NOTE ON CURVATURE OF FINSLER MANIFOLDS

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In his paper [1], L. Auslander has proved some theorems on curvatur of Finsler manifolds, but it seems to the author that the methods used in his papar contains faults in some points. In this note, the author will give, in connection with the Theorems 3.3, 3.4, 4.2 and 4.4 in [1], remarks on the sectional curvature and the mean curvature of Finsler manifolds and the relation between the metric connection of E. Cartan and the connection of Levi-Civita of the induced Riemannian manifold on a local cross section containing the lift of a geodesic of a Finsler manifold, based on the general theory of affine connections of the space of tangent directions in his paper [3].

1. We will use the notations in the author's paper [3] in the following. Let $\mathfrak{X}$ be an n-dimensional differentiable manifold with suitable differentiability which is necessary for our arguments.

Let $\{T(\mathfrak{X}), \mathfrak{X}, \tau\}$ and $\{\mathfrak{B}, \mathfrak{X}, \pi\}$ be the tangent vector bundle of $\mathfrak{X}$ which will be simply denoted by $T(\mathfrak{X})$ and its associated principal bundle. Let $\{T(\mathfrak{X}) \boxtimes T(\mathfrak{X}), T(\mathfrak{X}), \tilde{\tau}\}$ and $\{\tilde{\mathfrak{B}}, T(\mathfrak{X}), \tilde{\pi}\}$ be the induced bundles of $T(\mathfrak{X})$ and its principal bundle by the projection map $\tau : T(\mathfrak{X}) \to \mathfrak{X}$ and let τ_1 and τ_p be their induced bundle maps respectively. Let $y^i : \tilde{\mathfrak{B}} \to R$ (the real field) be the maps defined as follows :

For any $\tilde{b} \in \tilde{\mathfrak{B}}$, put $b = \tau_p(\tilde{b})$, $y = \tilde{\pi}(\tilde{b})$, then y is uniquely written as $y = y^i(\tilde{b})e_i(b)$, where $\{e_1(b), \cdots, e_n(b)\}$ is the frame at $x = \pi(b) \in \mathfrak{X}$ which defines the point b. Making use the canonical coordinates[1] for a local coordinate system $(U, u) = (U, u^1, \cdots, u^n)$ of $\mathfrak{X}$, we have

$$e_i(b) = \frac{\partial}{\partial u^j}(x)a^j_i, \qquad y = \frac{\partial}{\partial u^j}(x)\xi^j,$$

$$y^j = b^j_i \xi^i, \qquad (b^j_i) = (a^j_i)^{-1}.$$

Let $\mathfrak{F}$ be the vector bundle which is the portion on $T_0(\mathfrak{X}) = T(\mathfrak{X}) - \mathfrak{X}$ of $\{T(\mathfrak{X}) \boxtimes T(X), T(\mathfrak{X}), \tilde{\tau}\}$, then the bundle space of $\mathfrak{F}$ is $T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X})$.

Let $(\mathfrak{X}, L)$ be a positive regular Finsler manifold[2] and $\mathfrak{g}$ be its metric tensor of $\mathfrak{F}$ whose components are locally given by

$$g_{ij}(u, \xi) = \frac{1}{2}\frac{\partial^2 L^2(u, \xi)}{\partial \xi^i \partial \xi^j}. \tag{1.1}$$

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1) See [3], I, § 3.

2) See [3], III, § 25, that is, $g_{ij}(u, \xi)$ is positive definite everywhere.

The components $\tilde{g}_{ij}$ of the tensor $\tilde{\pi}^{\circ}\mathfrak{g}$ of the induced vector bundle $\tilde{\pi}^{\diamond}\mathfrak{F}$ with respect to its natural cross sections[3] are locally written as

$$\tilde{g}_{ij} = g_{hk}\, a^h_i\, a^k_j. \tag{1.2}$$

Let θ^j be the differential forms on $\mathfrak{B}$ which are the components of the $\pi^{\circ} \otimes \pi^*$-image of the identity transformation dp[4] of $T(\mathfrak{X})$ and are locally written as $\theta^j = b^j_i du^i$. We will denote also the differential forms $\tau^*_p \theta^j$ on $\tilde{\mathfrak{B}}$ by the same symbols θ^j.

