ON GENERATING ELEMENTS OF GALOIS EXTENSIONS OF DIVISION RINGS IV

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Let a division ring K be Galois over L. The aim of this paper is to investigate under which conditions all the intermediate subrings finite over L are simple over L. In what follows, our consideration will proceed in principle without assuming $[K: L] < \infty$. And our results for the special case $[K : L] < \infty$ will give several precisions of those cited in the previous papers [3], [4], [5], and [6]. Finally, as to notations and terminologies used here, we follow the previous ones mentioned just now.

1. Preliminaries.

Throughout the paper, K will be a division ring, and L a division subring of K. We set $V = V_K(L)$ and $H = V_K(V)$. Further, C, Z and C_0 will be the centers of K, L and V respectively.

Now let D be an intermediate subring of K/L, and $\mathfrak{M}$ be the L_r-K_r-module consisting of all the (module) homomorphisms of D into K. Then we set $\mathfrak{R} = \{m \in \mathfrak{M} \mid ml_r = l_r m \text{ for all } l_r \in L_r\}$. Under these conventions, there holds the next lemma. The proof proceeds as in the proof of [6 ; Lemma 1], and it may be left to readers.

Lemma 1. *For any subset $\mathfrak{S}$ of $\mathfrak{R}$, $\mathfrak{S}$ is linearly independent over V_r if and only if it is linearly independent over K_r.*

Corollary 1. *Let $\mathfrak{M}_0$, $\mathfrak{R}_0$ be arbitrary K_r-submodule of $\mathfrak{M}$ and V_r-submodule of $\mathfrak{R}$ respectively. Then :*

(1) $[\mathfrak{M}_0 \cap \mathfrak{R} : V_r]_r \leqq [\mathfrak{M}_0 : K_r]_r$.

(2) $[\mathfrak{R}_0 : V_r]_r = [\mathfrak{R}_0 K_r : K_r]_r$ and $\mathfrak{R}_0 K_r \cap \mathfrak{R} = \mathfrak{R}_0$.

Corollary 2. *Let K be Galois over L, and $\mathfrak{G}$ be a galois group of K/L, that is, the fixring of $\mathfrak{G}$ is L. If $\mathfrak{G}_D$ means the restriction of $\mathfrak{G}$ on D then :*

(1) $[\mathfrak{G}_D V_r : V_r]_r = [\mathfrak{G}_D K_r : K_r]_r$ *and* $\mathfrak{G}_D V_r = \mathfrak{G}_D K_r \cap \mathfrak{R}$.

(2) *If* $[D : L]_l < \infty$ *then* $[\mathfrak{G}_D V_r : V_r]_r = [D : L]_l$.

Proof. If $[D : L]_l < \infty$, then $\mathfrak{G}_D K_r = \mathfrak{R}' = \{m \in \mathfrak{M} \mid ml_l = l_l m$ for all $l_l \in L_l\}$ by Jacobson's density theorem, and so $[\mathfrak{R}' : K_r]_r = [D : L]_l$. Hence (2) is an easy consequence of (1).

Lemma 2. *Let K be Galois and locally finite over L. If* $[V : C_0] <$

∞, *then* K *is locally Galois over* L[1].

Proof. Our lemma is still valid for simple rings [8, Lemma 3 (iii)]. However, for the sake of convenience, we shall give here the proof. Let F be an arbitrary finite subset of K, and $D = L[F]$. As $[D : L]_l < \infty$, Corollary 1 (2) yields $\mathfrak{G}_D V_r = \sigma_{1D} V_r + \cdots + \sigma_{nD} V_r$ for some $\sigma_i \in \mathfrak{G}$, where $\mathfrak{G} = \mathfrak{G}(K/L)$. Now we set $D_1 = H[D, D\sigma_1, \cdots, D\sigma_n, V]$, then it is clear that $H\mathfrak{G} \subset H$, $V\mathfrak{G} \subset V$ and $D\sigma_i\mathfrak{G} \subset D(\sigma_{1D} V_r + \cdots + \sigma_{nD} V_r)$ for each i. Hence D_1 is $\mathfrak{G}$-normal. Noting that $H \supset C_0$, $[V : C_0] < \infty$, $[D : L]_l < \infty$ and that $[D\sigma_i : L]_l < \infty$ for all i, we obtain $[D_1 : H]_l < \infty$ and $[V_{D_1}(L) : V_{D_1}(D_1)] < \infty$ by [9, Lemma 2 and Theorem 1], whence D_1/L is locally Galois by [7, Theorem 1]. Since $D_1 \supset D$, we have proved our assertion.

