\delta_k^h f_{ji} - \delta_j^h f_{ki}).$$

Taking account of (1. 29), by means of $\tau'' = a$ we obtain in any regular neighbourhood

$$K_{kji}{}^{h} = (\bar{k} - a^2)\,(\delta_k^h f_{ji} - \delta_j^h f_{ki}), \tag{5. 8}$$

If $\bar{k} = a^2$, we see easily from (1. 29) and (5. 8) that manifold M is flat. If $\bar{k} \neq a^2$, then we obtain from (5. 8)

$$R_{kj}(P) = (\bar{k} - a^2)\,E_{kj}. \tag{5. 9}$$

by a suitable choice of coordinates at an ordinary point P The matrices E_{kj} span the Lie algebra of the orthogonal group $O(n-1)$. Therefore, the local homogeneous holonomy group H at P contains the special orthogonal group $SO(n-1)$ imbedded naturally in $SO(n)$.

It is well known that[6], provided $n \neq 4$, there exists no closed subgroup G of $O(n)$ such that

$$\frac{(n-1)(n-2)}{2} < \dim\, G < \frac{n(n-1)}{2}$$

and that there exists no proper closed subgroup of $O(4)$ which contains $O(3)$ as its proper subgroup[7]. Hence the local homogeneous holonomy group H is $SO(n-1)$ or $SO(n)$ for $n > 2$.

6) D. Montgomery and H. Samelson, Transformation groups of spheres, Ann. Math., vol. 44 (1943), 454—470.

7) See for example, S. Ishihara, Homogeneous Riemannian spaces of four dimentions, Jour. Math. Soc. Japan, vol. 7 (1955), 345—370.

However, the group H does not coincide with the group $SO(n-1)$. In fact, there exists a neighborhood U of the point P such that the homogeneous holonomy group of U coincides with the group H. We may suppose that the neighborhood U is a regular one, because the point P is an ordinary one. If the group H is $SO(n-1)$, the normal unit vectors i^κ of ρ-hypersurfacs form a parallel vector field in U, since the matrices E_{kj} given by (5. 9) generate the Lie algebra of the holonomy group H. Hence, by means of (1. 15), the function ψ vanishes identically in U. Therefore, taking account of (1. 10), we see that the constant a appearing in (5. 5) vanishes. This means that the given concircular transtormation is a homothetic one. Consequently, taking account of Lemma 10, we have the following

Theorem 6. *If a non-flat, conformally flat Riemannian manifold admits a concircular transformation, then the local homogeneous holonomy group at an ordinary point is the special orthogonal group $SO(n)$.*

If the manifold M is compact and τ''' vanishes identically in any regular neighborhood, then (5. 5) holds in any regular neighborhood. Since the stationary points are isolated, (5. 5) is necessarily valid throughout the manifold M. By the well known Stokes' theorem, we can easily see that the constant a appearing in (5. 5) is equal to zero and the function τ is constant in M. Hence the transformation is a homothety. Thus we have the following

Theorem 7. *If a compact Riemannian admits a concircular transformation, then the local homogeneous holonomy group at any point is the special orthogonal group $SO(n)$.*

DEPARTMENT OF MATHEMATICS,
TOKYO GAKUGEI UNIVERSITY
AND
DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY.

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