SOME REMARKS ON HOMOTOPY EQUIVALENCES AND H-SPACES

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1. Introduction

A (continuous) map $f: X \to Y$ is called a *homotopy equivalence* if, and only if, there is a map (called a *homotopy inverse* of f), $g: Y \to X$, such that the maps $g \circ f: X \to X$ and $f \circ g: Y \to Y$ are homotopic to the identity maps 1_X of X and 1_Y of Y, respectively, where X and Y are any two (topological) spaces. If so, using the homotopies

$$F_t = \{1_X \sim g \circ f\}^{1)}: X \to X, \qquad G_t = \{1_Y \sim f \circ g\}: Y \to Y,$$

the two homotopies $f \circ F_t$ and $G_t \circ f$, between f and $f \circ g \circ f$, are defined, and hence, a question arises whether or not these homotopies, considering as the maps $X \times I \to Y$, are homotopic each other, rel. $X \times \dot{I}$. By this reason, we shall describe a map f as a *strong homotopy equivalence* if, and only if, there is a homotopy inverse g of f and the homotopies $F_t = \{1_X \sim g \circ f\}$ and $G_t = \{1_Y \sim f \circ g\}$ can be so taken that

$$(1.1) \qquad \begin{aligned} &f \circ F_t \sim G_t \circ f : X \times I \to Y,^{2)} && \text{rel. } X \times \dot{I}, \\ &g \circ G_t \sim F_t \circ g : Y \times I \to X, && \text{rel. } Y \times \dot{I}. \end{aligned}$$

One of our purposes of this note is to remark a relation between these two homotopy equivalences.

Theorem 1.2. *If X and Y are CW-complexes and $f: X \to Y$ is a homotopy equivalence, then it is a strong one. In other words, the notions of the ordinary and the strong homotopy equivalences are equivalent in the category of all the CW-complexes.*

This theorem will be proved in § 2 below.

Now, let F be a *homotopy-associative H-space*, and also a CW-complex such that the product space $F \times F$ is a CW-complex. (Its H-structure will be denoted by $x \cdot y$ for $x, y \in F$.) Then, there is an *inversion* x^{-1}.[3] Applying Theorem 1.2, we shall prove, in § 4 below, the following

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1) This means that F_t, $0 \leqq t \leqq 1$, is a homotopy between 1_X and $g \circ f$, i.e., $F_0 = 1_X$ and $F_1 = g \circ f$.

2) We shall mean by this notation that the two homotopies, $f \circ F_t$, $G_t \circ f: X \to Y$, are homotopic, considering as the maps $X \times I \to Y$.

3) Cf. [3, Lemma 2.5].

lemmas, which are generalizations of [3, Lemmas 2. 14. 1—2] :[4)]

Lemma 1. 3. 1. *Let A be a space, and K_t, $G_t : A \to F$ be the homotopies and H_0, $H_1 : A \to F$ be the maps such that $K_t \cdot H_t^{-1} = G_t$*[5)] *for $t = 0, 1$.*

Then, there is a homotopy $H_t : A \to F$ between H_0 and H_1 such that

$$G_t \sim K_t \cdot H_t^{-1} : A \times I \to F,^{5)} \quad rel. \quad A \times \dot{I}.$$

Lemma 1. 3. 2. *Let A be a space, and G_t, $H_t : A \to F$ be the homotopies and K_0, $K_1 : A \to F$ be the maps such that $K_t \cdot H_t^{-1} = G_t$ for $t = 0, 1$.*

Then, there is a homotopy $K_t : A \to F$ between K_0 and K_1 such that

$$G_t \sim K_t \cdot H_t^{-1} : A \times I \to F, \quad rel. \quad A \times \dot{I}.$$

If we apply these lemmas instead of [3, Lemmas 2. 14. 1—2], we can prove [3, Lemma 5. 1] and hence [3, Theorems 4. 5 and 1. 9] by the completely same methods, without assuming [3, (2. 13)] on F.[6)] Therefore, combining with [3, Theorem 1. 4], we obtain the following theorem, which is our main result of this note.