Lemma 3. *Let* $x \in L$ *be transcendental over* Z, *and a submodule* M *of* K *be (right) finite over* V. *Then there exists some positive integer* k *such that* $\sum_{i=0}^{\infty} My^i = \sum_{i=0}^{\infty}\oplus My^i$ *for* $y = x^k$.

Proof. Let $\{d_1, \cdots, d_n\}$ be a (linearly independent) right V-basis of M. Recalling $L[V] = L \times_Z V$, it is clear that $\{x^j\}$ is linearly independent over V. And so, for each positive integer i, the division subring F_i generated by $V[x^{2^i}]$ is a quotient division ring of $V[x^{2^i}]$ which may be considered as a polynomial domain. Now we set $n(i) = [\sum_{j=1}^{n} d_j F_i : F_i]_r$ for each positive integer i. Since $n(i) \leqq n$, there exists some q with $\operatorname{Max}_{i \geqq 0} n(i) = n(q)$. In what follows, we shall prove $n(q) = n$. Suppose, on the contrary, $n(q) = m < n$. Then we may, and shall, assume that $\{d_1, \cdots, d_m\}$ is a linearly independent F_q-basis of $\sum_{j=1}^{n} d_j F_q$: $\sum_{j=1}^{n} d_j F_q = \sum_{j=1}^{m}\oplus d_j F_q$. We set here $d_{m+1} = \sum_{j=1}^{m} d_j f_j$ with $f_j \in F_q$, where, without loss of generality, we mey assume $f_1 \neq 0$. And so, we set $f_1 = (\sum_{i=0}^{n_1} y^i v_i)^{-1} (\sum_{j=0}^{n_2} y^j v_j')$, where $y = x^{2^q}$ and v_i's, v_j''s are elements in V. If t is an integer with $2^t > \operatorname{Max}(n_1, n_2)$ then $F_{q+t} \subset F_q$ and the maximality of m show that $\sum_{j=1}^{n} d_j F_{q+t} = \sum_{j=1}^{m}\oplus d_j F_{q+t}$. Thus, we have $d_{m+1} = \sum_{j=1}^{m} d_j f_j'$ with some $f_j' \in F_{q+t}$. Then $f_1 = f_1'$ yields $\sum_{j=0}^{n_2} y^j v_j' = (\sum_{i=0}^{n_1} y^i v_i) f_1'$. As is readily verified, $\{1, y, \cdots, y^{2^t-1}\}$ is linearly independent over the quotient division ring $V(y^{2^t})$ of $V[y^{2^t}]$, and so we have $f_1 = f_1' = v_p^{-1} v_p' \in V$ for each non-zero v_p. Similarly, we can prove that each f_i is contained in V. But this contradicts the fact that $\{d_1, \cdots, d_m\}$ is linearly independent over V. We have proved therefore $n(q) = n$.

1) See [7, Definition].

Accordingly, noting that $\{(x^{2^q})^s\,;\ s = 0,\ 1, \cdots\}$ $(\subset F_q)$ is linearly independent over V, we obtain $\sum_{j=1}^{n} \sum_{s=0}^{\infty} \oplus d_j(x^k)^s V = \sum_{s=0}^{\infty} \oplus M(x^k)^s$.

Proposition 1. *Let* $[K:L]_l < \infty$, $[L:Z] < \infty$, *and let* C *be finite and separable over* $L \cap C$. *If* $L \not\subset C$ *then* K/L *is simple.*

Proof. By [1, Theorem 7.9.1], we have $[K:C] < \infty$. Let M be a maximal subfield of K which is separable over C. Then M is finite and separable over $L \cap C$, and so $M = (L \cap C)[d]$ for some $d \in M$. Further, M containing only a finite number of subfields containing C, there exists only a finite number of division subrings containing $M = V_K(M)$. Now, the rest of the proof proceeds just as in the latter part of the proof of [5, Lemma 7], and the details may be left to readers.

