A CONDITION THAT A SPACE IS GROUP-LIKE[1)]

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1. Introduction

In the previous note [4], we studied a condition that a topological space is an H-space, and represented a necessary and sufficient one by the notions of the weak homotopy equivalence and the contractibility ([4], §1, Theorem 1). In the present note, we shall study conditions that a space is a homotopy-associative H-space or a group-like space,[1)] by discussing similarly with [4].

Our main result is

Theorem 1.1. *A countable CW-complex F is a group-like space if, and only if, there exist a countable CW-complex E, containing F as its subcomplex, and a topological space B and a (continuous) map p of E into B, satisfying the following properties* (1. 2) *and* (1. 3):

(1. 2) *E is contractible in itself to a vertex $\varepsilon \in F$, leaving ε fixed (throughout the contraction)*;

(1. 3) *$p(F)=b$, a point of B, and the map $p:(E, F) \to (B, b)$ is a weak homotopy equivalence, i. e., its induced homomorphism*

$$p_*:\pi_n(E, F)\to \pi_n(B, b),$$

of the relative homotopy groups, is an isomorphism onto for every integer $n>0$.

Concering to the homotopy-associativity of a *CW-complex F such that the product space $F\times F\times F$ is also a CW-complex*, we obtain the following two theorems.

Theorem 1.4.[2)] *We assume that there exist topological spaces $E_2 \supset E_1 \supset F$, $B_2 \supset B_1 \ni b$ (a point) and a map p: $(E_2, E_1, F)\to(B_2, B_1, b)$, satisfying the following properties* (1. 5)—(1. 8):

(1. 5) *F is contractible in E_1 to a vertex $\varepsilon \in F$, leaving ε fixed*;

(1. 6) *E_1 is a CW-complex containing F as its subcomplex, and $E_1\times F$ is also a CW-complex. Also, E_1 is contractible in E_2 to ε, leaving*

1) Cf. §§2—3 below, for the definitions of these spaces. The author notices here that the term "group-like space" is used, in this note, in the more restricted sense than [5].

2) The existence of $p:(E_1, F)\to(B_1, b)$ satisfying (1. 5) and (1. 7) is a necessary and sufficient condition for the fact that F is an H-space, ([4], §1, Theorem 1).

ε fixed;

(1. 7) *$p \mid E_1 : (E_1, F) \to (B_1, b)$ is a weak homotopy equivalence, i. e.,*
$(p \mid E_1)_* : \pi_n(E_1, F) \approx \pi_n(B_1, b)$, *for every integer* $n > 0$;

(1. 8) *$p : (E_2, F) \to (B_2, b)$ is also a weak homotopy eqivalence, i. e.,*
$p_* : \pi_n(E_2, F) \approx \pi_n(B_2, b)$, *for every integer* $n > 0$.

Then F is a homotopy-associative H-space having an inversion.

Theorem 1.9. *If F is a homotopy-associative H-space (and hence has an inversion by Lemma 2.5 below) and also satisfies the assumption (2. 13) below, then there exist topological sapces $E_2 \supset E_1 \supset F$, $B_2 \supset B_1 \ni b$ (a point) and a map $p : (E_2, E_1, F) \to (B_2, B_1, b)$ satisfying the properties (1. 5)—(1. 8).*

In the following two sections, H-spaces and group-like spaces are defined. Theorem 1. 9 and the necessity of Theorem 1. 1 are proved in §§4—7, constructing spaces and maps analogously with the constructions of [2, II]. The sufficiency of Theorem 1. 1 is proved in §8, and the last two sections are concerned with Theorem 1. 4.

2. Homotopy-associative H-spaces

A topological space F is called an *H-space*, if there is a (continuous) map $\mu : F \times F \to F$, called an *H-structure* or a *multiplication*, such that

(2. 1) $\mu(x, \varepsilon) = \mu(\varepsilon, x) = x$, for a fixed point ε (called an *unit*) and any point x of F.

An H-space F is said to be *homotopy-assciative*, if

(2. 2) the two maps $(x, y, z) \to \mu(\mu(x, y), z)$ and $(x, y, z) \to \mu(x, \mu(y, z))$ of $F \times F \times F = F^3$ into F are homotopic each other rel. $(\varepsilon, \varepsilon, \varepsilon) = \varepsilon^3 \in F^3$.

This homotopy will be denoted by $f_t : (F^3, \varepsilon^3) \to (F, \varepsilon)$, where

(2. 3) $f_0(x, y, z) = \mu(\mu(x, y), z)$, $f_1(x, y, z) = \mu(x, \mu(y, z))$, for $x, y, z \in F$.

An H-space F is said to have an *inversion*, if there is a (continuous) map $\sigma : (F, \varepsilon) \to (F, \varepsilon)$, called an inversion, such that

(2. 4) the maps $x \to \mu(x, \sigma(x))$ and $x \to \mu(\sigma(x), x)$ of F into itself are homotopic rel. ε to the constant map $x \to \varepsilon$, respectively.

We often write $x \cdot y$ or xy instead of $\mu(x, y)$ and x^{-1} instead of $\sigma(x)$.

By [4],[3] we have

Lemma 2.5. *Let F be a CW-complex such that $F \times F$ is also a CW-complex. If F is a homotopy-associative H-space, then it has an inversion.*

3) Cf. Lemma 6 of p. 118, Remark of p. 117 and Proof of Theorem 5 of p. 128 of [4].

In this note, we use the following notations concerning to the composition of homotopies:

Let X, Y be two spaces and $f_t^i : X \to Y$, $i = 1, \cdots, n$, be homotopies such that

$$f_0^1 = f^0;\ f_1^i = f_0^{i+1} = f^i \text{ for } i = 1, \cdots, n-1;\ f_1^n = f^n.$$

The homotopy $g_t : X \to Y$ between $g_0 = f$ and $g_1 = f^n$, defined by

$$g_t(x) = f_{nt-i+1}^i(x), \qquad \text{for } (i-1)/n \leqq t \leqq i/n,\ i=1, \cdots, n-1,\ x \in X,$$

will be denoted by

$$(2.6) \qquad g_t(x) = \left\{ f^0(x) \underset{f_t^1(x)}{\sim\sim} f^1(x) \underset{f_t^2(x)}{\sim\sim} \cdots \underset{f_t^n(x)}{\sim\sim} f^n(x) \right\}.$$

Now, let F be a homotopy-associative H-space having an inversion. Let $\varepsilon_t : (F, \varepsilon) \to (F, \varepsilon)$ be a first homotopy in (2. 4), such that

$$(2.7) \qquad \varepsilon_0(x) = \varepsilon, \quad \varepsilon_1(x) = x \cdot x^{-1}, \qquad \text{for } x \in F.$$

Let i_t, $\bar{\varepsilon}_t : (F, \varepsilon) \to (F, \varepsilon)$ and $j_t : (F \times F, (\varepsilon, \varepsilon)) \to (F, \varepsilon)$ be the homotopies, defined by using this and f_t of (2. 3) as follows: for $x, y \in F$,

$$(2.8) \quad i_t(x) = \left\{ x \underset{x \cdot \varepsilon_t}{\sim\sim} x \cdot (x^{-1} \cdot (x^{-1})^{-1}) \underset{f_{1-t}}{\sim} (x \cdot x^{-1}) \cdot (x^{-1})^{-1} \underset{\varepsilon_{1-t} \cdot (x^{-1})^{-1}}{\sim\sim\sim\sim\sim\sim} (x^{-1})^{-1} \right\};$$ [4]

$$(2.9) \qquad \bar{\varepsilon}_t(x) = \left\{ \varepsilon \underset{\varepsilon_t}{\sim} x^{-1} \cdot (x^{-1})^{-1} \underset{x^{-1} \cdot i_{1-t}}{\sim\sim\sim} x^{-1} \cdot x \right\};$$

$$(2.10) \qquad j_t(x, y) = \left\{ (xy)^{-1} \underset{(xy)^{-1}\varepsilon_t}{\sim\sim\sim} (xy)^{-1} \cdot (x \cdot x^{-1}) \underset{(xy)^{-1}(x(\varepsilon_t \cdot x^{-1}))}{\sim\sim\sim\sim\sim\sim\sim} \right.$$
$$(xy)^{-1} \cdot (x((yy^{-1})x^{-1})) \underset{(xy)^{-1}(xf_t)}{\sim\sim\sim\sim\sim} (xy)^{-1} \cdot (x(y(y^{-1}x^{-1}))) \underset{(xy)^{-1}f_{1-t}}{\sim\sim\sim\sim}$$
$$\left. (xy)^{-1} \cdot ((xy)(y^{-1}x^{-1})) \underset{f_{1-t}}{\sim} ((xy)^{-1}(xy)) \cdot (y^{-1}x^{-1}) \underset{\bar{\varepsilon}_{1-t} \cdot (y^{-1}x^{-1})}{\sim\sim\sim\sim\sim} y^{-1}x^{-1} \right\}.$$

Combining these homotopies, we obtain several homotopies, and

Lemma 2.11. *Among them, the following homotopies* $F^n \to F$, *for* $n = 1, 2$, *are homotopic rel.* $F^n \times \dot{I} \cup \varepsilon^n \times I$[5] *to the stationary homotopies, considering the homotopies as the maps* $F^n \times I \to F$:

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4) In these notations, $x \cdot \varepsilon_t$, f_{1-t} and $\varepsilon_{1-t} \cdot (x^{-1})^{-1}$ are abbreviations of $x \cdot \varepsilon_t(x^{-1})$, $f_{1-t}(x, x^{-1}, (x^{-1})^{-1})$ and $\varepsilon_{1-t}(x) \cdot (x^{-1})^{-1}$. We often use such abbreviations.

5) $F^n = F \times \cdots \times F$, the product space of n-copies of F, and $\varepsilon^n = (\varepsilon, \cdots, \varepsilon) \in F^n$. $I = [0,1]$, the closed interval, and $\dot{I} = \{0, 1\}$.

$$\left\{x \underset{\varepsilon_t\cdot x}{\sim\sim} (xx^{-1})x \underset{f_t}{\sim} x(x^{-1}x) \underset{x\cdot\bar{\varepsilon}_{1-t}}{\sim\sim\sim} x\right\},$$

$$\left\{\varepsilon \underset{\varepsilon_t}{\sim} (xy)(xy)^{-1} \underset{(xy)\cdot j_t}{\sim\sim\sim} (xy)(y^{-1}x^{-1}) \underset{f_{1-t}}{\sim\sim} ((xy)y^{-1})x^{-1} \underset{f_t\cdot x^{-1}}{\sim\sim} (x(yy^{-1}))x^{-1} \underset{(x\cdot\varepsilon_{1-t})\cdot x^{-1}}{\sim\sim\sim\sim\sim} xx^{-1} \underset{\varepsilon_{1-t}}{\sim\sim} \varepsilon\right\}.$$

This lemma follows directly from the definitions (2. 8)—(2. 10) and we shall omit its proofs.

Besides these homotopies, there are homotopies such that the author does not know whether they satisfy the properties of Lemma 2. 11, for examples,

$$\left\{x^{-1} \underset{i_t(x^{-1})}{\sim\sim\sim} ((x^{-1})^{-1})^{-1} \underset{(i_{1-t}(x))^{-1}}{\sim\sim\sim\sim} x^{-1}\right\},$$

$$\left\{((xy)z)w \underset{f_t\cdot w}{\sim\sim} (x(yz))w \underset{f_t}{\sim} x((yz)w) \underset{x\cdot f_t}{\sim\sim} x(y(zw)) \underset{f_{1-t}}{\sim\sim} (xy)(zw) \underset{f_{1-t}}{\sim\sim} ((xy)z)w\right\},$$

(2. 12. 1)
$$\left\{xy^{-1} \underset{x(\bar{\varepsilon}_t\cdot y)}{\sim\sim\sim} x((x^{-1}x)y^{-1}) \underset{xf_t}{\sim} x(x^{-1}(xy^{-1})) \underset{f_{1-t}}{\sim\sim} (xx^{-1})(xy^{-1}) \underset{\varepsilon_{1-t}\cdot(xy^{-1})}{\sim\sim\sim\sim} xy^{-1}\right\},$$

(2. 12. 2)
$$\left\{xy^{-1} \underset{(x\bar{\varepsilon}_t)y^{-1}}{\sim\sim\sim} (x(y^{-1}y))y^{-1} \underset{f_{1-t}\cdot y^{-1}}{\sim\sim\sim} ((xy^{-1})y)y^{-1} \underset{f_t}{\sim} (xy^{-1})(yy^{-1}) \underset{(xy^{-1})\varepsilon_{1-t}}{\sim\sim\sim\sim} xy^{-1}\right\}.$$

We consider the following assumptions for a homotopy-associative H-space F having an inversion, which is assumed in Theorem 1. 9 :

Assumption (2. 13). The homotopies f_t of (2. 3) and ε_t of (2. 7) can be so taken that the homotopies (2. 12. 1—2), of $F\times F$ into F, are homotopic rel. $F\times F\times \dot{I}\cup(\varepsilon,\varepsilon)\times I$ to the stationary homotopy, respectively, considering the homotopies as the maps $F\times F\times I\rightarrow F$.