Let Γ be the metric connection of E. Cartan with respect to $(\mathfrak{X}, L)$. Then, the corresponding connection $\tilde{\Gamma}$ for $\mathfrak{F}$ is locally given by the differential forms ω^j_i on $\tau^{-1}(U) - U$ as

$$\omega^j_i = \Gamma_i{}^j{}_k du^k + C_i{}^j{}_k\, d\xi^k, \tag{1.3}$$

which are also written as

$$\omega^j_i = \Gamma^*{}_i{}^j{}_k\, du^k + C_i{}^j{}_k\, D\xi^k, \tag{1.4}$$

where

$$\Gamma^*{}_i{}^j{}_k = \Gamma_i{}^j{}_k - C_i{}^j{}_m\, \Gamma^m_{hk}\, \xi^h, \tag{1.5}$$

$$D\xi^k = d\xi^k + \omega^k_h\, \xi^h. \tag{1.6}$$

Let θ^j_i be the differential forms on $\tilde{\mathfrak{B}}_0 = \tilde{\mathfrak{B}} - \mathfrak{B}$ for the induced connection $\tilde{\pi}^{\diamond}\tilde{\Gamma}$ which are locally written as

$$\theta^j_i = b^j_h(da^h_i + \omega^h_k\, a^k_i). \tag{1.7}$$

Let Ω^j, Ω^j_i be the torsion forms and the curvature forms of the affine connection $(\tilde{\Gamma}, d\mathfrak{p})$ in (U, u), where $d\mathfrak{p} = (\tau^c \otimes \tau^*)\, dp$, and let Θ^j, Θ^j_i be the torsion forms and the curvature forms of the induced affine connection $\tilde{\pi}^{\diamond}(\tilde{\Gamma}, d\mathfrak{p})$[5]. Then, Ω^j and Ω^j_i are written as

$$\Omega^j \equiv \omega^j_i \wedge du^i = -C_i{}^j{}_k\, du^i \wedge D\xi^k, \tag{1.8}$$

$$\begin{aligned} \Omega^j_i &\equiv d\omega^j_i + \omega^j_k \wedge \omega^k_i \\ &= \frac{1}{2} R_i{}^j{}_{hk}\, du^h \wedge du^k + P_i{}^j{}_{hk}\, du^h \wedge D\xi^k + \frac{1}{2} S_i{}^j{}_{hk}\, D\xi^h \wedge D\xi^k \\ &= \frac{1}{2} R_i{}^j{}_{hk}\, du^h \wedge du^k + L P_i{}^j{}_{hk}\, du^h \wedge Dl^k + \frac{1}{2} L^2 S_i{}^j{}_{hk}\, Dl^h \wedge Dl^k, \end{aligned} \tag{1.9}$$

3) See [3], I, §2.
4) See [3], I, §3.
5) See [3], I, §1.

where $l^j = \xi^j/L$[6]. Θ^j and Θ_i^j are also written as

$$\Theta^j \equiv d\theta^j + \theta_i^j \wedge \theta^i = -\widetilde{C}_i{}^j{}_k \theta^i \wedge Dy^k = b_i^j \Omega^i, \tag{1.10}$$

$$\begin{aligned}\Theta_i^j &\equiv d\theta_i^j + \theta_k^j \wedge \theta_i^k = b_k^j \Omega_h^k a_i^h \\ &= \frac{1}{2}\widetilde{R}_i{}^j{}_{hk}\, \theta^h \wedge \theta^k + \widetilde{P}_i{}^j{}_{hk}\, \theta^h \wedge Dy^k + \frac{1}{2} \widetilde{S}_i{}^j{}_{hk}\, Dy^h \wedge Dy^k \\ &= \frac{1}{2}\widetilde{R}_i{}^j{}_{hk}\, \theta^h \wedge \theta^k + L\widetilde{P}_i{}^j{}_{hk}\, \theta^h \wedge Df^k + \frac{1}{2} L^2 \widetilde{S}_i{}^j{}_{hk}\, Df^h \wedge Df^k,\end{aligned} \tag{11.1}$$

where $f^j = y^j/L$ and L also denotes the function $L\cdot\tilde{\pi}$ on $\widetilde{\mathfrak{B}}-\mathfrak{B}$. $\widetilde{R}_i{}^j{}_{hk}$, $\widetilde{P}_i{}^j{}_{hk}$, $\widetilde{S}_i{}^j{}_{hk}$ are the components of the $\tilde{\pi}^0$-images of the curvature tensors of the first, second and third kinds[7] with the local components $R_i{}^j{}_{hk}$, $P_i{}^j{}_{hk}$, $S_i{}^j{}_{hk}$ respectively and $\widetilde{C}_i{}^j{}_k$ correspond to $C_i{}^j{}_k$.

Now, for any curve $\bar{C}$ given by $f: I = [0, 1] \to T_0(\mathfrak{X})$ such that $C = \tau\bar{C}$ is of class C^1, we can define its length s by the integral along $\bar{C}$

$$s = \int_0^1 (\tilde{g}_{ij}\, \theta^i\, \theta^j)^{\frac{1}{2}} \tag{1.12}$$

The author proved the following theorems.

