TANGENT BUNDLES OF ORDER 2 AND GENERAL CONNECTIONS

TOMINOSUKE ŌTSUKI

The set of all tensor fields on a differentiable manifold $\mathfrak{X}$ may be regarded as a graded algebra with the contravariant and the covariant orders over the algebra $\mathfrak{A}(\mathfrak{X})$ of all scalar fields on $\mathfrak{X}$. Any affine connection of $\mathfrak{X}$ defines, as is well known, a differential operator of contravariant degree 0 and covariant degree 1 on the algebra which is called a covariant differentiation. On the other hand, in abstract theory of graded algebras, the so-called trivial differential operator which maps any element to zero can be always considered but it can not be regarded as a covariant differentiation derived from classical affine connections in the case of the above-mentioned algebra. It seems to the author that to construct a general theory of connections which can deal with these differential operators as special ones is an interesting problem in differential geometry.

The components of an affine connection with respect to local coordinates are well known as a distinguished example that they constitute a geometrical object but not a geometrical quantity, because for a coordinate transformation they are transformed on the whole in the same way as the components of a tensor of type (1, 2) but related with the terms including the partial derivatives of order 2 of the local coordinates. From this point of view, can we look on the components of an affine connection and the components of a tensor of type (1, 2) from a unificative standpoint and not as entirely different concepts ?

In this paper, the author will give an answer to the above-mentioned questions. In [4],[1] the author showed that an affine connection of $\mathfrak{X}$ may be regarded as a cross-section of the associated principal bundle of the tangent bundle $T(\mathfrak{X})$ into the one of the tangent bundle $\mathfrak{T}^2(\mathfrak{X})$ of order 2 and that the classical connections, for instance, the affine, projective, conformal connections, can be considered from a unificative standpoint by means of this idea. In this paper, the author will utilize also the tangent budle $\mathfrak{T}^2(\mathfrak{X})$ of order 2. An affine connection may be also regarded as a cross-section of the vector bundle $T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})$, where $\mathfrak{D}^2(\mathfrak{X})$ is the dual vector bundle of $\mathfrak{T}^2(\mathfrak{X})$. And so, he will consider as any cross-section of the vector bundle defines a general connection. Then, the classical affine connections and the tensors of type (1, 2) are remarkable as special

[1] The numbers in square brackets will show the numbers of the references at the end of this paper.

ones. The covariant differentiation corresponding to the latters is the above-mentioned trivial differential operator.

In §§ 1, 2, we shall state the group $\mathfrak{L}_n^2$ of all generalized infinitesimal isotropies, the tangent and cotangent bundles of order 2 and the associated principal bundle. In §§ 3, 4, 5, we shall define furthermore the tangent bundle of order 3 and the differentiation "d". Then, we shall extend the operation to the vector bundles $T^*(\mathfrak{X})^{\otimes m}$ and $T(\mathfrak{X})^{\otimes p} \otimes T^*(\mathfrak{X})^{\otimes q}$.

In §§ 6, 7, we shall study a general connection Γ and determine a corresponding operation μ_Γ so that it transforms $\mathfrak{T}^2(\mathfrak{X})$ into $T(\mathfrak{X})$ and $\mathfrak{D}^2(\mathfrak{X})$ into $T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X})$. Making use of this operation, we shall define a covariant differentiation which is the classical one when Γ is a classical affine connection.

In § 8, we shall prove the naturality for the induced general connections from general connections by mappings of base spaces and show some formulas on the Lie derivatives as its application. In § 9, we shall obtain analogies to the results obtained in [4] for general connections.

In §§ 10, 11, 12, we shall show that we can define a naturally determined affine connection of the induced vector bundle from $T(\mathfrak{X})$ by the projection of the associated principal bundle of $\mathfrak{T}^2(\mathfrak{X})$ which is called the universal affine connection of $T(\mathfrak{X})$ and investigate the relation between this connection and general connections of $T(\mathfrak{X})$.

§1. The group $\mathfrak{L}_n^2$ and the tangent bundles of order 2.

Let $\mathfrak{L}_n^2$ be the group of all generalized infinitesimal isotropies of order 2 at the origin of the n-dimensional coordinate space R^n, whose any element is written as a set of real numbers (a_i^j, a_{ih}^j) such that $|a_i^j| \neq 0$[2] and which was firstly used by the author in [4]. Its multiplication is given by the following formulas: For any $\alpha, \beta \in \mathfrak{L}_n^2$,

$$a_i^j(\alpha\beta) = a_k^j(\alpha)\, a_i^k(\beta),\text{[3]} \tag{1.1}$$

$$a_{ih}^j(\alpha\beta) = a_k^j(\alpha)\, a_{ih}^k(\beta) + a_{kl}^j(\alpha)\, a_i^k(\beta)\, a_h^l(\beta), \tag{1.2}$$

where we regard a_i^j, a_{ih}^j as coordinates of $\mathfrak{L}_n^2$. We may identify $L_n^1 = GL(n, R)$ with the subgroup of $\mathfrak{L}_n^2$ which consists of all elements α such that $a_{ih}^j(\alpha) = 0$ and we may also regard a_i^j as coordinates of L_n^1. Let σ be the natural homomorphism of $\mathfrak{L}_n^2$ onto L_n^1 given by

---

2) The set L_n^2 of all elements of $\mathfrak{L}_n^2$ such that $a_{ih}^j = a_{hi}^j$ is a subgroup of $\mathfrak{L}_n^2$ called the group of infinitesimal isotropies of order 2 at the origin of R^n according to C. Ehresmann [6].

3) We will use the Einstein's convention and the Latin letters $i, j, k, \cdots$ will run over 1, 2, $\cdots$, n.

$$a_i^j(\sigma(\alpha)) = a_i^j(\alpha). \tag{1.3}$$

Let $\mathfrak{N}_n^2$ be the kernel of σ, that is the subgroup consisting of all elements α such that $a_i^j(\alpha) = \delta_i^j$. Then we define a mapping $\gamma : \mathfrak{L}_n^2 \to \mathfrak{N}_n^2$ by

$$\gamma(\alpha) = \sigma(\alpha^{-1})\,\alpha. \tag{1.4}$$

Any element α of $\mathfrak{L}_n^2$ can be written uniquely as a product of an element of L_n^1 and an element of $\mathfrak{N}_n^2$ by

$$\alpha = \sigma(\alpha)\,\gamma(\alpha). \tag{1.5}$$

We can easily obtain the following formulas : (a) For any $\alpha,\ \beta \in \mathfrak{N}_n^2$

$$a_{ih}^j(\alpha\beta) = a_{ih}^j(\alpha) + a_{ih}^j(\beta), \tag{1.6}$$

(b) for any $\alpha \in \mathfrak{L}_n^2,\ \beta \in \mathfrak{N}_n^2$

$$a_{ih}^j(\alpha^{-1}\beta\alpha) = a_k^j(\alpha^{-1})\,a_{ml}^k(\beta)\,a_i^m(\alpha)\,a_h^l(\alpha), \tag{1.7}$$

(c) for any $\alpha \in \mathfrak{L}_n^2$

$$a_{ih}^j(\gamma(\alpha)) = a_k^j(\alpha^{-1})\,a_{ih}^k(\alpha), \tag{1.8}$$

and (d) for any $\alpha,\ \alpha_1 \in \mathfrak{L}_n^2$

$$a_{ih}^j(\gamma(\alpha\alpha_1)) = a_{ih}^j(\alpha_1^{-1}\gamma(\alpha)\,\alpha_1) + a_{ih}^j(\gamma(\alpha_1)). \tag{1.9}$$

Now let $\mathfrak{X}$ be any n-dimensional differentiable manifold with suitable differentiability for our purpose. With any coordinate neighborhood $(U,\ u^i)$[4], we associate $n + n^2$ fields of vectors defind on U which are denoted by $\partial u_i,\ \partial^2 u_{ih}$ for convenience' sake. Let $\partial v_i,\ \partial^2 v_{ih}$ be the vector fields associated with another coordinate neighborhood $(V,\ v^i)$. When $U \cap V \neq \emptyset$, we assume that they are related mutually as

$$\partial u_i = \frac{\partial v^j}{\partial u^i}\,\partial v_j, \tag{1.10}$$

$$\partial^2 u_{ih} = \frac{\partial^2 v^j}{\partial u^h \partial u^i}\,\partial v_j + \frac{\partial v^j}{\partial u^i}\,\frac{\partial v^k}{\partial u^h}\,\partial^2 v_{jk}. \tag{1.11}$$

Thus we obtain at each point x of $\mathfrak{X}$ an $(n + n^2)$-dimensional vector space independent of coordinate neighborhoods containing the point x, which we shall denote by $\mathfrak{T}_x^2(\mathfrak{X})$[5]. The union

$$\mathfrak{T}^2(\mathfrak{X}) = \bigcup_{x\in\mathfrak{X}} \mathfrak{T}_x^2(\mathfrak{X})$$

---

4) We use here the notation $(U,\ u^i)$ which represents an open neighborhood U and local coordinates u^i defined on U.

5) By putting $\partial^2 u_{ih} = \partial^2 u_{hi}$, we shall obtain the tangent space $T_x^2(\mathfrak{X})$ of order 2 at x in the sense of C. Ehresmann [6].

may be considered naturally as the total space of a vector bundle $\{\mathfrak{T}^2(\mathfrak{X}), \mathfrak{X}, \bar{\tau}\}$ with the natural projection $\bar{\tau}$, whose structure group is $\mathfrak{L}_n^2$ (in fact $L_n^2 \subset \mathfrak{L}_n^2$ but we take this group for our purposes) and the coordinate transformation $g_{VU} : U \cap V \rightarrow \mathfrak{L}_n^2$ is given by

$$a_i^j(g_{VU}) = \frac{\partial v^j}{\partial u^i}, \quad a_{ih}^j(g_{VU}) = \frac{\partial^2 v^j}{\partial u^h \partial u^i}. \tag{1.12}$$

For the sake of simplicity, we shall denote also this vector bundle over $\mathfrak{X}$ by the same notation $\mathfrak{T}^2(\mathfrak{X})$ and call it *the tangent bundle of order* 2 *of* $\mathfrak{X}$. By (1.10), we may identify the vector ∂u_i with the tangent vector $\partial/\partial u^i$ and so we may regard the tangent space $T_x(\mathfrak{X})$ at x as a subspace of $\mathfrak{T}_x^2(\mathfrak{X})$.

Let $\{\mathfrak{B}^2(\mathfrak{X}), \mathfrak{X}, \bar{\pi}\}$ be the associated principal bundle of $\mathfrak{T}^2(\mathfrak{X})$, which we will call *the principal bundle of order* 2 *of* $\mathfrak{X}$. Any point $\bar{b} \in \mathfrak{B}^2(\mathfrak{X})$ may be regarded as a frame of $\mathfrak{T}^2(\mathfrak{X})$ at the point $x = \bar{\pi}(\bar{b})$ such that

$$e_i(\bar{b}) = \partial u_j \, a_i^j(\bar{\beta}), \tag{1.13}$$

$$e_{ih}(\bar{b}) = \partial u_j \, a_{ih}^j(\bar{\beta}) + \partial^2 u_{jk} \, a_i^j(\bar{\beta}) \, a_h^k(\bar{\beta}). \tag{1.14}$$

where $\bar{\beta} \in \mathfrak{L}_n^2$. Corresponding to each $\bar{\alpha} \in \mathfrak{L}_n^2$, we define the right translation $r(\bar{\alpha})$ on $\mathfrak{B}^2(\mathfrak{X})$ by

$$e_i(\bar{b}\bar{\alpha}) = e_j(\bar{b}) \, a_i^j(\bar{\alpha}), \tag{1.15}$$

$$e_{ih}(\bar{b}\bar{\alpha}) = e_j(\bar{b}) \, a_{ih}^j(\bar{\alpha}) + e_{jk}(\bar{b}) \, a_i^j(\bar{\alpha}) \, a_h^k(\bar{\alpha}). \tag{1.16}$$

Now, let $\{\mathfrak{B}(\mathfrak{X}), \mathfrak{X}, \pi\}$ be the associated principal bundle of the tangent bundle $T(\mathfrak{X})$. As is well known, any point $b \in \mathfrak{B}(\mathfrak{X})$ may be regarded as a frame of $T(\mathfrak{X})$ at $\pi(b)$ such that

$$e_i(b) = a_i^j(\alpha) \frac{\partial}{\partial u^j}, \tag{1.17}$$

where we use also the same notation e_i for $\mathfrak{B}(\mathfrak{X})$ and $\mathfrak{B}^2(\mathfrak{X})$ from the abovementioned relation between $T(\mathfrak{X})$ and $\mathfrak{T}^2(\mathfrak{X})$. Then we obtain a natural homomorphism $\sigma = /\mathfrak{N}_n^2 : \mathfrak{B}^2(\mathfrak{X}) \rightarrow \mathfrak{B}(\mathfrak{X})$ by

$$e_i(\sigma(\bar{b})) = e_i(\bar{b}). \tag{1.18}$$

Furthermore for any $\bar{\alpha} \in \mathfrak{L}_n^2$, we have easily

$$\sigma \cdot r(\bar{\alpha}) = r(\sigma(\bar{\alpha})) \cdot \sigma. \tag{1.19}$$

§ 2. The cotangent bundles of order 2.

We shall denote the dual space of $\mathfrak{T}_x^2(\mathfrak{X})$ by $\mathfrak{D}_x^2(\mathfrak{X})$ in this paper. The union

$$\mathfrak{D}^2(\mathfrak{X}) = \bigcup_{x\in\mathfrak{X}} \mathfrak{D}^2_x(\mathfrak{X}) \tag{2.1}$$

may be considered as the total space of a differentiable vector bundle over $\mathfrak{X}$ called *the cotangent bundle of order* 2 *of* $\mathfrak{X}$ which we shall denote simply by the same notation. Now, we shall directly determine this bundle by means of local coordinates.

For any coordinate neighborhood (U, u^i) and at each $x \in U$, we consider the direct sum of the n-dimensional vector space spanned by *the differentials* d^2u^i *of order* 2 and the tensor product $T^*_x(\mathfrak{X}) \otimes T^*_x(\mathfrak{X})$ of the dual space $T^*_x(\mathfrak{X})$ of $T_x(\mathfrak{X})$ of which $du^i \otimes du^h$ form a base. Corresponding to another coordinate neighborhood (V, v^i), we consider analogously an $(n+n^2)$-dimensional vector space with the base $\{d^2v^i,\ dv^i \otimes dv^h\}$. Then, we relate the two vector spaces at x with each other by the equations

$$d^2v^j = \frac{\partial v^j}{\partial u^i} d^2u^i + \frac{\partial^2 v^j}{\partial u^h \partial u^i} du^i \otimes du^h, \tag{2.2}$$

$$dv^j \otimes dv^k = \frac{\partial v^j}{\partial u^i}\frac{\partial v^k}{\partial u^h} du^i \otimes du^h. \tag{2.3}$$

Comparing these equations with (1.10) and (1.11), we see that we can obtain an $(n+n^2)$-dimensional vector space $\mathfrak{D}^2_x(\mathfrak{X})$ at x which is dual to $\mathfrak{T}^2_x(\mathfrak{X})$ and its base $\{d^2u^i, du^i \otimes du^h\}$ is dual to the base $\{\partial u_i,\ \partial^2 u_{ih}\}$ of $\mathfrak{T}^2_x(\mathfrak{X})$. It is clear that $\mathfrak{D}^2_x(\mathfrak{X}) \supset T^*_x(\mathfrak{X}) \otimes T^*_x(\mathfrak{X})$.

Now in the following, for any vector bundle $\mathfrak{F} = \{\mathfrak{Z},\ \mathfrak{X}\}$ over $\mathfrak{X}$, we shall generally denote by $\Psi(\mathfrak{F})$ the vector space consisting of all cross-sections of $\mathfrak{F}$ over the algebra $\mathfrak{A}(\mathfrak{X})$ of all scalar fields on $\mathfrak{X}$ which is also considered as a vector space over the real field.

We define a natural differential operator of $\Psi(T^*(\mathfrak{X}))$ into $\Psi(\mathfrak{D}^2(\mathfrak{X}))$

$$d : \Psi(T^*(\mathfrak{X})) \to \Psi(\mathfrak{D}^2(\mathfrak{X})) \tag{2.4}$$

as follows : For any $\omega \in \Psi(T^*(\mathfrak{X}))$, which is locally written as $\omega = V_i du^i$, we put

$$d\omega = V_i d^2u^i + du^i \otimes dV_i. \tag{2.5}$$

Using (2.2), (2.3), we can easily prove that this definition does not depend on local coordinates.

§ 3. $\mathfrak{T}^3(\mathfrak{X})$ and $\mathfrak{D}^3(\mathfrak{X})$.

We can define analogously $\mathfrak{T}^3(\mathfrak{X})$ and $\mathfrak{D}^3(\mathfrak{X})$. With any coordinate neighborhood (U, u^i), we associate $n + n^2 + n^3$ vector fields ∂u_i, $\partial^2 u_{ih}$, $\partial^3 u_{ihl}$ defined on U. Let ∂v_i, $\partial^2 v_{ih}$, $\partial^3 v_{ihl}$ be the vector fields associated with another coordinate neighborhood (V, v^i) and then, at each point of $U \cap V = \emptyset$, we put

$$\begin{aligned} \partial u_i &= a_i^j \partial v_j, \\ \partial^2 u_{ih} &= a_{ih}^j \partial v_j + a_i^j a_h^k \partial^2 v_{jk}, \\ \partial^3 u_{ihl} &= a_{ihl}^j \partial v_j + (a_{ih}^j a_l^k + a_{il}^j a_h^k + a_i^j a_{hl}^k) \partial^2 v_{jk} \\ &\quad + a_i^j a_h^k a_l^m \partial^3 v_{jkm}, \end{aligned} \tag{3.1}$$

where

$$a_i^j = \frac{\partial v^j}{\partial u^i}, \quad a_{ih}^j = \frac{\partial^2 v^j}{\partial u^h \partial u^i}, \quad a_{ihl}^j = \frac{\partial^3 u^j}{\partial u^l \partial u^h \partial u^i}. \tag{3.2}$$

Thus, we can obtain a vector bundle over $\mathfrak{X}$ whose structure group is the group L_n^3 [6] of all infinitesimal isotropies of order 3 at the origin of R^n. Now let $\mathfrak{L}_n^3$ be the set of all $(a_i^j, a_{ih}^j, a_{ihl}^j)$ such that $|a_i^j| \neq 0$. $\mathfrak{L}_n^3$ may be considered as a group containing L_n^3 with the multiplication as follows: For any $\alpha, \beta \in \mathfrak{L}_n^3$,

$$\begin{aligned} a_i^j(\alpha\beta) &= a_k^j(\alpha)\, a_i^k(\beta), \\ a_{ih}^j(\alpha\beta) &= a_k^j(\alpha)\, a_{ih}^k(\beta) + a_{km}^j(\alpha)\, a_i^k(\beta)\, a_h^m(\beta), \\ a_{ihl}^j(\alpha\beta) &= a_k^j(\alpha)\, a_{ihl}^k(\beta) + a_{km}^j(\alpha)\,(a_{ih}^k(\beta)\, a_l^m(\beta) + a_{il}^k(\beta)\, a_h^m(\beta) + \\ &\quad + a_i^k(\beta)\, a_{hl}^m(\beta)) + a_{ikm}^j(\alpha)\, a_i^l(\beta)\, a_h^k(\beta)\, a_l^m(\beta). \end{aligned} \tag{3.3}$$

The first two equations show that $\mathfrak{L}_n^2$ may be regarded as a subgroup of $\mathfrak{L}_n^3$ which consists of all elements α such that $a_{ihl}^j(\alpha) = 0$. We shall denote this vector bundle over $\mathfrak{X}$ and its total space by the same symbol $\mathfrak{T}^3(\mathfrak{X})$ and its fibre over $x \in \mathfrak{X}$ by $\mathfrak{T}_x^3(\mathfrak{X})$.

