A NOTE ON CONJUGATES II

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In this note, we use the following conventions : By a ring we mean a ring with an identity, and by a subring we mean one which contains this identity. By a *simple ring* we shall mean a two-sided simple ring with minimum condition for left ideals, and by a *primary ring* [a *completely primary ring*] a ring such that the (Jacobson) radical is nilpotent and the residue class ring modulo the radical is a simple ring [a division ring]. For any non-empty subset B of a ring R, $V_R(B)$ will denote the centralizer of B in R. If, for each element of a subring S of R with an inverse in R, the inverse is always contained in S, then S will be called a *π-subring* of R. For example, $V_R(B)$ and each subring with minimum condition for left ideals are π-subrings of R. R^* means the multiplicative group consisting of all regular elements of a ring R. And for any set S, $\#(S)$ will signify the cardinal number of S.

Recently, W. R. Scott proved the following powerful lemma [3, p. 305]: *Let D be an infinite division ring, S a proper division subring of D. Then $(D^* : S^*) = \#(D)$, where $(D^* : S^*)$ is the group index of S^* in D^*.* And more recently, in [1], C. C. Faith has pointed out that the following fact given by F. Kasch in [2] is a direct consequence of Scott's lemma: *Let S be an infinite division subring of a division ring D not contained in the centre of D. Then every element $d \in D$ which is outside of $V_D(S)$ possesses infinitely many conjugates xdx^{-1} with $x \in S^*$.* On the other hand, in the previous note [4], the present author has obtained the following which contains Kasch's : *Let R be an infinite simple subring of a ring U, and T the set of conjugates of an element $t \in U$ by all regular elements in R. Then $\#(T) = \#(R)$ or 1.*

The purpose of this note is to prove a generalization of Scott's lemma, and to present, as its direct consequence, an extension of [4, Theorem].

Our fundamental lemma is the following :

Lemma 1. *Let R be a primary ring with the radical N such that $\bar{R} = R/N$ is infinite, and S a π-subring of R. If $(R^* : S^*) < \#(\bar{R})$, where $(R^* : S^*)$ is the group index of S^* in R^*, then $S = R$.*

Proof. Let $R = \sum^n Ce_{ij}$, where e_{ij}'s are matric units and $C = V_R(\{e_{ij}\})$ is a completely primary ring. If $n = 1$, then R (whence S) is completely primary, and let $\{\bar{r}_\alpha\}$ be a linearly independent left basis of $\bar{R}$ over $\bar{S} =$

$(S+N)/N$, where $\bar{r}_\alpha$ is the residue class of $r_\alpha \in R$. Then, it is clear that $r_\alpha r_\beta^{-1} \notin S^*$ if $\alpha \neq \beta$. Hence, by our assumption $(R^* : S^*) < \#(\bar{R})$, we obtain $\#(\bar{S}) = \#(\bar{R})$. And so, for each $r \in R$, we can choose suitable s_1, s_2, and $s \in S^*$ such that $s_1 \not\equiv s_2 \pmod N$, $r \not\equiv s_i \pmod N$ for $i = 1, 2$, and $r - s_1 = s(r - s_2)$. We obtain therefore $r = (1-s)^{-1}(s_1 - ss_2) \in S$, whence it follows $S = R$. Secondly, we shall prove the case $n > 1$. For each e_{ij} $(i \neq j)$, $\{c + e_{ij} \mid c$ runs over a fixed complete representative system of $\bar{C}^*$, where $\bar{C} = C/(C \cap N)\}$ forms a subset of R^* whose cardinal number is $\#(\bar{C}) = \#(\bar{R})$. And so, there exist some c_1, $c_2 \in C^*$ and $s \in S^*$ such that $c_1 \not\equiv c_2 \pmod N$, and $c_1 + e_{ij} = s(c + e_{ij})$, that is,

$$(*) \qquad s = (c_1 + e_{ij})(c_2 + e_{ij})^{-1} = c_1 c_2^{-1} + (1 - c_1 c_2^{-1}) c_2^{-1} e_{ij}.$$

Since, for each c, $c' \in C^*$, $c'c^{-1} \notin (C \cap S)^*$ yields $c'c^{-1} \notin S^*$, we obtain $(C^* : (C \cap S)^*) < \#(\bar{R}) = \#(\bar{C})$. And then, $C \cap S$ being evidently a π-subring of C, the proof for the case $n = 1$ shows $C \cap S = C$, that is, $S \supseteq C$. Hence, noting that $c_1 \not\equiv c_2 \pmod N$, from $(*)$ one will readily see that e_{ij} is contained in S, accordingly so are all e_{ij}'s. And then, S being $\sum^n (C \cap S)\, e_{ij}$ necessarily, we obtain our assertion $S = R$.

As an easy consequence of our theorem, we obtain the following extension of [4, Theorem].

Theorem 1. *Let R be a primary subring of a ring U such that the residue class ring $\bar{R}$ modulo its radical is infinite, and T the set of conjugates of an element $t \in U$ by all regular elements of R. Then either $\#(T) \geqq \#(R)$ or $\#(T) = 1$.*

Proof. Since $\#(T) = (R^* : V_R(t)^*)$, we can apply Lemma 1 to R and its π-subring $V_R(t)$. Hence our assertion is almost clear.

Of course, our theorem may be restated in the following way.

Theorem 1′. *Let R be a primary subring of U such that the residue class ring R modulo its radical is infinite, and T a subset of U which is transformed into itself by all regular elements of R. If $\#(T) < \#(R)$, then $T \subseteq V_U(R)$.*

Finally, as a special case of Theorem 1, we obtain

Corollary 1. *Let R be a primary subring of U which is of characteristic zero, and T the set of conjugates of an element $t \in U$ by all regular elements of R. Then $\#(T)$ is either infinite or 1.*

Remark. Let a be an element of a ring A. If $xax = x$ has no non-zero solutions in A, then a is called a *root element.* All the results of this note except Corollary 1 are still valid for such R that the set of all root elements of R coincides with the radical N and R/N is an infinite simple ring.

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