Theorem 1.[8] *In order that a curve $\bar{C}$ of class C^2 in $T_0(\mathfrak{X})$ has its length relative minimum as a sensed curve such that its image in $\mathfrak{X}$ under the projection $T_0(\mathfrak{X}) \to \mathfrak{X}$ has two fixed end points, it is necessary that the following equations hold good along $\bar{C}$:*

$$d\frac{\theta^j}{ds} + \theta_k^j \frac{\theta^k}{ds} + \widetilde{C}^j{}_{ik} \frac{\theta^i}{ds} Dy^k = 0, \tag{1.13}$$

$$\widetilde{C}_{ijk} \frac{\theta^i}{ds}\frac{\theta^j}{ds} = 0. \tag{1.14}$$

Furthermore, in order that it is so in a family $\bar{C}_\varepsilon$ of class C^2 including $\bar{C} = \bar{C}_0$, it must be

$$[\tilde{g}_{ij}\bar{\theta}^i \theta^j]_0^1 = 0 \tag{1.15}$$

where $\bar{\theta}^j$ are the forms corresponding to the variation $\delta\varepsilon$.

Especially, when $\bar{C}$ is an α-curve[9], the terms with coefficients $\widetilde{C}^j{}_{ik}$

6) See, [3], I, § 6 and III, § 24. Our symbols for the curvature tensors are slightly different from the ones of E. Cartan [4], because the theory in [3] is treated in a more general standpoint.

7) We also call them the basic, mixed and firmamental curvature tensors.

8) See [3], III, § 32 and § 3. This theorem is slightly modified from the one in [3].

9) See [3], I, § 7, that is, it is a curve piecewise composed of the tangent vectors of curves in $\mathfrak{X}$ or arcs in tangent spaces at some points of $\mathfrak{X}$.

in (1. 13) vanish and (1. 14) is satisfied automatically because Γ is the metric connection of E. Cartan. Thus we have

Theorem 2. *In order that a proper α-curve of class C^2 has its length relative minimum as a sensed curve in $T_0(\mathfrak{X})$ such that its image in $\mathfrak{X}$ under the projection has two fixed end points, it must be that $C = \tau\bar{C}$ is a geodesic arc in the Finsler manifold $(\mathfrak{X}, L)$.*

Furthermore, the author obtained the formula of the second variation for the family $\bar{C}_\varepsilon$ of curves in $T_0(\mathfrak{X})$ containing $\bar{C} = \bar{C}_0$ such that *it is the lift of $C = \tau\bar{C}$ in $T_0(\mathfrak{X})$ and C is a geodesic arc in $(\mathfrak{X}, L)$*[10]:

$$\delta^2 s = [\delta\bar{\theta}^n + \bar{\theta}^n_j \bar{\theta}^j]_0^1 + \int_0^1 \left(\sum_{\alpha=1}^{n-1} \frac{D\bar{\theta}^\alpha}{ds} \frac{D\bar{\theta}^\alpha}{ds} + \widetilde{R}_{nin j} \bar{\theta}^i \bar{\theta}^j \right) \frac{ds}{dt} dt, \tag{1. 16}$$

where we consider the equation only on the submanifold of the points in $\widetilde{\mathfrak{B}}$ at which $\tilde{g}_{ij} = \delta_{ij}$ and $y^j = \delta^j_n$. *If all $\bar{C}_\varepsilon$ are α-curves,* then the above equation is written as[11]:

$$\delta^2 s = [\delta\bar{\theta}^n + \bar{\theta}^n_j \bar{\theta}^j]_0^1 + \int_0^1 \left(\sum_{\alpha=1}^{n-1} \bar{\theta}^\alpha_n \bar{\theta}^\alpha_n + \widetilde{R}_{nin j} \bar{\theta}^i \bar{\theta}^j \right) \frac{ds}{dt} dt. \tag{1. 17}$$

2. For any 2-dimensional plane E in the fibre over any point $y \in T_0(\mathfrak{X})$ of the vector bundle $\mathfrak{F}$, we define *the basic sectional curvature* $R(y, E)$ and *the firmamental sectional curvature* $S(y, E)$ with respect to $\widetilde{\Gamma}$ by

$$R(y, E) = -\frac{R_{ijhk} v^i w^j v^h w^k}{A(v, w)} \tag{2. 1}$$

and

$$S(y, E) = -\frac{S_{ijhk} v^i w^j v^h w^k}{A(v, w)}, \tag{2. 2}$$

where $\left\{ v = v^i \frac{\partial}{\partial u^i},\ w = w^i \frac{\partial}{\partial u^i} \right\}$ is a base of E and $A(v, w) = g_{ij} v^i v^j g_{hk} w^h w^k - (g_{ij} v^i w^j)^2$, that is the square of the area of the parallelogam made by v and w in the fibre over y of $\mathfrak{F}$. When E contains y, it is clear that $S(y, E)$ vanishes[12].