Theorem 1. 4. *Let F be a CW-complex such that the product space $F \times F \times F$ is also a CW-complex. Then, F is a homotopy-associative H-space (having an inversion) if, and only if, there exist topological spaces $E_2 \supset E_1 (\supset F)$, $B_2 \supset B_1 \ni b$ (a point) and a map $p : (E_2, E_1, F) \to (B_2, B_1, b)$, satisfying the following properties* (1. 5)—(1. 8) :

(1. 5) *F is contractible in E_1 to a vertex $\varepsilon \in F$, leaving ε fixed throughout the contraction ;*

(1. 6) *E_1 is a CW-complex, containing F as its subcomplex, and $E_1 \times F$ is also a CW-complex. Furthermore, E_1 is contractible in E_2 to ε, leaving ε fixed throughout the contraction ;*

(1. 7) *$p | E_1 : (E_1, F) \to (B_1, b)$ is a weak homotopy equivalence, i. e.,*
$(p | E_1)_* : \pi_n(E_1, F) \approx \pi_n(B_1, b)$, *for every integer $n > 0$;*

(1. 8) *$p : (E_2, F) \to (B_2, b)$ is also a weak homotopy equivalence, i. e.,*
$p_* : \pi_n(E_2, F) \approx \pi_n(B_2, b)$, *for every integer $n > 0$.*

2. Proof of Theorem 1. 2

The following lemma is an immediate consequence of the definition

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4) In [3, Lemmas 2. 14. 1–2], these lemmas are proved, assuming that F satisfies the additional assumption [3, (2. 13)] (cf. the correction of §5 below).

5) $K \cdot H^{-1} : A \to F$ is the map or the homotopy, defined by $(H \cdot K^{-1})(a) = H(a) \cdot (K(a))^{-1}$ for $a \in A$.

6) In the proofs of [3, Lemma 5. 1, Theorems 4. 5 and 1. 9], the assumption [3, (2. 13)] is used only to prove [3, Lemmas 2. 14. 1—2] in [3, p. 134].

of the strong homotopy equivalence in §1.

Lemma 2.1. *If $f_1 \sim f_2: X \to Y$ and f_1 is a strong homotopy equivalence, then so is f_2.*

If $f: X \to Y$ and $h: Y \to Z$ are both strong homotopy equivalences, then so is $h \circ f: X \to Z$.

To prove Theorem 1.2, we use the following lemma about the deformation retract.

Lemma 2.2. *If a subspace X of a space Z is a deformation retract*[7] *of Z, then the inclusion map $i: X \to Z$ is a strong homotopy equivalence.*

Proof. Let $h_t: Z \to Z$ be a retracting deformation, i.e.,

$$h_0 = 1_Z, \quad h_1(Z) \subset X, \quad h_t | X = 1_X, \quad \text{for } 0 \leqq t \leqq 1.$$

$h_1: Z \to X$ is clearly a homotopy inverse of i. If we set $F_t = 1_X$ and $G_t = h_t$ for $0 \leqq t \leqq 1$, then $F_t = \{1_X \sim h_1 \circ i\}$ and $G_t = \{1_Z \sim i \circ h_1\}$. The first of (1.1) is clear, because $i \circ F_t = 1_X = h_t | X = G_t \circ i$.

Using the homotopy $h_s \circ G_t = h_s \circ h_t$, $(0 \leqq t,\ s \leqq 1)$,

$$\left\{ \underset{h_1 \circ G_t}{h_1 \sim\!\sim h_1 \circ i \circ h_1} \right\} = \left\{ \underset{h_1 \circ h_1 \circ G_t}{h_1 \circ h_1 \sim\!\sim\!\sim\!\sim h_1 \circ h_1 \circ i \circ h_1} \right\}$$

$$\sim \left\{ \underset{h_1 \circ h_{1-t}}{h_1 \circ h_1 \sim\!\sim\!\sim} \; \underset{h_1 \circ h_0 \circ G_t}{h_1 \circ h_0 \sim\!\sim\!\sim\!\sim} \; \underset{h_1 \circ h_t \circ i \circ h_1}{h_1 \circ h_0 \circ i \circ h_1 \sim\!\sim\!\sim\!\sim} \; h_1 \circ h_1 \circ i \circ h_1 \right\}^{8)},$$

