GALOIS THEORY OF SIMPLE RINGS III

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For Galois theory of simple rings, one of the present authors put forward his consideration under the assumption that the total group is l. f. d. and locally compact ([2] and [3]),[1] and obtained a somewhat satisfactory extension of Krull's theory given for fields, which contains Nakayama's for the case of finite degree as well.

Now, as is noted in [1], our next attention should be directed towards considering the theory only assuming that the total group is l. f. d. Regarding this problem, Prof. D. Zelinsky tried once to construct his theory, but his trial was not successful at that time. In this note, we shall present a slight generalization of the previous theory under the assumption that the total group is l. f. d.

§ **1** contains previous results as preliminaries, and our generalization will be found in § **2**. As to notations and terminologies used in this paper, we follow the previous ones [2] and [3].

1. Preliminaries

Throughout this section we assume that R is a simple ring which is Galois over a simple subring S, and $\mathfrak{G}$ will mean the total group $\mathfrak{G}(R/S)$.

In [3], essentially in [1], the notion of l. f. d. was introduced, and the following facts were obtained :

(**A**) Let R be locally finite over S. Then $\mathfrak{G}$ is l. f. d. if and only if either $\mathfrak{G}$ is outer or $[V_R(S) : V_S(S)] < \infty$[2] ([3 ; (**b***)]).

(**B**) If $\mathfrak{G}$ is l. f. d. then $H = V_R(V_R(S))$ is simple and $\mathfrak{G}(R/H) = \mathfrak{G}(H)$[3] is l. f. d. ([3 ; (**f***)]).

(**C**) Let $\mathfrak{G}$ be l. f. d. and non-outer, and F be an arbitrary finite subset of R. If N is a regular subring which is normal, finite over S and contains $S(F, V_R(S))$ then $T = V_R(V_R(N))$ is a regular subring normal over S such that $[T : H] < \infty$[4] and $[V_T(S) : V_T(T)] < \infty$ ([2; Lemma 11]).

We set here the following definition which, in case $[V_R(S) : V_R(R)]$

1) Numbers in brackets refer to the references cited at the end of this paper.

2) $V_R(S) = \{r \in R \mid rx = xr \text{ for all } x \in S\}$.

3) $\mathfrak{G}(H) = \{\sigma \in \mathfrak{G} \mid x^\sigma = x \text{ for all } x \in H\}$.

4) Throughout the paper, the dimension means the left dimension, and we alway consider left modules.

$<\infty$, coincides with that of (*)-regularity defined in [3].

Definition 1. Let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$, and set $V_{\mathfrak{H}}=V_R(J(\mathfrak{H}, R))$[5]. $\mathfrak{H}$ is said to be *$(*_f)$-regular* if $V_{\mathfrak{H}}$ is a simple subring over which $V_R(S)$ is finite and $\mathfrak{H}$ contains $\widetilde{V_{\mathfrak{H}}}$.[6]

In the rest of this section, we assume further that $\mathfrak{G}$ is l. f. d., and $\mathfrak{G}$ be considered as a topological group in the sense of [2] and [3]. Then also the following facts will be found in [2] and [3].

(D) If S' is an arbitrary regular subring of R that is finite over S then there holds $S'=J(\mathfrak{G}(S'), R)$, and any S-(ring) isomorphism ρ of S' into R can be extended to an automorphism contained in $\mathfrak{G}$, where we assume S'^{ρ} is regular ([2 ; Lemma 10] and [3 ; (**e***)]).

(E) $\mathfrak{G}$ is compact if and only if it is locally finite, or if it is almost outer. While it is discrete if and only if R is finite over S ([3 ; (**d***)]).

(F) $\mathfrak{G}$ is locally compact if and only if $[V_R(S):V_R(R)]<\infty$ ([3 ; (**h***)]).

(G) If $\mathfrak{G}$ is locally compact then there exists a 1-1 dual correspondence between closed (*)-regular subgroups $\mathfrak{H}$ of $\mathfrak{G}$ and intermediate regular subrings R' of R/S in the usual sense of Galois theory, and $\mathfrak{G}(R/R')$ is l. f. d. ([3 ; (**k***), (**c****) and Theorem 4]).

(H) If $\mathfrak{G}$ is locally compact then, for any intermediate regular subring R', each S-(ring) isomorphism ρ of R' into R can be extended to an automorphism contained in $\mathfrak{G}$, where we asume $V_R(R'^{\rho})$ is simple ([3; **l***)]).

It will be almost clear that the following lemma can be proved in the same way as in the proof of [3 ; Theorem 6], and the details may be left to readers.

