THEORY OF AFFINE CONNECTIONS OF THE SPACE OF TANGENT DIRECTIONS OF A DIFFERENTIABLE MANIFOLD, III

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Part III

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§ 24. Finsler manifolds.

When a positive valued function L on $T_0(\mathfrak{X})$ such that

$$L \cdot \lambda_k = kL, \qquad k > 0, \tag{24.1}$$

where λ_k are mappings defined in § 2, is given, we say $(\mathfrak{X}, L)$ is a *Finsler manifold*. This definition is slightly weaker than the ordinary one. If the condition (24. 1) is replaced with $L \cdot \lambda_k = |k| L$ for any k, then $(\mathfrak{X}, L)$ is a Finsler manifold in the ordinary sense.

Now, by means of (2. 13), $L \cdot \tilde{\tau}_0$, $L \cdot \tilde{\pi}_0$, $L \cdot \tilde{\pi}_0 \cdot \widetilde{\chi}_0$ are the functions with the same property as L on $T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X})$, $\widetilde{\mathfrak{B}}_0 = \widetilde{\mathfrak{B}} - \mathfrak{B}$, $\widetilde{\mathfrak{B}}_n - \mathfrak{B}_n$ respectively. We denote also these functions by the same symbol L. Then we define a map $f: \ T_0(\mathfrak{X}) \to T_0(\mathfrak{X})$ which covers the identity transformation of $\mathfrak{X}$ by

$$1 = L \cdot f. \tag{24.2}$$

By the same equation, we can define two bundle maps f_1, f_p on $T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X})$, $\widetilde{\mathfrak{B}}_0$ respectively which cover the map f. Furthermore, by (24. 1), (2. 13), we have the equations

$$f \cdot \lambda_k = f, \qquad f_\alpha \cdot \lambda_k = f_\alpha. \tag{24.3}$$

Then, by the following equations

$$f = \varphi \cdot \rho, \quad f_\alpha = \varphi_\alpha \cdot \rho_\alpha, \tag{24. 4}$$

we can define uniquely the map $\varphi : \mathfrak{S}(\mathfrak{X}) \to T_0(\mathfrak{X})$ and the bundle maps φ_α, $\alpha = 1, p$, which cover φ. As easily seen, we have

$$\rho \cdot \varphi = 1, \quad \rho_\alpha \cdot \varphi_\alpha = 1. \tag{24. 5}$$

Conversely, when a map $\varphi : \mathfrak{S}(\mathfrak{X}) \to T_0(\mathfrak{X})$, such that $\rho \cdot \varphi = 1$, we shall obtain the function L.

Nextly, we define n functions $f^i : \widetilde{\mathfrak{B}}_0 \to R$ by

$$y^i = L f^i. \tag{24. 6}$$

Proposition 24. 1. *For the induced connection* $\widetilde{\Gamma} = \rho^{\diamond} \Gamma$ *of* $\mathfrak{F} = \{T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X}),\ T_0(\mathfrak{X})\}$ *from a connection* Γ *of* $\{\mathfrak{Z}, \mathfrak{S}(\mathfrak{X})\}$, *we have* $f^{\diamond} \widetilde{\Gamma} = \widetilde{\Gamma}$, $\varphi^{\diamond} \widetilde{\Gamma} = \Gamma$.

Proof. This follows immediately from (24. 4), (24. 5). We shall have the same proposition for the affine connection (Γ, dp). For example, with respect to the first equation, since

$$\bar{\tau} \cdot f = \bar{\tau}, \quad \bar{\tau}_\alpha \cdot f_\alpha = \bar{\tau}_\alpha, \tag{24.7}$$

we have

$$(f^\circ \otimes f^*)(\bar{\tau}^\circ \otimes \bar{\tau}^*) dp = (f^\circ \cdot \bar{\tau}^\circ \otimes f^* \cdot \bar{\tau}^*) dp$$
$$= (\bar{\tau}^\circ \otimes \bar{\tau}^*) dp.$$

Proposition 24. 2. *In order that a tensor field* $\mathfrak{K}$ *of* $\mathfrak{F}$ *is a* ρ°-*image, it is necessary and sufficient that* $\mathfrak{K}$ *is invariant under* f.

