GALOIS THEORY OF SIMPLE RINGS IV[1)]

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Lemma 2 of the previous paper [3] should, as Mr. J. H. Walter has kindly pointed out (in a letter), be false, the separability of $\mathfrak{G}_\alpha$'s is needed to secure the fact that the inverse limit of $\mathfrak{G}_\alpha$'s is closed in the product space of them. Because of this fatal misunderstanding, all the results in [3, § 2] should be cancelled.

Throughout the present note, $R = \sum^n De_{ij}$ be a simple ring (with minimum condition), where e_{ij}'s are matric units and $D = V_R(\{e_{ij}\})$ is a division ring, and S a simple subring of R containing the identity element of R. Then we set $V = V_R(S)$, $H = V_R(V_R(S))$, $Z_0 = V \cap H$, and $S_1 = S(\{e_{ij}\})$. In this note, we shall consider several conditions, which enable us to extend [5, Theorem 4] and [5, Theorem 5] to a somewhat wider class of Galois extensions. Now we shall begin our study with the preliminary section.

1. Preliminaries. At first we shall extend [2, Lemma 1] to simple rings.

Lemma 1. *Let U be a subset of V, and T a subring of R containing S_1. Then $(U_l)_T$ is linearly right-independent over R_r if and only if U is linearly right-independent over $V_R(T)$ that is evidently a division subring of D.*

Proof. If $\{u_1, \cdots, u_m\} \subset U$ is linearly right-dependent over $V_R(T)$, then we have a non-trivial relation $\sum^m u_i v_i = 0$ with $v_i \in V_R(T)$. And so, $\sum^m \tilde{v}_i u_{il} v_{ir} = 0$, where we put $\tilde{v}_i = 0$ for $v_i = 0$. Since $(\tilde{v}_i)_T$ is identical, we have a non-trivial relation $\sum^m (u_{il})_T v_{ir} = 0$. Conversely, suppose $\{(u_{1l})_T, \cdots, (u_{ml})_T\}$ is linearly right-dependent over R_r, where $\{u_1, \cdots, u_m\}$ is a subset of U. Then there exists a non-trivial relation of the shortest length, say, $\sum^s (u_{il})_T y_{ir} = 0$ with (non-zero) $y_i \in R$. Here, recalling that $T \supset S_1$, we may, without loss of generality, assume that $y_1 = 1$ (cf. the proof of [4, Theorem 4]). And then the same argument as in the proof of [2, Lemma 1] shows that each y_i is in $V_R(T)$. We obtain therefore $0 = \sum^s (u_{il} y_{ir})_T = \sum^s (y_{il} u_{il})_T$, that is, $\sum^s u_i y_i = 0$.

1) As to notations and terminologies used in this note, we follow the previous ones [4] and [5].

Corollary 1. *If $S' \supset S$ is a regular subring with $[S'(\{e_{ij}\}) : S]_l < \infty$, then there holds $[V: V_R(S')]_r < \infty$.*

Proof. Let $T = S'(\{e_{ij}\})$. Since the R_r-module $\mathfrak{M}$ consisting of all the S_l-homomorphisms of T into R contains $(V_l)_T R_r$ and $[\mathfrak{M} : R_r]_r = [T: S]_l < \infty$, Lemma 1 implies $[V: V_R(S')]_r \leqq [V: V_R(T)]_r \leqq [T: S]_l < \infty$.

Next, by the light of [2, Theorem 2], we shall give several supplements to results cited in [5]. In the rest of this section, let R be Galois with the Galois group (total group) $\mathfrak{G}$ and locally finite over S, and Galois over S_1 too. Then, as is noted in [5, Remark 1], D is Galois with the Galois group $\mathfrak{G}(R/S_1)_D$ and locally finite over $D_1 = V_{S_1}(\{e_{ij}\})$. Accordingly, by [2, Theorem 2], we readily see that $J(\mathfrak{G}(S'), R) = S'$ for each subring S' containing S_1 with $[S' : S_1]_l < \infty$. We shall insert here the following definition (cf. [1]).