For the later purpose, we prove the following lemmas, where F is assumed to satisfy (2. 13):

Lemma 2.14.1. *Let A be a space and K_t, $G_t: A\rightarrow F$ be homotopies and H_0, H_1 : $A\rightarrow F$ be maps such that*

(2. 15) $$K_t(a)\cdot(H_t(a))^{-1} = G_t(a), \qquad \textit{for} \quad t=0,\ 1;\ a\in A.$$

Let $H_t : A \to F$ be the following composed homotopy between H_0 and H_1:

$$\left\{ H_0 \underset{i_t}{\sim} (H_0^{-1})^{-1} \underset{(\bar{\varepsilon}_t \cdot H_0^{-1})^{-1}}{\sim\sim\sim\sim} ((K_0^{-1}K_0)H_0^{-1})^{-1} \underset{(f_t)^{-1}}{\sim\sim} (K_0^{-1}(K_0H_0^{-1}))^{-1} = (K_0^{-1}G_0)^{-1} \underset{(K_t^{-1}\cdot G_t)^{-1}}{\sim\sim\sim\sim\sim} (K_1^{-1}G_1)^{-1} = (K_1^{-1}(K_1H_1^{-1}))^{-1} \underset{(f_{1-t})^{-1}}{\sim\sim\sim} ((K_1^{-1}K_1)H_1^{-1})^{-1} \underset{(\bar{\varepsilon}_{1-t}\cdot H_1^{-1})^{-1}}{\sim\sim\sim\sim\sim} (H_1^{-1})^{-1} \underset{i_{1-t}}{\sim\sim} H_1 \right\}. \tag{2.16}$$ [6]

Then, two homotopies G_t and $K_t \cdot H_t^{-1}$, considering as the maps $A \times I \to F$, are homotopic each other rel. $A \times \dot{I}$.

Lemma 2.14.2. *Let A be a space and $G_t, H_t : A \to F$ be homotopies and $K_0, K_1 : A \to F$ be maps such that $K_t \cdot H_t^{-1} = G_t$, for $t = 0, 1$.*

Let $K_t : A \to F$ be the following composed homotopy between K_0 and K_1:

$$\left\{ K_0 \underset{K_0\bar{\varepsilon}_t}{\sim\sim} K_0(H_0^{-1}H_0) \underset{f_{1-t}}{\sim} (K_0H_0^{-1})H_0 = G_0H_0 \underset{G_t\cdot H_t}{\sim\sim\sim} G_1H_1 = (K_1H_1^{-1})H_1 \underset{f_t}{\sim} K_1(H_1^{-1}H_1) \underset{K_1\bar{\varepsilon}_{1-t}}{\sim\sim} K_1 \right\}.$$

Then, two homotopies G_t and $K_t \cdot H_t^{-1}$, considering as the maps $A \times I \to F$, are homotopic each other rel. $A \times \dot{I}$.

Proof of Lemma 2.14.1. $K_t \cdot H_t^{-1}$ is homotopic rel. $A \times \dot{I}$ to

$$\left\{ K_0H_0^{-1} \underset{K_0 i^{-1}}{\sim\sim} K_0((H_0^{-1})^{-1})^{-1} \underset{K_0((\bar{\varepsilon}\cdot H_0^{-1})^{-1})^{-1}}{\sim\sim\sim\sim\sim\sim\sim} K_0(((K_0^{-1}K_0)H_0^{-1})^{-1})^{-1} \underset{K_0(f^{-1})^{-1}}{\sim\sim\sim\sim} K_0((K_0^{-1}(K_0H_0^{-1}))^{-1})^{-1} \underset{K((K^{-1}\cdot G)^{-1})^{-1}}{\sim\sim\sim\sim\sim\sim\sim} K_1((K_1^{-1}(K_1H_1^{-1}))^{-1})^{-1} \underset{K_1(f^{-1})^{-1}}{\sim\sim\sim\sim} K_1(((K_1^{-1}K_1)H_1^{-1})^{-1})^{-1} \underset{K_1((\bar{\varepsilon}\cdot H_1^{-1})^{-1})^{-1}}{\sim\sim\sim\sim\sim\sim\sim} K_1((H_1^{-1})^{-1})^{-1} \underset{K_1 i^{-1}}{\sim\sim} K_1H_1^{-1} \right\},$$

and the latter is homotopic to

$$\left\{ K_0H_0^{-1} \underset{K_0(\bar{\varepsilon}\cdot H_0^{-1})}{\sim\sim\sim\sim} K_0((K_0^{-1}K_0)H_0^{-1}) \underset{K_0 f}{\sim\sim} K_0(K_0^{-1}(K_0H_0^{-1})) \underset{K(K^{-1}\cdot G)}{\sim\sim\sim\sim} K_1(K_1^{-1}(K_1H_1^{-1})) \underset{K_1 f}{\sim\sim} K_1((K_1^{-1}K_1)H_1^{-1}) \underset{K_1(\bar{\varepsilon}\cdot H_1^{-1})}{\sim\sim\sim\sim} K_1H_1^{-1} \right\},$$

by using the homotopy i_t, and also to

$$\left\{ K_0H_0^{-1} \underset{K_0(\bar{\varepsilon}\cdot H_0^{-1})}{\sim\sim\sim\sim} K_0((K_0^{-1}K_0)H_0^{-1}) \underset{K_0 f}{\sim\sim} K_0(K_0^{-1}(K_0H_0^{-1})) \underset{f}{\sim} (K_0K_0^{-1})(K_0H_0^{-1}) \right.$$

6) $H^{-1} : A \to F$ is the map or the homotopy defined by $H^{-1}(a) = (H(a))^{-1}$ for $a \in A$.

$$\underset{\varepsilon\cdot(K_0H_0^{-1})}{\sim\!\sim\!\sim\!\sim} K_0H_0^{-1} = G_0 \underset{G}{\sim} G_1 = K_1H_1^{-1} \underset{\varepsilon\cdot(K_1H_1^{-1})}{\sim\!\sim\!\sim\!\sim} (K_1K_1^{-1})(K_1H_1^{-1})$$
$$\left.\underset{f}{\sim} K_1(K_1^{-1}(K_1H_1^{-1})) \underset{K_1f}{\sim\!\sim} K_1((K_1^{-1}K_1)H_1^{-1}) \underset{K_1(\bar\varepsilon\cdot H_1^{-1})}{\sim\!\sim\!\sim\!\sim} K_1H_1^{-1}\right\},$$

by using the homotopy $\left\{K(K^{-1}\cdot G) \underset{f}{\sim} (KK^{-1})\cdot G \underset{\varepsilon\cdot G}{\sim\!\sim} G\right\}$. Because the partial homotopies $\{K_0H_0^{-1} \sim K_0H_0^{-1}\}$ and $\{K_1H_1^{-1} \sim K_1H_1^{-1}\}$ in the last homotopy are those of (2. 12. 1), they are homotopic to the stationary homotopies by the assumption (2. 13). Therefore the last homotopy is homotopic to G_t, and the lemma is proved.

Lemma 2. 14. 2 is proved analogously, using the assumption (2. 13) that the homotopy (2. 12. 2) is homotopic to the stationary homotopy, and we shall omit its proofs.

Remark 2. 17. H-spaces are defined more generally by the weaker conditions that there is a map (H-structure) $\mu: F\times F\to F$ and a point $\varepsilon\in F$ (called a homotopy-unit), satisfying the following property instead of (2. 1):

(2. 18) the maps $x\to\mu(x,\varepsilon)$ and $x\to\mu(\varepsilon,x)$ of F into F are homotopic rel. ε to the identity map $x\to x$, respectively.

By Lemma 6. 4 of [1], the existence of a map having a homotopy-unit implies the existence of one having an unit, when F is a CW-complex such that $F\times F$ is also a CW-complex. In this note, we concern mainly with such cases.

The notions of the homotopy-associativity and the inversion are defined similarly for H-spaces of generally defined. The composition of loops in a space with fixed base point is a homotopy-associative H-structure, of generally defined, having an inversion.

3. Group-like spaces

Let F be a homotopy-associative H-space having an inversion. In this note, we consider the stronger condition than the assumption (2. 13), the notion of group-like spaces.

Let g be a map of $F^n=F\times\cdots\times F$ (n-times) into F, such that $g(u) = g(x_1,\cdots,x_n)$ is a form of the coordinates $x_1,\cdots,x_n$ of $u\in F^n$, given by iterating the multiplication and the inversion. Two such maps are said to be in a standard relation, if they are different only in one part, where they are given by the terminal maps of the homotopy f_t, ε_t, i_t, $\bar\varepsilon_t$, or j_t of (2. 3), (2. 7)—(2. 10).

For any two such maps g_0 and g_1, related by an iteration of the standard relations, there exist various homotopies g_t^i, $i = 1, 2, \cdots$, between $g_0^i = g_0$ and $g_1^i = g_1$, by using only the homotopies f_t, ε_t, i_t, $\bar{\varepsilon}_t$ and j_t. We assume first that

(3. 1) the homotopies f_t of (2. 3) and ε_t of (2. 7) can be so taken that any two g_t^i and g_t^j, considering as the maps $F^n \times I \to F$, are homotopic each other rel. $F^n \times \dot{I}$, by a homotopy $g_{t,s}^{ij}$, $0 \leqq t,\ s \leqq 1$, such that

$$g_{t,0}^{ij} = g_t^i,\ g_{t,1}^{ij} = g_t^j,\ g_{0,s}^{ij} = g_0,\ g_{1,s}^{ij} = g_1 .$$

We also set $g_{t,s}^{ii} = g_t^i$ for $0 \leqq s \leqq 1$.

Moreover, we assume that

(3. 2) $g_{t,s}^{ij}$, $i, j = 1, 2, \cdots$, can be so taken that any two $g_{t,s}^{ij}$ and $g_{t,s}^{kl}$, considering as the maps $F^n \times I^2 \to F$, are homotopic each other by a homotopy $g_{t,s,r}^{ijkl}$, $0 \leqq t, s, r \leqq 1$, such that

$$\begin{aligned} g_{t,s,0}^{ijkl} &= g_{t,s}^{ij}, & g_{t,s,1}^{ijkl} &= g_{t,s}^{kl}, & g_{t,0,r}^{ijkl} &= g_{t,r}^{ik}, \\ g_{t,1,r}^{ijkl} &= g_{t,r}^{jl}, & g_{0,s,r}^{ijkl} &= g_0, & g_{1,s,r}^{ijkl} &= g_1 . \end{aligned}$$

(3. 3) Further, we assume the same properties on $g_{t,s,r}^{ijkl}$, and so on.

Definition of group-like spaces. A topological space F is called a group-like space, if it is a homotopy-associative H-space having an inversion and it satisfies also the assumptions (3. 1)—(3. 3).

It is clear that group-like spaces satisfy the assumption (2. 13) and topological groups are group-like spaces.

Remark 3. 4. Group-like spaces are defined more generally for homotopy-associative H-spaces, of generally defined, having an inversion, which is noticed in Remark 2. 17, using the homotopies $\{x \sim \mu(x, \varepsilon)\}$ and $\{x \sim \mu(\varepsilon, x)\}$ of (2. 18), in addition to the fomotopies f_t and ε_t, in the above assumptions.