Now, let E_α, $\alpha = 1, 2, \cdots, n-1$, be a set of 2-dimensional planes

10) See, [3], III, § 33.

11) This formula is equivalent to the first one in the proof of Theorem 4. 3 in [1] but the terms in connection with P_{ijhk} must vanish since the metric connection of E. Cartan is a-proper, that is $\xi^h C_h{}^j{}_i = 0$, $\xi^h P_i{}^j{}_{hk} = 0$ by Theorem 25. 2 in [3], III.

12) Strictly speaking, let $\mathfrak{y}$ be the natural cross section of $\mathfrak{F}$ defined in [3], I, § 3, that is, $\mathfrak{y}(y) = (y, y)$. Here y is written in place of $\mathfrak{y}(y)$.

over y, containing $\mathfrak{y}(y)$ and being mutually orthogonal, then we can define *the mean curvature* at y by

$$M(y) = \frac{1}{n-1}\sum_{\alpha=1}^{n-1} R(y, E_\alpha). \tag{2.3}$$

For, if we take a unit vector $\underset{(\alpha)}{v}$ orthogonal to $\mathfrak{y}(y)$ in each E_α, then we get

$$M(y) = -\frac{1}{(n-1)L^2}\sum_\alpha R_{ijhk}\underset{(\alpha)}{v^i}\underset{(\alpha)}{v^h}\xi^j\xi^k = -\frac{1}{(n-1)L^2}R_{ijhk}g^{ih}\xi^j\xi^k. \tag{2.4}$$

This equation shows that $M(y)$ is a scalar field of $\mathfrak{F}$.

Now, we will prove a theorem which is a generalization of S. B. Myers' Theorem [5].

Theorem 3. *A complete n-dimensional positive regular Finsler manifold* $(\mathfrak{X}, L)$ *whose mean curvature* $M \geqq e^2$ *everywhere for some positive constant e is compact and has diameter less than or equal to* π/e.

This theorem was proved firstly by L. Auslander [1] but his mean curvature is defined by means of some local cross sections of $\mathfrak{F}$ and so it appers to the author that the non fine conditions (a) and (b) in connection with the assumption for his mean curvature in this theorem were written in [1]. Furthermore, generally speaking, the concept of geodesic coordinates along a geodesic is essentially different from the one in Riemannian manifolds, that is to say, such coordinates can be considered only in the Finsler manifolds which satisfy some conditions.[13] In order to prove this theorem, it may be desirable to do not use such coordinates. But we can prove this theorem analogously to those in [1] as follows:

Proof of Theorem 3. Let us suppose that diameter of $(\mathfrak{X}, L) > \pi/e$. Then, there exist two point $x_0, x_1 \in \mathfrak{X}$ and a geodesic are C joining x_0 to x_1 such that dist (x_0, x_1) = length of $C = l > \pi/e$, because the Finsler manifold $(\mathfrak{X}, L)$ is complete. For any family C_ε of class C^2 joining x_0 to x_1, in which $C = C_0$, using only the orthonormal frames such that $y^j = \delta^j_n$, we obtain from (1. 17) the equation

$$[\delta^2 J_\varepsilon]_{\varepsilon=0} = [\delta^2 J(C_\varepsilon)]_{\varepsilon=0} = \int_0^1\left(\sum_{\alpha=1}^{n-1}\bar\theta^\alpha_n\bar\theta^\alpha_n + \widetilde{R}_{ninj}\bar\theta^i\bar\theta^j\right)\frac{ds}{dt}\,dt, \tag{2.5}$$

where $J(C_\varepsilon)$ = length of C_ε.