Let $\mathfrak{D}^3(\mathfrak{X})$ and $\mathfrak{D}_x^3(\mathfrak{X})$ be the dual vector bundle of $\mathfrak{T}^3(\mathfrak{X})$ and its fibre over x. We shall denote also the total space of this bundle by $\mathfrak{D}^3(\mathfrak{X})$. We shall construct $\mathfrak{D}^3(\mathfrak{X})$ directly by means of $\mathfrak{D}^2(\mathfrak{X})$ in the following.

Firstly, we take the two tensor product bundles $\mathfrak{D}^2(\mathfrak{X}) \otimes T^*(\mathfrak{X})$ and $T^*(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})$ which contain $T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X})$[7]. Naturally extending the isomorphism of $T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X})$, which corresponds to the permutation of the second and third components, to $\mathfrak{D}^2(\mathfrak{X}) \otimes T^*(\mathfrak{X})$, we can obtain a vector bundle over $\mathfrak{X}$. We shall denote this vector bundle by $\mathfrak{D}^2(\mathfrak{X}) \dot{\otimes} T^*(\mathfrak{X})$. Then, we construct a vector bundle containing $T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X}) = T^*(\mathfrak{X})^{\otimes 3}$ [8] such that

$$\mathfrak{D}^2(\mathfrak{X}) \otimes T^*(\mathfrak{X}) + T^*(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X}) + \mathfrak{D}^2(\mathfrak{X}) \dot{\otimes} T^*(\mathfrak{X}), \tag{3.4}$$

6) See [6].

7) For any two vector bundles $\mathfrak{F}_1$ and $\mathfrak{F}_2$ over the same base space $\mathfrak{X}$, the author will denote by $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ and $\mathfrak{F}_1 \oplus \mathfrak{F}_2$ the vector bundles over $\mathfrak{X}$ whose fibres over each point of $\mathfrak{X}$ are the tensor product and the direct sum of the fibres of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ over the point respectively, in the following. See [1] and [2].

8) Generally, we shall denote the tensor product bundle $T^*(\mathfrak{X}) \otimes \cdots \otimes T^*(\mathfrak{X})$ of m $T^*(\mathfrak{X})$ by $T^*(\mathfrak{X})^{\otimes m}$.

where the summation symbol "+" means the direct sum of $T^*(\mathfrak{X})^{\otimes 3}$ and the above-mentioned vector bundles subtracted $T^*(\mathfrak{X})^{\otimes 3}$. Then, for any coordinate neighborhood (U, u^i), we take an $(n + 3n^2 + n^3)$-dimensional vector space at each point $x \in U$, which is the direct sum of the n-dimensional vector space spanned by *the differentials* d^3u^i *of order 3* and the $(3n^2 + n^3)$-dimensional fibre over x of the vector bundle over $\mathfrak{X}$ given by (3. 4). Relating this vector space with the one for another coordinate neighborhood (V, v^i) by the equations

$$d^3v^j = a_i^j d^3u^i + a_{ih}^j(d^2u^i \otimes du^h + du^i \otimes d^2u^h + d^2u^i \dot{\otimes} du^h) + a_{ihl}^j du^i \otimes du^h \otimes du^l, \tag{3. 5}$$

where a_i^j, a_{ih}^j, a_{ihl}^j are given by (3. 2), then we can define a vector space independent of local coordinates by virtue of (3. 3), which we shall denote by $\widetilde{\mathfrak{D}}_x^3(\mathfrak{X})$. Thus, we obtain a vector bundle over $\mathfrak{X}$ with the structure group L_n^3 which we shall denote by $\widetilde{\mathfrak{D}}^3(\mathfrak{X})$.

Now, on the other hand, we have easily from (1. 1) and (1. 2) the equations :

$$\begin{aligned} &d^2v^j \otimes dv^k + dv^j \otimes d^2v^k + d^2v^j \dot{\otimes} dv^k \\ &= a_i^j a_h^k (d^2u^i \otimes du^h + du^i \otimes d^2u^h + d^2u^i \dot{\otimes} du^h) \\ &\quad + (a_{ih}^j a_l^k + a_i^j a_{hl}^k + a_{il}^j a_h^k)\, du^i \otimes du^h \otimes du^l. \end{aligned} \tag{3. 6}$$

Hence (3. 5), (3. 6) and (3. 1) show clearly that the subspace spanned by d^3u^i, $(d^2u^i \otimes du^h + du^i \otimes d^2u^h + d^2u^i \dot{\otimes} du^h)$, $du^i \otimes du^h \otimes du^l$ at each point $x \in U$ is independent of local coordinates and dual to $\mathfrak{T}_x^3(\mathfrak{X})$ and these elements form the base dual to the base $\{\partial u_i, \partial^2 u_{ih}, \partial^3 u_{ihl}\}$. We shall denote this vector space by $\mathfrak{D}_x^3(\mathfrak{X})$. Thus, we have directly obtained the dual vector bundle $\mathfrak{D}^3(\mathfrak{X})$ to $\mathfrak{T}^3(\mathfrak{X})$.

Nextly, we shall define a differential operator

$$d : \Psi(\mathfrak{D}^2(\mathfrak{X})) \to \Psi(\widetilde{\mathfrak{D}}^3(\mathfrak{X})), \tag{3. 7}$$

which is analogous to the one in §2. For any $\xi \in \Psi(\mathfrak{D}^2(\mathfrak{X}))$, locally written as

$$\xi = U_i d^2u^i + U_{ih} du^i \otimes du^h, \tag{3. 8}$$

we put

$$d\xi = U_i d^3u^i + d^2u^i \otimes dU_i + U_{ih}(d^2u^i \dot{\otimes} du^h + du^i \otimes d^2u^h) + du^i \otimes du^h \otimes dU_{ih}. \tag{3. 9}$$

If we put

$$\xi = V_j d^2v^j + V_{jk} dv^j \otimes dv^k$$

in terms of another local coordinates v^i, then we have

$$U_i = \frac{\partial v^j}{\partial u^i} V_j, \quad U_{ih} = \frac{\partial^2 v^j}{\partial u^h \partial u^i} V_j + \frac{\partial v^j}{\partial u^i}\frac{\partial v^k}{\partial u^h} V_{jk}. \tag{3.10}$$

Putting (3. 10) into the right hand side of (3. 9) and using (2. 2), (2. 3) and (3. 5), we get easily

$$d\xi = V_j d^3 v^j + d^2 v^j \otimes dV_j + V_{jk}(d^2 v^j \dot{\otimes} dv^k + dv^j \otimes d^2 v^k) + dv^j \otimes dv^k \otimes dV_{jk}.$$

Hence, we see that we can naturally define a differential operator d by (3. 9) independent of local coordinates. We can easily prove the following

Proposition 3. 1. *The differential operator d is characterized by the following conditions : For any* $f,\ g \in \mathfrak{A}(\mathfrak{X})$, ξ, ξ_1, $\xi_2 \in \Psi(\mathfrak{D}^2(\mathfrak{X}))$,

$$\begin{aligned} &\text{(a)}\ d(d^2 f) = d^3 f, \\ &\text{(b)}\ d(df \otimes dg) = d^2 f \dot{\otimes} dg + df \otimes d^2 g, \\ &\text{(c)}\ d(\xi_1 + \xi_2) = d\xi_1 + d\xi_2, \\ &\text{(d)}\ d(f\xi) = fd\xi + \xi \otimes df. \end{aligned} \tag{3.11}$$

On the other hand, for any $\xi \in \Psi(\mathfrak{D}^2(\mathfrak{X}))$, we get from (3. 10)

$$\frac{\partial U_i}{\partial u^h} = \frac{\partial^2 v^j}{\partial u^h \partial u^i} V_j + \frac{\partial v^j}{\partial u^i}\frac{\partial v^k}{\partial u^h}\frac{\partial V_j}{\partial v^k}$$

and hence

$$\frac{\partial U_i}{\partial u^h} - U_{ih} = \frac{\partial v^j}{\partial u^i}\frac{\partial v^k}{\partial u^h}\left(\frac{\partial V_j}{\partial v^k} - V_{jk}\right). \tag{3.12}$$

Accordingly, we can define a natural transformation

$$\nu : \Psi(\mathfrak{D}^2(\mathfrak{X})) \to \Psi(T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X})) \tag{3.13}$$

by

$$\nu\xi = \left(\frac{\partial U_i}{\partial u^h} - U_{ih}\right) du^i \otimes du^h. \tag{3.14}$$

Now, (3. 9) is clearly rewritten as

$$d\xi = \{U_i d^3 u^i + U_{ih}(d^2 u^i \otimes du^h + du^i \otimes d^2 u^h + d^2 u^i \dot{\otimes} du^h) + du^i \otimes du^h \otimes dU_{ih}\} + \left(\frac{\partial U_i}{\partial u^h} - U_{ih}\right) d^2 u^i \otimes du^h. \tag{3.15}$$

This equation implies immediately

Proposition 3. 2. *The differential operator d defined by* (2.5) *and* (3. 9) *has the properties such that*

$$d(\Psi(T^*(\mathfrak{X}))) = \text{kernel of } \nu = d^{-1}(\Psi(\mathfrak{D}^3(\mathfrak{X}))). \qquad (3.16).$$

§ 4. The vector bundle $T^*(\mathfrak{X})\overline{\otimes}^m$.

Since the vector bundle $T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X})$ is a vector subbundle of $\mathfrak{D}^2(\mathfrak{X})$, we may put

$$\Psi(T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X})) \subset \Psi(\mathfrak{D}^2(\mathfrak{X})). \qquad (4.1)$$

By virtue of (3. 11), (b), (c), (d), we get

$$d(\Psi(T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X}))) \subset \Psi(\mathfrak{D}^2(\mathfrak{X}) \dot{\otimes} T^*(\mathfrak{X}) + T^*(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})), \qquad (4.2)$$

where the vector bundle $\mathfrak{D}^2(\mathfrak{X}) \dot{\otimes} T^*(\mathfrak{X}) + T^*(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})$ over $\mathfrak{X}$ is well defined as (3. 4) in the last section. In general, extending the isomorphism of $T^*(\mathfrak{X})\otimes^{(m+1)}$ which corresponds to the permutation $\begin{pmatrix} 1\cdots p(p+1) & (p+2)\cdots\cdots\cdots m(m+1) \\ 1\cdots p(p+1) & (p+3)\cdots(m+1)\,(p+2) \end{pmatrix}$ of the $m+1$ components[9] to $T^*(\mathfrak{X})\otimes^p \otimes \mathfrak{D}^2(\mathfrak{X}) \otimes T^*(\mathfrak{X})\otimes^{(m-p-1)}$, we can obtain a vector bundle over $\mathfrak{X}$ which we shall denote by $T^*(\mathfrak{X})\otimes^p \otimes \mathfrak{D}^2(\mathfrak{X}) \dot{\otimes} (T^*(\mathfrak{X})\otimes^{(m-p-1)})$. Then we can define naturally the differential operator

$$d : \Psi(T^*(\mathfrak{X})\otimes^m) \to \Psi(\sum_{p=0}^{m-1}+(T^*(\mathfrak{X})\otimes^p \otimes \mathfrak{D}^2(\mathfrak{X}) \dot{\otimes} (T^*(\mathfrak{X})\otimes^{(m-p-1)})) \quad (4.3)$$

with the following properties, where "$+$" is used in the sense as in (3. 4) : For any $f, f_1, \cdots, f_m \in \mathfrak{A}(\mathfrak{X})$, $\xi, \xi_1, \xi_2 \in \Psi(T^*(\mathfrak{X})\otimes^m)$,

$$\text{(a)} \quad d(df_1\otimes\cdots\otimes df_m) = \sum_{p=0}^{m-1} df_1 \otimes \cdots \otimes df_p \otimes d^2 f_{p+1} \dot{\otimes} (df_{p+2} \otimes\cdots\otimes df_m),$$

$$\text{(b)} \quad d(\xi_1 + \xi_2) = d\xi_1 + d\xi_2, \qquad (4.4)$$

$$\text{(c)} \quad d(f\xi) = fd\xi + \xi \otimes df.$$

Now, by virtue of (2. 2) and (2. 3), we see that at each point $x \in \mathfrak{X}$ all $d(df_1 \otimes \cdots \otimes df_m)$, $df \otimes df_1 \otimes \cdots \otimes df_m$, $f, f_1, \cdots f_m \in \mathfrak{A}(\mathfrak{X})$, span a linear subspace of the fibre of $\sum_{p=0}^{m-1} + (T^*(\mathfrak{X})\otimes^p \otimes \mathfrak{D}^2(\mathfrak{X}) \dot{\otimes} (T^*(\mathfrak{X})\otimes^{(m-p-1)})$ over x, of which

$$d(du^{i_1} \otimes \cdots \otimes du^{i_m}), \; du^{i_1} \otimes \cdots \otimes du^{i_{m+1}} \qquad (4.5)$$

make up a base, where u^i are local coordinates containing x. We shall denote the vector bundle over $\mathfrak{X}$ with this $n^m(n+1)$-dimensional subspace as its fibre over each point x of $\mathfrak{X}$ by $T^*(\mathfrak{X})\overline{\otimes}^{(m+1)}$. $T^*(\mathfrak{X})\otimes^{(m+1)}$ is a vector

---

9) That is $y_1 \otimes \cdots \otimes y_p \otimes y_{p+1} \otimes y_{p+2} \otimes y_{p+3} \otimes \cdots \otimes y_m \otimes y_{m+1} \to y_1 \otimes \cdots \otimes y_p \otimes y_{p+1} \otimes y_{p+3} \otimes y_{p+4} \otimes \cdots \otimes y_{m+1} \otimes y_{p+2}$.

subbundle of $T^*(\mathfrak{X})\overline{\otimes}^{(m+1)}$. We have easily that

$$d(\Psi(T^*(\mathfrak{X})\otimes^m)) \subset \Psi(T^*(\mathfrak{X})\overline{\otimes}^{(m+1)}). \tag{4.6}$$

On the other hand, for any $\xi \in \Psi(T^*(X)\overline{\otimes}^{(m+1)})$, we put

$$\begin{aligned}\xi &= U_{i_1\cdots i_m} d(du^{i_1}\otimes\cdots\otimes du^{i_m}) + U_{i_1\cdots i_{m+1}} du^{i_1}\otimes\cdots\otimes du^{i_{m+1}} \\ &= V_{j_1\cdots j_m} d(dv^{j_1}\otimes\cdots\otimes dv^{j_m}) + V_{j_1\cdots j_{m+1}} dv^{j_1}\otimes\cdots\otimes dv^{j_{m+1}}\end{aligned}$$

in terms of local coordinates u^i and v^j. Making use of (2. 2) and (4. 4), we get

$$\begin{aligned}d(dv^{j_1}\otimes\cdots\otimes dv^{j_m}) &= a^{j_1}_{i_1}\cdots a^{j_m}_{i_m} d(du^{i_1}\otimes\cdots\otimes du^{i_m}) \\ &+ \Big(\sum_{p=0}^{m-1} a^{j_1}_{i_1}\cdots a^{j_p}_{i_p} a^{j_{p+1}}_{i_{p+1}i_{m+1}} a^{j_{p+2}}_{i_{p+2}}\cdots a^{j_{m-1}}_{i_{m-1}} a^{j_m}_{i_m}\Big) du^{i_1}\otimes\cdots\otimes du^{i_{m+1}},\end{aligned} \tag{4.7}$$

where $a^j_i = \dfrac{\partial v^j}{\partial u^i}$, $a^j_{ih} = \dfrac{\partial^2 v^j}{\partial u^h \partial u^i}$. Hence it must be

$$U_{i_1\cdots i_m} = a^{j_1}_{i_1}\cdots a^{j_m}_{i_m} V_{j_1\cdots j_m}, \tag{4.8}$$

$$\begin{aligned}U_{i_1\cdots i_{m+1}} &= \Big(\sum_{p=0}^{m-1} a^{j_1}_{i_1}\cdots a^{j_{p+1}}_{i_{p+1}i_{m+1}} a^{j_{p+2}}_{i_{p+2}}\cdots a^{j_{m-1}}_{i_{m-1}} a^{j_m}_{i_m}\Big) V_{j_1\cdots j_m} \\ &+ a^{j_1}_{i_1}\cdots a^{j_{m+1}}_{i_{m+1}} V_{j_1\cdots j_{m+1}}.\end{aligned} \tag{4.9}$$

From (4. 8) and (4. 9), we obtain

$$\frac{\partial U_{i_1\cdots i_m}}{\partial u^{i_{m+1}}} - U_{i_1\cdots i_{m+1}} = \frac{\partial v^{j_1}}{\partial u^{i_1}}\cdots\frac{\partial v^{j_{m+1}}}{\partial u^{i_{m+1}}}\left(\frac{\partial V_{j_1\cdots j_m}}{\partial v^{j_{m+1}}} - V_{j_1\cdots j_{m+1}}\right). \tag{4.10}$$

Accordingly, we can define generally the natural transformation

$$\nu : \Psi(T^*(\mathfrak{X})\overline{\otimes}^{(m+1)}) \to \Psi(T^*(\mathfrak{X})\otimes^{(m+1)}) \tag{4.11}$$

by

$$\nu\xi = \left(\frac{\partial U_{i_1\cdots i_m}}{\partial u^{i_{m+1}}} - U_{i_1\cdots i_{m+1}}\right) du^{i_1}\otimes\cdots\otimes du^{i_{m+1}}, \tag{4.12}$$

but we must read $T^*(\mathfrak{X})\overline{\otimes}^{(m+1)}$ as $\mathfrak{D}^2(\mathfrak{X})$ when $m = 1$. Now, by means of (4. 8), we get a natural transformation

$$\tau : \Psi(T^*(\mathfrak{X})\overline{\otimes}^{(m+1)}) \to \Psi(T^*(\mathfrak{X})\otimes^m) \tag{4.13}$$

by

$$\tau\xi = U_{i_1\cdots i_m} du^{i_1}\otimes\cdots\otimes du^{i_m}. \tag{4.14}$$

On the other hand, for any $\xi \in \Psi(T^*(\mathfrak{X})\otimes^m)$, $\xi = U_{i_1\cdots i_p} du^{i_1}\otimes\cdots\otimes du^{i_m}$, we have

$$d\xi = U_{i_1\cdots i_m} d(du^{i_1} \otimes \cdots \otimes du^{i_m}) + \\ + \frac{\partial U_{i_1\cdots i_m}}{\partial u^{i_{m+1}}} du^{i_1} \otimes \cdots \otimes du^{i_{m+1}}. \tag{4.15}$$

Hence, it must be

$$\nu = d \cdot \tau - 1. \tag{4.16}$$

Theorem. 4.1. *For the differential operator d and the transformation ν, we get the following sequence*

$$\Psi(T^*(\mathfrak{X})) \xrightarrow{d} \Psi(\mathfrak{D}^2(\mathfrak{X})) \xrightarrow{\nu} \Psi(T^*(\mathfrak{X})\otimes^2) \xrightarrow{d} \Psi(T^*(\mathfrak{X})\overline{\otimes}^3) \xrightarrow{\nu} \cdots \\ \xrightarrow{\nu} \Psi(T^*(\mathfrak{X})\otimes^m) \xrightarrow{d} \Psi(T^*(\mathfrak{X})\overline{\otimes}^{(m+1)}) \xrightarrow{\nu} \Psi(T^*(\mathfrak{X})\otimes^{(m+1)}) \rightarrow \cdots \tag{4.17}$$

such that d is one-one and

$$\textit{image of } d = \textit{kernel of } \nu. \tag{4.18}$$

§5. The vector bundle $T(\mathfrak{X})\overline{\otimes}^{(p,q+1)}$.