Proof. The necessity is easily followed from (24. 4), (24. 5) and

$$\rho \cdot f = \rho, \quad \rho_\alpha \cdot f_\alpha = \rho_\alpha. \tag{24. 8}$$

The sufficiency can be proved by (24. 4).

Proposition 24. 3. *In order that a tensor field* $\mathfrak{K}$ *of* $\mathfrak{F}$ *is invariant under* f, *it is necessary and sufficient that* $\mathfrak{K}$ *is invariant under* λ_k $(k > 0)$.

Proof. The necessity is evident from (24. 3). If $\mathfrak{K}$ is invariant under λ_k $(k > 0)$, then $\mathfrak{K}$ is considered as a ρ°-image. Accordingly, it is invariant under f by Proposition 24. 2.

Now, for the canonical local coordinates (u^j, ξ^j) of $T(\mathfrak{X})$, we define l^j by

$$\xi^j = L\, l^j. \tag{24.9}$$

l^j are the local components of the vector field $\mathfrak{y}/L$ of $\mathfrak{F}$. f^j defined by (24.6) are the components of the $\bar{\pi}^0$-image of $\mathfrak{y}/L$. We have easily

$$D\xi^j = LDl^j + l^j dL, \quad Dy^j = LDf^j + f^j dL. \tag{24.10}$$

Accordingly, we have for $\widetilde{\Gamma} = \rho^{\diamond}\, \Gamma$ the equations

$$\omega_i^j = \Gamma^*{}_i{}^j{}_k\, du^k + LC_i{}^j{}_k\, M_h^k\, Dl^h = f^* \omega_i^j \tag{24.11}$$

from (3.8) and (4.10).

Proposition 24.4. *For an f-invartant tensor field $\mathfrak{K}$ of $\mathfrak{F}$, $D\mathfrak{K}$ is also f-invariant, that is*

$$(f^\circ \otimes f^*)\, D\mathfrak{K} = D\mathfrak{K}.$$

Proof. This is evident from the assumption and the formula (1.17).

Now, since L is a scalar, if we put

$$L\mu_i = L_{,i} = \frac{\partial L}{\partial u^i} - \frac{\partial L}{\partial \xi^h}\, \Gamma^*{}_k{}^h{}_i\, \xi^k, \quad l_i = L_{;i} = \frac{\partial L}{\partial \xi^i} \tag{24.12}$$

$$(L\varphi_i = L_{,i}, \quad f_i = L_{;i} \quad \text{on } \widetilde{\mathfrak{B}}_0),$$

we have

$$l_i\, l^i = 1 \qquad (f_i f^i = 1 \quad \text{on } \widetilde{\mathfrak{B}}_0), \tag{24.13}$$

since $\lambda_k^*\, L = kL$. For the connection $\widetilde{\Gamma}$ of $\mathfrak{F}$, we have locally from (24.9) and the above equations

$$\begin{aligned} dL &= L_{,i}\, du^i + L_{;h}\, M_i^h\, D\xi^i \\ &= L\mu_i\, du^i + L_{;h}\, M_i^h\, (LDl^i + l^i\, dL), \end{aligned}$$

that is

$$\mu_i\, du^i + l_h\, M_i^h\, Dl^i = 0 \quad (\varphi_i\, \theta^i + f_h\, \widetilde{M}_i^h\, Df^i = 0). \tag{24.14}$$

Proposition 24. 5. *The covariant derivative of the first kind and the one of the second kind multiplied with L of an f-invariant tensor field $\mathfrak{K}$ of $\mathfrak{F}$ are also f-invariant.*

