A NOTE ON GALOIS EXTENSIONS OF DIVISION RINGS

NOBUO NOBUSAWA

The purpose of this note is to prove a generalization of one of the theorems in Galois theory of commutative fields. Let a division ring P be Galois and locally finite[1] over its subring Φ. If Δ is a division subring of $V(V(\Phi))$ containing Φ such that it is a Galois extension of Φ and if Σ is an arbitrary division subring of P containing Φ, then $\Sigma\Delta$, the division subring of P generated by Σ and Δ, is an outer Galois extension of Σ and its Galois group is isomorphic to that of $\Delta/\Sigma \cap \Delta$.

In § **1**, it will be proved that, if a division ring P is locally finite over a division subring Φ, then P is also locally finite over $V(V(\Phi))$. In § 2, we shall prove that if P is Galois and locally finite over Φ then P is also Galois over any division subring which is finite over Φ. And finally in § 3, the above-mentioned theorem will be proved.

1. Throughout the present note, let P be a division ring and Φ a division subring of P. For any non-zero element ρ of P, we denote by $\tilde{\rho}$ the inner automorphism induced by ρ : $\tilde{\rho} = \rho_l \rho_r^{-1}$. Similarly $\widetilde{\Gamma}$ will mean the totality of inner automorphisms induced by non-zero elements of Γ where Γ is a subset of P. For a subring Σ of P, $(\tilde{\rho})_\Sigma$ and $\widetilde{\Gamma}_\Sigma$ mean the restrictions of $\tilde{\rho}$ and $\widetilde{\Gamma}$ onto Σ. $V(\Sigma)$ means as usual the centralizer of Σ in P. Then $\widetilde{V(\Phi)}_\Sigma P_r$ is naturally a P_r-right module.

Lemma 1. *Let $\alpha_1, \cdots, \alpha_n$ be non-zero elements of $V(\Phi)$. Then $(\tilde{\alpha}_1)_\Sigma, \cdots, (\tilde{\alpha}_n)_\Sigma$ are linearly right-independent over P_r if and only if $\alpha_1, \cdots, \alpha_n$ are linearly right-independent over $V(\Sigma)$.*

Proof. If $\alpha_1, \cdots, \alpha_n$ are linearly right-dependent over $V(\Sigma)$, we have a non-trivial relation $\sum_{i=1}^{n} \alpha_i \pi_i = 0$ with $\pi_i \in V(\Sigma)$. Then $\sum_{i=1}^{n} \tilde{\pi}_i \tilde{\alpha}_i \alpha_{ir} \pi_{ir} = 0$, where we put $\tilde{\pi}_i = 0$ in case $\pi_i = 0$. Since $(\tilde{\pi}_i)_\Sigma = 1$, we have a non-trivial relation $\sum_{i=1}^{n} (\tilde{\alpha}_i)_\Sigma (\alpha_i \pi_i)_r = 0$, which implies that $(\tilde{\alpha}_1)_\Sigma, \cdots, (\tilde{\alpha}_n)_\Sigma$ are linearly right-dependent over P_r. Conversely suppose that $(\tilde{\alpha}_1)_\Sigma, \cdots, (\tilde{\alpha}_n)_\Sigma$ are linearly right-dependent over P_r. Then we have their non-trivial relations and let one of the shortest relations among them be, for instance,

[1] As to notations and terminologies used in this note we follow [3] and [4].

$\sum_{i=1}^{s}(\tilde{\alpha}_i)_\Sigma\ \rho_{ir}=0$ with non-zero $\rho_i\in P$. From this we have $\sum_{i=1}^{s}(\alpha_{il})_\Sigma\ \rho'_{ir}=0$ where $\rho'_i=\alpha_i^{-1}\rho_i$. In the above relation we may assume $\rho'_1=1$ from the beginning. Then we shall show that each ρ'_i is in $V(\Sigma)$. For, if not, $\rho'_j\notin V(\Sigma)$ for some j, that is, there exists an element σ of Σ such that $\sigma\rho'_j\neq\rho'_j\sigma$. Clearly we have $\sum_{i=1}^{s}(\sigma_r(\alpha_{il}))_\Sigma\ \rho'_{ir}-\sum_{i=1}^{s}(\alpha_{il})_\Sigma\ \rho'_{ir}\sigma_r=0$, whence we have $\sum_{i=2}^{s}(\alpha_{il})_\Sigma(\sigma_r\rho'_{ir}-\rho'_{ir}\sigma_r)=0$. Thus we obtain a shorter non-trivial relation $\sum_{i=2}^{s}(\alpha_{il})_\Sigma\ \theta_{ir}=0$ with $\theta_i=\sigma\rho'_i-\rho'_i\sigma$, being a contradiction. Accordingly we have shown $\rho'_i\in V(\Sigma)$. Then $0=\sum_{i=1}^{s}(\alpha_{il}\rho'_{ir})_\Sigma=\sum_{i=1}^{s}(\rho'_{il}\alpha_{il})_\Sigma$, that is, $\sum_{i=1}^{s}\alpha_i\rho'_i=0$ with $\rho'_i\in V(\Sigma)$, which completes our proof.