Furthermore, we shall define the differential operator d for the vector bundles $T(\mathfrak{X})$ and $\mathfrak{T}^2(\mathfrak{X})$ such as

$$d : \begin{cases} \Psi(T(\mathfrak{X})) \rightarrow \Psi(\mathfrak{T}^2(\mathfrak{X}) \otimes T^*(\mathfrak{X})), & (5.1) \\ \Psi(\mathfrak{T}^2(\mathfrak{X})) \rightarrow \Psi(\mathfrak{T}^3(\mathfrak{X}) \otimes T^*(\mathfrak{X})). & (5.2) \end{cases}$$

For any $\xi \in \Psi(\mathfrak{T}^2(\mathfrak{X}))$, locally written as

$$\xi = U^i \partial u_i + U^{ih} \partial^2 u_{ih} \tag{5.3}$$

with respect to local coordinates u^i, we put

$$d\xi = \partial u_i \otimes dU^i + \partial^2 u_{ih} \otimes (U^i du^h + dU^{ih}) + \\ + \partial^3 u_{ihl} \otimes U^{ih} du^l. \tag{5.4}$$

Putting

$$\xi = V^j \partial v_j + V^{jk} \partial^2 v_{jk}$$

in terms of another local coordinates v^j, then we have from (3.1) the equations:

$$V^j = \frac{\partial v^j}{\partial u^i} U^i + \frac{\partial^2 v^j}{\partial u^h \partial u^i} U^{ih}, \quad V^{jk} = \frac{\partial v^j}{\partial u^i} \frac{\partial v^k}{\partial u^h} U^{ih}. \tag{5.5}$$

Substituting (3.1) and (5.5) into (5.4), we get easily

$$d\xi = \partial v_j \otimes dV^j + \partial^2 v_{jk} \otimes (V^j dv^k + dV^{jk}) + \partial^3 v_{jkm} \otimes V^{jk} dv^m.$$

Hence, we see that (5.4) is independent of the choice of local coordinates. The differential operator defined by (5.4) has the properties (5.1) and (5.2).

Now, in general, for a vector bundle over $\mathfrak{X}$, for instance,

$$T(\mathfrak{X})^{\otimes(p,q)} = T(\mathfrak{X})^{\otimes p} \otimes T^*(\mathfrak{X})^{\otimes q},$$

we can define naturally the operator

$$d: \Psi(T(\mathfrak{X})^{\otimes(p,q)}) \to \Psi((\sum_{s=0}^{p-1} + T(\mathfrak{X})^{\otimes s} \otimes \mathfrak{T}^2(\mathfrak{X}) \otimes T(\mathfrak{X})^{\otimes(p-s-1)}) \otimes T^*(\mathfrak{X})^{\otimes(q+1)}$$
$$+ T(\mathfrak{X})^{\otimes p} \otimes \sum_{t=0}^{q-1} + T^*(\mathfrak{X})^{\otimes t} \otimes \mathfrak{D}^2(\mathfrak{X}) \dot{\otimes} (T^*(\mathfrak{X})^{\otimes(q-t-1)}), \tag{5.6}$$

where the summation in the bracket in the right hand side must be read in the sense as in (3.4). For any $\xi \in \Psi(T(\mathfrak{X})^{\otimes(p,q)})$,

$$\xi = U^{i_1\cdots i_p}_{j_1\cdots j_q} \partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q},$$

$d\xi$ is in fact given by

$$d\xi = U^{i_1\cdots i_p}_{j_1\cdots j_q}\, d(\partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q})$$
$$+ \partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q} \otimes dU^{i_1\cdots i_p}_{j_1\cdots j_q}, \tag{5.7}$$

$$d(\partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q})$$
$$= \sum_{s=0}^{p-1} \partial u_{i_1} \otimes \cdots \otimes \partial u_{i_s} \otimes \partial^2 u_{i_{s+1}h} \otimes \partial u_{i_{s+2}} \otimes \cdots \otimes \partial u_{i_p}$$
$$\otimes du^{j_1} \otimes \cdots \otimes du^{j_q} \otimes du^h$$
$$+ \partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes d(du^{i_1} \otimes \cdots \otimes du^{i_q}). \tag{5.8}$$

By means of (2.2) and (3.1), we see easily that at each point $x \in \mathfrak{X}$

$$d(\partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{i_1} \otimes \cdots \otimes du^{j_q}),$$
$$\partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q} \otimes du^{j_{q+1}} \tag{5.9}$$

span a subspace of the fibre of the vector bundle in the right hand side of (5.6) over x which is independent of the choice of local coordinates. We shall denote the vector bundle over $\mathfrak{X}$ with this $n^{p+q}(n+1)$-dimensional subspace as the fibre over each point x of $\mathfrak{X}$ by $T(\mathfrak{X})^{\bar{\otimes}(p,q+1)}$. $T(\mathfrak{X})^{\otimes(p,q+1)}$ is a vector subbundle of this vector bundle over $\mathfrak{X}$. Then we have

Proposition 5.1. $d(\Psi(T(\mathfrak{X})^{\otimes(p,q)}) \subset \Psi(T(\mathfrak{X})^{\bar{\otimes}(p,q+1)})$ (5.10)

§6. Connections of $T(\mathfrak{X})$ and $T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})$.

We shall consider any cross-section Γ of the vector bundle $T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})$. In a coordinate neighborhood (U, u^i), let Γ be written as

$$\Gamma = \partial u_j \otimes (P^j_i\, d^2u^i + \Gamma^j_{ih}\, du^i \otimes du^h). \tag{6.1}$$

If we write this cross-section as

$$\Gamma = \partial v_j \otimes (Q_i^j \, d^2 v^i + \Delta_{ih}^j \, dv^i \otimes dv^h)$$

in another coordinate neighborhood (V, v^i), we get easily the relations

$$P_i^h = \frac{\partial u^h}{\partial v^k} Q_j^k \frac{\partial v^j}{\partial u^i}, \tag{6. 2}$$

$$\Gamma_{ih}^l = \frac{\partial u^l}{\partial v^m} \left(Q_j^m \frac{\partial^2 v^j}{\partial u^h \partial u^i} + \Delta_{jk}^m \frac{\partial v^j}{\partial u^i} \frac{\partial u^k}{\partial v^h} \right) \tag{6. 3}$$

on $U \cap V \neq \emptyset$, by means of (2. 2). Making use of (1. 1), (1. 2) and (1. 12), these equations are written simply as[10]

$$(\sigma \cdot g_{VU}) (P_i^j, \Gamma_{ih}^j) = (Q_i^j, \Delta_{ih}^j) g_{VU}, \tag{6. 4}$$

where σ is the natural homomorphism $\mathfrak{L}_n^2 \to L_n^1$. By virtue of (6. 1), we can define a natural transformation

$$\lambda : \Psi(T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})) \to \Psi(T(\mathfrak{X}) \otimes T^*(\mathfrak{X})) \tag{6. 5}$$

by

$$\lambda(\Gamma) = P_i^j \, \partial u_j \otimes du^i, \tag{6. 6}$$

which is analogous to τ defined by (4. 13). As is well known, any cross-section of $T(\mathfrak{X}) \otimes T^*(\mathfrak{X})$ may be considered as a homomorphism of $T(\mathfrak{X})$ covering the identity transformation of $\mathfrak{X}$. Accordingly, from (6. 3), we can easily obtain the following

Theorem 6. 1. *A classical connection of the tangent bundle $T(\mathfrak{X})$ of $\mathfrak{X}$ corresponds to a cross-section of $T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})$ such that its image under λ is the identity isomorphism of $T(\mathfrak{X})$*[11].

Now, for any $\Gamma \in \Psi(T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X}))$, we can define a homomorphism

$$\mu = \mu_\Gamma : \mathfrak{T}^2(\mathfrak{X}) \to T(\mathfrak{X}), \tag{6. 7}$$

which covers the identity transformation of the base space $\mathfrak{X}$, by the inner product

$$\mu(X) = \langle \Gamma, X \rangle, \quad X \in \mathfrak{T}^2(\mathfrak{X}), \tag{6. 8}$$

that is

$$\mu(\partial u_i) = P_i^j \partial u_j, \quad \mu(\partial^2 u_{ih}) = \Gamma_{ih}^j \partial u_j, \tag{6. 9}$$

---

10) Here, we consider the set of all (a_i^j, a_{ih}^j) as an algebraic set with the multiplication (1. 1) and (1. 2) satisfying the associative law. And so the transformation σ may be naturally extended on this set.

11) A classical connection of the tangent bundle $T(\mathfrak{X})$ is usually said an affine connection of $\mathfrak{X}$. But, the author will use this term as a pair (Γ, ψ) of a connection Γ of $T(\mathfrak{X})$ and a homomorphism ψ of $T(\mathfrak{X})$ covering the identity mapping of $\mathfrak{X}$. When ψ is the identity transformation, this becomes the one in the classical sense.

where $\partial u_i = \partial/\partial u^i$, $\partial^2 u_{ih}$ are the vector fields associated with local coordinates u^i, because $\{\partial u_i, \partial^2 u_{ih}\}$ and $\{d^2u^i, du^i \otimes du^h\}$ are dual to each other.

On the other hand, since $T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})$ is a vector subbundle of $\mathfrak{T}^2(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})$, we may regard Γ as a cross-section of $\mathfrak{T}^2(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})$. Hence we can analogously define a homomorphism of $\mathfrak{D}^2(\mathfrak{X})$ into itself by $\langle \omega, \Gamma \rangle$, $\omega \in \mathfrak{D}^2(\mathfrak{X})$, that is

$$\begin{cases} \langle d^2u^j, \Gamma \rangle = P_i^j d^2u^i + \Gamma_i{}^j{}_h du^i \otimes du^h, \\ \langle du^j \otimes du^k, \Gamma \rangle = 0. \end{cases} \tag{6.10}$$

Since $\lambda(\Gamma) \in \Psi(T(\mathfrak{X})\otimes^{(1,1)})$, it follows that $d(\lambda(\Gamma)) \in \Psi(T(\mathfrak{X})\overline{\otimes}^{(1,2)})$. According to §5, $T(\mathfrak{X})\overline{\otimes}^{(1,2)}$ is a vector subbundle of $\mathfrak{T}^2(\mathfrak{X}) \otimes T^*(\mathfrak{X})\otimes^2 + T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})$ and the latter is also a vector subbundle $\mathfrak{T}^2(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})$. Accordingly, we may regard as $\Psi(T(\mathfrak{X})\overline{\otimes}^{(1,2)}) \subset \Psi(\mathfrak{T}^2(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X}))$. From (6.6), (5.8), we have

$$d(\lambda(\Gamma)) = P_i^j(\partial^2 u_{jh} \otimes du^i \otimes du^h + \partial u_j \otimes d^2u^i)- + \frac{\partial P_i^j}{\partial u^h} \partial u_j \otimes du^i \otimes du^h. \tag{6.11}$$

Then, we obtain easily a homomorphism of $\mathfrak{D}^2(\mathfrak{X})$ into itself by the equation,

$\langle \omega, d(\lambda(\Gamma)) \rangle$, $\omega \in \mathfrak{D}^2(\mathfrak{X})$, that is

$$\begin{cases} \langle d^2u^j, d(\lambda(\Gamma)) \rangle = P_i^j d^2u^i + \dfrac{\partial P_i^j}{\partial u^h} du^i \otimes du^h, \\ \langle du^j \otimes du^h, d(\lambda(\Gamma)) \rangle = P_i^j du^i \otimes du^h. \end{cases} \tag{6.12}$$

From (6.10) and (6.12), we have the following

Lemma 6.2. *The transformation $\varphi = \varphi_\Gamma$ defined by*

$$\varphi(\omega) = \langle \omega, d(\lambda(\Gamma)) - \Gamma \rangle, \quad \omega \in \mathfrak{D}^2(\mathfrak{X}), \tag{6.13}$$

is a homomorphism of $\mathfrak{D}^2(\mathfrak{X})$ into $T^*(\mathfrak{X}) \otimes T^*(\mathfrak{X})$ which covers the identity transformation of the base space $\mathfrak{X}$. For the base d^2u^j, $du^j \otimes du^h$ of a fibre of $\mathfrak{D}^2(\mathfrak{X})$ associated with local coordinates u^i, we have

$$\begin{cases} \varphi(d^2u^j) = \left(\dfrac{\partial P_i^j}{\partial u^h} - \Gamma_i{}^j{}_h\right) du^i \otimes du^h \\ \varphi(du^j \otimes du^h) = P_i^j du^i \otimes du^h. \end{cases} \tag{6.14}$$

Now, in each fibre $\mathfrak{D}_x^2(\mathfrak{X})$, we can naturally extend the transformation ε corresponding to the permutation of components of $T_x^*(\mathfrak{X}) \otimes T_x^*(\mathfrak{X})$ to $\mathfrak{D}_x^2(\mathfrak{X})$ by

$$\varepsilon(du^i \otimes du^h) = du^h \otimes du^i, \quad \varepsilon(d^2u^j) = d^2u^j, \tag{6.15}$$

which is independent of local coordinates as is easily seen from (2. 2) and (2. 3). ε is a bundle map of $\mathfrak{D}^2(\mathfrak{X})$.

Theorem 6. 3. *The homomorphism* $\tilde{\mu} = \varepsilon\varphi\,\varepsilon\varphi : \mathfrak{D}^2(\mathfrak{X}) \to T^*(\mathfrak{X})^{\otimes 2}$ *is a natural extension to* $\mathfrak{D}^2(\mathfrak{X})$ *of the homomorphism* $\mu \otimes \mu$ *of the vector subbundle* $T^*(\mathfrak{X})^{\otimes 2}$ *of the vector bundle* $\mathfrak{D}^2(\mathfrak{X})$, *where* μ *is the homomorphism of* $T^*(\mathfrak{X})$ *on itself given by*

$$\mu(\omega) = \langle \omega, \lambda(\Gamma) \rangle \tag{6. 16}$$

and it is the dual of the restricted homomorphism (6. 7) *on* $T(\mathfrak{X})$.

Proof. Using local coordinates u^i, we get from (6. 6)

$$\mu(du^j) = P_i^j\, du^i, \tag{6. 17}$$

$$(\mu \otimes \mu)\,(du^j \otimes du^k) = P_i^j\, P_h^k\, du^i \otimes du^h. \tag{6. 18}$$

Moreover from (6. 14), we get

$$\varepsilon\varphi(d^2 u^j) = \left(\frac{\partial P_i^j}{\partial u^h} - \Gamma_i{}^j{}_h\right) du^h \otimes du^i,$$

$$\varepsilon\varphi(du^j \otimes du^k) = P_i^j\, du^k \otimes du^i$$

and

$$\tilde{\mu}(d^2 u_j) = \left(\frac{\partial P_i^j}{\partial u^k} - \Gamma_i{}^j{}_k\right) \varepsilon\varphi(du^k \otimes du^i)$$

$$= \left(\frac{\partial P_i^j}{\partial u^k} - \Gamma_i{}^j{}_k\right) P_h^k\, du^i \otimes du^h,$$

$$\tilde{\mu}(du^j \otimes du^k) = P_i^j\, \varepsilon\varphi(du^k \otimes du^i) = P_i^j P_h^k\, du^i \otimes du^h.$$

The proof is finished.

We shall denote $\tilde{\mu}$ simply by μ in the following. Then the last equations are written as useful formulas :

$$\mu(d^2 u^j) = \left(\frac{\partial P_i^j}{\partial u^k} - \Gamma_i{}^j{}_k\right) P_h^k\, du^i \otimes du^h, \tag{6. 19}$$

$$\mu(du^j \otimes du^k) = P_i^j\, P_h^k\, du^i \otimes du^h. \tag{6. 20}$$

We will call $\mu = \mu_\Gamma$ *the homomorphism iuduced from* Γ. According to Theorem 6. 3 and (6. 9), the homomorphism μ defined on $\mathfrak{T}^2(\mathfrak{X})$, $T(\mathfrak{X})$, $T^*(\mathfrak{X})$ and $\mathfrak{D}^2(\mathfrak{X})$ may be naturally extended to any tensor product bundle of these vector bundles over $\mathfrak{X}$. We shall denote these homomorphisms also by the same symbol μ. For instance, $\mu : \mathfrak{T}^2(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X}) \to T(\mathfrak{X})^{\otimes(1,2)}$ is given by

$$
\begin{aligned}
\mu(\partial^2 u_{ih} \otimes d^2 u^j) &= \Gamma_{ih}^m \left(\frac{\partial P_t^j}{\partial u^l} - \Gamma_t{}^j{}_l\right) P_s^l \, \partial u_m \otimes du^t \otimes du^s, \\
\mu(\partial u_i \otimes d^2 u^j) &= P_i^m \left(\frac{\partial P_t^j}{\partial u^l} - \Gamma_t{}^j{}_l\right) P_s^l \, \partial u_m \otimes du^t \otimes du^s, \\
\mu(\partial^2 u_{ih} \otimes du^j \otimes du^k) &= \Gamma_{ih}^m P_t^j P_s^k \, \partial u_m \otimes du^t \otimes du^s, \\
\mu(\partial u_i \otimes du^j \otimes du^k) &= P_i^m P_t^j P_s^k \, \partial u_m \otimes du^t \otimes du^s.
\end{aligned}
\tag{6.21}
$$

According to the above-mentioned circumstances, we will then call any cross-section Γ of $T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X})$ *a general connection of the tangent bundle* $T(\mathfrak{X})$ of $\mathfrak{X}$ and say that Γ is *of the type* P when $\lambda(\Gamma) = P \in \Psi(T(\mathfrak{X}) \otimes T^*(\mathfrak{X}))$, or a *P-connection of the tangent bundle* $T(\mathfrak{X})$ of $\mathfrak{X}$.

Remark. In the classical theory of differential geometry, as is well known, the suit of the components $\Gamma_i{}^j{}_h$ of an affine connection is one of the most handy and important geometric objects which are not geometric quantities,[12] that is, not a tensor of type (1, 2). In fact, the components of a connection in the classical sense resemble the ones of a tensor of type (1, 2) only in the point that they are of contravariant order 1 and covariant order 2, but they belong to entirely different notions. But in our generalized sense, they becomes relatives with each other, that is, the components $\Gamma_i{}^j{}_h$ of an affine connection of $\mathfrak{X}$ are the second components of an 1-connection and the components $T_i{}^j{}_h$ of a tensor of type (1, 2) are the ones of a 0-connection by means of (6. 3), where 1 and 0 denote the identity isomorphism and the zero homomorphism of $T(\mathfrak{X})$ respectively. We shall give two interesting general connections which are not of the above-mentioned types.