Proof. Let $K_{j_1 \cdots j_q}^{i_1 \cdots i_p}$ be the local components of $\mathfrak{K}$. Since we have $f^* M_i^j = M_i^j$, $f^* \xi^j = l^j$ by Proposition 24.4,

$$\begin{aligned} DK_{j_1 \cdots j_q}^{i_1 \cdots i_p} &= K_{j_1 \cdots j_q, h}^{i_1 \cdots i_p}\, du^h + K_{j_1 \cdots j_q; h}^{i_1 \cdots i_p}\, M_k^h D\xi^k \\ &= f^*(K_{j_1 \cdots j_q, h}^{i_1 \cdots i_p})\, du^h + f^*(K_{j_1 \cdots j_q; h}^{i_1 \cdots i_p})\, M_k^h\, Dl^k. \end{aligned}$$

Since $\mathfrak{K}$ is f-invariant, we have

$$\xi^h \partial K^{i_1\cdots i_p}_{j_1\cdots j_q}/\partial \xi^h = 0,$$

hence we get from (3.8) and (9.8)

$$K^{i_1\cdots i_p}_{j_1\cdots j_q;h}\,\xi^h = 0.$$

Accordingly, we have the equation

$$\{f^*(K^{i_1\cdots i_p}_{j_1\cdots j_q,h}) - K^{i_1\cdots i_p}_{j_1\cdots j_q,h}\}\,du^h + \{f^*(K^{i_1\cdots i_p}_{j_1\cdots j_q;h}) - LK^{i_1\cdots i_p}_{j_1\cdots j_q;h}\}\,M^h_k\,Dl^k = 0.$$

On the other hand, there exists only the relation (24.14) between du^j and Dl^j and we have

$$\{f^*(K^{i_1\cdots i_p}_{j_1\cdots j_q;h}) - LK^{i_1\cdots i_p}_{j_1\cdots j_q;h}\}\,l^h = f^*(K^{i_1\cdots i_p}_{j_1\cdots j_q;h}\,l^h) = 0.$$

Considering (24.13), we have

$$f^*(K^{i_1\cdots i_p}_{j_1\cdots j_q,h}) = K^{i_1\cdots i_p}_{j_1\cdots j_q,h}, \quad f^*(K^{i_1\cdots i_p}_{j_1\cdots j_q;h}) = LK^{i_1\cdots i_p}_{j_1\cdots j_q;h}. \tag{24.15}$$

Since

$$f^*L = 1, \qquad f^*l^j = l^j, \tag{24.16}$$

we have

$$f^*(LK^{i_1\cdots i_p}_{j_1\cdots j_q;h}) = LK^{i_1\cdots i_p}_{j_1\cdots j_q;h}. \qquad \text{q. e. d.}$$

Lastly, the curvature forms of $\widetilde{\Gamma}$ are written as

$$\Omega^j_i = \frac{1}{2}R_i{}^j{}_{hk}\,du^h \wedge du^k + LP_i{}^j{}_{ht}M^t_k\,du^h \wedge Dl^k + \frac{1}{2}L^2 S_i{}^j{}_{ts}\,M^t_h M^s_k\,Dl^h \wedge Dl^k$$

by means of (9.15), (9.16) and (24.10). Accordingly, we see easily that $R_i{}^j{}_{hk}$, $LP_i{}^j{}_{hk}$, $L^2S_i{}^j{}_{hk}$ are f-invariant. In order to denote them in terms of the f-invariant quantities, it is sufficient to cover (9.4′), (9.5′), (9.6′) with f^*. In fact, if we put

$$A_i{}^j{}_h = f^*C_i{}^j{}_h = LC_i{}^j{}_h, \tag{24.17}$$

these are f-invariant. Then, we have

$$\begin{cases} LC_i{}^j{}_{h,k} = A_i{}^j{}_{h,k} - A_i{}^j{}_h\,\mu_k, \\ L^2C_i{}^j{}_{h;k} = LA_i{}^j{}_{h;k} - A_i{}^j{}_h\,l_k. \end{cases} \tag{24.18}$$