Lemma 2. *Let Σ be a subring of P containing Φ. Then $[\Sigma:\Phi]_l\geqq[V(\Phi):V(\Sigma)]_r$*[2]. *Moreover, if $V(V(\Phi))=\Phi$, equality holds in the above relation.*

Proof. Let $\mathfrak{M}$ be the set of all homomorphisms of Φ_l-module Σ into P. Then $\mathfrak{M}$ is a P_r-right module and $[\Sigma:\Phi]_l=[\mathfrak{M}:P_r]_r$. Clearly $\mathfrak{M}\supseteq\widetilde{V(\Phi)_\Sigma}\,P_r$. Since $[\widetilde{V(\Phi)_\Sigma}\,P_r:P_r]_r=[V(\Phi):V(\Sigma)]_r$ by Lemma 1, we have $[\Sigma:\Phi]_l\geqq[V(\Phi):V(\Sigma)]_r$. If, moreover, $V(V(\Phi))=\Phi$, $\mathfrak{M}$ is the topological closure of $\widetilde{V(\Phi)_\Sigma}P_r$ by Jacobson's density theorem [1, p. 31]. Then $[\Sigma:\Phi]_l=[\mathfrak{M}:P_r]_r=[\widetilde{V(\Phi)_\Sigma}\,P_r:P_r]_r=[V(\Phi):V(\Sigma)]_r$.

Theorem 1. *If P is locally finite over Φ, then P is also locally finite over $V(V(\Phi))$.*

Proof. Let Φ_0 be $V(V(\Phi))$ and $\Phi_0(\alpha_1,\cdots,\alpha_n)$ a subring generated by Φ_0 and a finite number of elements $\alpha_1,\cdots,\alpha_n$ of P. Then $\infty>[\Phi(\alpha_1,\cdots,\alpha_n):\Phi]_l\geqq[V(\Phi):V(\Phi(\alpha_1,\cdots,\alpha_n))]_r=[V(V(\Phi(\alpha_1,\cdots,\alpha_n))):V(V(\Phi))]_l=[\Phi_0(\alpha_1,\cdots,\alpha_n):\Phi_0]_l$ by Lemma 2.[3]

2. When a subring Φ of P is the fixring of an automorphism group of P, that is, when Φ consists of all the elements left invariant by an automorphism group of P, we say that P is Galois over Φ or P/Φ is Galois.

2) Provided that we do not distinguish between two infinite dimensions.

3) Note that $V(\Phi(\alpha_1,\cdots,\alpha_n))=V(\Phi_0(\alpha_1,\cdots,\alpha_n))$ and $V(V(\Phi(\alpha_1,\cdots,\alpha_n)))\supseteq\Phi_0(\alpha_1,\cdots,\alpha_n)$.

Theorem 2. *Let P/Φ be locally finite and Galois. Then P/Σ is Galois for each subring Σ of P containing Φ which is finite over Φ.*

Proof. We may assume here $P \neq \Sigma$. Let ρ be an arbitrary element of P not contained in Σ, and Σ' a subring of P generated by Σ and ρ. We denote by $\mathfrak{M}'$ the set of all homomorphisms of Φ_l-module Σ' into P. $\mathfrak{M}'$ is a Σ'_r-P_r two-sided module. Then, by Jacobson's density theorem, $\mathfrak{M}' = \mathfrak{G}_{\Sigma'} P_r$ where $\mathfrak{G}$ is a regular automorphism group of P/Φ. Similarly let $\mathfrak{M}$ be the set of all homomorphisms of Σ_l -module Σ' into P. Clearly $\mathfrak{M}' \supseteq \mathfrak{M}$. Since $\mathfrak{M}'$ is a completely reducible Σ'_r-P_r two-sided module, $\mathfrak{M}$ is also a completely reducible Σ'_r-P_r two-sided module. Now we shall show that $\mathfrak{M} = (\mathfrak{G}_{\Sigma'} \cap \mathfrak{M})P_r$. Suppose, on the contrary, $\mathfrak{M} \neq (\mathfrak{G}_{\Sigma'} \cap \mathfrak{M})P_r$. Then $\mathfrak{M}$ contains an irreducible Σ'_r-P_r two-sided submodule $\mathfrak{N}$ which is not wholly contained in $(\mathfrak{G}_{\Sigma'} \cap \mathfrak{M})P_r$. As $\mathfrak{N}$ is contained in $\mathfrak{M}' = \mathfrak{G}_{\Sigma'} P_r$, a similar argument as in the proof of [4, Lemma 3] proves that $\mathfrak{N} = T_{\Sigma'} P_r$ where T is an element of $\mathfrak{G}$. This implies that $\mathfrak{N} = T_{\Sigma'} P_r \subseteq (\mathfrak{G}_{\Sigma'} \cap \mathfrak{M})P_r$, which is a contradiction. Thus, setting $\mathfrak{H}_{\Sigma'} = \mathfrak{G}_{\Sigma'} \cap \mathfrak{M}$ with a subgroup $\mathfrak{H}$ of $\mathfrak{G}$, we have $\mathfrak{M} = \mathfrak{H}_{\Sigma'} P_r$. Naturally $\mathfrak{H}$ is identical on Σ. We shall show $\rho\mathfrak{H} \neq \rho$. In fact, $\rho\mathfrak{H} = \rho$ implies $\mathfrak{H}_{\Sigma'} = 1$, that is, $\mathfrak{M} = 1P_r$. But this contradicts $[\mathfrak{M} : P_r]_r = [\Sigma' : \Sigma]_l > 1$. Since ρ is an arbitrary element of P not contained in Σ, we have proved P/Σ is Galois.