The following propositions are due to prof. M. Moriya who kindly permitted us to state them here.

Proposition 2. *Let* D *be a division subring of* K, *and* S *a subset of* D *such that* $[K: V_D(S)]_l < \infty$. *Then* $[K: V_K(S)]_l \geqq [D : V_D(S)]_l$.

Proof. If either S is empty or S contains only the zero element then our assertion is clear. Therefore we shall assume that S contains at most one non-zero element. Let $\{k_1, \cdots, k_n\}$ be an independent $V_D(S)$-basis of $V_K(S)$. If $\{k_i;\ i = 1, 2, \cdots, n\}$ is linearly dependent over D, then, without loss of generality, we may assume $\{k_1, \cdots, k_p\}$ $(p < n)$ is a minimal subset of $\{k_1, \cdots, k_n\}$ which is linearly dependent over D. Accordingly, $\sum_{i=1}^{p} k_i d_i = 0$ for some (non-zero) $d_i \in D$ $(i = 1, \cdots, p)$, where we may set $d_1 = 1$. Then for each non-zero $s \in S$, we have $\sum_{i=1}^{p} k_i s d_i s^{-1} = 0$, whence together with $\sum_{i=1}^{p} k_i d_i = 0$ it follows $\sum_{i=1}^{p} k_i(d_i - s d_i s^{-1}) = 0$, that is, $d_i = s d_i s^{-1} (i = 2, \cdots, p)$. Thus each d_i is contained in $V_D(S)$ but this is a contradiction. We have proved therefore $[K: D]_l \geqq [V_K(S): V_D(S)]_l$. Noting that $[K:D]_l\,[D: V_D(S)]_l = [K: V_K(S)]_l\,[V_K(S): V_D(S)]_l$, our assertion is evident.

Proposition 3. *Let* $[K:L]_l < \infty$. *Then:*

(1) *If* v *is an element of* $V_K(Z)$, *then there exists some* $k \in K$ *such that* $L[k] \ni v$ *and* $K = V_K(Z)[k]$.

(2) *If* $V_K(Z)$ *is simple over* L *then* K *is simple over* L.

Proof. By the light of Proposition 2, (1) and (2) will be proved just as in the proofs of [5, Lemma 6] and [5, Theorem 1] respectively.

2. Locally simple Galois extensions.

Throughout this section, K will be Galois and locally finite over L, $\mathfrak{G}$ the total Galois group of K/L, and D will denote an intermediate division subring. Further, we shall use the following conventions. Let $[D:L]_l < \infty$. Let m be the minimal number of elements in D such that D is obtained by ring adjunction of these m elements to L. Then m will be denoted by $n(D/L)$. In particular, if $n(D/L) = 1$ for all D with $[D:L]_l < \infty$, then we say that K/L is *locally simple*. Further, we set $n_0 = \text{Max}\ n(W/Z)$, where W runs over all the subrings of V with $[W:Z] < \infty$.

Lemma 4. *Let M be a right $\mathfrak{G}V_r$-submodule of K, $\{x_1, \cdots, x_t\}$ a finite subset of L such that $\sum_{i=1}^{t} Mx_i = \sum_{i=1}^{t}\oplus Mx_i$, and let $m_1, \cdots, m_s$ be a finite subset of M. Then:*

(1) *If $s \leqq t$, then $L[m_1, \cdots, m_s] = L[\sum_{i=1}^{s} m_i x_i]$.*

(2) *If $[L[m_1, \cdots, m_s, k] : L]_l = n$ and $s(n+1) \leqq t$, then there exists a subset $\{x_{\gamma_1}, \cdots, x_{\gamma_s}\}$ of $\{x_i\}$ such that $L[m_1, \cdots, m_s, k] = L[\sum_{i=1}^{s} m_i x_{\gamma_i} + k]$.*

Proof. (1) Since $L' = L[\sum_{i=1}^{s} m_i x_i] \subset L[m_1, \cdots, m_s]$ evidently, we shall prove only the converse inclusion. For each $\sigma \in \mathfrak{G}(K/L')$, we have $\sum_{i=1}^{s} m_i x_i = (\sum_{i=1}^{s} m_i x_i)^{\sigma} = \sum_{i=1}^{s} m_i^{\sigma} x_i$, whence $m_i = m_i^{\sigma} (i = 1, \cdots, s)$. Hence [9, Theorem 2] shows $L' \supset \{m_1, \cdots, m_s\}$.