Example 1. When $P = -1$, that is $-\Gamma$ is classical, (6. 3) becomes

$$
\Gamma_i{}^l{}_h = \frac{\partial u^l}{\partial v^m}\left(A_{jk}^m \frac{\partial v^j}{\partial u^i}\frac{\partial v^k}{\partial u^h} - \frac{\partial^2 v^m}{\partial u^h \partial u^i}\right)
\tag{6.22}
$$

and (6. 9), (6. 17) and (6. 19) become respectively

$$
\begin{aligned}
\mu(\partial u_i) &= -\partial u_i, & \mu(\partial^2 u_{ih}) &= \Gamma_i{}^j{}_h \, \partial u_j, \\
\mu(du^j) &= -du^j, & \mu(d^2 u^j) &= \Gamma_i{}^j{}_h \, du^i \otimes du^h.
\end{aligned}
\tag{6.23}
$$

Therefore $\mu^2 = \mu \cdot \mu$ is given by

$$
\begin{aligned}
\mu^2(\partial u_i) &= \partial u_i, & \mu^2(\partial^2 u_{ih}) &= -\Gamma_i{}^j{}_h \, \partial u_j, \\
\mu^2(du^j) &= du^j, & \mu^2(d^2 u^j) &= \Gamma_i{}^j{}_h \, du^i \otimes du^h,
\end{aligned}
$$

and these equations show that

$$
(\mu_\Gamma)^2 = \mu_{-\Gamma}. \tag{6.24}
$$

---

12) See [8], p. 68.

Example 2. Let $\mathfrak{X}$ be a complex analytic manifold of complex dimension n. Let the local coordinates u^{λ}, $\lambda = 1, 2, \cdots 2n$, be the n real parts and the n imaginary parts of local complex coordinates $z^j = u^j + \sqrt{-1}\, u^{n+j}$. Let P be the operator deterimined by the complex analytic structure of $\mathfrak{X}$, that is

$$P = I = \begin{pmatrix} 0 & -\delta^j_h \\ \delta^k_i & 0 \end{pmatrix}. \tag{6.25}$$

Then (6.9), (6.17) and (6.19) are written as

$$\mu(\partial u_i) = \partial u_{n+i}, \quad \mu(\partial u_{n+i}) = -\partial u_i, \quad \mu(\partial^2 u_{\lambda\rho}) = \Gamma^{\nu}_{\lambda\rho}\, \partial u_{\nu},$$
$$\mu(du^j) = -du^{n+j}, \quad \mu(du^{n+j}) = du^j,$$
$$\mu(d^2 u^{\nu}) = \Gamma_{\lambda}{}^{\nu}{}_{h}\, du^{\lambda} \otimes du^{n+h} - \Gamma_{\lambda}{}^{\nu}{}_{n+h}\, du^{\lambda} \otimes du^{h},$$

hence we have

$$\mu^2(\partial u_{\lambda}) = -\partial u_{\lambda}, \quad \mu^2(\partial^2 u_{\lambda\rho}) = \Gamma_{\lambda}{}^{j}{}_{\rho}\, \partial u_{n+j} - \Gamma^{n+j}_{\lambda\rho}\, \partial u_j,$$
$$\mu^2(du^{\nu}) = -du^{\nu},$$
$$\mu^2(d^2 u^{\nu}) = -\Gamma_{i}{}^{\nu}{}_{h}\, du^{n+i} \otimes du^{h} + \Gamma^{\nu}_{n+i,h}\, du^{i} \otimes du^{h} -$$
$$- \Gamma^{\nu}_{i,n+h}\, du^{n+i} \otimes du^{n+h} + \Gamma^{\nu}_{n+i,n+h}\, du^{i} \otimes du^{n+h}$$
$$= -\Gamma^{\nu}_{i\rho}\, du^{n+i} \otimes du^{\rho} + \Gamma^{\nu}_{n+i,\rho}\, du^{i} \otimes du^{\rho}$$

and

$$\mu^4(\partial u_{\lambda}) = \partial u_{\lambda}, \quad \mu^4(du^{\nu}) = du^{\nu},$$
$$\mu^4(\partial^2 u_{\lambda\rho}) = -\Gamma_{\lambda}{}^{j}{}_{\rho}\, \partial u_{n+j} + \Gamma^{n+j}_{\lambda\rho}\, \partial u_j,$$
$$\mu^4(d^2 u^{\nu}) = -\Gamma_{i}{}^{\nu}{}_{\rho}\, du^{n+i} \otimes du^{\rho} + \Gamma^{\nu}_{n+i,\rho}\, du^{i} \otimes du^{\rho}.$$

§7. Covariant differentiations and regular general connections.

Now, for any general connection $\Gamma \in \Psi(T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X}))$, we define a covariant differentiation $D = D_{\Gamma}$ by

$$D = \mu \cdot d : \Psi(T(\mathfrak{X})\otimes^{(p,q)}) \to \Psi(T(\mathfrak{X})\otimes^{(p,q+1)}), \tag{7.1}$$

where d and $\mu = \mu_{\Gamma}$ are the transformations in §5 and §6 respectively. Since μ is given by

$$\begin{aligned} \mu(\partial u_i) &= P^j_i\, \partial u_j, \quad \mu(\partial^2 u_{ih}) = \Gamma_{i}{}^{j}{}_{h}\, \partial u_j, \\ \mu(du^j) &= P^j_i\, du^i, \quad \mu(d^2 u^j) = -A_{i}{}^{j}{}_{k}\, P^k_h\, du^i \otimes du^h, \end{aligned} \tag{7.2}$$

where we put

$$A_{i}{}^{j}{}_{k} = \Gamma_{i}{}^{j}{}_{k} - \frac{\partial P^j_i}{\partial u^k}, \tag{7.3}$$

for any $\xi \in \Psi(T(\mathfrak{X})\otimes^{(p,q)})$ written locally as

$$\xi = U^{i_1 \cdots i_p}_{j_1 \cdots j_q}\, \partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q},$$

we get from (5. 7), (5. 8) and (4. 4)

$$D\xi = \frac{\partial U^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial u^l}\mu(\partial u_{i_1}\otimes\cdots\otimes\partial u_{i_p}\otimes du^{j_1}\otimes\cdots\otimes du^{j_q}\otimes du^l)$$
$$+U^{i_1\cdots i_p}_{j_1\cdots j_q}\mu\Big\{\sum_{s=0}^{p-1}\partial u_{i_1}\otimes\cdots\otimes\partial u_{i_s}\otimes\partial^2 u_{i_{s+1},l}\otimes\cdots\otimes\partial u_{i_p}$$
$$\otimes du^{j_1}\otimes\cdots\otimes du^{j_q}\otimes du^l$$
$$+\sum_{t=0}^{q-1}\partial u_{i_1}\otimes\cdots\otimes\partial u_{i_p}$$
$$\otimes du^{j_1}\otimes\cdots\otimes du^{j_t}\otimes d^2u^{j_{t+1}}\dot{\otimes}(du^{j_{t+2}}\otimes\cdots\otimes du^{j_q})\Big\}$$
$$=\Big[\frac{\partial U^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial u^l}P^{h_1}_{i_1}\cdots P^{h_p}_{i_p}P^{j_1}_{k_1}\cdots P^{j_q}_{k_q}P^l_m$$
$$+U^{i_1\cdots i_p}_{j_1\cdots j_q}\Big\{\sum_{s=0}^{p-1}P^{h_1}_{i_1}\cdots P^{h_s}_{i_s}\Gamma^{h_{s+1}}_{i_{s+1},l}P^{h_{s+2}}_{i_{s+2}}\cdots P^{h_p}_{i_p}P^{j_1}_{k_1}\cdots P^{j_q}_{k_q}P^l_m$$
$$-\sum_{t=0}^{q-1}P^{h_1}_{i_1}\cdots P^{h_p}_{i_p}P^{j_1}_{k_1}\cdots P^{j_t}_{k_t}\Lambda^{j_{t+1}}_{k_{t+1}l}P^{k_{t+2}}_{j_{t+2}}\cdots P^{j_q}_{k_q}P^l_m\Big\}\Big]\times$$
$$\times\partial u_{h_1}\otimes\cdots\otimes\partial u_{h_p}\otimes du^{k_1}\otimes\cdots\otimes du^{k_q}\otimes du^m,$$

that is

$$D\xi = U^{h_1\cdots h_p}_{k_1\cdots k_q,m}\partial u_{h_1}\otimes\cdots\otimes\partial u_{h_p}\otimes du^{k_1}\otimes\cdots\otimes du^{k_q}\otimes du^m,$$
$$U^{h_1\cdots h_p}_{k_1\cdots k_q,m} = P^{h_1}_{i_1}\cdots P^{h_p}_{i_p}\frac{\partial U^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial u^l}P^{j_1}_{k_1}\cdots P^{j_t}_{k_q}P^l_m$$
$$+\sum_{s=0}^{p-1}P^{h_1}_{i_1}\cdots P^{h_s}_{i_s}\Gamma^{h_{s+1}}_{i_{s+1}l}P^{h_{s+2}}_{i_{s+2}}\cdots P^{h_p}_{i_p}U^{i_1\cdots i_p}_{j_1\cdots j_q}P^{j_1}_{k_1}\cdots P^{j_q}_{k_q}P^l_m$$
$$-\sum_{t=0}^{q-1}P^{h_1}_{i_1}\cdots P^{h_p}_{i_p}U^{i_1\cdots i_p}_{j_1\cdots j_q}P^{j_1}_{k_1}\cdots P^{j_t}_{k_t}\Lambda^{j_{t+1}}_{k_{t+1}l}P^{j_{t+2}}_{k_{t+2}}\cdots P^{j_q}_{k_q}P^l_m. \tag{7.4}$$

Now let us consider only connections Γ such that $\lambda(\Gamma)$ is an isomorphism of $T(\mathfrak{X})$ and call them *to be regular*. Then $(P^j_i,\ \Gamma_i{}^j{}_h)$ belongs to $\mathfrak{L}^2_n$. Accordingly, we get from (1. 9), (1. 4) and (6. 4)

$$\eta((\sigma\cdot g_{VU})(P^j_i,\ \Gamma_i{}^j{}_h)) = \eta(P^j_i,\ \Gamma_i{}^j{}_h),$$
$$\eta((Q^j_i,\ \Lambda_i{}^j{}_h)\,g_{VU}) = (\sigma\cdot g_{UV})\,\eta(Q^j_i,\ \Lambda_i{}^j{}_h)\,g_{VU},$$

hence

$$(\sigma\cdot g_{VU})\,\eta(P^j_i,\ \Gamma_i{}^j{}_h) = \eta(Q^j_i,\ \Lambda_i{}^j{}_h)\,g_{VU}.$$

This equation shows that $\eta(P^j_i,\ \Gamma_i{}^j{}_h)$ is associated with an 1-connection $'\Gamma$ in local coordinates u^i. We will call this 1-connection $'\Gamma$ *the contravariant part of* Γ. Putting anew

$$(\lambda(\Gamma))^{-1} = Q^j_i\,\partial u_j\otimes du^i, \tag{7.5}$$
$$'\Gamma = \partial u_j\otimes(d^2u^j + {}'\Gamma_i{}^j{}_h\,du^i\otimes du^h) \tag{7.6}$$

and using (1.8), we get

$$'\Gamma_{i\ h}^{\ j} = Q_k^j \Gamma_{i\ h}^{\ k} \quad \text{or} \quad \Gamma_{i\ h}^{\ j} = P_k^j\, '\Gamma_{i\ h}^{\ k}. \tag{7.7}$$

On the other hand, we can consider the induced connection[13] $''\Gamma$ from the classical connection $'\Gamma$ by the isomorphism $(\lambda(\Gamma))^{-1}$ of the tangent bundle $T(\mathfrak{X})$ covering the identity mapping of $\mathfrak{X}$. We will call this 1-connection $''\Gamma$ *the covariant part of* Γ. Putting

$$''\Gamma = \partial u_j \otimes (d^2u^j + ''\Gamma_{i\ h}^{\ j}\, du^i \otimes du^h), \tag{7.8}$$

we have

$$''\Gamma_{i\ h}^{\ j} = P_k^j\left(\frac{\partial Q_i^k}{\partial u^h} + '\Gamma_{l\ h}^{\ k}\, Q_i^l\right)$$

$$= \left(\Gamma_{l\ h}^{\ j} - \frac{\partial P_l^j}{\partial u^h}\right) Q_i^l,$$

or

$$\Gamma_{i\ h}^{\ j} = ''\Gamma_{l\ h}^{\ j}\, P_i^l + \frac{\partial P_i^j}{\partial u^h}. \tag{7.9}$$

Using $\mu' = \mu_{'\Gamma}$ and $\mu'' = \mu_{''\Gamma}$, (7.2) can be written as

$$\begin{aligned}
\mu(\partial u_i) &= \mu\,\mu'(\partial u_i) = \mu\,\mu''(\partial u_i),\\
\mu(\partial^2 u_{ih}) &= P_k^j\, '\Gamma_{i\ h}^{\ k}\, \partial u_j = \mu('\Gamma_{i\ h}^{\ k}\, \partial u_k) = \mu\,\mu'(\partial^2 u_{ih})\\
&= \left(''\Gamma_{l\ h}^{\ j}\, P_i^l + \frac{\partial P_i^j}{\partial u^h}\right) \partial u_j,\\
\mu(du^j) &= \mu\,\mu'(du^j) = \mu\,\mu''(du^j)\\
\mu(d^2u^j) &= -''\Gamma_{l\ k}^{\ j}\, P_i^l\, P_h^k\, du^i \otimes du^h = \mu\,\mu''(d^2u^j)\\
&= -\left(P_l^j\, '\Gamma_{i\ k}^{\ l} - \frac{\partial P_i^j}{\partial u^k}\right) P_h^k\, du^i \otimes du^h.
\end{aligned} \tag{7.10}$$

We will call the restriction of μ_Γ on the tensor product bundles of $T(\mathfrak{X})$ and $T^*(\mathfrak{X})$ *the induced homomorphism of order* 1 *from the general connection* Γ which we denote by $\bar{\lambda} = \bar{\lambda}_\Gamma$. Then we have obtain the following

Theorem 7.1. *For any regular general connection $\Gamma \in \Psi(T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X}))$, the induced homomorphism μ_Γ of Γ is the product of $\bar{\lambda}_\Gamma$ and the homomorphism $\bar{\mu}_\Gamma$ which is the identity mapping on $T(\mathfrak{X})$ and $T^*(\mathfrak{X})$, $\mu_{'\Gamma}$ on $\mathfrak{T}^2(\mathfrak{X})$ and $\mu_{''\Gamma}$ on $\mathfrak{D}^2(\mathfrak{X})$, that is*

$$\mu_\Gamma = \bar{\lambda}_\Gamma \cdot \bar{\mu}_\Gamma. \tag{7.11}$$

We will call this homomorphism $\bar{\mu}_\Gamma$ *the basic homomorphism of the*

13) See [2], §1 and §23.

regular general cennection Γ. Then we can define a covariant differentiation by

$$\bar{D} = \bar{D}_\Gamma = \bar{\mu}_\Gamma \cdot d, \tag{7.12}$$

which we will call *the basic covariant differentiation of the regular connection Γ.* We get easily

$$D = \bar{\lambda} \cdot \bar{D}. \tag{7.13}$$

For any $\xi \in \Psi(T(\mathfrak{X})^{\otimes(p,q)})$, $\xi = U^{i_1\cdots i_p}_{j_1\cdots j_q}\, \partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q}$, we have

$$\bar{D}\xi = U^{i_1\cdots i_p}_{j_1\cdots j_q|l}\, \partial u_{i_1} \otimes \cdots \otimes \partial u_{i_p} \otimes du^{j_1} \otimes \cdots \otimes du^{j_q} \otimes du^l,$$

$$U^{i_1\cdots i_p}_{j_1\cdots j_q|l} = \frac{\partial U^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial u^l} + \sum_{s=1}^{p} {}'\Gamma^{i_s}_{ml}\, U^{i_1\cdots i_{s-1} m i_{s+1}\cdots i_p}_{j_1 j_2 \cdots\cdots\cdots j_q}$$
$$- \sum_{t=1}^{q} {}''\Gamma^{m}_{j_t l}\, U^{i_1 i_2 \cdots\cdots\cdots i_p}_{j_1\cdots j_{t-1} m j_{t+1}\cdots j_q}. \tag{7.14}$$

Accordingly, from (7.4) and (7.14), we obtain

$$U^{h_1\cdots h_p}_{k_1\cdots k_q, m} = P^{h_1}_{i_1} \cdots P^{h_p}_{i_p}\, U^{i_1\cdots i_p}_{j_1\cdots j_q|l}\, P^{j_1}_{k_1} \cdots P^{j_q}_{k_q}\, P^l_m. \tag{7.15}$$

Example 1. When $P = -1$, we have from (7.7) and (7.9)

$${}'\Gamma_i{}^j{}_h = -\Gamma_i{}^j{}_h, \quad {}''\Gamma_i{}^j{}_h = -\Gamma_i{}^j{}_h,$$

hence

$$\bar{\mu}_\Gamma = \mu_{-\Gamma}, \quad \mu_\Gamma = \bar{\lambda}_\Gamma \cdot \bar{\mu}_\Gamma = \bar{\lambda}_\Gamma \cdot \mu_{-\Gamma}.$$

Accordingly, we get (6.24), because

$$(\mu_\Gamma)^2 = \bar{\lambda}_\Gamma \cdot \mu_\Gamma = (\bar{\lambda}_\Gamma \cdot \bar{\lambda}_\Gamma) \cdot \mu_{-\Gamma} = \mu_{-\Gamma}.$$

Example 2. When $P = I$ in Example 2 in §6, we have

$${}'\Gamma_\lambda{}^j{}_\rho = \Gamma_\lambda{}^{n+j}{}_\rho, \quad {}'\Gamma_\lambda{}^{n+j}{}_\rho = -\Gamma_\lambda{}^j{}_\rho,$$
$${}''\Gamma_i{}^\nu{}_\rho = -\Gamma^\nu_{n+i,\rho}, \quad \Gamma^\nu_{n+i,\rho} = \Gamma_i{}^\nu{}_\rho,$$

by (7.7), (7.9). Hence it follows that

$${}'\Gamma_i{}^j{}_\rho = {}''\Gamma^{n+j}_{n+i,\rho} \quad {}'\Gamma^j_{n+i,\rho} = -{}''\Gamma_i{}^{n+j}{}_\rho,$$
$${}'\Gamma_i{}^{n+j}{}_\rho = -{}''\Gamma^j_{n+i,\rho}, \quad {}'\Gamma^{n+j}_{n+i,\rho} = {}''\Gamma_i{}^j{}_\rho.$$

Since $I^4 = 1$, we get $(\mu_\Gamma)^4 = \bar{\mu}_\Gamma$.

§ 8. The naturality and the Lie derivatives.

Firstly, we shall discribe some concepts on bundle homomorphisms of vector bundles.

Let $\mathfrak{F} = \{\mathfrak{Z}, \mathfrak{X}, \pi, \mathfrak{Y}\}$ and $\mathfrak{F}' = \{\mathfrak{Z}', \mathfrak{X}', \pi', \mathfrak{Y}'\}$ be two vector bundles and $h : \mathfrak{Z}' \to \mathfrak{Z}$ be a bundle homomorphism covering a given mapping $\psi : \mathfrak{X}' \to \mathfrak{X}$. We denote the dual vector bundle[14] of $\mathfrak{F}$ by $\mathfrak{F}^*$. Then, we can define naturally a transformation of $\Psi(\mathfrak{F}^*)$ into $\Psi(\mathfrak{F}'^*)$ denoted by $h^{\odot}$ as follows: For any $\xi \in \Psi(\mathfrak{F}^*)$, $z' \in \mathfrak{Z}'$, we put

$$<z', h^{\odot}\xi> = <hz', \xi>. \tag{8.1}$$

When $\mathfrak{F} = T(\mathfrak{X})$, $\mathfrak{F}' = T(\mathfrak{X}')$ and h is the differential mapping $d\psi$ of ψ[15], $h^{\odot} = (d\psi)^{\odot}$ is ψ^* in the ordinary sense.