As is easily seen from the above proof, we may restate Theorem 2 in the following way.

Theorem 2'. *Let P/Φ be Galois and locally finite, and let $\mathfrak{G}$ be a regular automorphism group of P/Φ. If Σ is an intermediate subring of P/Φ with $[\Sigma : \Phi]_l < \infty$ then there exists a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ such that Σ is the fixring of $\mathfrak{H}$.*

We may remark here the following : Let P/Φ be Galois. Then Theorem 2 shows that the assumtions (α) — (δ) introduced in [2; § 3] are fulfilled when and only when P is locally finite over Φ and $[V(\Phi) : V(P)] < \infty$.

3. For subrings Σ and Γ of P, we denote $V(V(\Sigma))$ by Σ_0 and denote by $\Sigma\Gamma$ the subring of P generated by Σ and Γ.

Lemma 3. *Let P/Φ be Galois and locally finite and let Σ be a subring of P containing Φ such that $[\Sigma\Phi_0 : \Phi_0]_l < \infty$. Then $\Sigma_0 = \Sigma\Phi_0$, and Σ_0/Σ is Galois and locally finite with Galois group which is isomorphic to that of $\Phi_0/\Phi_0 \cap \Sigma$.*

Proof. Since $V(\Sigma_0) = V(\Sigma\Phi_0)$, we obtain $\Sigma_0 \subseteq V(V(\Sigma\,\Phi_0))$ and

Lemma 2 shows $\infty > [\Sigma\Phi_0 : \Phi_0]_l = [V(\Phi_0) : V(\Sigma\Phi_0)]_r = [V(\Phi_0) : V(\Sigma_0)]_r = [\Sigma_0 : \Phi_0]_l$. Hence we have $\Sigma_0 = \Sigma\Phi_0 = \Phi_0(\alpha_1, \cdots, \alpha_n)$ with some $\alpha_1, \cdots, \alpha_n$ of Σ. Then $P/\Phi(\alpha_1, \cdots, \alpha_n)$ is Galois by Lemma 2, further $V(V(\Phi(\alpha_1, \cdots, \alpha_n))) = \Sigma_0$ implies that $\Sigma_0/\Phi(\alpha_1, \cdots, \alpha_n)$ is and hence Σ_0/Σ is outer Galois.[4] Since the Galois group $\mathfrak{H}'$ of $\Sigma_0/\Phi(\alpha_1, \cdots, \alpha_n)$ is locally finite, that of Σ_0/Σ is so, whence Σ_0 is locally finite over Σ.[5] Further, noting that $\mathfrak{H}'$ is the topological closure of $\mathfrak{G}'_{\Sigma_0}$ where $\mathfrak{G}'$ is the Galois group of $P/\Phi(\alpha_1, \cdots, \alpha_n)$ [4, Theorem 4], we readily see that $\Phi_0\mathfrak{H}' = \Phi_0$. In virtue of this fact, we shall prove that the Galois group $\mathfrak{H}$ of Σ_0/Σ is isomorphic with that of $\Phi_0/\Phi_0 \cap \Sigma$. Evidently $\mathfrak{H}_{\Phi_0}$ is an automorphism group of $\Phi_0/\Phi_0 \cap \Sigma$, and so $\mathfrak{H}_{\Phi_0}$ is dense in the Galois group $\mathfrak{H}_0$ of $\Phi_0/\Phi_0 \cap \Sigma$ by [4, Theorem 4]. Moreover, recalling that $\Sigma_0 = \Phi_0(\alpha_1, \cdots, \alpha_n)$, it is easy to see that $T \to T_{\Phi_0}$ $(T \in \mathfrak{H})$ is a continuous isomorphism of the compact group $\mathfrak{H}$ into the compact group $\mathfrak{H}_0$. Hence $\mathfrak{H}$ is isomorphic to $\mathfrak{H}_{\Phi_0} = \mathfrak{H}_0$.