Definition. R is said to be *locally Galois* over S if for each finite subset F of R there exists a simple subring S' such that S'/S is Galois and $[S' : S]_l < \infty$. (Here R is not necessarily assumed to be Galois over S.)

Since, in case $[V: V_R(R)] < \infty$ in addition, the conditions (1)—(4) introduced in [5, §3] are all fulfilled, the proof of [5, Lemma 15] shows that R is locally Galois over S. Hence Theorems 5 and 4 of [5] can be restated in the following way :

Theorem 1. *Let R be Galois and locally finite over S, Galois over S_1, and let $[V: V_R(R)] < \infty$.*

(i) *For each intermediate regular subrings R_1, R_2 of R/S, every S-(ring) isomorphism of R_1 onto R_2 can be extended to an automorphism of R.*

(ii) *For each intermediate regular subring R' of R/S, R is Galois and locally Galois over R'.*

Now we shall conclude this section with the following lemma, which will be required in the next section.

Lemma 2. *Let all the assumptions in Theorem 1 be satisfied, and $\mathfrak{G}^*$ a regular group of R/S such that $J(\mathfrak{G}^*(S_1), R) = S_1$*[2]*. Then there exists a subgroup $\hat{\mathfrak{G}}$ of $\mathfrak{G}^*$ such that $J(\hat{\mathfrak{G}}, R) = S$ and $(R/S, \hat{\mathfrak{G}})$ is l.f.d.*

Proof. It will be easy to see that the subgroup $\{\sigma \in \mathfrak{G}^* \mid M\sigma = M\}$ in the proof of [5, Lemma 15] may be adopted as our desired $\hat{\mathfrak{G}}$.

2. Galois theory for certain Galois extensions. In this section, we always assume R is Galois over S, and set $\mathfrak{G} = \mathfrak{G}(R/S)$. Now we shall

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2) By [2, Theorem 2], $J(\mathfrak{G}^*(S'), R) = S'$ for all subrings S' containing S_1 with $[S' : S_1]_l < \infty$.

begin this section with introducing the following conditions :

(1) *R is (Galois and) locally finite over S.*

(2) $[V : Z_0] < \infty$.

Evidently the condition (2) is fulfilled if either $\mathfrak{G}$ is l. f. d. or $[V : V_R(R)] < \infty$. Further, we shall prove the next

Lemma 3. *Let the conditions* (1), (2) *be satisfied.*

(i) *H is simpe.*

(ii) *For each finite subset F of R, there exists a $\mathfrak{G}$-normal regular subring N with the following properties*: (a) $N \supset V$, (b) $[V_N(S) : V_N(N)] < \infty$, *and* (c) $N = V_R(V_R(S^\S))$ *for some regular subring* $S^\S$ *cotaining* $S(F)$ *with* $[S^\S : S]_l < \infty$.

(iii) *For each intermediate regular subring T of R/S with* $[V : V_R(T)]_r < \infty$, *R is locally Galois over T.*

Proof. Let $\{w_1, \cdots, w_p\}$ be a linearly independent Z_0-basis of V. Then, setting $S^* = S_1(\{w_j\}, F)$, we have $H^* = V_R(V_R(S^*)) \supset H(V)$, whence it follows that $V_{H^*}(S) = V$. And so, there holds $[V_{H^*}(S) : V_{H^*}(H^*)] = [V : V_R(S^*)] < \infty$ by Corollary 1. Hence, H^* being simple, (i) is clear from $V_{H^*}(V_{H^*}(S)) = H$. Next, we shall prove (ii). Set $N' = S(H^*\mathfrak{G})$, which is evidently simple, $\mathfrak{G}$-normal, and Galois over S. Since $\mathfrak{G}_{S^*} \subset \sum^{v} \sigma_{S^*}^{(u)} V_r$ for some $\sigma^{(u)} \in \mathfrak{G}$ ([5, Lemma 3]), noting that $H^* \supset V$, we have $V_R(N') = \bigcap_{\sigma \in \mathfrak{G}} V_R(H^*\sigma) = \bigcap_{\sigma \in \mathfrak{G}} V_R(S^*)\sigma = \bigcap_{\sigma^{(u)}} V_R(H^*\sigma^{(u)}) = \bigcap_{\sigma^{(u)}} V_R(S^*\sigma^{(u)}) = V_R(S^*(\{S^*\sigma^{(u)}\}))$. Evidently $S^\S = S^*(\{S^*\sigma^{(u)}\})$ contains $S(F)$, and $[S^\S : S]_l < \infty$, moreover $N = V_R(V_R(S^\S))$ is $\mathfrak{G}$-normal, and there holds $\infty > [V : V_R(S^\S)] = [V_N(S) : V_N(N)]$ by Corollary 1. Thus we have proved (ii). Finally (iii) is a direct consequence of (ii) and Theorem 1 (ii), because we can find a finite subset F of T with $V_R(S(F)) = V_R(T)$, whence T is a regular subring of N treated in (ii).