Furthermore, when $\dim \mathfrak{Y} = \dim \mathfrak{Y}'$ and h is a bundle mapping, that is an isomorphism on each fibre of $\mathfrak{F}'$, we can define naturally a transformation of $\Psi(\mathfrak{F})$ into $\Psi(\mathfrak{F}')$ denoted by $h^{\ominus}$ as follows: For any $\xi \in \Psi(\mathfrak{F})$, we put

$$h^{\ominus}\xi(x') = (h \mid \mathfrak{Y}_{x'})^{-1}(\xi(\psi(x'))), \tag{8.2}$$

when h is the induced bundle mapping of ψ, we denote $h^{\ominus}$ by ψ°.

Now, for any $\psi : \mathfrak{X}' \to \mathfrak{X}$, we define a bundle homomorphism

$$d_2\psi : \mathfrak{T}^2(\mathfrak{X}') \to \mathfrak{T}^2(\mathfrak{X}) \tag{8.3}$$

covering ψ as follows: Let (U', u'^{α}) and (U, u^i) be coordinate neighborhoods of $x' \in \mathfrak{X}'$ and $x = \psi(x')$, then we put

$$d_2\psi(\partial u'_{\alpha}) = \frac{\partial u^i}{\partial u'^{\alpha}}\partial u_i, \tag{8.4}$$

$$d_2\psi(\partial^2 u'_{\alpha\beta}) = \frac{\partial^2 u^i}{\partial u'^{\beta}\partial u'^{\alpha}}\partial u_i + \frac{\partial u^i}{\partial u'^{\alpha}}\frac{\partial u^h}{\partial u'^{\beta}}\partial^2 u_{ih}. \tag{8.5}$$

We can easily see that this definition of $d_2\psi$ does not depend on local coordinates and $d_2\psi \mid T(\mathfrak{X}')$ is the differential mapping $d\psi$ of ψ in the ordinary sense. We will call $d_2\psi$ *the differential mapping of order 2 of* ψ. We shall denote $(d_2\psi)^{\odot}$ by ψ_2^*. For any $f, f_1, f_2 \in \mathfrak{A}(\mathfrak{X})$, we get immediately from (8. 4) and (8. 5) the equations

$$\psi_2^*(df_1 \otimes df_2) \equiv \frac{\partial(f_1\cdot\psi)}{\partial u'^{\alpha}}\frac{\partial(f_2\cdot\psi)}{\partial u'^{\beta}}du'^{\alpha}\otimes du'^{\beta} = d(f_1\cdot\psi)\otimes d(f_2\cdot\psi), \tag{8.6}$$

$$\psi_2^*(d^2 f) \equiv \frac{\partial(f\cdot\psi)}{\partial u'^{\alpha}}d^2u'^{\alpha} + \frac{\partial^2(f\cdot\psi)}{\partial u'^{\beta}\partial u'^{\alpha}}du'^{\alpha}\otimes du'^{\beta} = d^2(f\cdot\psi) \tag{8.7}$$

14) See [2], §1.

15) As is well known, $d\psi$ is usually written as ψ_*, the author will use also this notation in some sections in this paper.

and hence we have the following formula :

$$d \cdot \psi_r^* = \psi_{r2}^* \cdot d. \tag{8.8}$$

We get easily from (8.4) and (8.5) the following

Lemma 8.1. *If dim $\mathfrak{X}' =$ dim $\mathfrak{X}$ and $\psi_r : \mathfrak{X}' \to \mathfrak{X}$ is regular, then $d\psi_r : T(\mathfrak{X}') \to T(\mathfrak{X})$ and $d_2\psi_r : \mathfrak{T}^2(\mathfrak{X}') \to \mathfrak{T}^2(\mathfrak{X})$ are bundle mappings.*

In this section, from now on we assume that the conditions in this lemma are always satisfied. Here we shall explicitly show $(d\psi_r)\ominus$ and $(d_2\psi_r)\ominus$. For any $\xi \in \psi(\mathfrak{T}^2(\mathfrak{X}))$, $\xi = U^i\partial u_i + U^{ih}\partial^2 u_{ih}$, we get from (8.2), (8.4) and (8.5) the following equations :

$$U^i\partial u_i + U^{ih}\partial^2 u_{ih} = U^i\partial u_i + U^{ih}\Big(\frac{\partial u'^j}{\partial u^i}\frac{\partial u'^k}{\partial u^h}\, d_2\psi_r(\partial^2 u'_{jk}) - \frac{\partial u'^j}{\partial u^i}\frac{\partial u'^k}{\partial u^h}\frac{\partial^2 u^l}{\partial u'^k\partial u'^j}\,\partial u_l\Big)$$
$$= \frac{\partial u'^m}{\partial u^l}\Big(U^l - \frac{\partial^2 u^l}{\partial u'^k\partial u'^j}\,\frac{\partial u'^j}{\partial u^i}\,\frac{\partial u'^k}{\partial u^h}\,U^{ih}\Big)\, d_2\psi_r(\partial u'_m) +$$
$$+ \frac{\partial u'^j}{\partial u^i}\,\frac{\partial u'^k}{\partial u^h}\,U^{ih}\, d_2\psi_r(\partial^2 u'_{jk}),$$

hence

$$(d_2\psi_r)\ominus\xi = \Big(\frac{\partial u'^j}{\partial u^i}\,U^i + \frac{\partial^2 u'^j}{\partial u^h\partial u^i}\,U^{ih}\Big)\partial u'_j + \frac{\partial u'^j}{\partial u^i}\frac{\partial u'^k}{\partial u^h}\,U^{ih}\,\partial^2 u'_{jk}, \tag{8.9}$$

here we assume that ψ_r is homeomorphic on the coordinate neighborhood (U', u'^i).

Accordingly, for any $\xi \in \psi(T(\mathfrak{X}))$, $\xi = U^i\partial u_i$, we get

$$d((d\psi_r)\ominus\xi) = d\Big(\frac{\partial u'^j}{\partial u^i}\,U^i\,\partial u'_j\Big)$$
$$= \partial u'_j \otimes d\Big(\frac{\partial u'^j}{\partial u^i}\,U^i\Big) + \partial^2 u'_{jk} \otimes \frac{\partial u'^j}{\partial u^i}\,U^i du'^k,$$
$$((d_2\psi_r)\ominus \otimes \psi_r^*)(d\xi) = ((d_2\psi_r)\ominus \otimes \psi_r^*)(\partial u_i \otimes dU^i + \partial^2 u_{ih} \otimes U^i du^h)$$
$$= \frac{\partial u'^j}{\partial u^i}\,\partial u'_j \otimes dU^i +$$
$$+ \Big(\frac{\partial^2 u'^j}{\partial u^h\partial u^i}\,\partial u'_j + \frac{\partial u'^j}{\partial u^i}\,\frac{\partial u'^h}{\partial u^h}\,\partial^2 u'_{jk}\Big) \otimes U^i du^h$$
$$= \partial u'_j \otimes d\Big(\frac{\partial u'^j}{\partial u^i}\,U^i\Big) + \partial^2 u'_{jk} \otimes \frac{\partial u'^j}{\partial u^i}\,U^i du'^k,$$

where we omitted ψ_r^* for the calculations of the right hand sides, hence we have the formula

$$d \cdot (d\psi_r)\ominus = ((d_2\psi_r)\ominus \otimes \psi_r^*) \cdot d. \tag{8.10}$$

This formula is analogous to (1.17) in [2].

Now we take a general connection $\Gamma \in \psi(T(\mathfrak{X}) \otimes \mathfrak{D}^2(\mathfrak{X}))$ of $\mathfrak{X}$ given by (6.1). Then we will call $\Gamma' = ((d\psi_r)^{\ominus} \otimes \psi_{r2}^*)\Gamma \in \psi(T(\mathfrak{X}') \otimes \mathfrak{D}^2(\mathfrak{X}'))$ the induced general connection from Γ by the mapping ψ_r. We shall write Γ' explicitly. By means of (8.7) and (8.9), we get easily

$$\begin{aligned}\Gamma' &= ((d\psi_r)^{\ominus} \otimes \psi_{r2}^*)\{\partial u_l \otimes (P_i^l d^2u^i + \Gamma_{ih}^l\, du^i \otimes du^h)\} \\ &= \partial u'_m \otimes \frac{\partial u'^m}{\partial u^l}\left\{P_i^l \left(\frac{\partial^2 u^i}{\partial u'^k \partial u'^j} du'^j \otimes du'^k + \frac{\partial u^i}{\partial u'^k} d^2u'^k\right)\right. \\ &\quad \left. + \Gamma_{ih}^l \frac{\partial u^i}{\partial u'^j}\frac{\partial u^h}{\partial u'^k} du'^j \otimes du'^k\right\} \\ &= \partial u'_m \otimes (P'^m_j\, d^2u'^j + \Gamma'^m_{jk} du'^j \otimes du'^k),\end{aligned}$$

where we put

$$P'^m_j = \frac{\partial u'^m}{\partial u^l} P_i^l \frac{\partial u^i}{\partial u'^j}, \tag{8.11}$$

$$\Gamma'^m_{jk} = \frac{\partial u'^m}{\partial u^l}\left(P_i^l \frac{\partial^2 u^i}{\partial u'^k \partial u'^j} + \Gamma_{ih}^l \frac{\partial u^i}{\partial u'^j}\frac{\partial u^h}{\partial u'^k}\right). \tag{8.12}$$

Hence we put

$$\psi_{rUU'} = \left(\frac{\partial u^j \cdot \psi_r}{\partial u'^i},\ \frac{\partial^2 u^j \cdot \psi_r}{\partial u'^h \partial u'^i}\right) \tag{8.13}$$

on $U' \cap \psi_r^{-1}(U)$, these equations (8.11) and (8.12) can be written as

$$\sigma(\psi_{rUU'}) \cdot (P'^j_i,\ \Gamma'^j_{ih}) = (\psi_r^* P_i^j,\ \psi_r^* \Gamma_{ih}^j) \cdot \psi_{rUU'}, \tag{8.14}$$

which is analogous to (6.4).

Then, by means of (7.2), (8.9) and the above equations, we obtain easily

$$\begin{aligned}&\mu_{\Gamma'} \cdot (d_2\psi_r)^{\ominus}\, \partial u_i = \mu_{\Gamma'}\left(\frac{\partial u'^j}{\partial u^i}\, \partial u'_j\right) = \frac{\partial u'^j}{\partial u^i} P'^k_j\, \partial u'_k \\ &= \frac{\partial u'^k}{\partial u^h} P_i^h\, \partial u'_k = (d_2\psi_r)^{\ominus}(P_i^h\, \partial u_h) = (d\psi_r)^{\ominus} \cdot \mu_\Gamma\, \partial u_i, \\ &\mu_{\Gamma'} \cdot (d_2\psi_r)^{\ominus}\, \partial^2 u_{ih} = \mu_{\Gamma'}\left(\frac{\partial^2 u'^j}{\partial u^h \partial u^i}\, \partial u'_j + \frac{\partial u'^j}{\partial u^i}\frac{\partial u'^k}{\partial u^h}\, \partial^2 u'_{jk}\right) \\ &= \frac{\partial^2 u'^j}{\partial u^h \partial u^i} P'^l_j\, \partial u'_l + \frac{\partial u'^j}{\partial u^i}\frac{\partial u'^k}{\partial u^h} \Gamma'^l_{jk}\, \partial u'_l \\ &= \frac{\partial u'^l}{\partial u^j} \Gamma_{ih}^j\, \partial u'_l = (d\psi_r)^{\ominus} \cdot \mu_\Gamma\, \partial^2 u_{ih},\end{aligned}$$

hence

$$\mu_{\Gamma'} \cdot (d_2\psi_r)^{\ominus} = (d\psi_r)^{\ominus} \cdot \mu_\Gamma. \tag{8.15}$$

Nextly, by means of (7.2), (7.3) and (8.7), we obtain easily

$$\mu_{\Gamma'}\cdot\psi^* du^i = \mu_{\Gamma'}\left(\frac{\partial u^i}{\partial u'^j}\, du'^j\right) = \frac{\partial u^i}{\partial u'^j}\, P'^j_k du'^k$$

$$= P^i_h \frac{\partial u^h}{\partial u'^k}\, du'^k = \psi^*\cdot\mu_\Gamma\, du^i,$$

$$\mu_{\Gamma'}\cdot\psi_2^* d^2u^i = \mu_{\Gamma'}\left(\frac{\partial u^i}{\partial u'^j}\, d^2u'^j + \frac{\partial^2 u^i}{\partial u'^k \partial u'^j}\, du'^j\otimes du'^k\right)$$

$$= -\frac{\partial u^i}{\partial u'^j}\left(\Gamma'^j_{h\,l} - \frac{\partial P'^j_h}{\partial u'^l}\right) P'^l_k\, du'^h\otimes du'^k + \frac{\partial^2 u^i}{\partial u'^m \partial u'^l}\, P'^l_h P'^m_k\, du'^h\otimes du'^k$$

$$= \Big\{-\left(P^i_m \frac{\partial^2 u^m}{\partial u'^l \partial u'^h} + \Gamma^i_{mt}\frac{\partial u^m}{\partial u'^h}\frac{\partial u^t}{\partial u'^l} - \frac{\partial u^i}{\partial u'^j}\frac{\partial P'^j_h}{\partial u'^l}\right) P'^l_k +$$

$$+ \frac{\partial^2 u^i}{\partial u'^m \partial u'^l}\, P'^l_h P'^m_k\Big\}\, du'^h\otimes du'^k$$

$$= \Big\{-\Gamma^i_{mt}\frac{\partial u^m}{\partial u'^h}\, P^t_l \frac{\partial u^l}{\partial u'^k} - P^i_m \frac{\partial^2 u^m}{\partial u'^l \partial u'^h}\, P'^l_k + \frac{\partial u^i}{\partial u'^j}\frac{\partial P'^j_h}{\partial u'^l}\, P'^l_k$$

$$+ \frac{\partial^2 u^i}{\partial u'^l \partial u'^j}\, P'^j_h P'^l_k\Big\}\, du'^h\otimes du'^k$$

$$= \Big\{-\Gamma^i_{mt} P^t_l \frac{\partial u^m}{\partial u'^h}\frac{\partial u^l}{\partial u'^k} - P^i_m \frac{\partial^2 u^m}{\partial u'^l \partial u'^h}\, P'^l_k + \frac{\partial}{\partial u'^l}\left(P^i_m \frac{\partial u^m}{\partial u'^h}\right) P'^l_k\Big\}$$

$$du'^h\otimes du'^k$$

$$= \Big\{-\Gamma^i_{mt}\, P^t_l \frac{\partial u^m}{\partial u'^h}\frac{\partial u^l}{\partial u'^k} + \frac{\partial P^i_m}{\partial u^t}\frac{\partial u^t}{\partial u'^l}\frac{\partial u^m}{\partial u'^h}\, P'^l_k\Big\}\, du'^h\otimes du'^k$$

$$= -\left(\Gamma^i_{mt} - \frac{\partial P^i_m}{\partial u^t}\right) P^t_l\, \frac{\partial u^m}{\partial u'^h}\frac{\partial u^l}{\partial u'^k}\, du'^h\otimes du'^k$$

$$= -\left(\Gamma^i_{mt} - \frac{\partial P^i_m}{\partial u^t}\right) P^t_l\, \psi^* du^m\otimes\psi^* du^l = (\psi^*\otimes\psi^*)\cdot\mu_\Gamma\, d^2u^i,$$

hence

$$\mu_{\Gamma'}\cdot\psi^* = \psi^*\cdot\mu_\Gamma, \tag{8.16}$$

$$\mu_{\Gamma'}\cdot\psi_2^* = (\psi^*\otimes\psi^*)\cdot\mu_\Gamma. \tag{8.17}$$

Generally, let us put

$$\psi^{\otimes(p,q)} = ((d\psi)\ominus)^{\otimes p}\otimes(\psi^*)^{\otimes q} :\ \Psi(T(\mathfrak{X})^{\otimes(p,q)}) \to \Psi(T(\mathfrak{X}')^{\otimes(p,q)}) \tag{8.18}$$

and analogously consider the transformations

$$((d\psi)\ominus)^{\otimes s}\otimes(d_2\psi)\ominus\otimes((d\psi)\ominus)^{\otimes(p-s-1)}\otimes(\psi^*)^{\otimes(q+1)},$$
$$s = 0,\ 1,\cdots,\ p-1; \tag{8.19}$$

$$((d\psi)\ominus)^{\otimes p}\otimes(\psi^*)^{\otimes t}\otimes\psi_2^*\ \dot{\otimes}\ (\psi^*)^{\otimes(q-t-1)},$$
$$t = 0,\ 1,\cdots,\ q-1; \tag{8.20}$$

which operate on the spaces of cross-sections corresponding to the vector bundles over $\mathfrak{X}$ in the right hand sides of (5. 6). Furthermore, let

$$\psi\widetilde{\otimes}^{(p,q+1)} : \psi(\sum_{s=0}^{p-1} + T(\mathfrak{X})^{\otimes s} \otimes \dot{\mathfrak{T}}^2(\mathfrak{X}) \otimes T(\mathfrak{X})^{\otimes(p-s-1)} \otimes T^*(\mathfrak{X})^{\otimes(q+1)}$$
$$+\sum_{t=0}^{q-1} + T(\mathfrak{X})^{\otimes p} \otimes T^*(\mathfrak{X})^{\otimes t} \otimes \mathfrak{D}^2(\mathfrak{X}) \dot{\otimes} (T^*(\mathfrak{X})^{\otimes(q-t-1)})$$
$$\rightarrow \psi\,(\text{the corresponding vector bundle over } \mathfrak{X}') \tag{8.21}$$

be the transformation which operates on each parts of a decomposition of any element according to the structure of the vector bundle in the round parenthesis of the above equation as the operator (8. 19) or (8. 20) respectively. It is easily seen that this definition of $\psi\widetilde{\otimes}^{(p,q+1)}$ is well defined. And then we put

$$\psi\widetilde{\otimes}^{(p,q+1)} \mid \psi(T(\mathfrak{X})\overline{\otimes}^{(p,q+1)}) = \psi\overline{\otimes}^{(p,q+1)}. \tag{8.22}$$

After these transformations prepared, we shall obtain easily the following formulas by means of analogous calculations :

$$d \cdot \psi\otimes^{(p,q)} = \psi\overline{\otimes}^{(p,q+1)} \cdot d, \tag{8.23}$$

$$\mu_{\Gamma'} \cdot \psi\overline{\otimes}^{(p,q+1)} = \psi\otimes^{(p,q+1)} \cdot \mu_{\Gamma}. \tag{8.24}$$

Hence, we obtain the analogous formula on the covariant differentiations to (1. 17) of [2] :

$$D_{\Gamma'} \cdot \psi\otimes^{(p,q)} = \mu_{\Gamma'} \cdot d \cdot \psi\otimes^{(p,q)} = \mu_{\Gamma'} \cdot \psi\overline{\otimes}^{(p,q+1)} \cdot d$$
$$= \psi\otimes^{(p,q+1)} \cdot \mu_{\Gamma} \cdot d = \psi\otimes^{(p,q+1)} \cdot D_{\Gamma}.$$