We can prove easily

Lemma 3.5. *A space of loops in a given space with a fixed base point is a group-like space of generally defined, by the composition of loops.*

Now let F be a group-like space. For the later purpose, we define the map $N_n : (F^{n+1} \times I^{n-1}, \varepsilon^{n+1} \times I^{n-1}) \to (F, \varepsilon)$, for $n = 1, 2, \cdots$, by the induction, as follows:

(3. 6. 1) $N_1(x_0, x_1) = x_0 x_1^{-1}$;

(3. 6. 2) $N_2(x_0, x_1, x_2, t_2)$

$= N_1(x_0, x_1) = x_0 x_1^{-1}$, for $t_2 = 0$,

$= N_1(N_1(x_0, x_2),\ N_1(x_1, x_2)) = (x_0 x_2^{-1})\,(x_1 x_2^{-1})^{-1}$, for $t_2 = 1$,

and define, for $0 < t_2 < 1$, by the homotopy

$$\left\{\begin{array}{l} x_0x_1^{-1} \underset{(x_0e_t)x_1^{-1}}{\sim\sim\sim} (x_0(x_2^{-1}(x_2^{-1})^{-1}))x_1^{-1} \underset{f_{1-t}\cdot x_1^{-1}}{\sim\sim\sim} ((x_0x_2^{-1})(x_2^{-1})^{-1})x_1^{-1} \\ \qquad \underset{f_t}{\sim} (x_0x_2^{-1})\cdot((x_2^{-1})^{-1}x_1^{-1}) \underset{(x_0x_2^{-1})j_{1-t}}{\sim\sim\sim\sim} (x_0x_2^{-1})\cdot(x_1x_2^{-1})^{-1} \end{array}\right\};$$

(3. 6. n) $N_n(x_0, \cdots, x_n, t_2, \cdots, t_n)$

$$= N_{n-1}(x_0, \cdots, \hat{x}_t, \cdots, x_n, t_2, \cdots, \hat{t}_t, \cdots, t_n),^{7)} \qquad \text{for } t_i=0,$$

$$= N_{t-1}\Big(N_{n-t+1}(x_0, x_t, \cdots, x_n, t_{t+1}, \cdots, t_n), \cdots, N_{n-t+1}(x_{t-1}, x_t, \cdots, x_n, t_{t+1}, \cdots, t_n),\ t_2, \cdots, t_{t-1}\Big), \qquad \text{for } t_i=1,$$

and define, for $(t_2, \cdots, t_n) \in I^{n-1} - \dot{I}^{n-1}$, by the homotopies of the assumptions (3. 1)—(3. 3) of a group-like space.

Remark 3. 7. The maps N_1 and N_2 are defined, when F is only assumed to be an H-space having an inversion and a homotopy-associative one, respectively.

4. Constructions of spaces and maps

In this and the next sections, let F be a homotopy-associative H-space and also a CW-complex such that $F \times F \times F$ is a CW-complex. Then F has an inversion by Lemma 2. 5. The notations in §2 are used and also we define notations as follows:

(4. 1. 1) $E_1 = F \circ F =$ the join of two copies of F, i. e., the identification space obtained from $F \times F \times I$ by identifying each set of the form $x \times F \times 0$ with $x \in F$ and $F \times y \times 1$ with $(y, 1) \in F \times 1 = F_1$. The image of $(x_0, x_1, t) \in F \times F \times I$ will be denoted by $(x_0, x_1; t)$.

(4. 2. 1) $B_1 =$ the identification space obtained from $F \times I$ by shrinking each of spaces $F \times 0$ and $F \times 1$ to different points b and b_1, respectively. The image of $(x, t) \in F \times I$ will be denoted by $(x; t)$.

(4. 3. 1)[8] p^1 is the map of E_1 into B_1 defined by

$$p^1(x_0, x_1; t) = (x_0x_1^{-1}; t) = (N_1(x_0, x_1); t).$$

(4. 1. 2) $E_2 = F \circ F \circ F = E_1 \circ F =$ the join of E_1 and F, i. e., the identification space obtained from $E_1 \times F \times I$ by identifying each set of the form $u_1 \times F \times 0$ with $u_1 \in E_1$ and $E_1 \times y \times 1$ with $(y, 2) \in F \times 2 = F_2$. The image of $(u_1, x_2, t_2) = ((x_0, x_1; t_1), x_2, t_2) \in E_1 \times F \times I$ will be denoted by $(u_1, x_2; t_2)$ or $(x_0, x_1, x_2; t_1, t_2)$.

(4. 2. 2) $B_2 =$ the identification space obtained from $E_1 \times I$ by iden-

7) The notations $\hat{x}$ and $\hat{t}$ mean that x and t are removed.

8) This map p^1 is slightly different from the map p defined in the constructions of [4], §5.

tifying each point $(u_1, 0)=((x_0, x_1; t), 0)\in E_1\times 0$ with $p^1(u_1)=(x_0x_1^{-1}; t)$ $\in B_1$ and shrinking $E_1\times 1$ to a point $b_2(\neq b, b_1)$. The image of (u_1, t_2) $=((x_0, x_1; t_1), t_2)\in E_1\times I$ will be denoted by $(u_1; t_2)$ or $(x_0, x_1; t_1, t_2)$.

(4.3.2) p^2 is the map of E_2 into B_2 defined by

$$\begin{aligned} &p^2(x_0, x_1, x_2; t_1, t_2)\\ &= (x_0x_2^{-1}, x_1x_2^{-1}; t_1, 2t_2-1) = \Big(N_1(x_0, x_2), N_1(x_1, x_2); t_1, 2t_2-1\Big), \quad \text{for } 1/2\leqq t_2\leqq 1,\\ &= \Big((N_2(x_0, x_1, x_2, 2t_2); t_1\Big), \quad \text{for } 0\leqq t_2\leqq 1/2. \end{aligned}$$

In the above definitions, N_1 and N_2 are the maps of (3.6.1) and (3.6.2), which can be defined as noticed in Remark 3.7. The maps p^1 and p^2 are well defined and continuous, and we have also

Lemma 4.4. *$F\subset E_1\subset E_2$ and $b\in B_1\subset B_2$; $p^2\mid E_1=p^1$ and $p^1(F)=b$; and E_1 and E_2 satisfy the properties* (1.5) *and* (1.6).

To prove Theorem 1.9, it is sufficient to prove the following theorem, by this lemma.

Theorem 4.5. *If F satisfies* (2.13) *in addition, then the map p^2: $(E_2, E_1, F)\to(B_2, B_1, b)$ satisfies the properties* (1.7) *and* (1.8), *i.e.,*

$$\begin{aligned} &p^1_*:\pi_n(E_1, F)\approx\pi_n(B_1, b),\\ &p^2_*:\pi_n(E_2, F)\approx\pi_n(B_2, b), \end{aligned}\qquad \text{for every integer } n>0.$$

This theorem will be proved in the next section, and we define more elaborate notations here.

$$\text{(4.6)}\quad \begin{aligned} &U_0^2=\{u_2\mid 0\leqq t_1, t_2<1\},\ U_1^2=\{u_2\mid 0<t_1\leqq 1,\ 0\leqq t_2<1\},\\ &U_2^2=\{u_2\mid 3/4<t_2\leqq 1\};\ U_i^1=U_i^2\cap E_1, \quad \text{for } i=0, 1, \end{aligned}$$

where $u_2=(x_0, x_1, x_2; t_1, t_2)\in E_2$.

$$\text{(4.7)}\quad \begin{aligned} &V_0^2=\{v_2\mid 0\leqq t_1, t_2<1\},\ V_1^2=\{v_2\mid 0<t_1\leqq 1,\ 0\leqq t_2<1\},\\ &V_2^2=\{v_2\mid 1/2<t_2\leqq 1\};\ V_i^1=V_i^2\cap B_1, \quad \text{for } i=0, 1, \end{aligned}$$

where $v_2=(x_0, x_1; t_1, t_2)\in B_2$.

Lemma 4.8. *$\{U_i^k\mid 0\leqq i\leqq k\}$ is an open covering of E_k, $\{V_i^k\mid 0\leqq i\leqq k\}$ is an open covering of B_k, and $U_0^k\supset F$, $V_0^k\ni b$, for $k=1, 2$. Also $p^k(U_i^k)=V_i^k$, for $k=1, 2$, $0\leqq i\leqq k$.*

Further, we define maps $p_i^k: U_i^k\to F$ and $\phi_i^k: V_i^k\times F\to U_i^k$, for $k=1$, 2, $0\leqq i\leqq k$, as follows:

$$\text{(4.9)}\qquad p_i^k(x_0, x_1, x_2; t_1, t_2)=x_i.$$

(4.10.0) $\phi_0^2((x_0, x_1; t_1, t_2), x)$
$= (x, (x_0x_1^{-1})^{-1}\cdot x, x_0^{-1}x; t_1, (t_2+1)/2),$ for $1/2 \leqq t_2 < 1$,
$= (x, (x_0x_1^{-1})^{-1}\cdot x, x_0^{-1}x; t_1, 3t_2/2),$ for $0 \leqq t_2 \leqq 1/2$;

(4.10.1) $\phi_1^2((x_0, x_1; t_1, t_2), x)$
$= ((x_0x_1^{-1})x, x, x_1^{-1}x; t_1, (t_2+1)/2),$ for $1/2 \leqq t_2 < 1$,
$= ((x_0x_1^{-1})x, x, x_1^{-1}x; t_1, 3t_2/2),$ for $0 \leqq t_2 \leqq 1/2$;

(4.10.2) $\phi_2^2((x_0, x_1; t_1, t_2), x)$
$= (x_0x, x_1x, x; t_1, (t_2+1)/2),$ for $1/2 < t_2 \leqq 1$.

(4.11) $\phi_i^1 = \phi_i^2 \mid V_i^1 \times F$, for $i = 0, 1$.

These are well defined and continuous and we have

Lemma 4.12. *$\phi_i^2(V_i^1\times F)\subset U_i^1$, for $i = 0, 1$;*
$\phi_i^1((V_0^1\cap V_1^1)\times F)\subset U_0^1\cap U_1^1$;

$\phi_i^2((V_i^2\cap V_j^2\cap V_h^2)\times F)\subset U_i^2\cap U_j^2\cap U_h^2$, for $0\leqq i, j, h\leqq 2$.

$p_0^k \mid F: F\to F$ is the identity map, $\phi_0^k(b\times F)\subset F$ and $\phi_0^k \mid b\times F$: $b\times F\to F$ is also the identity map, for $k = 1, 2$.

5. Proofs of Theorems 4.5 and 1.9

By Lemmas 4.8 and 4.12, we obtain the two maps (p^k, p_i^k)[9] and ϕ_i^k:

$$(U_i^k, U_i^k\cap U_j^k\cap U_h^k, F) \underset{\phi_i^k}{\overset{(p^k, p_i^k)}{\rightleftarrows}} (V_i^k\times F, (V_i^k\cap V_j^k\cap V_h^k)\times F, b\times F),$$

for $k = 1, 2$ and $0\leqq i\leqq k$, where $0\leqq j$, $h\leqq k$, and F and $b\times F$ appear only for $i = 0$.

We prove the following lemma, concerning to these maps:

Lemma 5.1. *If F satisfies (2.13) in addition, then these maps (p^k, p_i^k) and ϕ_i^k are homotopy equivalences and the one is a homotopy inverse of the other. Further, a homotopy $\Phi_i^{k,t}: U_i^k\to U_i^k$, between $\Phi_0^{k,t} = \phi_i^k\circ(p^k, p_i^k)$ and $\Phi_1^{k,t}$ = the identity map, can be so taken that*

(5.2) $\Phi_i^{k,t}(U_i^k\cap U_j^k\cap U_h^k)\subset U_i^k\cap U_j^k\cap U_h^k$, $\Phi_i^{k,0}(F)\subset F$;

and a homotopy $\Psi_i^{k,t}: V_i^k\times F\to V_i^k\times F$, between $\Psi_0^{k,t} = (p^k, p_i^k)\circ\phi_i^k$ and $\Psi_1^{k,t}$ = the identity map, can be so taken that

(5.3) $\Psi_i^{k,t}((V_i^k\cap V_j^k\cap V_h^k)\times F)\subset(V_i^k\cap V_j^k\cap V_h^k)\times F$, $\Psi_i^{k,0}(b\times F)\subset b\times F$.

---

9) $(f, g): X\to Y\times Z$ is the map defined by $(f, g)(x) = (f(x), g(x))$, where $f: X\to Y$, $g: X\to Z$ are given maps.