On the other hand, (1. 16) is written as

$$d\theta^j + \theta^j_i \wedge \theta^i = -\widetilde{C}_i{}^j{}_k\,\theta^i \wedge \theta^k_n$$

13) See, Theorem 1 in [2], which is also true for $n \geqq 3$. It will be shown in a note.

for the above mentioned frames, because $Dy^k = \theta^k_j y^j = \theta^k_{,n}$. Hence, for the family $\bar{C}_\varepsilon$ which is the lift of C_ε, we get

$$d\bar{\theta}^j - \delta\theta^j + \theta^j_i \bar{\theta}^i - \bar{\theta}^j_{,n}\theta^n = -\widetilde{C}_i{}^j{}_\beta(\theta^i \bar{\theta}^\beta_n - \bar{\theta}^i \theta^\beta_{,n})$$
$$= \widetilde{C}_\gamma{}^j{}_\beta\, \bar{\theta}^\gamma\, \theta^\beta_{,n},$$

because we have $\theta^\alpha = 0$ along each $\bar{C}_\varepsilon$ and $\widetilde{C}_i{}^j{}_k\, y^k = \widetilde{C}_i{}^j{}_n = 0$. Along $\bar{C}_0$, we have $\theta^\alpha_{,n} = 0$ since C is a geodesic arc, hence the above equations follow

$$\begin{cases} d\bar{\theta}^\alpha + \theta^\alpha_\beta \bar{\theta}^\beta - \bar{\theta}^\alpha_{,n}\, ds = 0, \\ d\bar{\theta}^n - \delta\bar{\theta}^n = 0, \end{cases} \tag{2.6}$$

For each fixed α, we can take a family $\underset{(\alpha)}{C_\varepsilon}$ such that we have along C

$$\bar{\theta}^\alpha = \delta\varepsilon \sin\frac{\pi s}{l}, \quad \bar{\theta}^\beta = 0 \quad \alpha \neq \beta, \quad \theta^\alpha_\beta = 0,$$

hence we get from (2.6)

$$\bar{\theta}^\alpha_{,n}\, ds = d\bar{\theta}^\alpha = \frac{\pi\delta\varepsilon}{l}\cos\frac{\pi s}{l}\, ds, \quad \bar{\theta}^\beta_n\, ds = \theta^\beta_\alpha\, \bar{\theta}^\alpha = 0.$$

The second variation $[\delta^2 J_\varepsilon]_{\varepsilon=0}$ with respect to $\underset{(\alpha)}{C_\varepsilon}$ is written as

$$[\delta^2 J(\underset{(\alpha)}{C_\varepsilon})]_{\varepsilon=0} = \delta\varepsilon^2 \int_0^l \left(\frac{\pi^2}{l^2}\cos^2\frac{\pi s}{l} + \widetilde{R}_{n\alpha n\alpha}\sin^2\frac{\pi s}{l}\right) ds$$
$$= \delta\varepsilon^2 \int_0^l \sin^2\frac{\pi s}{l}\left(\frac{\pi^2}{l^2} + \widetilde{R}_{n\alpha n\alpha}\right) ds.$$

Summing over all α, we get by (2.4)

$$\sum_{\alpha=1}^{n-1} [\delta^2 J(\underset{(\alpha)}{C_\varepsilon})]_{\varepsilon=0} = \delta\varepsilon^2 \int_0^l \sin^2\frac{\pi s}{l}\left(\frac{(n-1)\pi^2}{l^2} + \sum_i \widetilde{R}_{nini}\right) ds$$
$$= \delta\varepsilon^2 \int_0^l \sin^2\frac{\pi s}{l}\left(\frac{(n-1)\pi^2}{l^2} + \frac{1}{L^2}\widetilde{R}_{ihjk}\, \tilde{g}^{ij}\, y^h\, y^k\right) ds$$
$$= \delta\varepsilon^2 \int_0^l (n-1)\sin^2\frac{\pi s}{l}\left(\frac{\pi^2}{l^2} - M(y)\right) ds.$$

By our assumption, we have $\frac{\pi^2}{l^2} < e^2 \leqq M(y)$, thus it must be

$$\sum_\alpha [\delta^2 J(\underset{(\alpha)}{C_\varepsilon})]_{\varepsilon=0} < 0.$$

Hence we have $[\delta^2 J(\underset{(\alpha)}{C_\varepsilon})]_{\varepsilon=0} < 0$ for at least one α. C is not relative minimum in this family $\underset{(\alpha)}{C_\varepsilon}$. This contradicts to our assumption that dist (x_0, x_1) = length of C.

3. In this section, we shall investigate the relation between L. Auslander's definition and the author's one of mean curvature.

