ON GALOIS THEORY OF DIVISION RINGS II

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It is the purpose of this note to present Galois theory for such a Galois extension K/L that K/L is locally Galois and $K/V_K(V_K(L))$ is of countable (countably finite or infinite) dimension. Although it seems that our present object is specially-fixed, our result really contains the previous one given in [2][1].

§ 1 contains several consequences from the assumption of being Galois and locally Galois, and our Galois theory is given in §2. As to notations and terminologies used in this note, we follow the previous paper [2].

1. Locally Galois extensions

Throughout this note, K be a division ring and L a division subring of K. Our first lemma is due to N. Nobusawa [3].

Lemma 1. *Let L' be an intermediate division subring of K/L.*

(i) *If $[L':L]_l < \infty$ then $[V_K(L):V_K(L')]_r < \infty$*[2]*, and $[V_{H'}(L):V_{H'}(H')] < \infty$ where $H' = V_K(V_K(L'))$.*

(ii) *Let $L = V_K(V_K(L))$. Then $[L':L]_l < \infty$ when and only when $[V_K(L): V_K(L')]_r < \infty$, and in the case $[L':L]_l = [V_K(L): V_K(L')]_r$.*

We set here the following definition.

Definition 1. K/L is said to be *locally Galois* if for any finite subset F of K there exists a division subring L' containing $L(F)$ that is Galois and finite over L.

For a while, we assume that K/L is Galois and locally finite and that $[V_K(L): V_K(K)] < \infty$, that is, the assumptions (α), (β), (γ) and (δ) in [2 ; § 3] are fulfilled by [3 ; Theorem 2]. Then [2; Lemma 10] shows that K/L is locally Galois, moreovere Theorems 9 and 11 of [2] give at once the following theorem.

Theorem 1. *Let K/L be Galois and locally finite. If $[V_K(L): V_K(K)] < \infty$ then for any intermediate subring K' of K/L there hold the follwing facts :*

(i) *K/K' is Galois and locally Galois.*

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1) Numbers in brackets refer to the references cited at the end of this note.

2) $[:]_l$ and $[:]_r$ denote the left and right dimensions respectively. And in case $[:]_l = [:]_r$, they are denoted as $[:]$.

(ii) *Each L-(ring) isomorphism of K' into K is contained in* $\mathfrak{G}(K/L)_{K'}$.

In the rest of this section, we always assume K/L is Galois and locally Galois, and set $H = V_K(V_K(L))$. We shall then investigate in what range being locally Galois is carried over to intermediate subrings. Our first result is the next, of which the proof is almost clear from [3 ; Theorem 2] and our definition.

Lemma 2. *Let K/L be Galois and locally Galois. If L' is an intermediate subring of K/L with $[L' : L] < \infty$ then K/L' is Galois and locally Galois.*

Our principal theorem of this section is stated as follows :

Theorem 2. *Let K/L be Galois and locally Galois. If K' is an intermediate subring of K/L with $[V_K(L) : V_K(K')]_r < \infty$ then K/K' is locally Galois, in particular, K/H is Galois and locally Galois.*

Proof. It is easy to see that there exists a subring L' of K' such that $[L' : L] < \infty$ and $V_K(K') = V_K(L')$. Now, for an arbitrary finite subset F of K, we can find a division suring L^* containing $L'(F)$ that is Galois and finite over L. Then $[V_{H^*}(L) : V_{H^*}(H^*)] < \infty$ by Lemma 1 (i), where $H^* = V_K(V_K(L^*))$. Noting that any $\sigma \in \mathfrak{G}(L^*/L)$ is contained in $\mathfrak{G}(K/L)_{L^*}$ by [1; Theorem 7. 4. 1] and that K/L^* is Galois (and so H^*/L^* is Galois) by Lemma 2, we can readily see that H^*/L is Galois. Thus H^*/L satisfies all the assumptions for K/L in Theorem 1, accordingly H^*/K' is Galois and locally Galois by Theorem 1 (i). Since $H^* \supset V_K(V_K(K'(F))) \supset K'(F) \supset K'$, we have proved K/K' is locally Galois.

Remark. In case K/L is locally Galois, it is easy to see that $[L' : L]_l < \infty$ yields $[L' : L]_r = [L' : L]_l$. Moreover let K/L be Galois and locally Galois, and let $K' \supset L$ be a division subring with $[V_K(L) : V_K(K')]_r < \infty$. Then, by Lemma 1 (ii), we obtain $[V_K(V_K(K')) : H]_l = [V_K(L) : V_K(K')]_r$. As K/H is locally Galois by Theorem 2, $[V_K(V_K(K')) : H]_l = [V_K(V_K(K')) : H]_r$, and a dual of Lemma 2 (ii) implies $[V_K(V_K(K')) : H]_r = [V_K(L) : V_K(K')]_l$, whence we have $[V_K(L) : V_K(K')]_r = [V_K(L) : V_K(K')]_l$. These remarks will be of use in the sequel.

We shall conclude this section with the following lemma, which will be required in the next section.

Lemma 3. *Let K/L be Galois and locally Galois. For any finite subset F of K, there exists a division subring K' containing $H(F)$ such that $[K' : H] < \infty$, $[V_{K'}(L) : V_{K'}(K')] < \infty$ and that K'/L is Galois*

Proof. In the proof of Theorem 2, if we set $L' = L$ then H^* may be adopted as our K'. Because, noting that $\infty > [V_K(L) : V_K(L^*)] = [V_K(H) : V_K(H^*)]$ by Lemma 1 (i) and $H = V_K(V_K(H))$, we obtain $[H^* : H] = [V_K(H) : V_K(H^*)] < \infty$ by Lemma 1 (ii).