(2) We set $L_j = L[\sum_{i=1}^{s} m_i x_{js+i} + k]$ $(j = 0, 1, \cdots, n)$. Now suppose that our assertion is false, and so that, for each j, we can choose such an $m_{j'}$ from m_i's that $m_{j'} \notin L_j$. Accordingly, by [9, Theorem 2] there exists some $\sigma_j \in \mathfrak{G}(K/L_j)$ with $m_{j'}^{\sigma_j} \neq m_{j'}$. Then $\sum_{i=1}^{s} m_i x_{js+i} + k = (\sum_{i=1}^{s} m_i x_{js+i} + k)^{\sigma_j} = \sum_{i=1}^{s} m_i^{\sigma_j} x_{js+i} + k^{\sigma_j}$ implies $\sum_{i=1}^{s} (m_i^{\sigma_j} - m_i) x_{js+i} = k(1_r - \sigma_j)$ for each j. Since $[L(k) : L]_l \leqq n$, we obtain, by Corollary 2 (2), $(\sum_{j=0}^{n} 1_r - \sigma_j)_{L[K]} v_{jr} = 0$ for not all zero $v_j \in V$. Thus we readily see that $0 = \sum_{j=0}^{n} \{\sum_{i=1}^{s} (m_i^{\sigma_j} - m_i) x_{js+i}\} v_{jr} = \sum_{j=0}^{n} \sum_{i=1}^{s} (m_i^{\sigma_j} - m_i) v_j x_{js+i}$. But the fact that $(m_{j'}^{\sigma_j} - m_{j'}) v_j \in M$ is nonzero for non-zero v_j contradicts $\sum_{i=1}^{t} Mx_i = \sum_{i=1}^{t}\oplus Mx_i$.

Theorem 1. *If $[L:Z] = \infty$ then K/L is locally simple.*

Proof. Let D be an intermediate subring with $[D:L]_l = n$. We shall distinguish two cases:

Case 1. *L is algebraic over Z.* Since L is not of bounded degree by [1, Theorem 7. 11. 1], there exits some intermediate subfield E of L/Z such that $n(n+1) \leqq [E:Z] < \infty$. We set here $L_0 = V_L(E)$, $K_0' = V_K(E)$. Then clearly $t = [L:L_0] = [E:Z]$, and K_0' is a right $\mathfrak{G}V_r$-module. If $\{x_1, \cdots, x_t\}$ is a linearly independent left L_0-basis of L, then there holds