Corollary 2. *Let the condition* (1) *be satisfied. Then the condition* (2) *is fulfilled if and only if H is simple and $\mathfrak{G}(R/H)$ is l. f. d.*

Proof. Our assertion is an easy consequence of Lemma 3 (i), (iii), and [5, (**b***)], because there holds $H(V) = H \times_{Z_0} V$.

In what follows, we always assume the conditions (1), (2), and introduce here the following condition :

(3) *R is Galois over* S_1.

Lemma 4. *Let the conditions* (1)—(3) *be satisfied.*

(i) *For each intermediate regular subring* S' *with* $[S' : S]_l < \infty$, *every S-(ring) isomorphism ρ of S′ into R is contained in* $\mathfrak{G}_{S'}$, *where we assume that* $V_R(S'\rho)$ *is simple.*

(ii) *For each intermediate regular subring* S' *with* $[S' : S]_l < \infty$, *R* *is Galois over* S'.

Proof. Let $V = \sum^{l} K g_{pq}$, $V_R(S') = \sum^{l'} K' g'_{p'q'}$, and $V_R(S'\rho) = \sum^{l''} K'' g''_{p''q''}$ be respecitive representations of V, $V_R(S')$, and $V_R(S'\rho)$ as complete matrix rings over division rings.

(i) By Lemma 3 (ii), there exists a $\mathfrak{G}$-normal regular subring N with the following properties: (a) $N \supset V$, (b) $[V_N(S) : V_N(N)] < \infty$, and (c) $N = V_R(V_R(S^{\S}))$ for some regular subring $S^{\S}$ with $[S^{\S} : S(\{e_{ij}\}, \{g_{pq}\}, \{g'_{p'q'}\}, \{g''_{p''q''}\}, S', S'\rho)]_l < \infty$. Then $\mathfrak{G}_N$ is evidently a regular group of N/S with $J(\mathfrak{G}_N(S_1), N) = S_1$. Thus for N/S and $\mathfrak{G}_N$ (instead of R/S and $\mathfrak{G}^*$ respectively) we can apply Lemma 2 to find a subgroup $\hat{\mathfrak{G}}$ of $\mathfrak{G}_N$ such that $J(\hat{\mathfrak{G}}, N) = S$ and $(N/S, \hat{\mathfrak{G}})$ is l. f. d. Considering $M = S(\{\{e_{ij}\}, \{g_{pq}\}, \{g'_{p'q'}\}, \{g''_{p''q''}\}, S', S'\rho\}\hat{\mathfrak{G}})$, we readily see that $\mathfrak{G}(M/S) = \mathfrak{G}_M \cdot \widetilde{V_M(S)}_M (\subset \mathfrak{G}_M)$, and that $V_M(S')$, $V_M(S'\rho)$ are both simple, whence our assertion will be easily seen (cf. the proof of [5, Lemma 16]).

(ii) Replacing $(\{e_{ij}\}, \{g_{pq}\}, \{g'_{p'q'}\}, \{g''_{p''q''}\}, S', S'\rho)$ in the proof of (i) by $(\{e_{ij}\}, \{g_{pq}\}, \{g'_{p'q'}\}, \{g''_{p''q''}\}, S', b)$ with arbitrary $b \in R \backslash S'$, we can readily see that there exists some $\tau \in \mathfrak{G}(M/S')$ $(\subset \mathfrak{G}_M)$ such that $b\tau \neq b$ (cf. the proof of [5, Lemma 15]). Thus we have proved (ii).