Let $\mathfrak{X}_1$ be a submanifold in $T_0(\mathfrak{X})$ which is a local cross section of $\mathfrak{F}$ over $\tau(\mathfrak{X}_1)$ and let $\iota_1: \mathfrak{X}_1 \to T_0(\mathfrak{X})$ be the imbedding map. It is clear that the induced affine connection $(\Gamma_1, \psi_{\Gamma_1}) = \iota_1^{\diamond}(\widetilde{\Gamma}, d\mathfrak{p})$ of the induced vector bundle $\mathfrak{F}_1 = \iota_1^{\diamond}\mathfrak{F}$ is metric with respect to the induced metric tensor $\mathfrak{g}_1$ of $\mathfrak{F}_1$ from $\mathfrak{g}$, where $\psi_{\Gamma_1} = ((\tau\cdot\iota_1)^{\circ} \otimes (\tau\cdot\iota_1)^*)\, dp$.[14] Since $\mathfrak{X}_1$ is a local cross section, the natural homomorphism $h: T(\mathfrak{X}_1) \to \mathfrak{F}_1 = \{\mathfrak{Z}_1, \mathfrak{X}_1, \tau_1\}$ given by

$$hX_1 = (\tau\cdot\iota_1)^{\circ} ((\tau\cdot\iota_1)_* X_1), \quad X_1 \in T(\mathfrak{X}_1) \tag{3.1}$$

is an isomorphism. Using local coordinates $u^i = u^i(v)$, $\xi^i = \xi^i(v)$, the homomorphism h is written generally as

$$h\frac{\partial}{\partial v^i} = \frac{\partial u^j}{\partial v^i}(\tau\cdot\iota_1)^{\circ}\frac{\partial}{\partial u^j}. \tag{3.2}$$

Furthermore, we have

$$dp_1 = (h^{\ominus} \otimes 1)\, \psi_{\Gamma_1} = \text{the identity transformation of } T(\mathfrak{X}_1)^{15)}.$$

And so, the affine connection $h^{\#}(\Gamma_1, \psi_{\Gamma_1}) = (h^{\#}\Gamma_1, dp_1)$ of $T(\mathfrak{X}_1)$ is an ordinary metric affine connection with respect to the metric tensor $\bar{\mathfrak{g}}$ induced from $\mathfrak{g}_1$ by h. Accordingly, we may identify $\mathfrak{F}_1$ with $T(\mathfrak{X}_1)$ and $(\Gamma_1, \psi_{\Gamma_1})$ with $h^{\#}(\Gamma_1, \psi_{\Gamma_1})$. Let us denote the restrictions of θ^j, Dy^j, θ_i^j on $\tilde{\pi}^{-1}(\iota_1(\mathfrak{X}_1)) \subset \widetilde{\mathfrak{B}}$ by π^j, φ^j, π_i^j.

On the other hand, let $\{\mathfrak{B}_1, \mathfrak{X}_1, \pi_1\}$ be the associated principal bundle of the tangent bundle $T(\mathfrak{X}_1)$, then h induces a natural isomorphism $h_1: \mathfrak{B}_1 \to \tilde{\pi}^{-1}(\iota_1(\mathfrak{X}_1))$ such that

$$h(\mathrm{e}_i(b_1)) = (\tau\cdot\iota_1)^{\circ}(\mathrm{e}_i(\tau_p\, h_1(b_1))). \tag{3.3}$$

Now, let $\bar{\Gamma}_1$ be the metric connection of the vector bundle $\mathfrak{F}_1$ whose induced connection $h^{\#}\bar{\Gamma}_1$ is the connection of Levi-Civita of the induced Riemannian manifold $(\mathfrak{X}_1, \bar{\mathfrak{g}})$[16]. We denote the differential forms on $\tilde{\pi}^{-1}(\iota_1(\mathfrak{X}_1))$ for the metric affine connection $(\bar{\Gamma}_1, \psi_{\Gamma_1})$ by $\bar{\pi}^j$, $\bar{\pi}_i^j$. It is clear that we may also consider π^j, π_i^j and $\bar{\pi}^j, \bar{\pi}_i^j$ as differential forms on $\mathfrak{B}_1$. As is well known, $\bar{\Gamma}_1$ is uniquely determined by the conditions that $\bar{\Gamma}_1$ is metric and symmetric and has no torsion. Hence we have

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14) See [3], III, §31, Proposition 31.1.

15) On the definition of $h^{\ominus}$ for a bundle map h, see [3], I, §1 and III, §31.