rel. $Z \times \dot{I}$. Because $h_1 \circ h_0 \circ G_t = h_1 \circ h_t$ and $h_1 \circ h_t \circ i \circ h_1 = h_1$, this is homotopic rel. $Z \times \dot{I}$ to $\left\{ \underset{h_1}{h_1 \sim h_1} \right\} = \left\{ \underset{F_t \circ h_1}{h_1 \sim\!\sim h_1 \circ i \circ h_1} \right\}$. Therefore, the second of (1.1) is proved, and we have Lemma 2.2.

Proof of Theorem 1.2. Let X, Y be CW-complexes and $f: X \to Y$ be a homotopy equivalence. Making use of a preliminary homotopy, if necessary, we assume that f is cellular. Let Z be the mapping cylinder of f. Then $X = X \times 0 \subset Z$ is a deformation retract of Z.[9] Therefore, the inclusion map $i: X \to Z$ is a strong homotopy equivalence, by Lemma 2.2.

It is clear that Y is a deformation retract of Z, and the map $\bar{f}: Z \to Y$, defined by

$$\bar{f}(x,\ t) = f(x), \quad \bar{f}(y) = y, \quad \text{for } x \in X,\ y \in Y,\ \ 0 \leqq t \leqq 1,$$

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7) We shall use this term when, and only when, there is a retracting deformation throughout which each point of X is held fixed.

8) This is the composed homotopy of $h_1 \circ h_{1-t} = \{h_1 \circ h_1 \sim h_1 \circ h_0\}$, $h_1 \circ h_0 \circ G_t = \{h_1 \circ h_0 \sim h_1 \circ h_0 \circ i \circ h_1\}$ and $h_1 \circ h_t \circ i \circ h_1 = \{h_1 \circ h_0 \circ i \circ h_1 \sim h_1 \circ h_1 \circ i \circ h_1\}$, cf. [3, (2.6)]. In this note, we shall often use the notations of [3].

9) Cf. [1], [2] and (J) of [4].

is a retraction, which is homotopic to 1_Z rel. Y. Hence, $\bar{f}$ is a strong homotopy equivalence, as be seen in the proof of Lemma 2. 2, and $f=\bar{f}\circ i$ is also so, by Lemma 2. 1. Therefore, Theorem 1. 2 is proved.

3. Auxiliary lemmas

To prove Lemmas 1. 3. 1—2, we use the auxiliary lemmas about the strong homotopy equivalence.

Lemma 3. 1. *Let $f: X\to Y$ be a strong homotopy equivalence, and $g: Y\to X$ be any its left homotopy inverse, i. e., $g\circ f\sim 1_X$, and $F_t=\{1_X\sim g\circ f\}$. Then g is a homotopy inverse of f, and there is such a homotopy $G_t=\{1_Y\sim f\circ g\}$ that (1. 1) is satisfied by these g, F_t and G_t.*

Proof. Because f is a strong homotopy equivalence, there are a homotopy inverse g' of f and homotopies $F'_t=\{1_X\sim g'\circ f\}$ and $G'_t=\{1_Y\sim f\circ g'\}$, such that

$$f\circ F'_t\sim G'_t\circ f,\quad \text{rel. } X\times\dot{I},\qquad g'\circ G'_t\sim F'_t\circ g',\quad \text{rel. } Y\times\dot{I}.$$