Substituting them in (9.5′) and (9.6′), we get

$$LP_i{}^j{}_{hk} = -L\frac{\partial\Gamma^*{}_i{}^j{}_h}{\partial\xi^k} + A_i{}^j{}_{k,h} - A_i{}^j{}_k\,\mu_h - A_i{}^j{}_t\,L\frac{\partial\Gamma^*{}^t{}_{sh}}{\partial\xi^k}\,l^s \tag{24.19}$$

$$
\begin{aligned}
L^2 S_i{}^j{}_{hk} = L A_i{}^j{}_{k;h} - L A_i{}^j{}_{h;k} - A_t{}^j{}_h A_i{}^t{}_k + A_t{}^j{}_k A_i{}^t{}_h \\
- A_i{}^j{}_t (A_h{}^t{}_k - A_k{}^t{}_h) + A_i{}^j{}_k l_h - A_i{}^j{}_h l_k.
\end{aligned} \tag{24.20}
$$

The following formulas are also easily proved :

$$
l^j{}_{,h} = -l^j \mu_h, \quad L l^j{}_{;h} = \Phi^j_h - l^j l_h. \tag{24.21}
$$

§ 25. Metric connections.

For any Finsler manifold $(\mathfrak{X}, L)$, we define a symmetric tensor field $\mathfrak{g}$ of the type (0, 2) by

$$
\mathfrak{g} = g_{ij} du^i \otimes du^j,
$$

$$
g_{ij} = \frac{\partial^2 F}{\partial u^i \partial u^j}, \quad F = \frac{1}{2} L^2. \tag{25.1}
$$

We put $\tilde{\mathfrak{g}} = \tilde{\pi}^\circ \mathfrak{g} = \tilde{g}_{ij} \mathfrak{z}^i \otimes \mathfrak{z}^j$. Since $\lambda_k{}^* F = k^2 F (k > 0)$, $\mathfrak{g}$ is f-invariant. Define the inner product of any two vectors in a fibre of $\mathfrak{F}$ or $\{\widetilde{\mathfrak{B}}_n - \mathfrak{B}_n, \widetilde{\mathfrak{B}}_0\}$ by virtue of

$$
g_{ij} = \frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u^j}, \quad \tilde{g}_{ij} = \mathfrak{z}_i \cdot \mathfrak{z}_j, \tag{25.2}
$$

then we get easily

$$
\mathfrak{y} \cdot \mathfrak{y} = L^2 \text{ or } g_{ij} l^i l^j = 1. \tag{25.3}
$$

When $\mathfrak{g}$ is non singular or positive definite everywhere, the Finsler manifold $(\mathfrak{X}, L)$ is called to be *regular* or *positive regular* respectively.

Now, with respect to a regular connection $\widetilde{\Gamma} = \rho^{\diamond} \Gamma$ of $\mathfrak{F}$, if $\mathfrak{g}$ is parallel, that is $D\mathfrak{g} = 0$, we say that Γ is *metric with respect to* $\mathfrak{g}$ (or $(\mathfrak{X}, L)$). This is locally written as $g_{ij,h} = g_{ij;h} = 0$, that is

$$
\frac{\delta g_{ij}}{\delta u^h} = g_{ik} \Gamma_j{}^k{}_h + g_{kj} \Gamma_i{}^k{}_h, \tag{25.4}
$$

$$
\frac{\partial g_{ij}}{\partial \xi^h} = g_{ik} C_j{}^k{}_h + g_{kj} C_i{}^k{}_h. \tag{25.5}
$$