By Lemma 3 (ii) and Lemma 4 (ii), we readily obtain the following

Corollary 3. *Let the conditions* (1)—(3) *be satisfied. Then, given any regular subring* S' *with* $[S' : S]_l < \infty$, *the conditions* (1)—(3) *are preserved for* R/S'.

And, in virtue of Lemma 4 (i), the proof of the next proceeds just as in the proof of [4, Lemma 15].

Corollary 4. *Let the conditions* (1)—(3) *be satisfied, and* R' *an intermediate simple subring of* H/S. *If* ρ *is an* S*-(ring) isomorphism of* R' *into* R *and* $R'\rho$ *is regular, then* $R'\rho$ *is also contained in* H.

We shall introduce here the final condition:

(4) $[R : H]_l \leqq \aleph_0$.

Lemma 5. *Under the conditions* (1)—(4), *there holds* $\mathfrak{G}(H/S) = \mathfrak{G}_H$.

Proof. If, $[R: H] < \aleph_0$, then our assertion is contained in Theorem 1 (i). And so, we shall restrict our attention to the case $[R : H]_l = \aleph_0$. Let $\{x_1, x_2, \cdots\cdots\}$ be a (countably infinite) left-independent H-basis of R. Now, by making use of a similar argument as in the proof of [1, Lemma 4], we shall prove that each $\sigma \in \mathfrak{G}(H/S)$ can be extended to an automorphism of R. By Lemma 3, there exists a regular subring $S_1{}^{\S}$ containing $S_1(x_1)$ such that $[S_1{}^{\S} : S]_l < \infty$, $N_1 = V_R(V_R(S_1{}^{\S}))$ contains $H(V)$, is $\mathfrak{G}$-normal (and so Galois over S), and that $[V_{N_1}(S) : V_{N_1}(N_1)] < \infty$. Then, by Theorem 1 (i), σ can be extended to some $\sigma_1 \in \mathfrak{G}(N_1/S)$. Next, let n_2

be the first integer such that $x_{\nu_2} \notin N_1$. Then, again by Lemma 3, we can find a regular subring $S_2\S$ containing $S_1\S(x_{\nu_2})$ such that $[S_2\S : S]_l < \infty$, $N_2 = V_R(V_R(S_2\S))$ is $\mathfrak{G}$-normal, and that $[V_{N_2}(S) : V_{N_2}(N_2)] < \infty$. And then σ_1 can be extended to some $\sigma_2 \in \mathfrak{G}(N_2/S)$. The rest of the proof proceeds just as in that of [1, Lemma 3], and the details may be left to readers.

Now we are at the position to prove our principal theorem.

Theorem 2. *Let the conditions* (1)—(4) *be satisfied, and let R' be an intermediate regular subring of R/S with* $[V : V_R(R')]_r < \infty$.

(i) *If ρ is an S-(ring) isomorphism of R' into R and $R'\rho$ is a regular subring with* $[V : V_R(R'\rho)]_r < \infty$, *then ρ is contained in* $\mathfrak{G}_{R'}$.

(ii) *R is Galois and locally Galois over R'.*

Proof. (i) By our assumption, there exists a simple subring S' of R' such that $[S' : S]_l < \infty$, $V_R(S') = V_R(R')$, and $V_R(S'\rho) = V_R(R'\rho)$. Then, by Lemma 4 (i), $\rho_{S'} = \sigma_{S'}$ for some $\sigma \in \mathfrak{G}$. Clearly $\rho\sigma^{-1}$ is an S'-isomorphism of R' into R and $V_R(R'\rho\sigma^{-1}) = V_R(R'\rho)\sigma^{-1}$ is a simple ring over which V is (right) finite. If we can prove that $\rho\sigma^{-1} = \tau_{R'}$ for some $\tau \in \mathfrak{G}(S')$, $\tau\sigma$ is evidently a required extension of ρ. To prove this, noting that $S' \subset R' \subset V_R(V_R(S'))$ and that all the conditions (1)—(4) are preserved for R/S' by Corollary 3, it suffices to prove our assertion for the case where R' is contained in H. Under this situation, $R'\rho$ is contained also in H by Corollary 4. Consequently, ρ can be extended to an automorphism of R by Theorem 1 (i) and Lemma 5.