If we set $H_t=\left\{g'\underset{F_t\circ g'}{\sim\sim}g\circ f\circ g'\underset{g\circ G'_{1-t}}{\sim\sim\sim}g\right\}$, then the homotopy $G_t=\left\{1_Y\underset{G'_t}{\sim}f\circ g'\underset{f\circ H_t}{\sim\sim}f\circ g\right\}$ shows that g is a homotopy inverse of f. Furthermore,

$$\left\{1_X\underset{F'_t}{\sim}g'\circ f\underset{H_t\circ f}{\sim\sim}g\circ f\right\}\sim\left\{1_X\underset{F'_t}{\sim}g'\circ f\underset{F_t\circ g'\circ f}{\sim\sim\sim}g\circ f\circ g'\circ f\underset{g\circ G'_{t-1}\circ f}{\sim\sim\sim\sim}g\circ f\right\}$$

$$\sim\left\{1_X\underset{F_t}{\sim}g\circ f\underset{g\circ f\circ F'_t}{\sim\sim\sim}g\circ f\circ g'\circ f\underset{g\circ f\circ F'_{1-t}}{\sim\sim\sim\sim}g\circ f\right\}\sim\left\{1_X\underset{F_t}{\sim}g\circ f\right\},$$

rel. $X\times\dot{I}$. Therefore,

$$\left\{f\underset{f\circ F_t}{\sim\sim}f\circ g\circ f\right\}\sim\left\{f\underset{f\circ F'_t}{\sim\sim}f\circ g'\circ f\underset{f\circ H_t\circ f}{\sim\sim\sim}f\circ g\circ f\right\}$$

$$\sim\left\{f\underset{G'_t\circ f}{\sim\sim}f\circ g'\circ f\underset{f\circ H_t\circ f}{\sim\sim\sim\sim}f\circ g\circ f\right\}=\left\{f\underset{G_t\circ f}{\sim\sim}f\circ g\circ f\right\},$$

$$\left\{g\underset{g\circ G_t}{\sim\sim}g\circ f\circ g\right\}=\left\{g\underset{g\circ G'_t}{\sim\sim}g\circ f\circ g'\underset{g\circ f\circ H_t}{\sim\sim\sim}g\circ f\circ g\right\}$$

$$\sim\left\{g\underset{H_{1-t}}{\sim\sim}g'\underset{g'\circ G'_t}{\sim\sim\sim}g'\circ f\circ g'\underset{H_t\circ f\circ g'}{\sim\sim\sim}g\circ f\circ g'\underset{g\circ f\circ H_t}{\sim\sim\sim}g\circ f\circ g\right\}$$

$$\sim\left\{g\underset{H_{1-t}}{\sim\sim}g'\underset{F'_t\circ g'}{\sim\sim\sim}g'\circ f\circ g'\underset{H_t\circ f\circ g'}{\sim\sim\sim}g\circ f\circ g'\underset{g\circ f\circ H_t}{\sim\sim\sim}g\circ f\circ g\right\}$$

$$\sim\left\{g\underset{H_{1-t}}{\sim\sim}g'\underset{F_t\circ g'}{\sim\sim\sim}g\circ f\circ g'\underset{g\circ f\circ H_t}{\sim\sim\sim}g\circ f\circ g\right\}\sim\left\{g\underset{F_t\circ g}{\sim\sim}g\circ f\circ g\right\}.$$

It is clear that the terminal maps are held fixed throughout these homotopies.

Therefore, g, F_t and G_t satisfy (1. 1), and Lemma 3. 1 is proved.

Lemma 3.2. *Let $f: X \to Y$ be a strong homotopy equivalence, and $g: Y \to X$, $F_t = \{1_X \sim g\circ f\}$ and $G_t = \{1_Y \sim f\circ g\}$ be a homotopy inverse of f and homotopies such that they satisfy* (1.1).

Let A be a space, and $\mu_0, \mu_1: A \to X$, $\nu_t: A \to Y$ be maps and a homotopy such that $f\circ\mu_t = \nu_t$ for $t = 0, 1$.