Lemma 1. *Any $(*_f)$-regular subgroup of $\mathfrak{G}$ is a regular subgroup.*

Further noting that, in the proof of [2 ; Theorem 7], the essential assumption is not $[V_R(S):V_R(R)]<\infty$ but $[V_R(S):V_{\mathfrak{H}}]<\infty$, the proof and Lemma 1 give at once the next corollary.

Corollary 1. *Any closed $(*_f)$-regular subgroup of $\mathfrak{G}$ is a regular total subgroup.*

2. Galois correspondence and extension theorem

It is the purpose of this section to generalize (G) and (H) of § 1. In the sequel, we assume again that R is a simple ring which is Galois over

5) $J(\mathfrak{H}, R)=\{x\in R \mid x^{\sigma}=x \text{ for all } \sigma\in\mathfrak{H}\}$.

6) $\widetilde{V_{\mathfrak{H}}}$ means the totality of inner automorphisms generated by regular elements of $V_{\mathfrak{H}}$.

a simple subring S and that the total group $\mathfrak{G}=\mathfrak{G}(R/S)$ is l. f. d. Further, as $H=V_R(V_R(S))$ is simple by (B), we set $H=\sum_{h,k=1}^{m} Cd_{hk}$ where d_{hk}'s are matric units and $C=V_H(\{d_{hk}\})$ is a division ring.

Now we shall begin our course with introducing a new (not necessarily separable) topology $\mathscr{T}$ into $\mathfrak{G}$:

($\mathscr{T}$) A fundamental system of neighborhoods of the identity is $\{\mathfrak{G}(H')\}$ where H' runs over all the subrings of H which are normal, finite over S and contain $\{d_{hk}\}$.

Evidently, in case $\mathfrak{G}$ is outer, the topology $\mathscr{T}$ coincides with the finite topology considered in § 1, and so $\mathfrak{G}$ is compact by (E). On the other hand, if $\mathfrak{G}$ is non-outer then $[V_R(S):V_S(S)]<\infty$ by (A), and so we have $R=\bigcup_\alpha R_\alpha$ where R_α runs over all the simple subrings which are normal, finite over S and contain $V_R(S)$ as well as $\{d_{hk}\}$. Here we set $\mathfrak{G}_\alpha=\mathfrak{G}(R_\alpha/S)$, and topologize them in the following way:

($\mathscr{T}_\alpha$) A fundamental system of neighborhoods of the identity is $\{\mathfrak{G}_\alpha(H')\}$ where H' runs over all the subrings of $H\cap R_\alpha$ which are normal, finite over S and contain $\{d_{hk}\}$.

Noting that $H\cap R_\alpha/S$ is outer Galois and $\mathfrak{G}_\alpha(H\cap R_\alpha)$ is the least neighborhood of the identity, one can readily see that there exists only a finite number of open subsets of $\mathfrak{G}_\alpha$. Hence $\mathfrak{G}_\alpha$ is compact. Since $\mathfrak{G}$ may be considered naturally as the inverse limit of the compact groups $\mathfrak{G}_\alpha$'s, it is compact too. While we can readily see that the last topology of $\mathfrak{G}$ is equivalent to $\mathscr{T}$, we have proved therefore the following lemma.

Lemma 2. *$\mathfrak{G}$ is compact with respect to the topology $\mathscr{T}$.*

Now we can prove the next lemma which is essential in the present investigation.

Lemma 3. $\mathfrak{G}(H/S)=\mathfrak{G}_H$.[7)]

Proof. We set $H=\bigcup_\alpha H_\alpha$ where H_α runs over all the subrings of H which are normal, finite over S and contain $\{d_{hk}\}$. Given any $\sigma\in\mathfrak{G}(H/S)$, as $V_R(H_\alpha^\sigma)=V_R(S)$, σ_{H_α} can be extended to an automorphism contained in $\mathfrak{G}$ by (D). We shall denote here by $\mathfrak{N}_\alpha$ the totality of these extensions of σ_{H_α}. Then clearly $\mathfrak{N}_\alpha$ is closed (with respect to $\mathscr{T}$), and

7) $\mathfrak{G}_H$ means the restriction of $\mathfrak{G}$ onto H. Similarly, for $\sigma\in\mathfrak{G}$ and a subset T, σ_T means the restriction of σ onto T. It may not be meaningless to remark here the following: Combining (C) and (H), one will readily see that each S-automorphism of H can be extended to an automorphism of any finite-demensional normal extension over H. Accordingly, in the particular case where R is of countably infinite dimension over H, the assertion $\mathfrak{G}(H/S)=\mathfrak{G}_H$ is a consequence of (C) and (H).

further the collection $\{\mathfrak{N}_\alpha\}$ possesses the finite intersection property. Since $\mathfrak{G}$ is compact by Lemma 2, we obtain $\bigcap_\alpha \mathfrak{N}_\alpha$ is non-empty. Evidently any $\sigma^* \in \bigcap_\alpha \mathfrak{N}_\alpha$ is a desired extension of σ, which proves $\mathfrak{G}(H/S) \subset \mathfrak{G}_H$. As the converse inclusion is trivial, our proof is complete.