Proof. $\Phi_t^{2,0}$ is defined as follows: for $(x_0, x_1, x_2; t_1, t_2) \in U_0^2$,

$$\Phi_t^{2,0}(x_0, x_1, x_2; t_1, t_2) = (x_0, y_1, y_2; t_1, s_2);$$

$$\begin{aligned}
y_1 &= \Big(N_2(x_0, x_1, x_2, 1-2t)\Big)^{-1}\cdot x_0, && \text{for } 1/2 \leqq t_2 < 1,\ 0 \leqq t \leqq 1/2,\\
&= O^1_{2-2t}(x_0, x_1), && \text{for } 1/2 \leqq t_2 < 1,\ 1/2 \leqq t \leqq 1,\\
&= \Big(N_2(x_0, x_1, x_2, 2(t_2-t))\Big)^{-1}\cdot x_0, && \text{for } 0 \leqq t \leqq t_2 \leqq 1/2,\\
&= O^1_{(1-t)/(1-t_2)}(x_0, x_1), && \text{for } 0 \leqq t_2 \leqq t \leqq 1,\ t_2 \leqq 1/2;\\
y_2 &= O^1_{1-t}(x_0, x_2);\\
s_2 &= t_2, && \text{for } 3/4 \leqq t_2 < 1,\ 0 \leqq t \leqq 1,\\
&= \mathrm{Max.}\,(3(2t_2+t-1)/2(2t+1),\ 0), && \text{for } 0 \leqq t_2 \leqq 3/4,\ 0 \leqq t \leqq 1;
\end{aligned}$$

where $O_t^1(x, y) = \left\{ y \underset{i_t\cdot\bar{\varepsilon}_t}{\sim\!\sim} (y^{-1})^{-1}(x^{-1}x) \underset{f_{1-t}}{\sim} ((y^{-1})^{-1}x^{-1})x \underset{j_{1-t}\cdot x}{\sim\!\sim} (xy^{-1})^{-1}x \right\}$, by the notations of (2. 6). Using (3. 6. 2), simple calculations show that this is well defined and $\Phi_t^{2,0}(U_0^1) \subset U_0^1$, and $\Phi_t^{1,0} = \Phi_t^{2,0} \mid U_0^1$ and $\Phi_t^{2,0}$ satisfy (5. 2), and further

$$\Phi_0^{k,0} = \phi_0^k \circ (p^k, p_0^k), \quad \Phi_1^{k,0} = \text{the identity map.}$$

$\Phi_t^{2,1}$ and hence $\Phi_t^{1,1}$ are defined analogously, by using the homotopy

$$O_t^2(x, y) = \left\{ x \underset{x\bar{\varepsilon}_t}{\sim} x(y^{-1}y) \underset{f_{1-t}}{\sim} (xy^{-1})y \right\}, \tag{5. 4}$$

and we shall omit their definitions. $\Phi_t^{2,2}$ is defined more plainly by

$$\Phi_t^{2,2}(x_0, x_1, x_2; t_1, t_2) = (O_{1-t}^2(x_0, x_2), O_{1-t}^2(x_1, x_2), x_2; t_1, t_2),$$

for $3/4 < t_2 \leqq 1$.

Now, we define the homotopies K_t, $G_t: F\times F\times F \to F$ as follows: for $x_0, x_1, x \in F$,

$$\begin{aligned}
G_t(x_0, x_1, x) &= N_2(x, (x_0x_1^{-1})^{-1}x, x_0^{-1}x, 1-3t), && \text{for } 0 \leqq t \leqq 1/3,\\
&= O^3_{(3t-1)/2}(x, x_0x_1^{-1}), && \text{for } 1/3 \leqq t \leqq 1;\\
K_t(x_0, x_1, x) &= O_t^3(x, x_0);
\end{aligned}$$

where

$$O_t^3(x, y) = \left\{ x(y^{-1}x)^{-1} \underset{xj_t}{\sim} x(x^{-1}(y^{-1})^{-1}) \underset{f_{1-t}}{\sim} (xx^{-1})(y^{-1})^{-1} \underset{\varepsilon_{1-t}\cdot i_{1-t}}{\sim\!\sim\!\sim} y \right\}. \tag{5. 5}$$

Then $H_0(x_0, x_1, x) = ((x_0x_1^{-1})^{-1}x)\cdot(x_0^{-1}x)^{-1}$, $H_1(x_0, x_1, x) = x_1$ and G_t, K_t satisfy (2. 15). Let H_t be the homotopy defined by (2. 16).

Let $\bar{\Psi}_t^{2,0}: V_0^2 \times F \to V_0^2$ be defined as follows: for $(x_0, x_1; t_1, t_2) \in V_0^2$,

$$\begin{aligned}
&\bar{\Psi}_t^{2,0}((x_0, x_1; t_1, t_2),\ x)\\
&= (K_t(x_0, x_1, x),\ H_t(x_0, x_1, x); t_1, t_2), \qquad \text{for } 1/2 \leqq t_2 < 1,\ 0 \leqq t \leqq 1,
\end{aligned}$$

$=(K_t(x_0, x_1, x),\ H_t(x_0, x_1, x)\,;\, t_1,\ (3t_2+t-1)/(2t+1)),$

for $(1-t)/3 \leqq t_2 \leqq 1/2,\ 0 \leqq t \leqq 1,$

$=(G_{(1-3t_2)/3}(x_0, x_1, x)\,;\, t_1),$ for $0 \leqq t_2 \leqq 1/3,\ t=0,$

$=(G_{(2t+1)/3}(x_0, x_1, x)\,;\, t_1),$ for $t_2=0,\ 0 \leqq t \leqq 1.$

This is well defined and, by (4. 2. 2),

$$\bar{\Psi}_t^{2,0} = (K_t \cdot H_t^{-1}\,;\, t_1), \qquad \text{for } 3t_2+t=1.$$

Hence $\bar{\Psi}_t^{2,0}$ can be extended for $0 \leqq 3t_2 \leqq 1-t \leqq 1$, using the homotopy $\{G_t \sim K_t \cdot H_t^{-1}\}$ of Lemma 2. 14. 1.

By simple calculations, it follows immediately that the homotopy $\Psi_t^{2,0}: V_0^2 \times F \to V_0^2 \times F$, defined by

$$\Psi_t^{2,0}(v_2, x) = (\bar{\Psi}_t^{2,0}(v_2, x), x), \qquad \text{for } v_2 \in V_0^2,\ x \in F,$$

and $\Psi_t^{1,0} = \Psi_t^{2,0} \mid V_0^1 \times F$ have the desired properties.

$\Psi_t^{2,1}$ and hence $\Psi_t^{1,1}$ are defined analogously, using Lemma 2. 14. 2.

$\Psi_t^{2,2}$ is defined more plainly by

$$\Psi_t^{2,2}((x_0, x_1\,;\, t_1, t_2), x) = ((O_{1-t}^1(x_1, x)\,;\, t_1, t_2), x), \qquad \text{for } 1/2 < t_2 \leqq 1,$$

where

$$(5.6) \qquad O_t^1(x, y) = \left\{ x \underset{x\varepsilon_{1-t}}{\sim\!\sim} x(yy^{-1}) \underset{f_{1-t}}{\sim} (xy)y^{-1} \right\}.$$

Remark 5. 7. In the above definitions of $\Phi_t^{k,i}$, the assumption (2. 13) is not used, and it is used for $\Psi_t^{k,i}$.

On the other hand, by [4], [10] we have

Proposition 5.8. *Let $p:(E, F) \to (B, b)$ be a given map. We asuume that there are open coverings $\{U_i \mid i=0, 1, \cdots\}$ of E and $\{V_i \mid i=0, 1, \cdots\}$ of B, and maps $p_i: U_i \to F$ and $\phi_i: V_i \times F \to U_i$ and homotopies $\Phi_t^i: U_i \to U_i$ and $\Psi_t^i: V_i \times F \to V_i \times F$, for $i=0, 1, \cdots$, such that*

(5. 9)

$U_0 \supset F,\ V_0 \ni b,\ U_i \cap F = \emptyset,\ V_i \not\ni b,$ *for* $i=1, \cdots;\ p(U_i) \subset V_i;$

$\phi_0(b \times F) \subset F,\ \phi_i((V_i \cap V_{i_1} \cap \cdots \cap V_{i_l}) \times F) \subset U_i \cap U_{i_1} \cap \cdots \cap U_{i_l};$

$\Phi_0^i = \phi_i \circ (p, p_i),\ \Phi_1^i$ *is the identity map of* $U_i,\ \Phi_t^0(F) \subset F,$

$\Phi_t^i(U_i \cap U_{i_1} \cap \cdots \cap U_{i_l}) \subset U_i \cap U_{i_1} \cap \cdots \cap U_{i_l};$

$\Psi_0^i = (p, p_i) \circ \phi_i,\ \Psi_1^i$ *is the identity map of* $V_i \times F,\ \Psi_t^0(b \times F) \subset b \times F,$

$\Psi_t^i((V_i \cap V_{i_1} \cap \cdots \cap V_{i_l}) \times F) \subset (V_i \cap V_{i_1} \cap \cdots \cap V_{i_l}) \times F.$

Then $p:(E, F) \to (B, b)$ is a weak homotopy equivalence, i.e.,

$$p_*: \pi_n(E, F) \approx \pi_n(B, b), \qquad \textit{for every integer } n > 0.$$

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10) Cf. Remark of p. 124—125 of [4], §6.

Proofs of Theorems 4.5 *and* 1.9. Theorem 4.5 follows immediately from Lemma 5.1 and Proposition 5.8. Therefore, by Lemma 4.4 and Theorem 4.5, Theorem 1.9 is obtained completely.

6. Analogous constructions for group-like spaces

In the present and next sections, let F be a group-like space, and also a countable CW-complex. Then F satisfies (2.13), and hence the discussions of the former sections are applicable for F. Further, we define notations continuing to (4.1)—(4.3), by the induction, for $n \geqq 3$.

(6.1. n) $E_n = F \circ \cdots \circ F((n+1)\text{-times}) = E_{n-1} \circ F =$ the join of E_{n-1} and F, i.e., the identification space obtained from $E_{n-1} \times F \times I$ by identifying each set of the form $u_{n-1} \times F \times 0$ with $u_{n-1} \in E_{n-1}$ and $E_{n-1} \times y \times 1$ with $(y, n) \in F \times n = F_n$. The image of $(u_{n-1}, x_n, t_n) = ((x_0, \cdots, x_{n-1}; t_1, \cdots, t_{n-1}), x_n, t_n) \in E_{n-1} \times F \times I$ will be denoted by $(u_{n-1}, x_n; t_n)$ or $(x_0, \cdots, x_{n-1}, x_n; t_1, \cdots, t_{n-1}, t_n)$. We often call x_i, $i = 0, \cdots, n$, the coordinate of $(x_0, \cdots, x_n; t_1, \cdots, t_n)$.

(6.2. n) $B_n =$ the identification space obtained from $E_{n-1} \times I$ by identifying each point $(u_{n-1}, 0) \in E_{n-1} \times 0$ with $p^{n-1}(u_{n-1}) \in B_{n-1}$ and shrinking $E_{n-1} \times 1$ to a point $b_n (\neq b, b_1, \cdots, b_{n-1})$. The image of $(u_{n-1}, t_n) = ((x_0, \cdots, x_{n-1}; t_1, \cdots, t_{n-1}), t_n) \in E_{n-1} \times I$ will be denoted by $(u_{n-1}; t_n)$ or $(x_0, \cdots, x_{n-1}; t_1, \cdots, t_{n-1}, t_n)$. We often call x_i the coordinate of $(x_0, \cdots, x_{n-1}; t_1, \cdots, t_n)$.

(6.3. n) p^n is the map of E_n into B_n defined by

$$\begin{aligned} &p^n(x_0, \cdots, x_n; t_1, \cdots, t_n) \\ &= \Big(N_{n-i+1}(x_0, x_i, \cdots, x_n; 2t_{i+1}, \cdots, 2t_n), \cdots, \\ &\quad N_{n-i+1}(x_j, x_i, \cdots, x_n; 2t_{i+1}, \cdots, 2t_n), \cdots, \\ &\quad N_{n-i+1}(x_{i-1}, x_i, \cdots, x_n; 2t_{i+1}, \cdots, 2t_n); t_1, \cdots, t_{i-1}, 2t_i - 1\Big) \in B_i \subset B_n, \\ &\qquad \text{for } i = 2, \cdots, n, 0 \leqq t_n, \cdots, t_{i+1} \leqq 1/2, 1/2 \leqq t_i \leqq 1, 0 \leqq t_{i-1}, \cdots, t_1 \leqq 1, \\ &= (N_n(x_0, \cdots, x_n; 2t_2, \cdots, 2t_n); t_1) \in B_1 \subset B_n, \\ &\qquad \text{for } 0 \leqq t_n, \cdots, t_2 \leqq 1/2, \ 0 \leqq t_1 \leqq 1, \end{aligned}$$

where N_k is the map defined by (3.6. k).

This map p^n is rewritten as follows:

(6.4. n)
$$\begin{aligned} &p^n(u_{n-1}, x_n; t_n) \\ &= (u_{n-1} \cdot x_n^{-1}; 2t_n - 1), && \text{for } 1/2 \leqq t_n \leqq 1, \\ &= M^{n-1}_{2t_n}(u_{n-1}, x_n), && \text{for } 0 \leqq t_n \leqq 1/2, \end{aligned}$$

where

$$u_{n-1}\cdot y=(x_0, \cdots, x_{n-1}; t_1, \cdots, t_{n-1})\cdot y=(x_0\cdot y, \cdots, x_{n-1}\cdot y; t_1, \cdots, t_{n-1}),$$

and $M_t^{n-1}: E_{n-1}\times F \to B_{n-1}$ is the homotopy such that

$$(6.5)\quad \begin{aligned} M_1^{n-1}(u_{n-1}, x_n) &= p^{n-1}(u_{n-1}\cdot x_n^{-1}),\\ M_0^{n-1}(u_{n-1}, x_n) &= p^{n-1}(u_{n-1}). \end{aligned}$$

By (3.6.n), Lemma 4.4 and the assumptions that F is a countable CW-complex, it is immediately to see that the maps p^n is well defined and continuous and

Lemma 6.6. *$F\subset E_1\subset \cdots \subset E_{n-1}\subset E_n\subset \cdots$, $b\in B_1\subset \cdots \subset B_{n-1}\subset B_n\subset\cdots$; $p^n \mid E_{n-1} = p^{n-1}$, for $n=2, 3, \cdots$; and E_n is a countable CW-complex.* [11]

By this lemma, the following spaces and map can be defined:

(6.1.∞) E_∞ = the direct limit space of the sequence $F\subset E_1\subset\cdots\subset E_n\subset\cdots$.