Theorem 8. 2. *For any regular mapping $\psi : \mathfrak{X}' \rightarrow \mathfrak{X}$, $dim\ \mathfrak{X}' = dim\ \mathfrak{X}$, and any general connection Γ of $T(\mathfrak{X})$, the following equation of the naturality holds good*

$$D_{\Gamma'} \cdot \psi\otimes^{(p,q)} = \psi\otimes^{(p,q+1)} \cdot D_{\Gamma}. \tag{8.25}$$

Now we assume furthermore that $\mathfrak{X}' = \mathfrak{X}$, $\psi = \psi_t$ is depend on the real parameter t and $\psi_0 = 1$. Then, we may consider as $u'^i = u^i$, and so we put $u^i \cdot \psi_t = \psi_t^i$,

$$\psi_t^j(u) = u^j + \left(\frac{\partial \psi_t^j}{\partial t}\right)_{t=0} t + \cdots,$$

$$\left(\frac{\partial \psi_t^j}{\partial t}\right)_{t=0}(u) = \xi^j(u).$$

Then we get from (8. 11) and (8. 12)

$$P'^m_j = \left(\delta^m_l - \frac{\partial \xi^m}{\partial u^l} t + \cdots\right)\left(P^l_i + \frac{\partial P^l_i}{\partial u^k} \xi^k t + \cdots\right)\left(\delta^i_j + \frac{\partial \xi^i}{\partial u^j} t + \cdots\right)$$

$$= P^m_j + \left(\frac{\partial P^m_j}{\partial u^k}\xi^k - \frac{\partial \xi^m}{\partial u^l} P^l_j + P^m_i \frac{\partial \xi^i}{\partial u^j}\right) t + \cdots,$$

$$\Gamma'^m_{jk} = \left(\delta^m_l - \frac{\partial \xi^m}{\partial u^l} t + \cdots\right)\left\{\left(P^l_i + \frac{\partial P^l_i}{\partial u^h}\xi^h t + \cdots\right)\left(\frac{\partial^2 \xi^i}{\partial u^k \partial u^j} t + \cdots\right)\right.$$

$$\left. + \left(\Gamma^l_{i\,h} + \frac{\partial \Gamma^l_{ih}}{\partial u^s}\xi^s t + \cdots\right)\left(\delta^i_j + \frac{\partial \xi^i}{\partial u^j} t + \cdots\right)\left(\delta^h_k + \frac{\partial \xi^h}{\partial u^k} t + \cdots\right)\right\}$$

$$= \Gamma^m_{jk} + \left(P^m_i \frac{\partial^2 \xi^i}{\partial u^k \partial u^j} + \frac{\partial \Gamma^m_{jk}}{\partial u^h}\xi^h - \frac{\partial \xi^m}{\partial u^l}\Gamma^l_{jk} + \Gamma^m_{ik}\frac{\partial \xi^i}{\partial u^j}\right.$$

$$\left. + \Gamma^m_{jh}\frac{\partial \xi^h}{\partial u^k}\right) t + \cdots.$$

Therefore, we have the following formula on the Lie derivative of a general connection :

$$\pounds_\xi \Gamma \equiv \lim_{t\to 0} \frac{((d\psi_t)\ominus \otimes (\psi_t)^*_2)\,\Gamma - \Gamma}{t}$$

$$= \partial u_j \otimes \left(\frac{\partial P^j_i}{\partial u^k}\xi^k - \frac{\partial \xi^j}{\partial u^k} P^k_i + P^j_k \frac{\partial \xi^k}{\partial u^i}\right) d^2 u^i$$

$$+ \left(P^j_k \frac{\partial^2 \xi^k}{\partial u^h \partial u^i} + \frac{\partial \Gamma_i{}^j{}_h}{\partial u^k}\xi^k - \frac{\partial \xi^j}{\partial u^k}\Gamma_i{}^k{}_h + \Gamma_k{}^j{}_h \frac{\partial \xi^k}{\partial u^i} + \Gamma_i{}^j{}_k \frac{\partial \xi^k}{\partial u^h}\right)\times$$

$$du^i \otimes du^h$$

$$\equiv \partial u_j \otimes \{\pounds_\xi(P^j_i)\, d^2u^i + \pounds_\xi(\Gamma_i{}^j{}_h)\, du^i \otimes du^h\}. \tag{8.26}$$

Analogously, we can define the Lie derivative of any cross-section of a vector bundle over $\mathfrak{X}$, which is constructed from the vector bundles $T(\mathfrak{X})$, $\mathfrak{T}^2(\mathfrak{X})$, $T^*(\mathfrak{X})$, $\mathfrak{D}^2(\mathfrak{X})$ by means of the tensor product and the sum of vector bundles, with respect to the tangent vector field $\xi = \xi^i \partial u_i$, for instance, for $V \in \Psi(T(\mathfrak{X})\otimes^{(p,q)})$, we define $\pounds V$ by

$$\pounds_\xi V \equiv \lim_{t\to 0} \frac{\psi_t \otimes^{(p,q)} V - V}{t}. \tag{8.27}$$

§9. The canonical mapping ρ_Γ for Γ.

In the following sections, we shall deal exclusively with regular general connections. Let Γ be such a general connection. For coordinate neighborhoods (U, u^i) and (V, v^i), $U \cap V \neq \emptyset$, we have

$$\partial^2 u_{ih} = \frac{\partial^2 v^j}{\partial u^h \partial u^i}\partial v_j + \frac{\partial v^j}{\partial u^i}\frac{\partial v^k}{\partial u^h}\partial^2 v_{jk}.$$

Making use of $\mu' = \mu_\Gamma (= \bar\lambda_\Gamma^{-1} \cdot \mu_\Gamma$ on $\mathfrak{T}^2(\mathfrak{X}))$ in §7, we get

$$\mu'\partial^2 u_{ih} = \frac{\partial^2 v^j}{\partial u^h \partial u^i}\mu'\partial v_j + \frac{\partial v^j}{\partial u^i}\frac{\partial v^k}{\partial u^h}\mu'\partial^2 v_{jk}$$

$$= \frac{\partial^2 v^j}{\partial u^h \partial u^i} \partial v_j + \frac{\partial v^j}{\partial u^i} \frac{\partial v^k}{\partial u^h} \mu' \partial^2 v_{jk},$$

hence

$$(1 - \mu') \partial^2 u_{ih} = \frac{\partial v^j}{\partial u^i} \frac{\partial v^k}{\partial u^h} (1 - \mu') \partial^2 v_{jk}, \tag{9. 1}$$

which can be written as

$$(1 - \mu') \partial^2 u_{ih} = (1 - \mu') \partial^2 v_{jk}\, a^j_i(g_{VU})\, a^k_h(g_{VU}). \tag{9. 1'}$$

For any $b \in \pi^{-1}(U)$, $e_i(b) = a^j_i({}_l\beta) \partial u_j$, we put $\bar{b} = \rho(b)$ by

$$e_i(\bar{b}) = e_i(b), \tag{9. 2}$$

$$\begin{aligned} e_{ih}(\bar{b}) &= (1 - \mu') \partial^2 u_{jk}\, a^j_i({}_l\beta)\, a^k_h({}_l\beta) \\ &= (\partial^2 u_{jk} - \Gamma^l_{jk} Q^m_l \partial u_m)\, a^j_i({}_l\beta)\, a^k_h({}_l\beta). \end{aligned} \tag{9. 3}$$

By means of (9. 1), we see easily that ρ is well defined on the whole space $\mathfrak{B}(\mathfrak{X})$ stated in §1 by (9. 2) and (9. 3). We will call this mapping $\rho = \rho_\Gamma$: $\mathfrak{B}(\mathfrak{X}) \to \mathfrak{B}^2(\mathfrak{X})$ *the canonical mapping* for the general connection which is a cross-section of the fibre bundle $\{\mathfrak{B}^2(\mathfrak{X}), \mathfrak{B}(\mathfrak{X}), \sigma\}$.

Since we have $e_i(b\alpha) = e_j(b)\, a^j_i(\alpha)$ for any $\alpha \in L^1_n$, we get easily

$$e_i(\rho(b\alpha)) = e_i(b\alpha) = e_j(b)\, a^j_i(\alpha) = e_j(\rho(b))\, a^j_i(\alpha)^{16)}$$

and

$$e_{ih}(\rho(b\alpha)) = (1 - \mu') \partial^2 u_{jk}\, a^j_i({}_l\beta\alpha)\, a^k_h({}_l\beta\alpha) = e_{jk}(\rho(b))\, a^j_i(\alpha)\, a^k_h(\alpha).$$

Since $a^j_{ih}(\alpha) = 0$ for $\alpha \in L^1_n$, we obtain immediately the formula :

$$\rho_\Gamma \cdot r(\alpha) = r(\alpha) \cdot \rho_\Gamma \tag{9. 4}$$

by means of (1. 15) and (1. 16).

Conversely, let be given a cross-section ρ of $\{\mathfrak{B}^2(\mathfrak{X}), \mathfrak{B}(\mathfrak{X}), \sigma\}$ satisfying (9. 4) and an isomorphism P of $T(\mathfrak{X})$ covering the identity mapping of $\mathfrak{X}$. For (U, u^i), $b \in \pi^{-1}(U)$, $\bar{b} = \rho(b)$, we put

$$e_{ih}(\bar{b}) = \partial u_j\, a^j_{ih}({}_l\bar{\beta}) + \partial^2 u_{jk}\, a^j_i({}_l\bar{\beta})\, a^k_h({}_l\bar{\beta}).$$

By (9. 4), we have $\rho(b\alpha) = \rho(b)\alpha$ for any $\alpha \in L^1_n$. Hence, by means of (1. 9) and (1. 7), we get

$$\begin{aligned} a^j_{ih}(\gamma_l(({}_l\bar{\beta}\alpha)^{-1})) &= a^j_{ih}(\gamma_l(\alpha^{-1}{}_l\bar{\beta}^{-1})) \\ &= a^j_{ih}({}_l\bar{\beta}\gamma_l(\alpha^{-1}){}_l\bar{\beta}^{-1}) + a^j_{ih}(\gamma_l({}_l\bar{\beta}^{-1})) = a^j_{ih}(\gamma_l({}_l\bar{\beta}^{-1})), \end{aligned}$$

which shows that

$${}'\Gamma^j_{ih} = a^j_{ih}(\gamma_l({}_l\bar{\beta}^{-1}))$$

---

16) The two e_i in the middle of this equation may be regarded as e_i : $\mathfrak{B}(\mathfrak{X}) \to T(\mathfrak{X})$.

depend only on the coordinate neighborhood (U, u^i). Hence ${}'\Gamma_i{}^j{}_h$ are the components of an 1-connection ${}'\Gamma$ of the tangent bundle $T(\mathfrak{X})$ with respect to the local coordinates u^i. By (7.7), putting $\Gamma_i{}^j{}_h = P^j_k\,{}'\Gamma_i{}^k{}_h$, we can obtain a P-connection Γ and we can easily prove thst $\rho_\Gamma = \rho_{'\Gamma} = \rho$. Hence, we have

Theorem 9.1. *Any regular general connection Γ of the tangent bundle $T(\mathfrak{X})$ of $\mathfrak{X}$ determines a cross-section ρ_Γ of $\{\mathfrak{B}^2(\mathfrak{X}), \mathfrak{B}(\mathfrak{X}), \sigma\}$ invariant under the right translations of $\mathfrak{B}(\mathfrak{X})$. Conversely, such a cross-section ρ and an isomorphisme P of $T(\mathfrak{X})$ covering the identity mapping of $\mathfrak{X}$ determine a general connection Γ such that $\rho = \rho_\Gamma$ and $P = \lambda(\Gamma)$.*

§ 10. The universal affine connection.

Let us consider the induced vector bundles[17)]

$$\{\mathfrak{B}_n(\mathfrak{X}), \mathfrak{B}(\mathfrak{X})\} = \pi^\diamond\{T(\mathfrak{X}), \mathfrak{X}\}, \tag{10.1}$$

$$\{\mathfrak{B}^2_n(\mathfrak{X}), \mathfrak{B}^2(\mathfrak{X})\} = \bar{\pi}^\diamond\{\mathfrak{T}^2(\mathfrak{X}), \mathfrak{X}\} \tag{10.2}$$

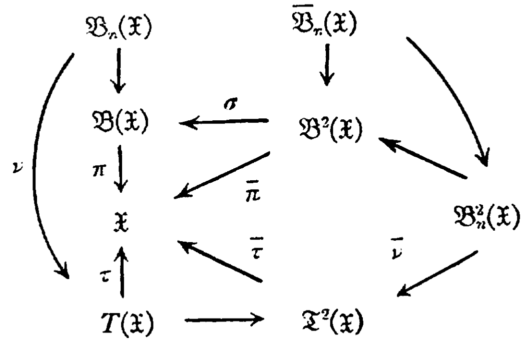
from the tangent bundles $T(\mathfrak{X})$ of order 1 and $\mathfrak{T}^2(\mathfrak{X})$ of order 2 by the projections $\pi: \mathfrak{B}(\mathfrak{X}) \to \mathfrak{X}$ and $\bar{\pi}: \mathfrak{B}^2(\mathfrak{X}) \to \mathfrak{X}$ of their principal bundles $\{\mathfrak{B}(\mathfrak{X}), \mathfrak{X}, \pi\}$ and $\{\mathfrak{B}^2(\mathfrak{X}), \mathfrak{X}, \bar{\pi}\}$ respectively. Let $\{\mathfrak{e}_i\}$ and $\{\bar{\mathfrak{e}}_i, \bar{\mathfrak{e}}_{ih}\}$ be the natural cross-sections of the induced vector bundles, that is

$$\nu(\mathfrak{e}_i(b)) = e_i(b) \tag{10.3}$$

and

$$\bar{\nu}(\bar{\mathfrak{e}}_i(\bar{b})) = e_i(\bar{b}), \quad \bar{\nu}(\bar{\mathfrak{e}}_{ih}(\bar{b})) = e_{ih}(\bar{b}), \tag{10.4}$$

where $\nu: \mathfrak{B}_n(\mathfrak{X}) \to T(\mathfrak{X})$ and $\bar{\nu}: \mathfrak{B}^2_n(\mathfrak{X}) \to \mathfrak{T}^2(\mathfrak{X})$ are the induced bundle mappings by π and $\bar{\pi}$. We may regard the induced vector bundle



$$\mathfrak{U}(\mathfrak{X}) = \bar{\pi}^\diamond\{T(\mathfrak{X}), \mathfrak{X}\} = \{\bar{\mathfrak{B}}_n(\mathfrak{X}), \mathfrak{B}^2(\mathfrak{X})\}$$

as a vector subbundle of $\bar{\pi}^\diamond\{\mathfrak{T}^2(\mathfrak{X}), \mathfrak{X}\}$, which we will call *the universal vector bundle of* $\mathfrak{X}$. We may regard $\bar{\mathfrak{e}}_i$ as the natural cross-sections of

17) See [2], §1.

the universal vector bunde of $\mathfrak{X}$.

On the other hand, if we regard a_i^j as functions defined on U and take b_i^j such that $b_k^j a_i^k = \delta_i^j$ for a moment, we obtain from (5. 3) the following equations :

$$\begin{aligned} d\mathrm{e}_i &= d(a_i^h \partial u_h) \\ &= a_i^h \partial^2 u_{hk} \otimes du^k + \partial u_h \otimes da_i^h \\ &= a_i^h(b_{hk}^j \mathrm{e}_j + b_h^j b_k^l \mathrm{e}_{jl}) \otimes du^k + b_h^j \mathrm{e}_j \otimes da_i^h \\ &= \mathrm{e}_j \otimes \{b_h^j da_i^h + b_{hk}^j a_i^h du^k\} + \mathrm{e}_{ij} \otimes b_k^j du^k, \end{aligned}$$

where $(b_i^j,\ b_{ih}^j) = (a_i^j,\ a_{ih}^j)^{-1}$. Based on these equations, we can define a sort of differentiation

$$d : \Psi(\bar{\pi}^{\diamond}(T(\mathfrak{X}))) \rightarrow \Psi(\bar{\pi}^{\diamond}(\mathfrak{T}^2(\mathfrak{X})) \otimes T^*(\mathfrak{B}^2(\mathfrak{X}))) \tag{10. 5}$$

by

$$d\bar{\mathrm{e}}_i = \bar{\mathrm{e}}_j \otimes \pi_i^j + \bar{\mathrm{e}}_{ij} \otimes \theta^j, \tag{10. 6}$$

and

$$d(f^i \bar{\mathrm{e}}_i) = f^i d\bar{\mathrm{e}}_i + \bar{\mathrm{e}}_i \otimes df^i, \tag{10. 7}$$

where f^i are any functions on $\mathfrak{B}^2(\mathfrak{X})$ and

$$\theta^j = b_i^j du^i. \tag{10. 8}$$

$$\pi_i^j = b_h^j da_i^h + b_{hk}^j a_i^h du^k. \tag{10. 9}$$

θ^j are differential forms on $\mathfrak{B}(\mathfrak{X})$ but they may be considered on $\mathfrak{B}^2(\mathfrak{X})$ as the differential forms transformed by σ^*. At the end of this section, we shall prove that π_i^j are well defined on the whole space $\mathfrak{B}^2(\mathfrak{X})$.

Furthermore, by virtue of (10. 6) and (10. 7), we can define a natural affine connection for the universal vector bundle $\mathfrak{U}(\mathfrak{X})$ whose covariant differentiation is given by

$$D\bar{\mathrm{e}}_i = \bar{\mathrm{e}}_j \otimes \pi_i^j, \tag{10. 10}$$

and whose developing part is given by

$$d\mathfrak{p} = \bar{\mathrm{e}}_j \otimes \theta^j \in \Psi(\mathfrak{U}(\mathfrak{X}) \otimes T^*(\mathfrak{B}^2(\mathfrak{X}))).\ ^{18)} \tag{10. 11}$$

We will call this connection *the universal affine connection of* $\mathfrak{X}$.

Now, let Γ be a regular general connection of $\mathfrak{X}$ and $\rho = \rho_\Gamma$ be the canonical mapping for Γ defined in the last section. Making use of (9. 3) and (10. 9), we shall calculate $\rho^*\pi_i^j$. Writing (9. 3) as

$$\mathrm{e}_{ih}(\bar{b}) = a_{ih}^j(\bar{\beta})\, \partial u_j + a_i^j(\bar{\beta})\, a_h^k(\beta)\, \partial^2 u_{jk},$$

we get

$$a_i^j(\bar{\beta}) = a_i^j(\beta),\ a_{ih}^j(\bar{\beta}) = -{}'\Gamma_l{}^j{}_k\, a_i^l(\beta)\, a_h^k(\beta),$$

18) See [2], §1.

hence

$$b^j_{ih}(\bar{\beta}) = a^j_{ih}(\bar{\beta}^{-1}) = -b^j_l(\beta)\, a^l_{mk}(\bar{\beta})\, b^m_i(\beta)\, b^k_h(\beta)$$
$$= b^j_k(\beta)\, {}'\Gamma_i{}^k{}_h.$$

Accordingly, we have

$$\rho^*\pi^j_i = b^j_h(da^h_i + {}'\Gamma_l{}^h{}_k\, a^l_i\, du^k) = \theta^j_i \tag{10.12}$$

which are the differential forms on $\mathfrak{B}(\mathfrak{X})$ for the classical connection $'\Gamma$ as is well known. Hence, we have

Theorem 10.1. *For any regular general connection Γ, $\rho_\Gamma{}^*\pi^j_i$ are the differential forms on $\mathfrak{B}(\mathfrak{X})$ for its contravariant part $'\Gamma$.*

Proposition 10.2. *For any coordinate neighborhood (U, u^i) and any $\bar{b} \in \bar{\pi}^{-1}(U)$, let $\bar{b} = \mathrm{r}(\beta)(\partial u_i, \partial^2 u_{ih})$, $\beta \in \mathfrak{L}^2_n$, then π^j_i can be written as*

$$\pi^j_i = a^j_i(\beta^{-1}\, d\beta) - a^j_{ih}(\gamma(\beta))\, \theta^h, \tag{10.13}$$

and they are independent of local coordinates.