Lemma 4. *Let R' be an intermediate regular subring of R/S with $[V_R(S) : V_R(R')] < \infty$. Then $J(\mathfrak{G}(R'), R) = R'$.*

Proof. By the assumption $[V_R(S) : V_R(R')] < \infty$, we can find a simple subring S' of R' finite over S and satisfying $V_R(R') = V_R(S')$. By (D), R/S' is Galois and $\mathfrak{G}' = \mathfrak{G}(R/S')$ is l. f. d. Noting that $S' \subset R' \subset V_R(V_R(S')) = H'(= V_R(V_R(H')))$, the preceding lemma and (G) imply $J(\mathfrak{G}(R'), R) = J(\mathfrak{G}'_{H'}(R'), H') = R'$.

In the rest of this section, $\mathfrak{G}$ will be considered always as a topological group in the sense of § 1. Combining Lemma 4 with Corollary 1, we obtain the first part of the following theorem which contains (G).

Theorem 1. *Let $\mathfrak{G}$ be l. f. d. Then there exists a* 1-1 *dual correspodence between closed $(*_f)$-regular subgroups $\mathfrak{H}$ of $\mathfrak{G}$ and intermediate regular subrings R' with $[V_R(S) : V_R(R')] < \infty$ in the usual sense of Galois theory, and $\mathfrak{G}(R/R')$ is l. f. d.*

Proof. Since R/R' is Galois by Lemma 4, it suffices to show that R is locally finite over R'. In case $\mathfrak{G}$ is outer, our assertion is contained in (G). Thus hereafter we shall restrict our attention to the case where $\mathfrak{G}$ is non-outer. By our assumption, there exists a simple subring S' of R' finite over S with $V_R(S') = V_R(R')$. Now let F be an arbitrary finite subset of R, and N a regular subring which is normal, finite over S and contains $S'(V_R(S), F)$. Then, by (C), $T = V_R(V_R(N))$ is a normal regular subring such that $[V_T(S) : V_T(T)] < \infty$. Noting that $T = V_R(V_R(T)) = V_R(V_R(N)) \supset V_R(V_R(S'(F))) = V_R(V_R(R'(F))) \supset R'(F)$, our assertion is an easy consequence of (G).

Our next task is to present a generalization of (H) corresponding to Theorem 1, which is stated as follows :

Theorem 2. *Let $\mathfrak{G}$ be l. f. d., and R_1, R_2 arbitrary intermediate regular subrings of R/S with $[V_R(S) : V_R(R_i)] < \infty$ $(i = 1, 2)$. Then any S-(ring) isomorphism ρ of R_1 onto R_2 can be extended to an automorphism contained in $\mathfrak{G}$.*

Proof. By our assumption, there exists a simple subring S_1 of R_1 finite over S with $V_R(S_1) = V_R(R_1)$ and $V_R(S_1{}^\rho) = V_R(R_2)$. Then, by (D), there exists some $\sigma \in \mathfrak{G}$ such that $\rho_{S_1} = \sigma_{S_1}$. Clearly $\rho\sigma^{-1}$ is an S_1-

isomorphism of R_1 into R and $V_R(R_1^{\rho\sigma^{-1}}) = (V_R(R_2))^{\sigma^{-1}}$ is a simple ring over which $V_R(S)$ is finite. If we can prove that $\rho\sigma^{-1} = \tau_{R_1}$ for some $\tau \in \mathfrak{G}(S_1)$, $\tau\sigma$ is a required extension of ρ. Thus, noting that $S_1 \subset R_1 \subset V_R(V_R(S_1))$, R/S_1 is Galois and that $\mathfrak{G}(R/S_1)$ is l. f. d. by (**D**), it suffices to prove our theorem for the case where R_1 is contained in H. Under this situation, the argument in the proof of [2 ; Lemma 15] shows that $R_2 \subset H$. Consequently ρ can be extended to an automorphism in $\mathfrak{G}$ by (**H**) and Lemma 3.

References

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