(6.2.∞) B_∞ = the direct limit space of the sequence $b\in B_1\subset\cdots\subset B_n\subset\cdots$.

(6.3.∞) p^∞ is the map of E_∞ into B_∞, defined by

$$p^\infty \mid E_n = p^n, \qquad \text{for} \quad n = 1, 2, \cdots.$$

Lemma 6.7. *E_∞ is a countable CW-complex and is contractible in itself to a vertex $\varepsilon\in F$ leaving ε fixed.* [12]

To prove the necessity of Theorem 1.1, it is sufficient to prove the following theorem, by this lemma.

Theorem 6.8. *The map $p^\infty: (E_\infty, F)\to (B_\infty, b)$ is a weak homotopy equivalence.*

Also this theorem is an immediate consequence of

Theorem 6.9. *The map $p^n: (E_n, F)\to (B_n, b)$ is a weak homotopy equivalence, for every integer $n>0$.*

We shall prove this theorem, using Proposition 5.8. Continuing to (4.6)—(4.9), we define the following notations inductively, for $n\geqq 3$:

$$(6.10)\quad \begin{aligned} U_i^n &= \{(u_{n-1}, x_n; t_n) \mid u_{n-1}\in U_i^{n-1}, 0\leqq t_n<1\}, \quad \text{for } i = 0, \cdots, n-1,\\ U_n^n &= \{(u_{n-1}, x_n; t_n) \mid u_{n-1}\in E_{n-1}, 3/4<t_n\leqq 1\}. \end{aligned}$$

$$(6.11)\quad \begin{aligned} V_i^n &= \{(u_{n-1}; t_n) \mid u_{n-1}\in U_i^{n-1}, 0\leqq t_n<1\}, \quad \text{for } i=0, \cdots, n-1,\\ V_n^n &= \{(u_{n-1}; t_n) \mid u_{n-1}\in E_{n-1}, 1/2<t_n\leqq 1\}. \end{aligned}$$

11) This is a consequence of Lemma 2.1 of [2, I].

12) By Lemma 2.3 of [2, II], E_∞ is ∞-connected, and hence contractible as it is a CW-complex.

(6. 12) $p_i^n : U_i^n \to F$, $i = 0, \cdots, n$, is the map defined by

$$p_i^n(x_0, x_1, \cdots, x_n ; t_1, \cdots, t_n) = x_i .$$

This definition is rewritten as follows:

$$p_i^n(u_{n-1}, x_n ; t_n) = p_i^{n-1}(u_{n-1}), \qquad \text{for } i = 0, \cdots, n-1,$$
$$p_n^n(u_{n-1}, x_n ; t_n) = x_n.$$

Lemma 6.13. *$\{U_i^n \mid i = 0, \cdots, n\}$ and $\{V_i^n \mid i = 0, \cdots, n\}$ are open coverings of E_n and B_n, respectively. $U_0^n \supset F$ and $V_0^n \ni b$. Also $p^n(U_i^n) = V_i^n$, for $i = 0, \cdots, n$.*

7. Proofs of Theorem 6.9 and the necessity of Theorem 1.1

In the above section, the map (p^n, p_i^n):

$$(U_i^n, U_i^n \cap U_{i_1}^n \cap \cdots \cap U_{i_l}^n, F) \to (V_i^n \times F, (V_i^n \cap V_{i_1}^n \cap \cdots \cap V_{i_l}^n) \times F, b \times F)$$

is defined for $i=0, \cdots, n$, where $0 \leqq i_1, \cdots, i_l \leqq n$.

Similarly to Lemma 5. 1, we prove

Lemma 7.1. *A map $\phi_i^n : V_i^n \times F \to U_i^n$ and homotopies $\Phi_i^{n,t} : U_i^n \to U_i^n$, $\Psi_i^{n,t} : V_i^n \times F \to V_i^n \times F$, for $i = 0, \cdots, n$, can be so defined that they satisfy the properties* (5. 9). *Also they are defined by using only iterations of the multiplication and the inversion of the coordinates of $u_n \in U_i^n$ or those of $v_n \in V_i^n$ and $x \in F$ and the homotopies f_t, ε_t, i_t, $\bar{\varepsilon}_t$ and j_t of* §2, *and the homotopies of the assumptions* (3. 1)—(3. 3) *of group-like spaces.*

Proof. We shall prove this lemma by the induction, constructing inductively $\phi_i^k : V_i^k \times F \to U_i^k$, $\Phi_i^{k,t} : U_i^k \to U_i^k$ and $\Psi_i^{k,t} : V_i^k \times F \to V_i^k \times F$, for $i = 0, \cdots, k$.

Let $\phi_i^n : V_i^n \times F \to U_i^n$, $i=0, \cdots, n-1$, be the map, defined inductively by

$$(7.2.i) \quad \phi_i^n\Big((u_{n-1} ; t_n), x\Big) = \phi_i^n\Big((x_0, \cdots, x_{n-1} ; t_1, \cdots, t_n), x\Big)$$
$$= \Big(u_{n-1} \cdot (x^{-1}x_i)^{-1},\ (x^{-1}x_i)^{-1};\ (t_n+1)/2\Big), \qquad \text{for } 1/2 \leqq t_n < 1,$$
$$= \Big(u_{n-1} \cdot (x^{-1}x_i)^{-1},\ (x^{-1}x_i)^{-1};\ 3(3t_n-1)/2\Big), \qquad \text{for } 1/3 \leqq t_n \leqq 1/2,$$
$$= \Phi_{6t_n-1}^{n-1,i}(u_{n-1} \cdot (x^{-1}x_i)^{-1}), \qquad \text{for } 1/6 \leqq t_n \leqq 1/3,$$
$$= \phi_i^{n-1}\Big(M_{6t_n}^{n-1}(u_{n-1}, x^{-1}x_i),\ O_{1-6t_n}^3(x_i, x)\Big), \qquad \text{for } 0 \leqq t_n \leqq 1/6,$$

where M_t^{n-1} and O_t^3 are the homotopies of (6. 5) and (5. 5). By (5. 9), (6. 12), (6. 5) and (5. 5), this is well defined.

Let $\phi_n^n : V_n^n \times F \to U_n^n$ be the map defined by

$$(7.2.n) \qquad \phi^n_n((u_{n-1}\,;t_n),\,x) = (u_{n-1}\cdot x,\,x\,;(t_n+1)/2), \qquad \text{for } 1/2 < t_n \leqq 1.$$

Then it is easy to see that ϕ^n_i, $i = 0, \cdots, n$, satisfy the properties of the second row of (5.9).

We define $\Phi^{n,i}_t : U^n_i \to U^n_i$, for $i = 0, \cdots, n-1$, as follows:

$$\Phi^{n,i}_t(u_{n-1}, x_n\,; t_n) = \Phi^{n,i}_t(x_0, \cdots, x_n\,; t_1, \cdots, t_n) = (w_{n-1}, y_n\,; s_n)\,;$$

$$\begin{aligned}
w_{n-1} &= O^5_{(2-3t)/3}(u_{n-1}, x_n), && \text{for } 2/3 \leqq t_n < 1,\ 0 \leqq t \leqq 2/3,\\
&= \Phi^{n-1,i}_{3t-2}(u_{n-1}), && \text{for } 2/3 \leqq t_n < 1,\ 2/3 \leqq t \leqq 1,\\
&= O^5_{t_n-t}(u_{n-1}, x_n), && \text{for } 0 \leqq t \leqq t_n \leqq 2/3,\\
&= \Phi^{n-1,i}_{(t-t_n)/(1-t_n)}(u_{n-1}), && \text{for } 0 \leqq t_n \leqq t \leqq 1,\ t_n \leqq 2/3,
\end{aligned}$$

$$\begin{aligned}
O^5_t(u_{n-1}, x_n) &= \Phi^{n-1,i}_{12t-7}\Big((u_{n-1}\cdot x_n^{-1})\cdot(x_i^{-1}(x_i x_n^{-1}))^{-1}\Big), && \text{for } 7/12 \leqq t \leqq 2/3,\\
&= \phi^{n-1}_i\Big(M^{n-1}_{6(2t-1)}(u_{n-1}\cdot x_n^{-1},\, x_i^{-1}(x_i x_n^{-1})),\ O^3_{7-12t}(x_i x_n^{-1}, x_i)\Big), && \\
& && \text{for } 1/2 \leqq t \leqq 7/12,\\
&= \phi^{n-1}_i\Big(M^{n-1}_{2t}(u_{n-1}, x_n),\, x_i\Big), && \text{for } 0 \leqq t \leqq 1/2\,;
\end{aligned}$$

$$y_n = O^6_t(x_i, x_n),$$

$$O^6_t(x, y) = \Big\{(x^{-1}(xy^{-1}))^{-1} \underset{(f_{1-t})^{-1}}{\sim\!\sim\!\sim} ((x^{-1}x)y^{-1})^{-1} \underset{(\bar{\varepsilon}_{1-t}\cdot y^{-1})^{-1}}{\sim\!\sim\!\sim\!\sim\!\sim} (y^{-1})^{-1} \underset{j_{1-t}}{\sim} y\Big\}\,;$$

$$\begin{aligned}
s_n &= t_n, && \text{for } 3/4 \leqq t_n < 1,\ 0 \leqq t \leqq 1,\\
&= \text{Max.}\,(3(3t_n+2t-2)/(8t+1),\ 0), && \text{for } 0 \leqq t_n \leqq 3/4,\ 0 \leqq t \leqq 1.
\end{aligned}$$

These are well defined, and simple calculations show that these have the desired properties.

$\Phi^{n,n}_t$ is defined more plainly by

$$\Phi^{n,n}_t(u_{n-1}, x_n\,; t_n) = (O^2_{1-t}(u_{n-1}, x_n),\, x_n\,; t_n),\text{[13]} \qquad \text{for } 3/4 < t_n \leqq 1.$$

$\Psi^{n,n}_t$ is defined by

$$\Psi^{n,n}_t((u_{n-1}\,; t_n),\ x) = \Big((O^4_{1-t}(u_{n-1}, x)\,; t_n),\, x\Big), \qquad \text{for } 1/2 < t_n \leqq 1.$$

Let $q^n : B_n \times F \to B_n$, $r^n : B_n \times F \to F$ be the natural projections; and we define $r^n \circ \Psi^{n,i}_t : V^n_i \times F \to F$, for $i = 0, \cdots, n-1$, as follows:

$$\begin{aligned}
r^n \circ \Psi^{n,i}_t\Big((u_{n-1}\,; t_n),\, x\Big) &= r^n \circ \Psi^{n,i}_t\Big((x_0, \cdots, x_{n-1}\,; t_1, \cdots, t_n),\, x\Big)\\
&= O^7_{(1-3t)/3}(u_{n-1}, x), && \text{for } 1/3 \leqq t_n < 1,\, 0 \leqq t \leqq 1/3,\\
&= r^{n-1} \circ \Psi^{n-1,i}_{(3t-1)/2}\Big(p^{n-1}(u_{n-1}),\, x\Big), && \text{for } 1/3 \leqq t_n < 1,\, 1/3 \leqq t \leqq 1,\\
&= O^7_{t_n-t}(u_{n-1}, x), && \text{for } 0 \leqq t \leqq t_n \leqq 1/3,\\
&= r^{n-1} \circ \Psi^{n-1,i}_{(t-t_n)/(1-t_n)}\Big(p^{n-1}(u_{n-1}),\ x\Big), && \text{for } 0 \leqq t_n \leqq t \leqq 1,\ t_n \leqq 1/3,
\end{aligned}$$

13) $O^2_t : E_{n-1} \times F \to E_{n-1}$ is the homotopy defined by using $O^2_t : F \times F \to F$ of (5.4) in each coordinate.