2. Galois theory for certain locally Galois extensions

In their paper [4], one of the present authors and N. Nobusawa proved the following: *Let K/L be Galois and $\mathfrak{G}(K/L)$ be l.f.d. If K' is an arbitrary intermediate subring of K/L with $[V_K(L) : V_K(K')] < \infty$ then K/K' is Galois and $\mathfrak{G}(K/K')$ is l.f.d.* In case $\mathfrak{G}(K/L)$ is l.f.d., K/L is locally Galois of course. Concerning the above result, it is probable to suppose that under the assumptions of Theorem 2 K/K' is Galois, but it is an open question. However, in what follows, we shall give an affirmative answer for a certain class of Galois extensions.

In this section, we assume again that K/L is Galois and locally Galois, and further that K is of countable dimension over $H = V_K(V_K(L))$. Since our consideration for the case where $[K : H] < \infty$ is nothing but that given in [2; § 3], our present investigation will present a slight progress from the previous one in [2].

Lemma 4. *Let K/L be Galois and locally Galois, and $[K : H]_l \leqslant \aleph_0$. Then $\mathfrak{G}(H/L) = \mathfrak{G}(K/L)_H$.*

Proof. In virtue of Theorem 1, it suffices to prove our assertion for the case where $[K : H]_l = \aleph_0$. Let $\{k_1, k_2, \cdots\}$ be a (countably infinite) independent basis of K/H. Now, given any $\sigma \in \mathfrak{G}(H/L)$, we shall construct an automorphism σ^* of K/L such that $\sigma^*_H = \sigma$. By Lemma 3, we can find a division subring K_1 containing $H(k_1)$ such that $[K_1 : H] < \infty$, $[V_{K_1}(L) : V_{K_1}(K_1)] < \infty$ and that K_1/L is Galois. Then, by Theorem 1 (ii), σ can be extended to some $\sigma_1 \in \mathfrak{G}(K_1/L)$. Next let n_2 be the first integer such that $k_{n_2} \notin K_1$. Then, again by Lemma 3, we can find a division subring K_2 containing $K_1(k_{n_2})$ such that $[K_2 : H] < \infty$, $[V_{K_2}(L) : V_{K_2}(K_2)] < \infty$ and that K_2/L is Galois. And then σ_1 can be extended to some $\sigma_2 \in \mathfrak{G}(K_2/L)$ by Theorem 1 (ii) again. Repeating the same procedures, we obtain an ascending chain of subrings $K_1 \subsetneqq K_2 \subsetneqq K_3 \subsetneqq \cdots$ and an ascending chain of positive integers $1 < n_2 < n_3 < \cdots$ such that k_j is contained in K_i if and only if $j < n_{i+1}$, and obtain a sequence of automorphisms of K_i's each of which is an extension of the preceding one: $\sigma_1, \sigma_2, \sigma_3, \cdots$. Now we can define an extension σ^* of σ in the following way: $k_j^{\sigma^*} = k_j^{\sigma_i}$ for $j < n_{i+1}$ and $h^{\sigma^*} = h^\sigma$ for $h \in H$. Thus we have proved $\mathfrak{G}(H/L) \subset \mathfrak{G}(K/L)_H$, and so our lemma.

Now our first principal theorem is easily shown.

Theorem 3. *Let K/L be Galois and locally Galois, and $[K:H]_l \leqslant \aleph_0$. If K' is an intermediate division subring of K/L with $[V_K(L): V_K(K')] < \infty$ then K/K' is Galois and locally Galois.*

Proof. Choose a subring L' of K' such that $[L':L] < \infty$ and $V_K(L') = V_K(K')$. Then K/L' is Galois and locally Galois by Lemma 2, and there holds $[K:H']_l \leqslant \aleph_0$ where $H' = V_K(V_K(L'))$. As H'/L' is outer Galois, H'/K' is Galois by Theorem 1 (i). Further noting that $V_K(V_K(H')) = H'$ and $\mathfrak{G}(H'/L') = \mathfrak{G}(K/L')_{H'}$ by Lemma 4, we can readily see that K/K' is Galois. The rest of the proof is contained in Theorem 2.

Our next task is to present the following extension theorem.

Theorem 4. *Let K/L be Galois and locally Galois, and $[K:H]_l \leqslant \aleph_0$. Then for any intermediate division subrings K_1, K_2 with $[V_K(L): V_K(K_i)] < \infty$ $(i = 1, 2)$, each L-(ring) isomophism of K_1 onto K_2 is contained in $\mathfrak{G}(K/L)_{K_1}$.*

Proof. By the validity of Theorem 1, Lemma 4 and [1 ; Theorem 7.4.1] the proof proceeds as in the proof of [4 ; Theorem 2], and it may be left to readers.

Example. We consider here $K = K_1 \times K_2 \times \cdots \times K_n \times \cdots$ treated in [2 ; § 4, (a)]. Then the division subring $N = K_1 \times M_2 \times \cdots \times K_{2n-1} \times M_{2n} \times \cdots$ is evidently Galois and locally Galois over Z (the rational number field), and $[N: V_N(V_N(Z))] = [N: V_N(N)] = \aleph_0$. Thus N/Z satisfies all the assumptions for K/L in Theorem 3, while $\mathfrak{G}(N/Z)$ is not l. f. d. by [2 ; Theorem 3].

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