Proof. From (10.8) and (10.9), we have

$$\begin{aligned}
\pi^j_i &= a^j_i(\beta^{-1}d\beta) + a^j_{hk}(\beta^{-1})\, a^h_i(\beta)\, du^k \\
&= a^j_i(\beta^{-1}d\beta) + a^j_{hk}(\beta^{-1})\, a^h_i(\beta)\, a^k_l(\beta)\, \theta^l \\
&= a^j_i(\beta^{-1}d\beta) + a^j_{ih}(\beta^{-1}\sigma(\beta))\, \theta^h \\
&= a^j_i(\beta^{-1}d\beta) - a^j_{ih}(\sigma(\beta^{-1})\beta)\, \theta^h \\
&= a^j_i(\beta^{-1}d\beta) - a^j_{ih}(\gamma(\beta))\, \theta^h.
\end{aligned}$$

For another coordinate neighborhood (V, v^i), $U \cap V \neq \emptyset$, let $\bar{b} = \mathrm{r}(\bar{\beta})(\partial v_i, \partial^2 v_{ih})$, then we have easily

$$\bar{\beta} = g_{VU}\, \beta \tag{10.14}$$

because (1.10) and (1.11) can be written as

$$(\partial u_i, \partial^2 u_{ih}) = \mathrm{r}(g_{VU})((\partial v_i, \partial^2 v_{ih})).$$

Accordingly, we have

$$\bar{\beta}^{-1}\, d\bar{\beta} = (\beta^{-1}\, g_{UV})\; d(g_{VU}\, \beta),$$
$$a^j_i(\bar{\beta}^{-1}\, d\bar{\beta}) = a^j_i(\beta^{-1}\, d\beta) + a^j_k(\beta^{-1})\, a^k_h(g_{UV}\, dg_{VU})\, a^h_i(\beta)$$

and by (1.8) and (1.9)

$$\begin{aligned}
a^j_{ih}(\gamma(\bar{\beta})) &= a^j_{ih}(\gamma(g_{VU}\,\beta)) \\
&= a^j_{ih}(\gamma(\beta)) + a^j_{ih}(\beta^{-1}\gamma(g_{VU})\,\beta),
\end{aligned}$$

$$\begin{aligned}
a^j_{ih}(\gamma(\bar{\beta}))\, \theta^h &= a^j_{ih}(\gamma(\beta))\, \theta^h + a^j_k(\beta^{-1})\, a^k_{lm}(\gamma(g_{VU}))\, a^l_i(\beta)\, a^m_h(\beta)\, \theta^h \\
&= a^j_{ih}(\gamma(\beta))\theta^h + a^j_k(\beta^{-1}) a^k_{lm}(\gamma(g_{VU}))\, du^m a^l_i(\beta)
\end{aligned}$$

$$= a^j_{ih}(\gamma(_i\beta))\theta^h + a^j_k(_i\beta^{-1})\, a^k_l(g_{UV}\, dg_{VU})\, a^l_i(_i\beta).$$

Hence we have

$$a^j_i(_i\bar{\beta}^{-1}d_i\bar{\beta}) - a^j_{ih}(\gamma(_i\bar{\beta}))\theta^h = a^j_i(_i\beta^{-1}d_i\beta) - a^j_{ih}(\gamma(_i\beta))\theta^h.$$

Theorem 10.3. *For the right translation* $\mathrm{r}(\alpha)$ *of* $\mathfrak{B}^2(\mathfrak{X})$, $\alpha \in \mathfrak{L}^2_n$, *we have*

$$\mathrm{r}(\alpha)^*\theta^j = a^j_i(\alpha^{-1})\,\theta^i, \tag{10.15}$$

$$\begin{aligned} \mathrm{r}(\alpha)^*\pi^j_i &= a^j_k(\alpha^{-1})\,\{\pi^k_h + a^k_{hl}(\gamma(\alpha^{-1}))\,\theta^l\}\, a^h_i(\alpha) \\ &= a^j_k(\alpha^{-1})\, a^h_i(\alpha)\pi^k_h - a^j_{ik}(\gamma(\alpha))\, a^k_l(\alpha^{-1})\,\theta^l. \end{aligned} \tag{10.16}$$

Proof. (10.15) holds good evidently. For (10.16), we have

$$\begin{aligned} \mathrm{r}(\alpha)^*\pi^j_i &= a^j_i(\alpha^{-1}{}_i\beta^{-1}d(_i\beta\alpha)) - a^j_{ih}(\gamma(_i\beta\alpha))\,\mathrm{r}(\alpha)^*\theta^h \\ &= a^j_k(\alpha^{-1})\, a^k_h(_i\beta^{-1}d_i\beta)\, a^h_i(\alpha) \\ &\quad - \{a^j_{ih}(\alpha^{-1}\gamma(_i\beta)\alpha) + a^j_{ih}(\gamma(\alpha))\}\, a^h_k(\alpha^{-1})\theta^k \\ &= a^j_k(\alpha^{-1})\{a^k_h(_i\beta^{-1}d_i\beta) - a^k_{hl}(\gamma(_i\beta))\theta^l\}\, a^h_i(\alpha) - a^j_{ih}(\gamma(\alpha))\, a^h_k(\alpha^{-1})\theta^k \\ &= a^j_k(\alpha^{-1})\, a^h_i(\alpha)\,\pi^k_h - a^j_{ih}(\ (\alpha))\, a^h_k(\alpha^{-1})\theta^k, \end{aligned}$$

$$\begin{aligned} - a^j_{ih}(\gamma(\alpha))\, a^h_k(\alpha^{-1})\theta^k &= a^j_{ih}(\alpha^{-1}\sigma(\alpha))\, a^h_k(\alpha^{-1})\theta^k \\ &= a^j_k(\alpha^{-1})\, a^k_{hl}(\sigma(\alpha)\,\alpha^{-1}) a^h_i(\alpha)\theta^l, \end{aligned}$$

hence

$$\mathrm{r}(\alpha)^*\pi^j_i = a^j_k(\alpha^{-1})\,\{\pi^k_h + a^k_{hl}(\gamma(\alpha^{-1}))\,\theta^l\}\, a^h_i(\alpha).$$

Corollary 10.4. *We have especially for* $\alpha \in L^1_n$

$$\mathrm{r}(\alpha)^*\pi^j_i = a^j_k(\alpha^{-1})\,\pi^k_h\, a^h_i(\alpha) \tag{10.17}$$

and for $\alpha \in \mathfrak{N}^2_n$

$$\mathrm{r}(\alpha)^*\pi^j_i = \pi^j_i + a^j_{ik}(\alpha^{-1})\theta^k. \tag{10.18}$$

§ 11. The vertical tangent vector fields of $\mathfrak{B}^2(\mathfrak{X})$.

In this section, we shall consider the vertical tangent vector fields on $\mathfrak{B}^2(\mathfrak{X})$ defined as follows.

Let Y^i_j, Y^{ih}_j be the tangent vectors at the neutral element of $\mathfrak{L}^2_n$ such that

$$Y^i_j = \partial/\partial a^j_i, \qquad Y^{ih}_j = \partial/\partial a^j_{ih} \tag{11.1}$$

and at any point $b \in \mathfrak{B}^2(\mathfrak{X})$, we define the corresponding tangent vectors to $\mathfrak{B}^2(\mathfrak{X})$ by

$$Q^i_j(b) = db(Y^i_j)^{19)}, \quad Q^{ih}_j(b) = db(Y^{ih}_j), \tag{11.2}$$

19) The notation Q^i_j will not be confused with the one used for the inverse of $\lambda(\Gamma) = P$ in §7 in the following sections.

where b is regarded as the admissible mapping of $\{\mathfrak{B}^2(\mathfrak{X}), \mathfrak{X}, \bar{\pi}\}$, $b: \mathfrak{L}_n^2 \to \bar{\pi}^{-1}(\bar{\pi}(b))$. We will call Q_j^i and Q_j^{ih} *the first and second basic vertical tangent vector fields* of the principal fibre bundle $\mathfrak{B}^2(\mathfrak{X})$ respectively. We shall write Q_j^i, Q_j^{ih} in terms of local coordinates of $\mathfrak{B}^2(\mathfrak{X})$. Making use of (1. 1) and (1. 12), for the left translation $l(\beta)$ of $\mathfrak{L}_n^2$, we obtain

$$
\begin{aligned}
l(\beta)_* Y_j^i &= \left[\frac{\partial(a_l^k(\beta)\, a_h^l(\alpha))}{\partial a_i^j(\alpha)}\right]_{\alpha=e} \left(\frac{\partial}{\partial a_h^k}\right)_\beta \\
&\quad + \left[\frac{\partial(a_l^k(\beta)\, a_{hm}^l(\alpha) + a_{lt}^k(\beta)\, a_h^l(\alpha)\, a_m^t(\alpha))}{\partial a_i^j(\alpha)}\right]_{\alpha=e} \left(\frac{\partial}{\partial a_{hm}^k}\right)_\beta \\
&= a_j^k(\beta)\left(\frac{\partial}{\partial a_i^k}\right)_\beta + a_{jm}^k(\beta)\left(\frac{\partial}{\partial a_{im}^k}\right)_\beta + a_{hj}^k(\beta)\left(\frac{\partial}{\partial a_{hi}^k}\right)_\beta
\end{aligned}
$$

and

$$
\begin{aligned}
l(\beta)_* Y_j^{ih} &= \left[\frac{\partial(a_t^m(\beta)\, a_{kl}^t(\alpha) + a_{ts}^m(\beta)\, a_k^t(\alpha)\, a_l^s(\alpha))}{\partial a_{ih}^j(\alpha)}\right]_{\alpha=e} \left(\frac{\partial}{\partial a_{kl}^m}\right)_\beta \\
&= a_j^m(\beta)\left(\frac{\partial}{\partial a_{ih}^m}\right)_\beta .
\end{aligned}
$$

Hence, regarding u^j, a_i^j, a_{ih}^j as local coordinates of $\mathfrak{B}^2(\mathfrak{X})$, the tangent vector fields Q_j^i and Q_j^{ih} can be written as

$$
Q_j^i = a_j^k\, \partial/\partial a_i^k + a_{jh}^k\, \partial/\partial a_{ih}^k + a_{hj}^k\, \partial/\partial a_{hi}^k, \tag{11. 3}
$$

$$
Q_j^{ih} = a_j^k\, \partial/\partial a_{ih}^k. \tag{11. 4}
$$

Conversely, from the above equations we obtain easily

$$
\partial/\partial a_{ih}^j = b_j^k\, Q_k^{ih}, \tag{11. 5}
$$

$$
\partial/\partial a_i^j = b_j^k(Q_k^i - a_{km}^l\, b_l^h\, Q_h^{im} - a_{mk}^l\, b_l^h\, Q_h^{mi}). \tag{11. 6}
$$

Furthermore, for the right translation $\mathrm{r}(\alpha)$ of the principal bundle $\mathfrak{B}^2(\mathfrak{X})$ of order 2, $\alpha \in \mathfrak{L}_n^2$, and $b = \mathrm{r}(\beta)(\partial u_i, \partial^2 u_{ih})$, making use of (11. 3) and (11. 4), we obtain

$$
\begin{aligned}
\mathrm{r}(\alpha)_* Q_j^i(b) &= a_j^k(\beta)\, \mathrm{r}(\alpha)_*\, (\partial/\partial a_i^k)_\beta + a_{jh}^k(\beta)\, \mathrm{r}(\alpha)_*\, (\partial/\partial a_{ih}^k)_\beta \\
&\quad + a_{hj}^k(\beta)\, \mathrm{r}(\alpha)_*\, (\partial/\partial a_{hi}^k)_\beta \\
&= a_j^k(\beta)\left\{\left[\frac{\partial(a_l^h(\gamma\alpha))}{\partial a_i^k(\gamma)}\right]_{\gamma=\beta}\left(\frac{\partial}{\partial a_l^k}\right)_{\beta\alpha} + \left[\frac{\partial(a_{ml}^t(\gamma\alpha))}{\partial a_i^k(\gamma)}\right]_{\gamma=\beta}\left(\frac{\partial}{\partial a_{ml}^t}\right)_{\beta\alpha}\right\} \\
&\quad + \left\{a_{jh}^k(\beta)\left[\frac{\partial(a_{ml}^t(\gamma\alpha))}{\partial a_{ih}^k(\gamma)}\right]_{\gamma=\beta} + a_{hj}^k(\beta)\left[\frac{\partial(a_{ml}^t(\gamma\alpha))}{a_{hi}^k(\gamma)}\right]_{\gamma=\beta}\right\}\left(\frac{\partial}{\partial a_{ml}^t}\right)_{\beta\alpha} \\
&= a_j^k(\beta)\, \{a_l^i(\alpha)\, (\partial/\partial a_l^k)_{\beta\alpha} + a_{ml}^i(\alpha)\, (\partial/\partial a_{ml}^k)_{\beta\alpha}\} \\
&\quad + \{a_{jh}^k(\beta)\, a_m^i(\alpha)\, a_l^h(\alpha) + a_{hj}^k(\beta)\, a_m^h(\alpha)\, a_l^i(\alpha)\}\, (\partial/\partial a_{ml}^k)_{\beta\alpha} \\
&= a_j^k(\beta)\, a_l^i(\alpha)\, b_k^m(\beta\alpha)\, \{Q_m^l(b\alpha) - a_{mh}^s(\beta\alpha)\, b_s^t(\beta\alpha)\, Q_t^{lh}(b\alpha) \\
&\qquad - a_{hm}^s(\beta\alpha)\, b_s^t(\beta\alpha)\, Q_t^{hl}(b\alpha)\}
\end{aligned}
$$

$$+ \{a^k_j(\beta)\,a^i_{ml}(\alpha) + a^k_{jh}(\beta)\,a^i_m(\alpha)a^h_l(\alpha) + a^k_{hj}(\beta)\,a^h_m(\alpha)\,a^i_l(\alpha)\}\;\; b^l_k(\beta\alpha)Q^{ml}_t(b\alpha)$$
$$= a^i_l(\alpha)\,a^m_j(\alpha^{-1})\,Q^l_m(b\alpha)$$
$$-\; a^i_l(\alpha)\,a^m_j(\alpha^{-1})\,b^t_s(\beta\alpha)\;\{a^s_k(\beta)\,a^k_{mh}(\alpha) + a^s_{kw}(\beta)\,a^k_m(\alpha)\,a^w_h(\alpha)\}\;\; Q^{lh}_t(b\alpha)$$
$$-\; a^i_l(\alpha)\,a^m_j(\alpha^{-1})\,b^t_s(\beta\alpha)\;\{a^s_k(\beta)\,a^k_{hm}(\alpha) + a^s_{kw}(\beta)\,a^k_h(\alpha)\,a^w_m(\alpha)\}\;\; Q^{hl}_t(b\alpha)$$
$$+ a^i_{ml}(\alpha)\,a^l_j(\alpha^{-1})\,Q^{ml}_t(b\alpha)$$
$$+\; b^t_k(\beta\alpha)\;\{a^k_{jh}(\beta)\,a^i_m(\alpha)\,a^h_l(\alpha) + a^k_{hj}(\beta)\,a^h_m(\alpha)\,a^i_l(\alpha)\}\;\; Q^{ml}_t(b\alpha)$$
$$= a^i_l(\alpha)\,a^m_j(\alpha^{-1})\,Q^l_m(b\alpha) + a^i_{ml}(\alpha)\,a^l_j(\alpha^{-1})\,Q^{ml}_t(b\alpha)$$
$$-\; a^i_l(\alpha)\;\{a^m_j(\alpha^{-1})\,a^l_k(\alpha^{-1})\,a^k_{mh}(\alpha) + b^l_s(\beta\alpha)\,a^s_{jm}(\beta)\,a^m_h(\alpha)\}\;\; Q^{lh}_t(b\alpha)$$
$$-\; a^i_l(\alpha)\;\{a^m_j(\alpha^{-1})\,a^l_k(\alpha^{-1})\,a^k_{hm}(\alpha) + b^l_s(\beta\alpha)\,a^s_{kj}(\beta)\,a^k_h(\alpha)\}\;\; Q^{hl}_t(b\alpha)$$
$$+\; b^t_k(\beta\alpha)\;\{a^k_{jh}(\beta)\,a^i_m(\alpha)\,a^h_l(\alpha) + a^k_{hj}(\beta)\,a^h_m(\alpha)\,a^i_l(\alpha)\}\;\; Q^{ml}_t(b\alpha)$$
$$= a^i_l(\alpha)\,a^m_j(\alpha^{-1})\,Q^l_m(b\alpha) + a^i_{ml}(\alpha)\,a^l_j(\alpha^{-1})\,Q^{ml}_t(b\alpha)$$
$$-\; a^i_l(\alpha)\,a^m_j(\alpha^{-1})\;\{a^t_{mh}(\gamma(\alpha))\,Q^{lh}_t(b\alpha) + a^t_{hm}(\gamma(\alpha))\,Q^{hl}_t(b\alpha)\}$$
$$= a^i_h(\alpha)\,a^k_j(\alpha^{-1})\;\{Q^h_k(b\alpha) - a^l_{km}(\gamma(\alpha))\,Q^{hm}_l(b\alpha) - a^l_{mk}(\gamma(\alpha))\,Q^{mh}_l(b\alpha)\}$$
$$+\; a^i_{hl}(\alpha)\,a^k_j(\alpha^{-1})\,Q^{hl}_k(b\alpha)$$

and

$$\mathrm{r}(\alpha)_*\,Q^{ih}_j(b) = a^k_j(\beta)\;\mathrm{r}(\alpha)_*(\partial/\partial a^k_{ih})_\beta$$
$$= a^k_j(\beta)\left[\frac{\partial(a^t_{ml}(\gamma\alpha))}{\partial a^k_{ih}(\gamma)}\right]_{\gamma=\beta}\left(\frac{\partial}{\partial a^t_{ml}}\right)_{\beta\alpha}$$
$$= a^k_j(\beta)\,a^i_m(\alpha)\,a^h_l(\alpha)\;(\partial/\partial a^k_{ml})_{\beta\alpha}$$
$$= a^k_j(\beta)\,a^i_m(\alpha)\,a^h_l(\alpha)\,b^t_k(\beta\alpha)\,Q^{ml}_t(b\alpha)$$
$$= a^i_m(\alpha)\,a^h_l(\alpha)\,a^k_j(\alpha^{-1})\,Q^{ml}_k(b\alpha).$$

Theorem 11. 1. *The vertical tangent vector fields* Q^i_j *and* Q^{ih}_j *of the principal bundle* $\mathfrak{B}^2(\mathfrak{X})$ *of order 2 of* $\mathfrak{X}$ *are operated by the right translation* $\mathrm{r}(\alpha)$ *of* $\mathfrak{B}^2(\mathfrak{X})$, $\alpha \in \mathfrak{L}^2_n$, *as follows :*

$$\mathrm{r}(\alpha)_*\,Q^i_j = a^i_h(\alpha)\,a^k_j(\alpha^{-1})\;\{Q^h_k - a^l_{km}(\gamma(\alpha))\;Q^{hm}_l - a^l_{mk}(\gamma(\alpha))\;Q^{mh}_l\}$$
$$+\; a^i_{hl}(\alpha)\,a^k_j(\alpha^{-1})\,Q^{hl}_k, \tag{11. 7}$$
$$\mathrm{r}(\alpha)_*\,Q^{ih}_j = a^i_m(\alpha)\,a^h_l(\alpha)\,a^k_j(\alpha^{-1})\,Q^{ml}_k. \tag{11. 8}$$

Corollary 11. 2. *For any* $\alpha \in L^1_n$, *we have*

$$\mathrm{r}(\alpha)_*\,Q^i_j = a^i_h(\alpha)\,a^k_j(\alpha^{-1})\,Q^h_k. \tag{11. 9}$$

Lastly, from (10. 8), (10. 9), (11. 3) and (11. 4), we get easily the following equations :

$$\langle Q^i_j,\ \theta^k\rangle = 0, \tag{11. 10}$$
$$\langle Q^i_j,\ \pi^k_h\rangle = \delta^i_h\,\delta^k_j, \tag{11. 11}$$
$$\langle Q^{ih}_j,\ \theta^k\rangle = \langle Q^{ih}_j,\ \pi^k_l\rangle = 0. \tag{11. 12}$$

§12. The field of universal tangent planes and the modified differential mappings of right translations.