$\sum_{i=1}^{t} K_0'x_i = \sum_{i=1}^{t}\oplus K_0'x_i$. For, if not, there hold among $\{x_1, \cdots, x_n\}$ non-trivial relations with coefficients in K_0'. Therefore, we may assume without loss generality that $x_1 + \sum_{i=2}^{q} k_i x_i = 0$ is such a non-trivial relation of the shortest length q where some k_i, say k_2, does not belong to L_0. Since the restriction of every automorphism in $\mathfrak{G}$ on K_0 is an automorphism of K_0' the restriction of $\mathfrak{G}$ on K_0' has L_0 as the fixring in K_0. Accordingly, there is some $\sigma \in \mathfrak{G}$ with $k_2^{\sigma} \neq k_2$ so that we obtain a non-trivial relation of the length less than q : $\sum_{i=1}^{q}(k_i - k_i^{\sigma})x_i = 0$, but this is a contradiction. Hence $\sum_{i=1}^{t} K_0'x_i = \sum_{i=1}^{t}\oplus K_0'x_i$. We set here $K_0 = V_K(Z)$, $D_0 = D \cap K_0 = V_D(Z)$, and $D_0' = D \cap K_0' = V_D(E)$. Then we shall prove $s = [D_0' : L_0]_l = [D_0 : L]_l \leqq n(\leqq t/(n+1))$ and $D_0 = \sum_{i=1}^{t}\oplus D_0'x_i$. Noting that $D \supset D_0 = V_D(Z) \supset L$ and $[D : L]_l < \infty$, we readily see that the center of D_0 coincides with $Z[C^*]$, where C^* is the center of D, further that $D_0' = V_{D_0}(E[C^*])$. Noting that $L[V] = L \times_Z V$, we obtain $[D_0 : D_0']_l = [E[C^*] : Z[C^*]] = [E \times_Z Z[C^*] : Z[C^*]] = [E : Z] = [L : L_0]$. And so, $[D_0 : D_0']_l$ $[D_0' : L_0]_l = [D_0 : L]_l$ $[L : L_0]_l$ implies $s = [D_0' : L_0]_l = [D_0 : L]_l \leqq [D : L] = n$. Further $[D_0 : D_0']_l = [L : L_0] = t$ and $\sum_{i=1}^{t}\oplus D_0'x_i \subset D_0$ show that $D_0 = \sum_{i=1}^{t}\oplus D_0'x_i$. Let $\{a_1, \cdots, a_s\}$ be a linearly independent L_0-basis of $D_0'(\subset K_0')$. Then $D_0 = L[a_1, \cdots, a_s]$ eventually. On the other hand, D is Galois and finite over $D_0 = V_D(Z)$ and $V_D(D_0) = V_D(V_D(Z)) \subset V_D(Z) = D_0$. Hence, by [2, Satz 14], we have $D = D_0[k] = L[a_1, \cdots, a_s, k]$ for some k. Now our assertion is a direct consequence of Lemma 4 (2).

Case 2. *L is not algebraic over Z.* Let $x \in L$ be transcendental over Z, and $\{a_1, \cdots, a_n\}$ a linearly independent L-basis of D. We consider the module $M = \sum_{i=1}^{n} a_i \mathfrak{G} V_r = \sum_{i=1}^{n} a_i \mathfrak{G}_D V_r$. Then, $[\mathfrak{G}_D V_r : V_r]_r$ being finite by Corollary 2 (2). we have $[M : V]_r < \infty$. Hence, by Lemma 3, there holds $\sum_{i=0}^{\infty} My^i = \sum_{i=0}^{\infty}\oplus My^i$, where $y = x^k$ for some positive integer k. As evidently $M \supset \{a_1, \cdots, a_n\}$, we have $D = L[a_1, \cdots, a_n] = L[\sum_{i=1}^{n} a_i y^i]$ by Lemma 4 (1).

Theorem 2. *Let $n_0 < \infty$. Then :*

(1) *$n(D/L) \leqq n_0$ for each D with $[D : L]_l < \infty$.*

(2) *K/L is locally simple if and only if $[L : Z] \geqq n_0$.*

Proof. Since, in case $[L : Z] = \infty$, K/L is locally simple by Theorem 1, we shall restrict our attention to the case where $[L : Z] < \infty$.

Then, by [1, Theorem 7. 9. 1], D is also finite over its center C^*, so that D is (finite and) inner Galois over $L[C^*]$, that is, $V_D(V_D(L)) = L[C^*]$. Noting that $L[V] = L \times_Z V$ and $V \supset V_D(L) \supset Z[C^*]$, we can readily see that $V_D(L[V_D(L)]) = L[C^*] \cap V_D(L) = Z[C^*]$, whence we have $V_D(Z) = L[V_D(L)] = L \times_Z V_D(L)$. Now let $V_D(L) = Z[v_1, \cdots, v_s]$, where $s = n(V_D(L)/Z)$. Then, of course, $s \leqq n_0$ and $V_D(Z) = L[v_1, \cdots, v_s]$. By Proposition 3 (1), there exists some $d \in D$ such that $D = V_D(Z)[d]$ and $L[d] \ni v_1$. Hence $L[d, v_2, \cdots, v_s] = L[d][v_2, \cdots, v_s] = D$, which proves our assertion (1) $n(D/L) \leqq s \leqq n_0$. Next we shall prove (2). Let $[L : Z] = t$ and $\{x_1, \cdots, x_t\}$ a linearly independent Z-basis of L. Then, by our assumption, there exists some W with $n(W/Z) = n_0$. If K/L is locally simple then $L \times_Z W = L[d]$ with some $d \in L \times_Z W (\subset L \times_Z V)$. Accordingly, $d = \sum_{i=1}^{t} w_i x_i$ for some $w_i \in W$, whence we obtain $L[w_1, \cdots, w_t] = L[d]$. Hcnce $W = Z[w_1, \cdots, w_t]$ which means $t \geqq n_0$. Conversely, let $t \geqq n_0$. Since, as is remarked above, $V_D(Z) = L \times_Z V_D(L) \subset L \times_Z V$ and $V_D(L) = Z[v_1, \cdots, v_s]$ with $s \leqq t$, Lemma 4 (1) proves $V_D(Z) = L[v_1, \cdots, v_s]$ $L[\sum_{i=1}^{s} v_i x_i]$. Consequently, by Proposition 3 (2), D/L is simple.