$$O_t^i(u_{n-1}, x) = p_i^{n-1} \circ \Phi_{6t-1}^{n-1,i}(u_{n-1} \cdot (x^{-1}x_i)^{-1}), \qquad \text{for } 1/6 \leqq t \leqq 1/3,$$
$$= p_i^{n-1} \circ \phi_i^{n-1}\Big(M_{6t}^{n-1}(u_{n-1}, x^{-1}x_i),\ O_{1-6t}^{3}(x_i, x)\Big), \qquad \text{for } 0 \leqq t \leqq 1/6.$$

We also define $q^n \circ \Psi_t^{n,i}: V_i^n \times F \to V_i^n$, for $i = 0, \cdots, n-1$, as follows:

$$q^n \circ \Psi_t^{n,i}\Big((u_{n-1}; t_n), x\Big) = q^n \circ \Psi_t^{n,i}\Big((x_0, \cdots, x_{n-1}; t_1, \cdots, t_n), x\Big)$$
$$= \Big(O_{1-t}^i(u_{n-1}, (x^{-1}x_i)^{-1}); t_n\Big), \qquad \text{for } 1/2 \leqq t_n < 1,\ 0 \leqq t \leqq 1,$$
$$= \Big(O_{1-t}^i(u_{n-1}, (x^{-1}x_i)^{-1}); (9t_n + 4t - 4)/(8t+1)\Big),$$
$$\text{for } 4(1-t) \leqq 9t_n \leqq 9/2,\ 0 \leqq t \leqq 1,$$
$$= M_{3(3t_n-1)}^{n-1}\Big(u_{n-1} \cdot (x^{-1}x_i)^{-1}; (x^{-1}x_i)^{-1}\Big), \qquad \text{for } 1/3 \leqq t_n \leqq 4/9,\ t=0,$$
$$= p^{n-1} \circ \Phi_{6t_n-1}^{n-1,i}\Big(u_{n-1} \cdot (x^{-1}x_i)^{-1}\Big), \qquad \text{for } 1/6 \leqq t_n \leqq 1/3,\ t=0,$$
$$= p^{n-1} \circ \phi_i^{n-1}\Big(M_{6t_n}^{n-1}(u_{n-1}, x^{-1}x_i),\ O_{1-6t_n}^{3}(x_i, x)\Big), \qquad \text{for } 0 \leqq t_n \leqq 1/6,\ t=0,$$
$$= q^{n-1} \circ \Psi_t^{n-1,i}\Big(p^{n-1}(u_{n-1}), x\Big), \qquad \text{for } t_n = 0,\ 0 \leqq t \leqq 1.$$

Further, we shall extend this homotopy for $\Delta = \{(t_n, t) \mid 0 \leqq 9t_n \leqq 4(1-t) \leqq 4\}$, as follows. We shall concern with the case $i \geqq 2$, and omit the analogous processes for $i = 0, 1$.

Here, we notice that, by these inductive definitions, each coordinate of $\phi_i^k((x_0, \cdots, x_{k-1}; t_1, \cdots, t_k), x)$ is independent on t_i for $1/2 < t_i \leqq 3/4$ and on t_l for $1/2 \leqq t_l < 1$, $l = i+1, \cdots, k$, and each coordinate of $\Phi_t^{k,i}(x_0, \cdots, x_k; t_1, \cdots, t_k)$ is so on t_i for $3/4 < t_i \leqq 7/8$ and on t_l for $3/4 \leqq t_l < 1$, $l = i+1, \cdots, k$. Also we can assume for $\Psi_t^{k,i}$ samely as ϕ_i^k.

By the above definitions, $q^n \circ \Psi_t^{n,i}$ is defined for $\dot{\Delta}$, the boundary of Δ, and its image is contained in V_i^{n-1} and hence we can write it as

$$q^n \circ \Psi_t^{n,i}\Big((u_{n-1}; t_n), x\Big) = q^n \circ \Psi_t^{n,i}\Big((x_0, \cdots, x_{n-1}; t_1, \cdots, t_n), x\Big)$$
$$= v_{n-1} = (y_0, \cdots, y_{n-2}; s_1, \cdots, s_{n-1}) \in V_i^{n-1}.$$

Because $u_{n-1} \in U_i^{n-1}$, we consider v_{n-1} as the map defined on $(x_0, \cdots, x_{n-1}, t_1, \cdots, t_n, t, x) \in F^n \times I^{i-1} \times (3/4, 1] \times [0, 1)^{n-i-1} \times \dot{\Delta} \times F$.

Clearly, s_j can be so extended for $(t_n, t) \in \Delta$ that the image of s_j for $(t_n, t) \in \Delta$ is equal to that for $(t_n, t) \in \dot{\Delta}$.

For $u_{n-1} \in U_i^i$, v_{n-1} is contained in V_i^i and hence $v_{n-1} = (y_0, \cdots, y_{i-1}; s_0, \cdots, s_i)$ and $1/2 < s_i \leqq 1$. The map y_j is defined on $F^{i+1} \times I^{i-1} \times (3/4, I] \times \dot{\Delta} \times F$ by using only iterations of the multiplication and the inversion of $x_0, \cdots, x_i$ and x and the homotopies of (3. 1)—(3. 3), by the inductive assumtions. Therefore, because F is a group-like space, we can extend y_j on $F^{i+1}\times I^{i-1}\times[7/8,1]\times\Delta\times F$, firstly on $F^{i+1}\times$ (the vertices of $I^{i-1}\times[7/8,1])\times\Delta\times F$, secondly on $F^{i+1}\times$(the edges of $I^{i-1}\times[7/8,1])\times\Delta\times F$, and then on the side rectangles, and so on, using the homotopies of (3. 1)—(3. 3) defined on some x_l and x appearing in each case. Further, we can extend y_j on $F^{i+1}\times I^{i-1}\times(3/4,7/8]\times\Delta\times F$ by the same map for $F^{i+1}\times I^{i-1}\times 7/8\times\Delta\times F$, because y_j is independent on t_i for $3/4<t_i\leqq 7/8$ by the above notices. Therefore v_{n-1} can be extended for $\{((u_{n-1};t_n),t,x)\mid u_{n-1}\in U_i^i,(t_n,t)\in\Delta\}$.

We assume that v_{n-1} is extended on $\{((u_{n-1};t_n),t,x)\mid u_{n-1}\in U_i^k,(t_n,t)\in\Delta\}$. If $u_{n-1}\in U_i^{k+1}$, then $v_{n-1}=(y_0,\cdots,y_k;s_1,\cdots,s_{k+1})\in V_i^{k+1}$. By the same method as above, we can extend y_j on $F^{k+2}\times I^{i-1}\times[7/8,1]\times[0,3/4]^{k-i}\times 3/4\times\Delta\times F$, using the homotopies of (3. 1)—(3. 3), so that it uses only concerning x_l and x on each vertex, edge and side of $I^{i-1}\times[7/8,1]\times[0,3/4]^{k-i}\times 3/4$.

The map $v'=(u_k';s_{k+1})=v_{n-1}\mid F^{k+2}\times I^{i-1}\times[7/8,1]\times[0,3/4]^{k-i}\times(3/4\times\Delta\cup[0,3/4]\times\dot{\Delta})\times F$ is homotopic to $p^k\circ u_k'=(y_0',\cdots,y_{k-1}';s_1',\cdots,s_k')$, by tending s_{k+1} to 0. $p^k\circ u_k'$ is equal to $v_{n-1}\mid F^{k+2}\times I^{i-1}\times[7/8,1]\times[0,3/4]^{k-i}\times 0\times\Delta\times F=(y_0,\cdots,y_{k-1};s_1,\cdots,s_k)$ on $0\times\dot{\Delta}$, and they are defined by using only the homotopies of (3. 1)—(3. 3). Hence, by (3. 1)—(3. 3), y_j and y_j' are homotopic by a homotopy using only concering x_l on each vertex, edge and side of $I^{i-1}\times[7/8,1]\times[0,3/4]^{k-i}$. Therefore, we can extend v_{n-1} on $F^{k+2}\times I^{i-1}\times[7/8,1]\times[0,3/4]^{k-i}\times[0,3/4]\times\Delta\times F$; and also on $I^{i-1}\times(3/4,1]\times[0,1)^{k-i+1}\times\Delta$, because the coordinates of v_{n-1} are independent on t_i for $3/4<t_i\leqq 7/8$ and on t_l for $3/4\leqq t_l<1$, $l=i+1,\cdots,k+1$. By the above constructions, v_{n-1} is extended on $\{((u_{n-1};t_n),t,x)\mid u_{n-1}\in U_i^{k+1},(t_n,t)\in\Delta\}$.

Therefore, v_{n-1} is extended for $(t_n,t)\in\Delta$, and we obtain $q^n\circ\Psi_i^{n,i}$ and hence $\Psi_i^{n,i}$, completely.

Simple calculations show that $\Psi_i^{n,i}$, $i=0,\cdots,n$, thus defined, have the desired properties.

Therefore, we obtain ϕ_i^n, $\Phi_i^{n,i}$ and $\Psi_i^{n,i}$, satisfying the properties of Lemma 7. 1, and Lemma 7. 1 is proved by the induction.

Proofs of Theorem 6. 9 *and the necessity of Theorem* 1. 1. By Lemma 7. 1, Proposition 5. 8 is applicable, and hence $p^n:(E_n,F)\to(B_n,b)$ is a homotopy equivalence. Therefore $p^\infty:(E_\infty,F)\to(B_\infty,b)$ is also so, and, by Lemma 6. 7, the proofs of the necessity of Theorem 1. 1 are finished.

8. Proofs of the sufficiency of Theorem 1.1

Now we shall prove thc sufficiency of Theorem 1. 1.

Let E be a countable CW-complex, containg F as its subcomplex, and B be a space and p be a map of E into B, satisfying the properties (1. 2) and (1. 3). Then, by [4], [14] we have

Lemma 8.1. *F is a homotopy-associative H-space having an inversion, and there is an H-homomorphism h, which is also a weak homotopy equivalence, of the H-space F into the H-space $\Lambda(B)$ (of generally defined[15]), the space of loops in B with the base point b.*

In this lemma, the term "H-homomorphism" is used in the following sense:

For H-spaces (of generally defined) F and F', with the multiplications μ and μ' and the (homotopy-) units ε and ε' respectively, a map $h:(F, \varepsilon) \to (F', \varepsilon')$ is called an H-homomorphism, if the two maps $(x_1, x_2) \to h\circ\mu(x_1, x_2)$ and $(x_1, x_2) \to \mu'(h(x_1), h(x_2))$ of $F\times F$ into F' are homotopic each other rel. $(\varepsilon, \varepsilon)$. If F and F' have the inversions σ and σ' respectively, one requires that the two maps $x \to h\circ\sigma(x)$ and $x \to \sigma'\circ h(x)$ of F into F' are homotopic each other rel. ε, in addition.

We shall prove that F is a group-like space by the H-structure of Lemma 8. 1, by the following propositions:

Proposition 8.2. *Let F be a countable CW-complex and an H-space (of generally defined) having an inversion and F' be a group-like space (of generally defined[16]). If there is an H-homorphism h, which is also a weak homotopy equivalence, of F into F', then F is also a group-like space (of generally defined) by the given H-structure.*

Proof. We denote the multiplications, the inversions and the (homotopy-) units of F and F' by $x\cdot y$ or xy and $x'\cdot y'$ or $x'y'$, x^{-1} and x'^{-1} and ε and ε' respectively. Because F' is a group-like space, we take the homotopies, of F', f'_t of (2. 3), ε'_t of (2. 7) (and the homotopies of (2. 18) if it is necessary) and $g'^{ij}_{t,s}, g'^{ijkl}_{t,s,r}, \cdots$ of (3. 1)—(3. 3), so that (3. 1)—(3. 3) are satisfied.

Let $\widetilde{E}$ be the mapping cylinder of h, the identification space obtained from $F\times I\cup F'$ by identifying each point $(x, 1)\in F\times 1$ with $h(x)$

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14) Cf. Lemma 9 of §7 and Theorem 5 of §8 of [4]. It was not proved there that the map h in question satisfies the condition of H-homomorphisms concerning with the inversions, but it can be proved analogously, by the same method as §5 of [3].

15) Cf. Remark 2. 17.

16) Cf. Remark 3. 4.

$\in F'$ and each point $(\varepsilon, t) \in \varepsilon \times I$ with $h(\varepsilon) = \varepsilon' \in F'$. We also identify $(x, 0) \in F \times 0$ with $x \in F$, and hence $\varepsilon \in F$ and $\varepsilon' \in F'$ are identified.