(ii) Let S' be a simple subring of R' such that $[S' : S]_l < \infty$ and $V_R(S') = V_R(R')$. Then, by Corollary 3, the conditions (1)—(4) are preserved for R/S'. Noting that $S' \subset R' \subset V_R(V_R(S'))$, Theorem 1 (ii) and Lemma 5 will yield at once that R is Galois over R'. The rest of our assertion is contained in Lemma 3 (iii).

In case $\mathfrak{G}$ is l. f. d., the conditions (1)—(3) are necessarily fulfilled ([5, (**b***)]), and so we readily obtain the following corollary.

Corollary 5.[3] *Let $\mathfrak{G}$ be l. f. d., and the condition* (4) *be satisfied.*

(i) *Let R' be an intermediate regular subring of R/S with* $[V : V_R(R')]_r < \infty$. *If ρ is an S-(ring) isomorphism of R' into R, and $R'\rho$ is a regular subring with* $[V : V_R(R'\rho)]_r < \infty$, *then ρ is contained in* $\mathfrak{G}_{R'}$.

(ii) *There exists a* 1—1 *dual correspondence between closed* $(*_f)$-

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3) In Galois theory of fields, most important results were extension theorem of isomorphisms, existence theorem of Galois correspondence, and normality theorem. In our present stage too, the validity of them is desirable, in fact Corollary 5 may be regarded as extensions of them. Further, we presuppose the validity of Corollary 5 without assuming the condition (4), or equivalently, the validity of [3, Lemma 3] cancelled at the beginning of this paper.

regular subgroups $\mathfrak{H}$ *of* $\mathfrak{G}$ *and intermediate regular subrings* R' *with* $[V: V_R(R')]_r < \infty$ *in the usual sense of Galois theory, and* $\mathfrak{G}(R/R')$ *is l.f.d.* (cf. [3, Corollary 1]).

(iii) *Let* T *be an intermediate regular subring with* $[V: V_R(T)]_r < \infty$. *If* $\mathfrak{T}^*$ *is the composite of* $\mathfrak{T} = \{\sigma \in \mathfrak{G} \mid T\sigma = T\}$ *and the totality of* $J(\mathfrak{T}, R)$*-inner automophisms, and* $\mathfrak{H}$ *is the group of* S*-automorphisms of* T, *then* $J(\mathfrak{H}, T) = S$ *if and only if* $\mathfrak{T}^*$ *is dense in* $\mathfrak{G}$ (cf. [5, (**m***)]).

Remark. In case R is a division ring, the condition (3) is superfluous of course. However, in [1], we have obtained more general results, that is, Theorem 2 is valid under the conditions (1), (4) and (2') R is locally Galois over S[4]. It seems to the authors that most difficulties in extending the fact cited above to simple rings arise from the following unsolved problem : Is $\mathfrak{G}R_r$ dense in the S_l-endomorphism ring of R ?

Example. We shall consider again $K = K_1 \times K_2 \times \cdots \times K_n \times \cdots$ treated in [1, Example]. Then the division subring $R = K_1 \times M_2 \times \cdots \times K_{2n-1} \times M_{2n} \times \cdots$ is Galois and locally finite over $S = M_1 \times Z \times \cdots \times M_{2n-1} \times Z \times \cdots$ (Z the rational number field), and $V_R(S) = M_1 \times M_2 \times \cdots \times M_n \times \cdots$. Thus the conditions (1)—(4) are all satisfied, however $[V_R(S): V_R(R)] = \aleph_0$ and $\mathfrak{G}(R/S)$ is not l. f. d.

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4) The assumption $[V_K(L) : V_K(K'_2)] < \infty$ in [1, Theorem 4] is really needless, although it was assumed. However, for simple rings, the corresponding assumption will be needed (Theorem 2 and Corollary 5).