16) $\bar{\mathfrak{g}}$ is locally written as $\bar{\mathfrak{g}} = g_{hk}(u(v), \xi(v))\, \dfrac{\partial u^h}{\partial v^i}\dfrac{\partial u^k}{\partial v^j}\, dv^i \otimes dv^j$.

$$d\tilde{g}_{ij} + \pi^k_i \tilde{g}_{kj} + \pi^k_j \tilde{g}_{ik} = 0, \tag{3.4}$$

$$d\tilde{g}_{ij} + \bar{\pi}^k_i \tilde{g}_{kj} + \bar{\pi}^k_j \tilde{g}_{ik} = 0, \tag{3.5}$$

$$\pi^j = \bar{\pi}^j = b^j_i \, du^i \text{ [17]}. \tag{3.6}$$

From (1. 10), the equations of structure of Γ_1 are written as

$$\begin{cases} d\pi^j + \pi^j_i \wedge \pi^i = -\widetilde{C}_i{}^j{}_k \, \pi^i \wedge \varphi^k \\ d\pi^j_i + \pi^j_k \wedge \pi^k_i = \dfrac{1}{2} \widetilde{R}_i{}^j{}_{hk} \, \pi^h \wedge \pi^k + \widetilde{P}_i{}^j{}_{hk} \, \pi^h \wedge \varphi^k + \dfrac{1}{2} \widetilde{S}_i{}^j{}_{hk} \, \varphi^h \wedge \varphi^k \end{cases} \tag{3.7}$$

and the ones of $\bar{\Gamma}_1$ are written as

$$\begin{cases} d\pi^j + \bar{\pi}^j_i \wedge \pi^i = 0, \\ d\bar{\pi}^j_i + \bar{\pi}^j_k \wedge \bar{\pi}^k_i = \dfrac{1}{2} \bar{R}_i{}^j{}_{hk} \, \pi^h \wedge \pi^k, \end{cases} \tag{3.8}$$

where $\bar{R}_i{}^j{}_{hk}$ are the components of the Riemannian curvature of $\bar{\Gamma}_1$. Then we get from the first equations of (3. 7) and (3. 8)

$$(\bar{\pi}^j_i - \pi^j_i) \wedge \pi^i = \widetilde{C}_i{}^j{}_k \, \pi^i \wedge \varphi^k.$$

If we put

$$\varphi^k = L^k_j \, \pi^j \tag{3.9}$$

and substitute them into the above equations, we get

$$(\bar{\pi}^j_i - \pi^j_i) \wedge \pi^i = \widetilde{C}_i{}^j{}_k \, L^k_h \, \pi^i \wedge \pi^h,$$

hence by a E. Cartan's lemma $\bar{\pi}^j_i$ can be written as

$$\bar{\pi}^j_i = \pi^j_i - \widetilde{C}_i{}^j{}_k \, L^k_h \, \pi^h + F_i{}^j{}_h \, \pi^h, \tag{3.10}$$

$$F_i{}^j{}_h - F_h{}^j{}_i = 0. \tag{3.11}$$

On the other hand, we get from (3. 4) and (3. 5) the equations :

$$(\bar{\pi}^k_i - \pi^k_i)\tilde{g}_{kj} + (\bar{\pi}^k_j - \pi^k_j)\tilde{g}_{ik} = 0,$$

and substituting (3. 10) into these equations, we get

$$F_{ijh} + F_{jih} = \widetilde{C}_{ijk} \, L^k_h + \widetilde{C}_{jik} \, L^k_h = 2\widetilde{C}_{ijk} \, L^k_h. \tag{3.12}$$

From (3. 11) and (3. 12), we obtain

$$F_{ihj} = \widetilde{C}_{ihk} \, L^k_j + \widetilde{C}_{jhk} \, L^k_i - \widetilde{C}_{ijk} \, L^k_h \tag{3.13}$$

or

$$F_i{}^h{}_j = \widetilde{C}_i{}^h{}_k \, L^k_j + \widetilde{C}_j{}^h{}_k L^k_i - \widetilde{C}_{ijk} \, \tilde{g}^{hm} \, L^k_m. \tag{3.13'}$$

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17) In this case, we may consider u^i as local coordinates of $\mathfrak{X}_1$.

Thus $\bar{\pi}_i^j$ can be written as

$$\bar{\pi}_i^j = \pi_i^j + \widetilde{C}_h{}^j{}_k L_i^k \pi^h - \widetilde{C}_{ihk} \tilde{g}^{jm} L_m^k \pi^h. \tag{3.14}$$

Now, *if we use only the orthonormal frames of* $\mathfrak{F}_1$ *such that* $y^i = \delta_n^i$, then we have the relations

$$\varphi^k = \pi_n^k = L_j^k \pi^j, \quad L_j^n = 0, \quad \widetilde{C}_{nij} = 0. \tag{3.15}$$