Because h is a weak homotopy equivalence, $\pi_n(\tilde{F}, F) = 0$, for every integer $n > 0$, and we obtain therefore, as is well known,

Lemma 8.3. *Let K be a CW-complex and L be its subcomplex. Then every map $k : (K, L) \to (\tilde{F}, F)$ is homotopic rel. L to a map $k' : K \to F$.*

Let $f_0, f_1 : F^3 \to F$ be the map of (2. 3), i. e.,

$$f_0(x, y, z) = (x \cdot y) \cdot z, \quad f_1(x, y, z) = x \cdot (y \cdot z), \qquad \text{for} \quad x, y, z \in F,$$

and let $\tilde{f}_t : F^3 \to \tilde{F}$ be the following composed homotopy, by the notation of (2. 6):

$$\Big\{(xy)z \underset{\text{I}}{\sim} ((xy)z, 1) = h((xy)z) \underset{\text{II}}{\sim} ((h(x)h(y))h(z)$$
$$\underset{f_t'}{\sim} h(x)(h(y)h(z)) \underset{\text{II}}{\sim} h(x(yz) = (x(yz), 1) \underset{\text{I}}{\sim} x(yz)\Big\},$$

where I is given by (x, t), and II is given by the fact that h is an H-homomorphism.

Then, considering as the map of $F^3 \times I$ into $\tilde{F}$, this homotopy is homotopic rel. $F^3 \times \dot{I}$ to a homotopy f_t such that $f_t(F^3) \subset F$, by Lemma 8. 3. This shows that F is homotopy-associative and we take this f_t as a homotopy of (2. 3) for F.

Let $\tilde{\varepsilon}_t : F \to \tilde{F}$ be

$$\Big\{\varepsilon = \varepsilon' \underset{\varepsilon_t'}{\sim} h(x)(h(x))^{-1} \underset{\text{II}}{\sim} h(xx^{-1}) = (xx^{-1}, 1) \underset{\text{I}}{\sim} xx^{-1}\Big\},$$

and we define $\varepsilon_t : F \to F$ by using Lemma 8. 3 as above, and take this as a homotopy of (2. 7) for F.

(If it is necessary, we take homotopies of (2. 18) by the same way).

Let g_0, g_1 and g_t^i be such forms and a homotopy of F^n into F as in the definitions of group-like spaces of §3. Then, by the above constructions of the homotopies, there is a map $\tilde{g}^i : F^n \times I^2 \to \tilde{F}$ such that, for $u \in F^n$, $t, t' \in I$,

$$\tilde{g}^i(u, t, 0) = g_t^i(u), \qquad \tilde{g}^i(u, t, 1) = g'^i_t \circ h^n(u),^{17)}$$

$$\text{(8. 4)} \quad \tilde{g}^i(u, t, t') = \tilde{g}_{t,t'}(u) = \Big\{g_t(u) \underset{\text{I}}{\sim} h \circ g_t(u) \underset{\text{II}}{\sim} g_t'^i \circ h^n(u)\Big\}, \qquad \text{for } t = 0, 1,$$

17) $h^n(x_1, \cdots, x_n) = (h(x_1), \cdots, h(x_n))$.

where g'_0, g'_1 and g'^i_t are the corresponding forms and homotopy, of F'^n into F', with g_0, g_1 and g^i_t. Because F' is a group-like space, any two g'^i_t and g'^j_t are homotopic rel. $F'^n \times \dot{I}$ by the homotopy $g'^{ij}_{t,s}$, by (3.1). Let $\tilde{g}^{ij} : F^n \times (\dot{I}^2 \times I \cap I^2 \times 1) \to \widetilde{F}$ be the map defined by

$$\begin{aligned}
(8.5) \qquad \tilde{g}^{ij}(u, t, s, t') &= \tilde{g}^i(u, t, t') && \text{for } s = 0,\\
&= \tilde{g}^j(u, t, t'), && \text{for } s = 1,\\
&= \tilde{g}_{t,t'}(u), && \text{for } t = 0, 1,\\
&= g'^{ij}_{t,s} \circ h^n(u), && \text{for } t' = 1.
\end{aligned}$$

Because F is a countable CW-complex, F^n and $F^n \times I^3$ are so. Therefore, by Lemma 8.3, $\tilde{g}^{ij}$ is so extended on $F^n \times I^3$ that $\tilde{g}^{ij}(F^n \times I^2 \times 0) \subset F$. The homotopy $g^{ij}_{t,s}$, defined by $g^{ij}_{t,s}(u) = \tilde{g}^{ij}(u, t, s, 0)$, satisfies (3.1), which follows immediately from the above definitions.

Let $\tilde{g}^{ijkl}: F^n \times (\dot{I}^3 \times I \cup I^3 \times 1) \to \widetilde{F}$ be the map defined by

$$\begin{aligned}
\tilde{g}^{ijkl}(u, t, s, r, t') &= \tilde{g}^{ij}(u, t, s, t'), && \text{for } r = 0,\\
&= \tilde{g}^{kl}(u, t, s, t'), && \text{for } r = 1,\\
&= \tilde{g}^{ik}(u, t, r, t'), && \text{for } s = 0,\\
&= \tilde{g}^{jl}(u, t, r, t'), && \text{for } s = 1,\\
&= \tilde{g}_{t,t'}(u), && \text{for } t = 0, 1,\\
&= g'^{ijkl}_{t,s,r} \circ h^n(u), && \text{for } t' = 1,
\end{aligned}$$

where $g'^{ijkl}_{t,s,r}$ is the homotopy of (3.2) for F'. This is well defined, by (8.4) and (8.5). By Lemma 8.3, $\tilde{g}^{ijkl}$ is so extended on $F^n \times I^4$ that $\tilde{g}^{ijkl}(F^n \times I^3 \times 0) \subset F$. Therefore, the homotopy $g^{ijkl}_{t,s,r}$ defined by $g^{ijkl}_{t,s,r}(u) = \tilde{g}^{ijkl}(u, t, s, r, 0)$, satisfies (3.2).

Further, we can prove the analogous properties on $g^{ijkl}_{t,s,r}$ for F, by the properties of $g'^{ijkl}_{t,s,r}$ for F', and so on. Therefore, F is a group-like space and the proofs of Proposition 8.2 are finished.

Proof of the sufficiency of Theorem 1.1. It follows immediately from Lemma 8.1 and Proposition 8.2 that F of Theorem 1.1 is a group-like space. Therefore, the sufficiency of Theorem 1.1 is proved, and so Theorem 1.1 is obtained completely.

9. The condition (A_1) and an auxiliary lemma

The remaining sections will be concerned with proofs of Theorem 1.4.

Here, we restate the condition (A_1) of [4], §2, concerning to a given map $p : (E, F) \to (B, C)$, which is a necessary and sufficient condition for the fact that p is a weak homotopy equivalence between the two pairs (Cf. Theorem 3 of [4], §3), as it is used often.

(A_1) *Let K be a CW-complex, L its subcomplex, and M a subcomplex of the product complex $K\times I$. Let*

$$\xi' : K\times 0 \cup L\times I \to E, \qquad \eta : K\times I \to B$$

be given maps such that $\xi'(M')\subset F$, $(M' = (K\times 0 \cup L\times I)\cap M)$, and $\eta(M)\subset C$, and the two maps $p\circ\xi'$ and $\eta \mid K\times 0\cup L\times I$ are homotopic each other by a homotopy

$$Y'_t : (K\times 0 \cup L\times I,\ M') \to (B, C),$$

with $Y'_0 = p\circ\xi'$ and $Y'_1 = \eta \mid K\times 0\cup L\times I$.

From these assumptions, it follows that ξ' has an extension

$$\xi : K\times I \to E, \quad \text{being} \quad \xi(M)\subset F,$$

and the two maps $p\circ\xi$ and η are homotopic each other by a homotopy

$$Y_t : (K\times I,\ M) \to (B, C), \quad \text{with} \quad Y_0 = p\circ\xi,\ Y_1 = \eta,$$

and also this homotopy Y_t is taken as an extension of the given homotopy Y'_t, i. e., $Y_t \mid K\times 0\cup L\times I = Y'_t$.

In this section, we prove the following lemma:

Lemma 9.1. *Let F be a CW-comlex such that $F\times F\times F = F^3$ is also so, E be a space containing F, B be a space containing a point b, and p be a map of E into B such that $p(F) = b$. We assume that*

(9.2) *F is contractible in E to a vertex $\varepsilon\in F$ leaving ε fixed;*

(9.3) *$p : (E, F)\to(B, b)$ is a weak homotopy equivalence.*

We also assume that there is a map $\bar\mu : E\times F\to E$ satisfying the following properties:

(9.4) *$\bar\mu(F\times F)\subset F$ and $\bar\mu(u, \varepsilon) = u$, $\bar\mu(\varepsilon, x) = x$, for $u\in E$, $x\in F$;*

(9.5) *the map $p\circ\bar\mu : (E\times F,\ F\times F)\to(B, b)$ is homotopic rel. $F\times F$ to the map $\bar p : (E\times F,\ F\times F)\to(B, b)$, defined by $\bar p(u, x) = p(u)$.*

Then the H-structure $\bar\mu \mid F\times F$ of F is a homotopy-associative one (having an inversion).

Proof. By (9.4) and (2.1), $\bar\mu \mid F\times F$ is an H-structure, and we write $x\cdot y$ or xy instead of $\bar\mu(x, y)$ for $x, y\in F$. It is necessary to prove (2.2), i. e., the two maps $f_0, f_1 : (F^3, \varepsilon^3)\to(F, \varepsilon)$, such that

(9.6) $\qquad f_0(x, y, z) = (xy)z,\quad f_1(x, y, z) = x(yz), \qquad$ for $x, y, z\in F$,

are homotopic each other rel. ε^3.

Let a homotopy $k_t : F\to E$ be a contraction of F in E, given by the assumption (9.2), such that

$$k_0 = \text{the identity map},\ k_1(F) = \varepsilon,\ k_t(\varepsilon) = \varepsilon \ \text{ for }\ 0\leqq t\leqq 1.$$

Let $\xi' : F^3\times I\times 0 \cup F^3\times \dot{I}\times I \cup \varepsilon^3\times I\times I \to E$ be the map defined by

$$\begin{aligned}
\xi'(x, y, z, t, 0) &= \bar{\mu}(\bar{\mu}(k_{2t}(x), y), z), && \text{for } 0 \leqq t \leqq 1/2,\\
&= \bar{\mu}(k_{2-2t}(x), yz), && \text{for } 1/2 \leqq t \leqq 1,\\
\xi'(x, y, z, 0, s) &= (xy)z = f_0(x, y, z), && \\
\xi'(x, y, z, 1, s) &= x(yz) = f_1(x, y, z), && \text{for } 0 \leqq s \leqq 1,\\
\xi'(\varepsilon, \varepsilon, \varepsilon, t, s) &= \varepsilon, && \text{for } 0 \leqq s,\ t \leqq 1.
\end{aligned}$$

This map is well defined, by (9. 4). Also

(9. 7) $$\xi'((F^3\times \dot{I} \cup \varepsilon^3\times I)\times 1) \subset F.$$

Let $P_t : E\times F \to B$ be the homotopy such that

$$P_0 = p\circ\bar{\mu}, \quad P_1 = \bar{p}, \quad P_t(F\times F) = b,$$

given by the assumption (9. 5), and let $\eta : F^3\times I\times I \to B$ be the map defined by

$$\begin{aligned}
\eta(x, y, z, t, s) &= P_{4s}(\bar{\mu}(k_{2t}(x), y), z), && \text{for } 0 \leqq t \leqq 1/2,\ 0 \leqq s \leqq 1/4,\\
&= P_{4s-1}(k_{2t}(x), y), && \text{for } 0 \leqq t \leqq 1/2,\ 1/4 \leqq s \leqq 1/2,\\
&= P_{2s}(k_{2-2t}(x), yz), && \text{for } 1/2 \leqq t \leqq 1,\ 0 \leqq s \leqq 1/2,\\
&= p\circ k_{2t(2-2s)}(x), && \text{for } 0 \leqq t \leqq 1/2,\ 1/2 \leqq s \leqq 1,\\
&= p\circ k_{(2-2t)(2-2s)}(x), && \text{for } 1/2 \leqq t \leqq 1,\ 1/2 \leqq s \leqq 1.
\end{aligned}$$

This map η is well defined, and

(9. 8) $$\eta(F^3\times I\times 1) = b,$$

and the two maps ξ' and η are related by

(9. 9) $$p\circ\xi' = \eta \mid F^3\times I\times 0 \cup (F^3\times \dot{I} \cup \varepsilon^3\times I)\times I.$$

On the other hand, by the assumption (9. 3), the map $p : (E, F) \to (B, b)$ is a weak homotopy equivalence. Therefore the condition (A_1) is satisfied by this map p, by Theorem 3 of [4], §3.