At any point $b \in \mathfrak{B}^2(\mathfrak{X})$, we denote by N_b the tangent subspace of all tangent vectors X such that

$$\pi^i_j(X) = \langle X, \pi^j_i \rangle = 0, \tag{12.1}$$

which is of dimension $n + n^2$. Let N be the field of N_b on $\mathfrak{B}^2(\mathfrak{X})$. We will say any tangent vector belonging to N *to be proper* and call N_b *the universal tangent plane at b.*

For any $X \in N_b$ and any $\alpha \in \mathfrak{L}^2_n$, by means of (10.16) and (11.11), we get

$$\begin{aligned}\langle \mathrm{r}(\alpha)_* X, \pi^j_i \rangle &= \langle X, \mathrm{r}(\alpha)^* \pi^j_i \rangle \\ &= \langle X, a^j_k(\alpha^{-1}) a^h_i(\alpha) \pi^k_h - a^j_{ik}(\gamma(\alpha))\, a^k_l(\alpha^{-1})\, \theta^l \rangle \\ &= -\, a^j_{ik}(\gamma(\alpha))\, a^k_l(\alpha^{-1})\, \theta^l(X) \\ &= -\, a^k_{hm}(\gamma(\alpha))\, a^m_l(\alpha^{-1})\, \theta^l(X) \langle Q^h_k, \pi^j_i \rangle,\end{aligned}$$

hence it follows that

$$\mathrm{r}(\alpha)_* X + a^k_{hm}(\gamma(\alpha))\, a^m_l(\alpha^{-1})\, \theta^l(X)\, Q^h_k(b\alpha) \in N_{b\alpha}.$$

Owing to the above result, we define a mapping on $T(\mathfrak{B}^2(\mathfrak{X}))$ as follows: For any $X \in T_b(\mathfrak{B}^2(\mathfrak{X}))$,

$$\bar{\mathrm{r}}(\alpha) X = \mathrm{r}(\alpha)_* X + a^k_{hm}(\gamma(\alpha))\, a^m_l(\alpha^{-1})\, \theta^l(X)\, Q^h_k(b\alpha). \tag{12.2}$$

We will call $\bar{\mathrm{r}}(\alpha)$ *the modified differential mapping of the right translation* $\mathrm{r}(\alpha)$.

Theorem 12.1. *The modified differential mapping* $\bar{\mathrm{r}}(\alpha)$ *of the right translation* $\mathrm{r}(\alpha)$ *of* $\mathfrak{B}^2(\mathfrak{X})$ *has the following properties:*

i) $\bar{\mathrm{r}}(\alpha)$ *is an isomorphism of the tangent bundle* $T(\mathfrak{B}^2(\mathfrak{X}))$ *covering* $\mathrm{r}(\alpha)$,

ii) $\bar{\mathrm{r}}(\alpha)$ *operates for vertical tangent vectors to* $\mathfrak{B}^2(\mathfrak{X})$ *in the same way as* $\mathrm{r}(\alpha)_*$ *and* $\bar{\mathrm{r}}(\alpha) = \mathrm{r}(\alpha)_*$ *if and only if* $\alpha \in L^1_n$,

iii) $\bar{\mathrm{r}}(\alpha)\, N_b = N_{b\alpha}$, (12.3)

and

iv) *for any* $\alpha, \alpha_1 \in \mathfrak{L}^2_n$, $X \in T_b(\mathfrak{B}^2(\mathfrak{X}))$,

$$\begin{aligned}\bar{\mathrm{r}}(\alpha_1) \cdot \bar{\mathrm{r}}(\alpha) X = \bar{\mathrm{r}}(\alpha\alpha_1) X + a^j_{il}(\alpha_1^{-1}\gamma(\alpha)\,\alpha_1)\, \theta^l(\mathrm{r}(\alpha\alpha_1)_* X) \times \\ \times \{a^i_{hk}(\gamma(\alpha_1))\, Q^{hk}_j(b\alpha\alpha_1) - a^h_{jk}(\gamma(\alpha_1))\, Q^{ik}_h(b\alpha\alpha_1) - a^h_{kj}(\gamma(\alpha_1))\, Q^{ki}_h(b\alpha\alpha_1)\}\end{aligned} \tag{12.4}$$

Proof. We have proved that $\bar{\mathrm{r}}(\alpha)\, N_b \subset N_{b\alpha}$ and from (11.2) we see easily that $\bar{\mathrm{r}}(\alpha) = \mathrm{r}(\alpha)_*$ if and only if $a^j_{ih}(\gamma(\alpha)) = 0$, that is $\gamma(\alpha) \in L^1_n \cap \mathfrak{N}^2_n = \{e\}$. The first part of ii) is evident from (12.2), (11.10) and (11.12).

We shall prove iv) and then i). By means of (12. 2), we have

$$\bar{r}(\alpha_1)(\bar{r}(\alpha)X) = r(\alpha_1)_*\{r(\alpha)_* X + a^j_{il}(\gamma(\alpha))\theta^l(r(\alpha)_* X)Q^i_j(b\alpha)\}$$
$$+ a_h{}^k{}_m(\gamma(\alpha_1))\theta^m(r(\alpha_1)_*\{r(\alpha)_*X + a^j_{il}(\gamma(\alpha))\theta^l(r(\alpha)_*X)Q^i_j(b\alpha)\})Q^h_k(b\alpha\alpha_1)$$
$$= r(\alpha_1)_* r(\alpha)_*X + a^j_{il}(\gamma(\alpha))\theta^l(r(\alpha)_* X)\, r(\alpha_1)_* Q^i_j(b\alpha)$$
$$+ a^k_{hm}(\gamma(\alpha_1))\theta^m(r(\alpha_1)_* r(\alpha)_* X)Q^h_k(b\alpha\alpha_1)$$
$$+ a^k_{hm}(\gamma(\alpha_1))a^j_{il}(\gamma(\alpha))\theta^l(r(\alpha)_* X)\theta^m(r(\alpha_1)_* Q^i_j(b\alpha))Q^h_k(b\alpha\alpha_1).$$

Using (11. 7), (10. 15) and (11. 10), we have

$$= r(\alpha\alpha_1)_* X$$
$$+ a^j_{il}(\gamma(\alpha))\theta^l(r(\alpha)_*X)\,[a^i_h(\alpha_1)a^k_j(\alpha_1^{-1})\{Q^h_k(b\alpha\alpha_1) - a^t_{ks}(\gamma(\alpha_1))Q^{hs}_t(b\alpha\alpha_1)$$
$$- a^t_{sk}(\gamma(\alpha_1))Q^{sh}_t(b\alpha\alpha_1)\} + a^t_{hk}(\alpha_1)a^i_j(\alpha_1^{-1})Q^{hk}_t(b\alpha\alpha_1)]$$
$$+ a^k_{hm}(\gamma(\alpha_1))\theta^m(r(\alpha\alpha_1)_* X)Q^h_k(b\alpha\alpha_1)$$
$$+ a^k_{hm}(\gamma(\alpha_1))a^j_{il}(\gamma(\alpha))\theta^l(r(\alpha)_* X)a^m_t(\alpha_1^{-1})\theta^t(Q^i_j(b\alpha))Q^h_k(b\alpha\alpha_1).$$

Furthermore, using (1. 7), (10. 15) and (1. 8), we have

$$= r(\alpha\alpha_1)_* X$$
$$+ a^k_{hl}(\alpha_1^{-1}\gamma(\alpha)\alpha_1)\theta^l(r(\alpha\alpha_1)_* X)\{Q^h_k(b\alpha\alpha_1) - a^t_{ks}(\gamma(\alpha_1))Q^{hs}_t(b\alpha\alpha_1)$$
$$- a^t_{sk}(\gamma(\alpha_1))Q^{sh}_t(b\alpha\alpha_1)\}$$
$$+ a^i_{tl}(\alpha_1^{-1}\gamma(\alpha)\alpha_1)\theta^l(r(\alpha\alpha_1)_*X)a^t_{hk}(\gamma(\alpha_1))Q^{hk}_t(b\alpha\alpha_1)$$
$$+ a^k_{hm}(\gamma(\alpha_1))\theta^m(r(\alpha\alpha_1)_* X)Q^h_k(b\alpha\alpha_1)$$
$$= r(\alpha\alpha_1)_* X$$
$$+ \{a^k_{hl}(\alpha_1^{-1}\gamma(\alpha)\alpha_1) + a^k_{hl}(\gamma(\alpha_1))\}\theta^l(r(\alpha\alpha_1)_* X)Q^h_k(b\alpha\alpha_1)$$
$$+ a^j_{il}(\alpha_1^{-1}\gamma(\alpha)\alpha_1)\theta^l(r(\alpha\alpha_1)_* X)\{a^i_{hk}(\gamma(\alpha_1))Q^{hk}_j(b\alpha\alpha_1)$$
$$- a^h_{jk}(\gamma(\alpha_1))Q^{ik}_h(b\alpha\alpha_1) - a^h_{kj}(\gamma(\alpha_1))Q^{ki}_h(b\alpha\alpha_1)\},$$

hence, using (1. 9), we obtain the formula (12. 4).

Nextly, we suppose that $\bar{r}(\alpha)X = 0$, $X \in T_b(\mathfrak{B}^2(\mathfrak{X}))$. Then, by (12. 4) in which we put $\alpha_1 = \alpha^{-1}$, we get

$$0 = X + a^j_{il}(\alpha\gamma(\alpha)\alpha^{-1})\theta^l(X) \times \{a^i_{hk}(\gamma(\alpha^{-1}))Q^{hk}_j(b)$$
$$- a^h_{jk}(\gamma(\alpha^{-1})Q^{ik}_h(b) - a^h_{kj}(\gamma(\alpha^{-1})Q^{ki}_h(b)\}$$
$$= X - a^j_{il}(\gamma(\alpha^{-1}))\theta^l(X)\{a^i_{hk}(\gamma(\alpha^{-1}))Q^{hk}_j(b)$$
$$- a^h_{jk}(\gamma(\alpha^{-1}))Q^{ik}_h(b) - a^h_{kj}(\gamma(\alpha^{-1}))Q^{hk}_j(b)\},$$

since $\bar{r}(e) = r(e)_* =$ the identity mapping of $T(\mathfrak{B}^2(\mathfrak{X}))$ by ii). It must be

$$X = a^j_{il}(\gamma(\alpha^{-1}))\theta^l(X)\{a^i_{hk}(\gamma(\alpha^{-1}))Q^{hk}_j(b) - a^h_{jk}(\gamma(\alpha^{-1}))Q^{ik}_h(b)$$
$$- a^h_{kj}(\gamma(\alpha^{-1}))Q^{ki}_h(b)\},$$

hence X must be a vertical tangent vector of $\mathfrak{B}^2(\mathfrak{X})$. In accordance with ii), we have $\bar{\mathrm{r}}(\alpha)X = \mathrm{r}(\alpha)_* X$. As is well known, $\mathrm{r}(\alpha)_*$ is an isomorphism of the tangent bundle $T(\mathfrak{B}^2(\mathfrak{X}))$ of $\mathfrak{B}^2(\mathfrak{X})$. Accordingly, it must be $X = 0$. Thus, i) has been proved.

Lastly, iii) follows immediately from i) and the fact that $\bar{\mathrm{r}}(\alpha) N_b \subset N_{b\alpha}$. The proof is finished.

Now at each point $b \in \mathfrak{B}^2(\mathfrak{X})$, let Q_b^2 the n^3 dimensional tangent subspace spanned by $Q_j^{ih}(b)$.

Corollary 12.2. *The mapping* $\bar{\mathrm{r}}(\alpha)$ *is a representation of* $\mathfrak{L}_n^2$ *as transformation mod* Q_b^2.

Now, we shall determine $(\bar{\mathrm{r}}(\alpha))^{\odot} = \mathrm{r}'(\alpha)$ which was generally de-defined for any bundle homomorphism of a vector bundle into another one in §8. For any $\alpha \in \mathfrak{L}_n^2$, $\omega \in \Psi(T^*(\mathfrak{B}^2(\mathfrak{X})))$ and $X \in T_b(\mathfrak{B}^2(\mathfrak{X}))$, by definition, we have

$$\begin{aligned}(\mathrm{r}'(\alpha)\omega)(X) &= \omega(\bar{\mathrm{r}}(\alpha)X) \\ &= \omega(\mathrm{r}(\alpha)_* X + a_{il}^j(\gamma(\alpha))\,\theta^l(\mathrm{r}(\alpha)_* X)\,Q_j^i(b\alpha)) \\ &= \omega(\mathrm{r}(\alpha)_* X) + a_{il}^j(\gamma(\alpha))\,\theta^l(\mathrm{r}(\alpha)_* X)\,\omega(Q_j^i(b\alpha)) \\ &= (\mathrm{r}(\alpha)^*\omega)(X) + a_{il}^j(\gamma(\alpha))\,(\mathrm{r}(\alpha)^*\theta^l)(X)\omega(Q_j^i)(b\alpha) \\ &= (\mathrm{r}(\alpha)^*\omega + a_{il}^j(\gamma(\alpha))\,\omega(Q_j^i)(b\alpha)\,\mathrm{r}(\alpha)^*\theta^l)(X),\end{aligned}$$

hence we get the formula

$$\mathrm{r}'(\alpha)\,\omega = \mathrm{r}(\alpha)^*\,\{\omega + a_{il}^j(\gamma(\alpha))\,\omega(Q_j^i)\,\theta^l\}. \tag{12.5}$$

Especially, for θ^j, π_i^j, we obtain

$$\mathrm{r}'(\alpha)\,\theta^j = \mathrm{r}(\alpha)^*\theta^j = a_i^j(\alpha^{-1})\,\theta^i \tag{12.6}$$

by means of (11.10) and

$$\begin{aligned}\mathrm{r}'(\alpha)\pi_i^j &= \mathrm{r}(\alpha)^*\,\{\pi_i^j + a_{hl}^k(\gamma(\alpha))\,\pi_i^j(Q_k^h)\,\theta^l\} \\ &= \mathrm{r}(\alpha)^*\pi_i^j + a_{il}^j(\gamma(\alpha))\,\mathrm{r}(\alpha)^*\,\theta^l \\ &= a_k^j(\alpha^{-1})\,a_i^h(\alpha)\,\pi_h^k - a_{il}^j(\gamma(\alpha))\,a_k^l(\alpha^{-1})\theta^k + a_{il}^j(\gamma(\alpha))\,\mathrm{r}(\alpha)^*\,\theta^l \\ &= a_k^j(\alpha^{-1})\,a_i^h(\alpha)\,\pi_h^k,\end{aligned}$$

that is

$$\mathrm{r}'(\alpha)\pi_i^j = a_k^j(\alpha^{-1})\,a_i^h(\alpha)\pi_h^k \tag{12.7}$$

by means of (11.11), (10.16) and (10.15).

Lastly, we shall show a relation between the field of universal tangent planes and regular general connections.

Let Γ be any regular general connection of $T(\mathfrak{X})$ and $\rho = \rho_\Gamma$ be the canonical mapping for Γ defined in §9. At each point $b \in \mathfrak{B}(\mathfrak{X})$, we define a tangent subspace by

$$H_b = T_b(\mathfrak{B}(\mathfrak{X})) \cap \rho_*^{-1}(N_{\rho(b)}), \tag{12.8}$$

that is $X \in T_b(\mathfrak{B}(\mathfrak{X}))$ belongs to H_b if and only if $\rho_* X \in N_{\rho(b)}$, namely

$$\langle \rho_* X, \pi^j_i \rangle = \langle X, \rho^* \pi^j_i \rangle = \langle X, \theta^j_i \rangle = 0.$$

Hence we have

Theorem 12.3. *H_b is the n-dimensional horizontal plane at b for the* 1*-connection $'\Gamma$ which is the contravariant part of Γ.*[20]

The well known property that for any $\alpha \in L^1_n$

$$\mathrm{r}(\alpha)_* H_b = H_{b\alpha} \tag{12.9}$$

is implied immediately from (9.4) and (12.3) as follows :

$$\begin{aligned} \mathrm{r}(\alpha)_* H_b &= \mathrm{r}(\alpha)_* T_b(\mathfrak{B}(\mathfrak{X})) \cap \mathrm{r}(\alpha)_* \rho_*^{-1}(N_{\rho(b)}) \\ &= T_{b\alpha}(\mathfrak{B}(\mathfrak{X})) \cap \rho_*^{-1}(\mathrm{r}(a)_* N_{\rho(b)}) \\ &= T_{b\alpha}(\mathfrak{B}(\mathfrak{X})) \cap \rho_*^{-1}(N_{\rho(b\alpha)}) = H_{b\alpha}, \end{aligned}$$

since $\bar{\mathrm{r}}(\alpha) = \mathrm{r}(\alpha)^*$ for any $\alpha \in L^1_n$.

REFERENCES

[1] T. ŌTSUKI, Geometries of connections (in Japanese), Kyōritsu Shuppan Co. (1957).

[2] ———, Theory of affine connections of the space of tangent directions of a differentiable manifold I, II, III, Math. J. Okayama Univ., Vol. 7 (1957), 1—49, 50—74, 95—122.

[3] ———, Note on homotopies of some curves in tangent bundles, Math. J. Okayama Univ., Vol. 7 (1957), 191—194.

[4] ———, On tangent bundles of order 2 and affine cnnnections, Proc. Japan Acad., Vol. 34 (1958), 325—330.

[5] G. EHRESMANN, Les connexions infinitésimales dans un espace fibré différentiable, Colloque de Topologie (Espaces fibrés), (1950), 29—55.

[6] ———, Les prolongements d'une variété differentiable I. Calcul des jets, prolongement principal, C. R. Acad. Sci., Paris, 233 (1951), 598—600.

[7] S. S. CHERN, Lecture note on differential geometry, Chicago University (1950).

[8] J. A. SCHOUTEN, Ricci-Calculus, Springer (1954).

[9] W. AMBROSE and I. SINGER, A theorem on holonomy, Trans. American Math. Soc., Vol. 75 (1953), 428—443.

[10] K. NOMIZU, Lie groups and differential geometry, Pub. Math. Soc. Japan (1956).

DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

(Received September 11, 1958)

20) See [9] or [10], Ch. II.