Corollary 3. *If V is commutative, then K/L is locally simple.*

Proof. Since the commutative field V is (algebraic and) separable over Z, $n_0 = 1$ evidently. Consequently, our assertion is clear by Theorem 2 (2).

Corollary 4. *Let K be of characteristic zero. Then K/L is locally simple if and only if either $L \supsetneqq Z$ or V is commutative.*

Proof. Evidently, each intermeditate subring W of V/Z with $[W : Z] < \infty$ is a separable algebra over Z. And so, $n(W/Z) = 1$ or 2 by [10, Theorem 2]. Now, our assertion is clear by Theorem 2 (2) and corollary 3.

Remark. If $[K : L] < \infty$, then $n_0 < \infty$ evidently. And so, Theorem 2 is applicable to the case where $[K : L] < \infty$.

Corollary 5. *Let K be of characteristic zero, and finite over L. Then, for any intermediate subriug D of K/L, D/L is simple if and only if either $L \not\subset V_D(D)$ or D is commutalive.*

Proof. Since the only if part is evident, we shall prove the if part. If either D is commutative or $L \supsetneqq Z$ then our assertion is clear by Corollary 4. And so, we shall restrict our attention to the case where $L = Z$ and $L \not\subset V_D(D)$. Then K is finite over $L \cap C$[2] and so $V_D(D)$ is finite (and separable) over $L \cap V_D(D)$ ($\supset V_L(K) = L \cap C$). Hence, our assertion is a direct consequence of Proposition 1.

2) Cf. [6, Footnote 6].

Lemma 6. *If $L \subset C$ and $[K:C] < \infty$, then $n_0 \leqq [K:C] < \infty$.*

Proof. Let D be an arbitrary subring with $[D:L] < \infty$ and $\{d_1 = 1, \cdots, d_m\}$ a linearly independent $C \cap D$-basis of D. Then $\{d_1, \cdots, d_m\}$ is linearly independent over C. For, if not, we have a non-trivial relation of the shortest length : $d_{t_1} = c_2 d_{t_2} + \sum_{i=3}^{q} c_i d_{t_i}$, where $c_i \in C$, $c_2 \in C \setminus D$. Since $J(\mathfrak{G}(K/D)_C, C) = C \cap D$ and $C^\sigma = C$ for each $\sigma \in \mathfrak{G}(K/D)$, there exists $\sigma \in \mathfrak{G}(K/D)$ with $c_2^\sigma \neq c_2$. Then $c_2 d_{t_2} + \sum_{i=3}^{q} c_i d_{t_i} = c_2^\sigma d_{t_2} + \sum_{i=3}^{q} c_i^\sigma d_{t_i}$ gives a contradiction $0 = (c_2 - c_2^\sigma) + \sum_{i=3}^{q} (c_i - c_i^\sigma) d_{t_i}$. Hence $m \leqq [K:C] < \infty$. And then, noting that $C \cap D$ is separable over L, it will be easily seen that $n(D/L) \leqq m \leqq [K:C] < \infty$.

Now we shall conclude our study with the follwoing

Theorem 3. *Let $[V:C_0] < \infty$, and let D be an arbitrary subring with $[D:L] < \infty$.*

(1) $n_0 \leqq [V:C_0] < \infty$.