By (9. 7)—(9. 9) and the assumption that F^3 is a CW-complex, the above defined map ξ' and η satisfy the assumptions of (A_1) for $p : (E, F) \to (B, b)$, by taking

$$K = F^3\times I, \quad L = F^3\times \dot{I} \cup \varepsilon^3\times I, \quad M = F^3\times I\times 1,$$

and $Y'_t = p\circ\xi' = \eta \mid K\times 0 \cup L\times I$. Therefore, by the conclusions of (A_1), ξ' has an extension $\xi : F^3\times I\times I \to E$ such that $\xi(F^3\times I\times 1) \subset F$, by the property $\xi(M) \subset F$ of (A_1), and

$$\xi(x, y, z, t, 1) = \xi'(x, y, z, t, 1) = f_t(x, y, z), \qquad \text{for } t = 0, 1,$$

and $\xi(\varepsilon, \varepsilon, \varepsilon, t, 1) = \varepsilon$ for $0 \leqq t \leqq 1$.

Hence, the homotopy $f_t:(F^{ii}, \varepsilon^{ii}) \to (F, \varepsilon)$, defined by $f_t(x, y, z) = \xi(x, y, z, t, 1)$, is a homotopy between f_0 and f_1 of (9.6), and (2.2) is proved, as desired. Showing that F has also an inversion by Lemma 2.5, we have Lemma 9.1.

10. Proofs of Theorem 1.4

We prove the following lemma, by the analogous method with the proof of Lemma 9 of [4], §7.

Lemma 10.1. *Let F be a CW-complex. We assume that there are topological spaces $E_2 \supset E_1 \supset F$, $B_2 \supset B_1 \ni b$ (a point), and a map p: $(E_2, E_1, F) \to (B_2, B_1, b)$, satisfying the properties* (1.5)—(1.8).

Then, there is a map $\bar{\mu}: E_1 \times F \to E_1$ satisfying the properties (9.4) *and* (9.5) *of Lemma* 9.1.

Proof. Let homotopies $k_t':(F, \varepsilon) \to (E_1, \varepsilon)$ and $k_t'':(E_1, \varepsilon) \to (E_2, \varepsilon)$ be the contractions, given by (1.5) and (1.6), such that k_0', k_0'' are the identity maps of F, E_1, respectively, and $k_1'(F) = k_1''(E_1) = \varepsilon$.

By (1.6), E_1 is a CW-complex and F is its subcomplex, and hence we can extend the homotopy $k_t': F \to E_1$ to a homotopy $k_t': E_1 \to E_1$, such that k_0' is is the identity map of E_1, by the homotopy extension theorem for CW-complexes.

Let $k_t: E_1 \to E_2$ be the homotopy defined by, for $u \in E_1$,

$$\begin{aligned} k_t(u) &= k_{2t}'(u), && \text{for } 0 \leqq t \leqq 1/2, \\ &= k_{2t-1}'' \circ k_1'(u), && \text{for } 1/2 \leqq t \leqq 1. \end{aligned}$$

Clearly, this is well defined, and

(10.2) $k_0 =$ the identity map of E_1, $k_1(E_1) = \varepsilon$, $k_t(F) \subset E_1$, $k_t(\varepsilon) = \varepsilon$.

Starting from this contraction of E_1 in E_2, we shall follow analogous processes of the proof of Lemma 9 of [4], §7.

We define a map $g_0: E_1 \times F \to E_2$ by $g_0(u, x) = x$, and a homotopy $g_t': E_1 \vee F \to E_2$ by[18]

$$g_t'(u, \varepsilon) = k_{1-t}(u), \quad g_t'(\varepsilon, x) = x, \qquad \text{for } u \in E_1, \ x \in F.$$

Then g_t' ia a homotopy of $g_0 \mid E_1 \vee F$, and also $g_0(F \times F) \subset E_1$ and $g_t'(F \vee F) \subset E_1$ by (10.2). Therefore, we can extend this homotopy g_t' by the homotopy extension theorem of the form of Lemma 1 of [4], §2, noticing that $E_1 \times F$ is a CW-complex, by (1.6), and $E_1 \vee F$ is its

18) $E_1 \vee F = E_1 \times \varepsilon \cup \varepsilon \times F \subset E_1 \times F$.

subcomplex; and we obtain a homotopy $g_t : E_1 \times F \to E_2$ such that

(10. 3) $g_t(F\times F)\subset E_1, \quad g_t(u, \varepsilon) = k_{1-t}(u), \quad g_t(\varepsilon, x) = x, \quad g_0(u, x) = x.$

Let $h' : E_1 \times F \times I \to B_2$ be the map defined by

$$\begin{aligned} h'(u, x, t) &= p\circ g_{1-2t}(u, x), && \text{for } 0 \leqq t \leqq 1/2, \\ &= p\circ k_{2-2t}(u), && \text{for } 1/2 \leqq t \leqq 1. \end{aligned}$$

By (10. 2) and (10. 3), this is well defined and

$$(10.4) \quad \begin{aligned} h'(u, x, 0) = p\circ g_1(u, x), \quad h'(u, x, 1) = p(u) = \bar{p}(u, x), \\ h'(\varepsilon\times F\times I) = b, \quad h'(F\times F\times I)\subset B_1. \end{aligned}$$

Also, let $h'_s : (E_1\vee F)\times I \cup E_1\times F\times \dot{I} \to B_2$ be the homotopy defined by

$$\begin{aligned} & h'_s(\varepsilon\times F\times I) = b, \\ & h'_s(u, \varepsilon, t) = p\circ k_{2(1-s)t}(u), && \text{for } 0 \leqq t \leqq 1/2, \\ & \qquad\qquad = p\circ k_{(1-s)(2-2t)}(u), && \text{for } 1/2 \leqq t \leqq 1, \\ & h'_s(u, x, 0) = p\circ g_1(u, x), \\ & h'_s(u, x, 1) = p(u) = \bar{p}(u, x); && \text{for } u\in E_1, \ x\in F. \end{aligned}$$

By (10. 2)—(10. 4), this is well defined and

$$h'_0 = h' \mid (E_1\vee F)\times I\cup E_1\times F\times \dot{I}, \quad h'_s((F\vee F)\times I\cup F\times F\times \dot{I})\subset B_1.$$

By these properties and the last one of (10. 4), we can extend the homotopy h'_s, by using Lemma 1 of [4], §2, again, and obtain a homotopy $h_s : E_1\times F\times I \to B_2$ and hence a map $h_1 : E_1\times F\times I\to B_2$ satisfying the following properties:

$$(10.5) \quad \begin{aligned} & h_1(F\times F\times I)\subset B_1, \\ & h_1(u, x, 0) = h'_1(u, x, 0) = p\circ g_1(u, x), \qquad \text{for } u\in E_1, x\in F, \\ & h_1(u, x, t) = \bar{p}(u, x) = p(u), \\ & \qquad \text{for } t=1, \ (u, x)\in E_1\times F, \text{ or } 0\leqq t\leqq 1, (u, x)\in E_1\vee F. \end{aligned}$$

Let $g' : E_1\times F\times 0\cup(E_1\vee F)\times I\to E_2$ be the map defined by

(10. 6) $\quad g'(u, x, 0) = g_1(u, x), \quad g'(u, \varepsilon, t) = u, \quad g'(\varepsilon, x, t) = x;$

and set $g'' = g' \mid F\times F\times 0\cup(F\vee F)\times I$. Then the image of g'' is contained in E_1, by (10. 3). Also, by (10. 5),

(10. 7) $\quad p\circ g' = h_1 \mid E_1\times F\times 0\cup(E_1\vee F)\times I.$

Therefore the assumptions of (A_1) for the map $p \mid E_1 : (E_1, F)\to(B_1, b)$, restated in §9, are satisfied, by taking the CW-complexes $K=F\times F$, $L=F\vee F$, $M=F\times F\times 1$, and the maps

$$\xi' = g'' : F\times F\times 0\cup(F\vee F)\times I\to E_1, \quad \eta = h_1 \mid F\times F\times I : F\times F\times I\to B_1,$$

and the homotopy $Y'_t = p \circ g''$, because $g''((F \vee F) \times 1) \subset F$ by (10.6) and $h_1(F \times F \times 1) = p(F) = b$ by (10.5).

On the other hand, by the assuption (1.7), $p \mid E_1 : (E_1, F) \to (B_1, b)$ is a weak homotopy equivalence, and hence it satisfies the condition (A_1), by Theorem 3 of [4], §3. Therefore, by the conclusions of (A_1), there are an extension $\bar{g}'' : F \times F \times I \to E_1$ of g'' and a homotopy $Y''_t : F \times F \times I \to B_1$, satisfying the following properties:

(10.8)
$$\bar{g}''(F \times F \times 1) \subset F; \quad Y''_0 = p \circ \bar{g}'', \quad Y''_1 = h_1 \mid F \times F \times I,$$
$$Y''_t(F \times F \times 1) = b, \quad Y''_t \text{ is stationary on } F \times F \times \dot{I} \cup (F \vee F) \times I.$$

Using these results, we can define $\bar{g}' : E_1 \times F \times 0 \cup (E_1 \vee F \cup F \times F) \times I \to E_2$ by
$$\bar{g}' \mid E_1 \times F \times 0 \cup (E_1 \vee F) \times I = g', \quad \bar{g}' \mid F \times F \times I = \bar{g}'',$$
and $Y'_t : E_1 \times F \times 0 \cup (E_1 \vee F \cup F \times F) \times I \to B_2$, by
$$Y'_t \mid E_1 \times F \times 0 \cup (E_1 \vee F) \times I = p \circ g', \quad Y'_t \mid F \times F \times I = Y''_t.$$
Then, by (10.7) and (10.8),
$$p \circ \bar{g}' = Y'_0, \quad h_1 \mid E_1 \times F \times 0 \cup (E_1 \vee F \cup F \times F) \times I = Y'_1.$$

Therefore the maps $\bar{g}', h_1$ and the homotopy Y'_t satisfy the assumtions of (A_1) for the map $p : (E_2, E_1) \to (B_2, B_1)$, by taking $K = E_1 \times F$, $L = E_1 \vee F \cup F \times F$ and $M = E_1 \times F \times 1$, because $\bar{g}'(M') = g'((E_1 \vee F) \times 1) \cup \bar{g}''(F \times F \times 1) \subset E_1$, $h_1(E_1 \times F \times 1) \subset B_1$, and $Y'_t(M') = p \circ g'((E_1 \vee F) \times 1) \cup Y''_t(F \times F \times 1) \subset B_1$, by (10.5), (10.6) and (10.8).

On the other hand, by the assumptions (1.7) and (1.8), $p : (E_2, F) \to (B_2, b)$ and $p \mid E_1 : (E_1, F) \to (B_1, b)$ are both weak homotopy equivalences, and hence $p : (E_2, E_1) \to (B_2, B_1)$ is also so, which can be shown immediately by the five lemma. Therefore $p : (E_2, E_1) \to (B_2, B_1)$ satisfies (A_1), by Theorem 3 of [4], §3, again, and we can apply (A_1) to $\bar{g}'$ and h_1, and obtain $g : E_1 \times F \times I \to E_2$ and $Y_t : E_1 \times F \times I \to B_2$, such that
$$g \text{ and } Y_t \text{ are extensions of } \bar{g}' \text{ and } Y'_t, \text{ respectively},$$
$$g(E_1 \times F \times 1) \subset E_1, \quad Y_t(E_1 \times F \times 1) \subset B_1, \quad p \circ g = Y_0, \quad h_1 = Y_1.$$

Let $\bar{\mu} : E_1 \times F \to E_1$ be the map defined by $\bar{\mu}(u, x) = g(u, x, 1)$, and $P_t : E_1 \times F \to B_1$ be the homotopy defined by $P_t(u, x) = Y_t(u, x, 1)$, for $u \in E_1$, $x \in F$. Then they have the following properties:
$$\bar{\mu}(F \times F) = \bar{g}''(F \times F \times 1) \subset F, \quad P_t(F \times F) = Y''_t(F \times F \times 1) = b, \text{ by (10.8)},$$
$$\bar{\mu}(u, \varepsilon) = g'(u, \varepsilon, 1) = u, \quad \bar{\mu}(\varepsilon, x) = g'(\varepsilon, x, 1) = x, \text{ by (10.6)},$$
$$P_0 = p \circ \bar{\mu}, \quad P_1(u, x) = h_1(u, x, 1) = \bar{p}(u, x), \text{ by (10.5)}.$$

These relations show that the map $\bar{\mu}$ satisfies (9.4) and (9.5), and the proofs of Lemma 10.1 are finished.

Proof of Theorem 1.4. By Lemma 10.1 and the assuptions that $F \times F \times F$ is a *CW*-complex, Lemma 9.1 is applicable. Therefore F is a homotopy-associative *H*-space having an inversion by the *H*-structure $\bar{\mu} \mid F \times F$ of Lemma 10.1; and the proofs of Theorem 1.4 are finished completely.

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