Lemma 4. *Let P/Φ be Galois and locally finite. Then, for any subring Σ of P containing Φ, $\Phi_0\Sigma/\Sigma$ is outer Galois and locally finite with the Galois group isomorphic with that of $\Phi_0/\Phi_0 \cap \Sigma$.*

Proof. Since P/Φ_0 is locally finite by Theorem 1, $\Phi_0\Sigma = \bigcup_\nu \Gamma_\nu$ where Γ_ν are all the subrings of the form $\Gamma_\nu = \Phi_0(\sigma_1, \cdots, \sigma_n)$ for some $\sigma_i \in \Sigma$. Put $\Sigma_\nu = \Gamma_\nu \cap \Sigma$. Then it is easy to see that $\Gamma_\nu = \Phi_0\Sigma_\nu$ and $\Sigma_\nu \cap \Phi_0 = \Sigma \cap \Phi_0$. If T is an element of the Galois group $\mathfrak{H}_0$ of $\Phi_0/\Phi_0 \cap \Sigma$ then, as is seen from the proof of Lemma 3, T can be uniquely extended to an element $T^{(\nu)}$ of the Galois group $\mathfrak{H}^{(\nu)}$ of Γ_ν/Σ_ν and $\mathfrak{H}^{(\nu)}_{\Phi_0} = \mathfrak{H}_0$. Now if $\Gamma_\nu \subseteq \Gamma_\mu$ then $\Gamma_\nu T^{(\mu)} = (\Phi_0 T)(\Sigma_\nu T^{(\mu)}) = \Phi_0\Sigma_\nu = \Gamma_\nu$, that is, $T^{(\mu)}_{\Gamma_\nu} = T^{(\nu)}$. Thus we can define an automorphism $T^{(0)}$ of $\Phi_0\Sigma$ in the following way: $\rho T^{(0)} = \rho T^{(\nu)}$ if $\rho \in \Gamma_\nu$. We denote here by $\mathfrak{S}'$ the totality of these extended automorphisms of automorphisms in $\mathfrak{H}_0$. Then evidently $(\Phi_0\Sigma)\mathfrak{S}' = (\bigcup_\nu \Gamma_\nu)\mathfrak{S}' = \bigcup_\nu \Gamma_\nu = \Phi_0\Sigma$ and the fixring of $\mathfrak{S}'$ is $\bigcup_\nu \Sigma_\nu = \Sigma$. Clearly $V(\Sigma) = V(\Phi_0\Sigma)$, and hence $\Phi_0\Sigma/\Sigma$ is outer Galois. Noting that $\mathfrak{S}'$ is identical on Σ, we readily see that $\mathfrak{S}'$ is a locally finite group of $\Phi_0\Sigma/\Sigma$, whence $\Phi_0\Sigma/\Sigma$ is locally finite [4, p. 43] and the Galois group $\mathfrak{S}$ of $\Phi_0\Sigma/\Sigma$ is the topological closure of $\mathfrak{S}'$ [4, Theorem 4]. Accordingly $\Phi_0\mathfrak{S} = \Phi_0$ and $\Gamma_\nu\mathfrak{S} = \Gamma_\nu$, and so our assertion will be easily seen by considering the mapping $S \to S_{\Phi_0}$ $(S \in \mathfrak{S})$. (In fact, $\mathfrak{S}'$ coincides with $\mathfrak{S}$.)

4) See [1, Proposition 7.6.3]. Cf. also [2], [3] and [4].

5) See [4, p. 43].

Finally we shall prove the following:

Theorem 3. *Let P/Φ be Galois and locally finite. If Λ is a subring of Φ_0 containing Φ, and Λ/Φ is Galois and if Σ is any subring of P containing Φ, then $\Lambda\Sigma/\Sigma$ is outer Galois and locally finite with the Galois group which is isomorphic to that of $\Lambda/\Lambda\cap\Sigma$.*

Proof. Noting that Λ is normal over Φ as a subring of Φ_0 by [4, Theorem 5], our assertion will be easily seen from the proof of Lemma 4.

REFERENCES

[1] N. JACOBSON, Structure of rings, New York, (1956).
[2] T. NAGAHARA and H. TOMINAGA, On Galois theory of division rings, Math. J. Okayama Univ., 6 (1956) 1—26.
[3] N. NOBUSAWA, An extension of Krull's Galois theory to division rings, Osaka Math. J., 7 (1955) 1—6.
[4] ———, On compact Galois groups of division rings, Osaka Math. J., 8 (1956) 43—50.

DEPARTMENT OF MATHEMATICS,
OSAKA UNIVERSITY

(*Received December 2, 1957*)