(2) *K/L is locally simple if and only if $[L:Z] \geqq n_0$.*

(3) $n(D/L) \leqq n_0$.

(4) *Every D can be embeded in some subring D^* that is simple over L if and only if either $L \not\subset C$ or K is commutative.*

(5) *D is embedded in $L[k, vkv^{-1}]$ for some $k \in K$ and $v \in V$.*

Proof. Evidently, V is Galois and locally finite over Z (and $V_F(Z) = V$), and so (1) is a consequence of Lemma 6. Further (2), (3) are contained in Theorem 2. Finally, K being locally Galois over by Lemma 2, (4), (5) are consequences of [6, Theorem 3] and [3, Theorem 1] respectively.

REFERENCES

[1] N. JACOBSON, Structure of rings, Amer. Math. Soc. Colloq. Publ., vol. 37 (1956).

[2] F. KASCH, Über den Endomorphismenring eines Vektorraumes und den Satz von der Normalbasis, Math. Ann., 129 (1953) 447—463.

[3] M. MORIYA and T. NAGAHARA, On generating elements of Galois extensions of division rings II, Math, J. Okayama Univ., 7 (1957) 89—94.

[4] T. NAGAHARA, On primitive elements of Galois extensions of division rings, Math. J. Okayama Univ., 6 (1956) 23—28.

[5] T. NAGAHARA, On generating elements of Galois extensions of divisions rings, Math. J. Okayama Univ., 6 (1957) 181—190.

[6] T. NAGAHARA, On generating elements of Galois extensions of divisions rings III, Math. J. Okayama Univ., 7 (1957) 173—178.

[7] T. NAGAHARA and H. TOMINAGA, On Galois theory of division rings II, Math. J. Okayama Univ., 7 (1957) 169—172.

[8] T. NAGAHARA, N. NOBUSAWA and H. TOMINAGA, Galois theory of simple rings

IV, Math. J. Okayama Univ., 8 (1958)
[9] N. NOBUSAWA, A note on Galois extensions of division rings, Math. J. Okayama Univ., 7 (1957) 179—183.
[10] H. TOMINAGA and F. KASCH, On generating elements of simple rings, Proc. Japan Acad., 33 (1957) 187—189.

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Added in Proof. Let K be Galois and not always locally finite over L. Then we obtain the following that contains Theorem 1.

Theorem 1*. *If* $[L:Z]=\infty$ *then* D/L *is simple for each subring* D *with* $[D:L]_l<\infty$, *that is,* K/L *is locally simple.*

To prove this, we shall require the following chain of lemmas, the first of which is the next whose proof will be obtained in the similar way as in that of [9, Theorem 2].

Lemma 7. *Let* D *be an intermediate subring of* K/L *with* $[D:L]_l<\infty$. *If* D_0 *is an intermediate subring of* D/L *then* $J(\mathfrak{G}(K/D_0), K)\cap D=D_0$.

By making use of Lemma 7, we can prove the next whose proof is analogous to that of Lemma 4.

Lemma 4*. *Let* M *be a right* $\mathfrak{G}V_r$*-submodule of* K, $\{x_1, \cdots, x_t\}$ *a finite subset of* L *such that* $\sum^t Mx_i=\sum^t \oplus Mx_i$, *and let* $\{m_1, \cdots, m_s\}$ *be a finite subset of* M.

(1) *If* $s\leqq t$, *and* $[L[m_1, \cdots, m_s]:L]_l<\infty$ *then* $L[m_1, \cdots, m_s]=L[\sum^s m_ix_i]$.

(2) *If* $[L[m_1, \cdots, m_s, k]:L]_l=n<\infty$ *and* $s(n+1)\leqq t$, *then there exists a subset* $\{x_{\gamma_1}, \cdots, x_{\gamma_s}\}$ *of* $\{x_i\}$ *such that* $L[m_1, \cdots, m_s, k]=L[\sum^s m_ix_{\gamma_i}+k]$.

In virtue of the validity of Lemma 7 and Lemma 4*, our proof of Theorem 1* will proceed just as in that of Theorem 1.

Moreover, one will readily see that Theorem 2 can be modified corresponding to Theorem 1*, too.