SOME REMARKS ON GALOIS EXTENSIONS OF DIVISION RINGS

Dedicated to the 60th birthday of Dr. ZYOITI SUETUNA

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1°. Let K be a division ring, and L a division subring of K. The left dimension $[K: L]_l$ means the cardinal number of any linearly left-independent basis of K over L. The right dimension $[K: L]_r$ will be similarly defined, and in case the left dimension coincides with the right one they will be denoted as $[K: L]$. In general, the cardinal number of any set S will be denoted as $\#(S)$. If for any finite subset F of K the ring $L[F]$ generated by F over L is of finite left (right) dimension, we say that K is *left (right) locally finite* over L.

Let K be Galois over L. Then the following pretty theorem is shown by N. Nobusawa [5] : *Let K be left locally finite over L. If K' is an intermediate subring of K/L with $[K': L]_l < \infty$ then K is Galois over K'.* Our principal aim of this note is to prove Nobusawa's theorem without assuming the left local finiteness of K/L (Theorem 1).

On the other hand, the following fact is known [1, Theorem 7. 9. 2]: *Let K be Galois over L. If K' is an intermediate subring of K/L with $[K': L]_l < \infty$ and $[K': L]_r < \infty$, then the two dimensions are equal.* It is an open question weither $[K': L]_l < \infty$ yields $[K': L]_r < \infty$ or not. However, in case K is left locally finite over V_K $(V_K(L))$, we can prove, as a by-product in the course of proving Theorem 1, that the answer is affirmative (Theorem 2).

Finally, as to notations and terminologies used in this paper we follow our previous papers [2], [3] and [6].

2°. In the rest of this paper, we always assume that K is a division ring which is Galois over a division subring L. And we set $\mathfrak{G} = \mathfrak{G}(K/L)$, $V = V_K(L)$, and $H = V_K(V_K(L))$.

Lemma 1. *Let K' be an intermediate subring of K/L, and $\mathfrak{H}$ a Galois group of K/L.*

(i) *If $\#(\{k'\}\mathfrak{H}) < \infty$ for every $k' \in K'$, then $L(K'\mathfrak{H})$ is left as well as right locally finite over L, where $L(K'\mathfrak{H})$ is the division subring generated by $K'\mathfrak{H}$ over L.*

(ii) *Let K/L be outer Galois. If $\#(\{k'\}\mathfrak{H}) \leqq n$ for every $k' \in K'$, then $[K' : L] \leqq n$.*

Proof. (i) Let F be an arbitrary finite subset of K'. Then $L' = L(F\mathfrak{H})$ (the division subring generated by $F\mathfrak{H}$ over L) is Galois over L

with the finite group $\mathfrak{H}_{L'}$ as a Galois group. Hence, as is well-known, $[L': L] \leqq \#(\mathfrak{H}_{L'}) < \infty$, whence our assertion will be almost clear.

(ii) In virtue of (i), $L[K' \mathfrak{H}]$ is (left and right) locally finite over L. If $[K': L]_l > n$ then, by [2, Corollary 3], there exists some $k' \in K'$ such that $[L[k']: L]_l > n$. Now we form $L' = L[\{k'\} \mathfrak{H}]$, then L' is (outer) Galois over L with the finite group $\mathfrak{H}_{L'}$ as the Galois group. Noting that $[L': L[k']] = \#(\mathfrak{H}_{L'}(L[k']))$, we readily see that $\#(\{k'\} \mathfrak{H}) = \#(\mathfrak{H}_{L[k']}) = [L[k']: L] > n$. But this is a contradiction. Hence, $L[K' \mathfrak{H}]$ being Galois and finite over L, we have $[K': L]$ $(= [K': L]_l) \leqq n$.

Remark 1. Let $N = \{h \in H \mid \#(\{h\} \mathfrak{G})\} < \infty\}$. Then, by Lemma 1 and [6, (a*)], one will readily see that N is the unique maximal $\mathfrak{G}$-normal subring of H that is (left and right) locally finite over L.

The proof of the next lemma will be obtained in the similar way as in that of [5, Theorem 2], and so the details may be left to readers.

Lemma 2. *Let K' be an intermediate subring of K/L which is left locally finite over L. If L' is an intermediate subring of K'/L with $[L': L]_l < \infty$ then $K' \cap J(\mathfrak{G}(L'), K) = L'$.*

The next will play an essential rôle in our present study.

Lemma 3. *Let K' be an intermediate subring of K/L, and let $K_0 = J(\mathfrak{G}(K'), K)$, $H_0 = K_0 \cap H$.*

(i) *$[K_0: H_0]_l \leqq [H[K_0]: H]_l$ and $[K_0: H_0]_r \leqq [H[K_0]: H]_r$.*

(ii) *If $[K': L]_l < \infty$ then $[H_0: L] < \infty$.*

Proof. (i) Let $\{k_i\} \subset K_0$ be linearly left-independent over H_0. If there holds a non-trivial relation $\sum_{i=1}^{n} h_i k_i = 0$ $(h_i \in H)$ among the $\{k_i\}$ then, without loss of generality, we may assume that $k_1 + \sum_{i=2}^{n} h_i k_i = 0$ $(h_i \in H$ and $h_2 \notin H_0)$ is of the shortest length. Since there exists some $\sigma \in \mathfrak{G}(K')$ such that $h_2 \sigma \neq h_2$, there holds $k_1 + \sum_{i=2}^{n} (h_i \sigma) k_i = 0 = k_1 + \sum_{i=2}^{n} h_i k_i$. But this yields the contradiction $\sum_{i=2}^{n} (h_i \sigma - h_i) k_i = 0$. Hence, we have $[K_0: H_0]_l \leqq [H[K_0]: H]_l$. The second will be similarly shown.