Furthermore, let $\mu_t: A \to X$ be the following homotopy between μ_0 and μ_1:

$$(3.3) \qquad \mu_t = \left\{\mu_0 \underset{F_t\circ\mu_0}{\sim\!\sim} g\circ f\circ\mu_0 = g\circ\nu_0 \underset{g\circ\nu_t}{\sim\!\sim} g\circ\nu_1 = g\circ f\circ\mu_1 \underset{F_{1-t}\circ\mu_1}{\sim\!\sim\!\sim} \mu_1\right\}.$$

Then, the two homotopies $f\circ\mu_t$ and ν_t are homotopic each other:

$$f\circ\mu_t \sim \nu_t: A\times I \to Y, \quad rel. \quad A\times \dot{I}.$$

Proof.

$$f\circ\mu_t = \left\{f\circ\mu_0 \underset{f\circ F_t\circ\mu_0}{\sim\!\sim\!\sim} f\circ g\circ f\circ\mu_0 \underset{f\circ g\circ\nu_t}{\sim\!\sim\!\sim} f\circ g\circ f\circ\mu_1 \underset{f\circ F_{1-t}\circ\mu_1}{\sim\!\sim\!\sim\!\sim} f\circ\mu_1\right\}$$

$$\sim \left\{\nu_0 \underset{G_t\circ\nu_0}{\sim\!\sim} f\circ g\circ\nu_0 \underset{f\circ g\circ\nu_t}{\sim\!\sim\!\sim} f\circ g\circ\nu_1 \underset{G_{1-t}\circ\nu_1}{\sim\!\sim\!\sim} \nu_1\right\}$$

$$\sim \left\{\nu_0 \underset{\nu_t}{\sim} \nu_1\right\} = \nu_t, \quad \text{rel.} \quad A\times\dot{I}.$$

4. Proof of Lemmas 1.3.1—2.

Now, we consider about a homotopy-associative H-space F (having an inversion[3]), which is a CW-complex such that $F\times F$ is also a CW complex, and use the following notations of [3]:

$\varepsilon \in F$ is the unit, i.e., $\varepsilon\cdot x = x\cdot\varepsilon = x$, [3, (2.1)];

$f_t: F\times F\times F \to F\times F\times F$ is the homotopy such that $f_t(x, y, z) = \{(x\cdot y)\cdot z \sim x\cdot(y\cdot z)\}$, [3, (2.3)];

$\bar{\varepsilon}_t$, $i_t: F\to F$ are the homotopies such that $\bar{\varepsilon}_t(x) = \{\varepsilon \sim x^{-1}\cdot x\}$, $i_t(x) = \{x \sim (x^{-1})^{-1}\}$, [3, (2.9), (2.8)];

$j_t: F\times F \to F$ is the homotopy such that $j_t(x, y) = \{(x\cdot y)^{-1} \sim y^{-1}\cdot x^{-1}\}$, [3, (2.10)].

Let l_1, $m_1: F\times F \to F\times F$ be the maps defined by

$$l_1(x, y) = (x, x\cdot y^{-1}), \qquad m_1(x, y) = (x, y^{-1}\cdot x).$$

Then, it is easy to see that $1_{F\times F} \sim m_1\circ l_1$, $1_{F\times F} \sim l_1\circ m_1$, using the above homotopies. Therefore, l_1 is a strong homotopy equivalence, by Theorem 1.2.

Let $L_t^1: F\times F \to F$, $\bar{L}_t^1: F\times F \to F\times F$ be the homotopies defined by

$$L_t^1(x, y) = \left\{y \underset{i_t\cdot\bar{\varepsilon}_t}{\sim\!\sim} (y^{-1})^{-1}\cdot(x^{-1}\cdot x) \underset{f_{1-t}}{\sim\!\sim} ((y^{-1})^{-1}\cdot x^{-1})\cdot x \underset{j_{1-t}\cdot x}{\sim\!\sim} (x\cdot y^{-1})^{-1}\cdot x\right\},$$

and $\bar{L}_t^1(x, y) = (x, L_t^1(x, y))$. Then, $\bar{L}_t^1 = \{1_{F\times F} \sim m_1 \circ l_1\}$. Applying Lemma 3.1 to m_1 and $\bar{L}_t^1$, we obtain a homotopy $\bar{M}_t^1 = \{1_{F\times F} \sim l_1 \circ m_1\}$ such that these satisfy (1.1). Let $q_2 : F\times F \rightarrow F$ be the natural projection of $F\times F$ onto F of the second factor, and $M_t^1 = q_2 \circ \bar{M}_t^1$.