Hence from (3.14), we get

$$\begin{cases} \bar{\pi}_n^\beta = \pi_n^\beta + \widetilde{C}_h{}^\beta{}_\alpha L_n^\alpha \pi^h = (L_h^\beta + \widetilde{C}_h{}^\beta{}_\alpha L_n^\alpha)\pi^h, \\ d\bar{\pi}_n^\beta = d\pi_n^\beta + d(\widetilde{C}_\gamma{}^\beta{}_\alpha L_n^\alpha) \wedge \pi^\gamma + \widetilde{C}_\gamma{}^\beta{}_\alpha L_n^\alpha d\pi^\gamma, \\ \bar{\pi}_\gamma^\beta \wedge \bar{\pi}_n^\gamma = (\pi_\gamma^\beta + \widetilde{C}_\sigma{}^\beta{}_\rho L_\gamma^\rho \pi^\sigma - \widetilde{C}_{\gamma\sigma\rho} L_\beta^\rho \pi^\sigma) \wedge (\pi_n^\gamma + \widetilde{C}_\mu{}^\gamma{}_\tau L_n^\tau \pi^\mu), \end{cases} \tag{3.16}$$

where greek indices α, β, γ, $\cdots$ run over $1, 2, \cdots, n-1$.

Now, we *assume that* $\mathfrak{X}_1$ *contains the lift* $\bar{C}$ *in* $T_0(\mathfrak{X})$ *of a geodesic arc* C *of the Finsler manifold* $(\mathfrak{X}, L)$. Along $\bar{C}$, we have $\theta^\alpha = \theta_n^\alpha = 0$, hence it follows that

$$L_n^\alpha = 0 \quad \text{on } \bar{\pi}^{-1}(\bar{C}). \tag{3.17}$$

Accordingly, at any point of $\bar{\pi}^{-1}(\bar{C})$, (3.16) is written as

$$\begin{cases} \bar{\pi}_n^\beta = \pi_n^\beta = L_j^\beta \pi^j \\ d\bar{\pi}_n^\beta = d\pi_n^\beta + \widetilde{C}_\gamma{}^\beta{}_\alpha dL_n^\alpha \wedge \pi^\gamma \\ \bar{\pi}_\gamma^\beta \wedge \bar{\pi}_n^\gamma = (\pi_\gamma^\beta + \widetilde{C}_\sigma{}^\beta{}_\rho L_\gamma^\rho \pi^\sigma - \widetilde{C}_{\gamma\sigma\rho} L_\beta^\rho \pi^\sigma) \wedge \pi_n^\gamma. \end{cases}$$

Hence, at any point of $\bar{\pi}^{-1}(\bar{C})$, we have

$$\bar{\theta}_n^\beta = \theta_n^\beta + \widetilde{C}_\gamma{}^\beta{}_\alpha dL_n^\alpha \wedge \pi^\gamma + (\widetilde{C}_\sigma{}^\beta{}_\rho L_\gamma^\rho - \widetilde{C}_{\gamma\sigma\rho} L_\beta^\rho) L_\tau^\gamma \pi^\sigma \wedge \pi^\tau$$

and

$$\theta_n^\beta = \frac{1}{2} \widetilde{R}_n{}^\beta{}_{hk} \pi^h \wedge \pi^k + \widetilde{P}_n{}^\beta{}_{hm} L_k^m \pi^h \wedge \pi^k + \frac{1}{2} \widetilde{S}_n{}^\beta{}_{lt} L_h^l L_k^t \pi^h \wedge \pi^k.$$

Furthermore, if we put $dL_n^\alpha = M_i^\alpha \pi^i$, then $M_n^\alpha = 0$ at any point of $\bar{\pi}^{-1}(\bar{C})$ because we have $dL_n^\alpha = 0$ along $\bar{\pi}^{-1}(\bar{C})$ and $\bar{\pi}^{-1}(\bar{C})$ is given by $\theta^\alpha = 0$, $\theta_n^\alpha = 0$. Hence, it must hold at any point of $\bar{\pi}^{-1}(\bar{C})$

$$\bar{R}_n{}^\beta{}_{n\gamma} = \widetilde{R}_n{}^\beta{}_{n\gamma} + \widetilde{P}_n{}^\beta{}_{nk} L_\gamma^k + \widetilde{C}_\gamma{}^\beta{}_\alpha M_n^\alpha = \widetilde{R}_n{}^\beta{}_{n\gamma}$$

since $\widetilde{P}_n{}^\beta{}_{nk} = M_n^\alpha = 0$. Thus, we obtain the following

Theorem 4. *For any local cross section* $\mathfrak{X}_1$ *of* $T_0(\mathfrak{X})$ *which contains the lift* $\bar{C}$ *of a geodesic arc* C *of the Finsler manifold* $(\mathfrak{X}, L)$, *the sectional curvatures* $R(y, E)$ *and* $\bar{R}(y, E)$ *with respect to* Γ *and* $\bar{\Gamma}_1$ *for any two dimentional plane* E *containing the tangent direction* y *at any* *point of C coincide with each other.*

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