(ii) Let h be an element of H such that $\#(\{h\} \mathfrak{G}) > [K': L]_l$. Then h possesses p different images $h\sigma_1, \ldots, h\sigma_p$ $(\sigma_i \in \mathfrak{G})$, where $p > [K': L]_l$. Since $[\mathfrak{G}_{K'} K_r: K_r]_r = [K': L]_l < p$, the same argument as in the proof of [4, Lemma 3] proves that there exist some different σ_i, σ_j and some $v \in V$ such that $\sigma_{i K'} = \sigma_{j K'} \tilde{v}$. We consider here the automorphism $\tau = \sigma_j \sigma_i^{-1} \tilde{w}$ where $w = v \sigma_i^{-1} \in V$. Then it is easy to see that $k' \tau = k'$ for every $k' \in K'$, that is, $\tau \in \mathfrak{G}(K')$. On the other hand, noting that H is $\mathfrak{G}$-normal, we obtain $h\tau = h \sigma_j \sigma_i^{-1} \neq h$. Thus we see that h is not contained in H_0, or what is the same, that $\#(\{h_0\} \mathfrak{G}) \leqq [K': L]_l$ for every $h_0 \in H_0$. Our assertion follows therefore from Lemma 1 (ii).

Now we shall prove our principal theorem which contains [5, Theorem 2].

Theorem 1. *Let K be Galois over L. If K' is an intermediate subring of K/L with $[K' : L]_l < \infty$ then K is Galois over K'.*

Proof. By [5, Lemma 2] and Lemma 3 (i), we obtain $\infty > [K' : L]_l \geqq [V : V_K(K')]_r = [V : V_K(K_0)]_r = [H[K_0] : H]_l \geqq [K_0 : H_0]_l$, where $K_0 = J(\mathfrak{G}(K'), K)$ and $H_0 = K_0 \cap H$. Combining this with Lemma 3 (ii), we see that $[K_0 : L]_l = [K_0 : H_0]_l \cdot [H_0 : L] < \infty$. Hence, by Lemma 2, $K_0 = K_0 \cap J(\mathfrak{G}(K'), K) = K'$.

Remark 2. In [3], the present authors contructed Galois theory of division rings under the conditions (α), (β), (γ) and (δ). However, the validity of Theorem 1 enables us to see that (δ) is a consequence of (γ).

3°. Our next task is to present a theorem which is related to [1, Theorem 7. 9. 2]. To this end, we set the following lemma, which we believe may be of interest for itself.

Lemma 4. *Let K be inner Galois over L. If K is left locally finite over L, then it is right locally finite too.*

Proof. Let L' be an intermediate subring of K/L with $[L' : L]_l < \infty$. We set here $V' = V_K(L')$, and $\{v_1, \dots, v_m\} \subset V$ be linearly left independent over V'. If $L'' = L'[v_1, \dots, v_m]$ then, by [5, Lemma 2], $\infty > [L'' : L]_l \geqq [V_{L''}(L) : V_{L''}(L'')] \geqq [V_{L''}(L) : (\text{center of } V_{L''}(L))]$. Hence, by [1, Proposition 7. 1. 3] and [5, Lemma 2], $[V_{L''}(L) : V_{L''}(L')]_l = [V_{L''}(L) : V_{L''}(L')]_r \leqq [L' : L]_l$. Consequently, noting that $\{v_1, \dots, v_m\} \subset V_{L''}(L)$ and $V_{L''}(L') \subset V'$, we obtain $m \leqq [L' : L]_l$, which means $\infty > [V : V']_l = [L' : L]_r$ again by [5, Lemma 2].

Theorem 2. *Let K be Galois over L, and left locally finite over $H = V_K(V_K(L))$. If L' is an intermediate subring of K/L with $[L' : L]_l < \infty$ then $[L' : L]_l = [L' : L]_r$.*

Proof. Let $H' = V_K(V_K(L'))$. Then, by [5, Lemma 2], Lemma 4 and [1, Theorem 7. 9. 2], there holds $\infty > [L' : L]_l \geqq [V : V_K(L')]_r = [H' : H]_l = [H' : H]_r = [H[L'] : H]_r$. Since $J(\mathfrak{G}(L'), K) = L'$ by Theorem 1, Lemma 3 (i) gives $\infty > [H[L'] : H]_r \geqq [L' : (L' \cap H)]_r$. Further, $[(L' \cap H) : L] < \infty$ by Lemma 3 (ii), and so we have $[L' : L]_r < \infty$. Now, our assertion is clear by [1, Theorem 7. 9. 2].

Corollary 1. *If K is left locally finite over L, then it is right locally finite too.*

Proof. By [5, Theorem 1], K is left locally finite over H. Our corollary is therefore a direct consequence of Theorem 2.

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(Received February 21, 1959)