Now, let K_t, G_t, H_0, $H_1 : A \rightarrow F$ be the homotopies and the maps such that $K_t \cdot H_t^{-1} = G_t$, for $t = 0, 1$, as in Lemma 1.3.1.

Let μ_0, μ_1, $\nu_t : A \rightarrow F\times F$ be defined as follows:

$$\mu_t = (K_t, H_t), \quad \text{for } t = 0, 1; \quad \nu_t = (K_t, G_t), \quad \text{for } 0 \leqq t \leqq 1.$$

Then, $l_1 \circ \mu_t = \nu_t$, for $t = 0, 1$, and we can define the homotopy μ_t between μ_0 and μ_1 by (3.3), using m_1, $\bar{L}_t^1$ and $\bar{M}_t^1$. It is clear that $q_1 \circ \mu_t = \{K_0 \sim K_1\}$, $q_2 \circ \mu_t = \{H_0 \sim H_1\}$, where $q_1 : F\times F \rightarrow F$ be the natural projection onto F of the first factor. Furthermore, by (3.3),

$$q_1 \circ \mu_t = \left\{K_0 \underset{K_0}{\sim} K_0 \underset{K_t}{\sim} K_1 \underset{K_1}{\sim} K_1\right\} \sim K_t : A\times I \rightarrow F, \quad \text{rel. } A\times \dot{I},$$

because $q_1 \circ \bar{L}_t^1(x, y) = x$, and hence $q_2 \circ l_1 \circ \mu_t \sim K_t \cdot (q_2 \circ \mu_t)^{-1}$, rel. $A\times \dot{I}$.

On the other hand, by Lemma 3.2,

$$l_1 \circ \mu_t \sim \nu_t : A\times I \rightarrow F\times F, \quad \text{rel. } A\times \dot{I}.$$

Therefore, projecting by q_2, we have

$$K_t \cdot H_t^{-1} \sim G_t : A\times I \rightarrow F, \quad \text{rel. } A\times \dot{I},$$

where $H_t = q_2 \circ \mu_t$. This proves Lemma 1.3.1.

Lemma 1.3.2 is proved similarly, using the maps l_2, $m_2 : F\times F \rightarrow F\times F$, defined by

$$l_2(x, y) = (x\cdot y^{-1}, y), \quad m_2(x, y) = (x\cdot y, y),$$

and the homotopy

$$L_t^2(x, y) = \left\{x \underset{x\cdot \bar{\varepsilon}_t}{\sim\sim} x\cdot(y^{-1}\cdot y) \underset{f_{1-t}}{\sim\sim} (x\cdot y^{-1})\cdot y\right\} : F\times F \rightarrow F,$$

instead of l_1, m_1 and L_t^1 in the above proofs.

5. Corrections to [3].

The author takes this opportunity of correcting the following errata in [3]. (These errata are concerned only with [3, (2.13)], which may be omitted by the results of this note.)

p. 126, l. 9 — The homotopy "$\left\{x^{-1} \underset{i_t(x^{-1})}{\sim\sim} ((x^{-1})^{-1})^{-1} \underset{(i_{1-t}(x))^{-1}}{\sim\sim\sim\sim} x^{-1}\right\}$" is numbered by (2.12.0).

l. 8 up — For "(2. 12. 1—2)" read "(2. 12. 0—2)".
— For "$F\times F$" read "F or $F\times F$".

l. 7 up — For "$F\times F\times \dot{I}\cup(\varepsilon, \varepsilon)\times I$" read "$F\times \dot{I}\cup \varepsilon\times I$ or $F\times F \times \dot{I}\cup(\varepsilon, \varepsilon)\times I$".

l. 6 up — For "$\cdots$ maps $F\times F\times I\to F$" read "$\cdots$ maps of $F\times I$ or $F\times F\times I$ into F".

p. 127, l. 2 up — For "$\cdots i_t$, and $\cdots$" read "$\cdots i_t$ and (2. 13) that the homotopy (2. 12. 0) is homotopic to the stationary homotopy, and $\cdots$".

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