We get easily

$$
l_i = g_{ij} l^j \qquad f_i = \tilde{g}_{ij} f^j. \tag{25.6}
$$

It follows from (25.3), (25.4) and (25.5) that

$$
l_i \, D l^i = 0. \tag{25.7}
$$

Accordingly, making use of (24.14), we have

$$
\mu_i = 0, \qquad l_k \, M^k_i = l_i. \tag{25.8}
$$

These equations are equivalent to

$$\frac{\partial L}{\partial u^i} = L\Gamma^{*}{}_{k}{}^{h}{}_{i} l^k l_h, \qquad A_{k}{}^{h}{}_{i} l^k l_h = 0. \tag{25.8'}$$

Lemma 25.1. *For a regular metric connection* $\tilde{\Gamma}$ *with respect to a regular Finsler manifold* $(\mathfrak{X}, L)$, *if we have*

$$\tilde{C}_{ijk} = \tilde{g}_{jh}\tilde{C}_{i}{}^{h}{}_{k} = \tilde{C}_{jik}.$$

then $\Phi_\Gamma = 1$.

Proof. We get from (25.5) the equation

$$C_{ijk} = \frac{1}{2}\,\frac{\partial^3 F}{\partial\xi^i\partial\xi^j\partial\xi^k}, \tag{25.9}$$

hence $C_{o}{}^{i}{}_{j} = 0$.

Theorem 25.2. *The metric connection of E. Cartan*[1] *with respect to a regular Finsler manifold* $(\mathfrak{X}, L)$ *is a-proper, that is* $^*\Gamma = \Gamma$.

Proof. As is well known, the metric connection of E. Cartan is uniquely determined by means of (25.4), (25.5) and the conditions

$$C_{ijk} = C_{jik}, \qquad \Gamma^{*h}_{ij} = \Gamma^{*h}_{ji}. \tag{25.10}$$

Hence, by virtue of Lemma 25.1, we get $\Phi_\Gamma = 1$. According to the book of E. Cartan, putting

$$2G_i = \frac{\partial^2 F}{\partial\xi^i\partial u^k}\xi^k - \frac{\partial F}{\partial u^i}, \quad G^i = g^{ij}G_j, \tag{25.11}$$

we get

$$\xi^i\Gamma_{i}{}^{h}{}_{j} = \xi^i\Gamma^{*h}_{ij} = \frac{\partial G^h}{\partial\xi^j} \tag{25.12}$$

and

$$\begin{cases} \Gamma_{ihj} = g_{hk}\Gamma^{k}_{ij} = \dfrac{1}{2}\left(\dfrac{\partial g_{ih}}{\partial u^j} + \dfrac{\partial g_{hj}}{\partial u^i} - \dfrac{\partial g_{ij}}{\partial u^h}\right) + C_{ijk}\dfrac{\partial G^k}{\partial\xi^h} - C_{hjk}\dfrac{\partial G^k}{\partial\xi^i}, \\ \Gamma^{*}_{ihj} = g_{hk}\Gamma^{*k}_{ij} = \Gamma_{ihj} - C_{ihk}\dfrac{\partial G^k}{\partial\xi^j}, \end{cases} \tag{25.13}$$

$$\frac{\partial\Gamma^{*h}_{kl}}{\partial\xi^j}\xi^k = \frac{\partial^2 G^h}{\partial\xi^i\partial\xi^j} - \Gamma^{*h}_{ji} = A_{i}{}^{h}{}_{j,k}l^k. \tag{25.14}$$

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1) Cf. E. Cartan, Les espace de Finsler, Hermann, Paris, 1934.

Now, making use of these equations, (24. 9) is written as

$$LP_i{}^j{}_{hk} = -L\frac{\partial \Gamma^*{}_i{}^j{}_h}{\partial \xi^k} + A_i{}^j{}_{k,h} - A_i{}^j{}_t A_h{}^t{}_{k,s} l^s. \tag{24.15}$$

Then, from (25. 9) and (25. 10), we get the equation

$$C_{ijh;k} = C_{ijk;h}.$$

Hence, (24. 20) is written as

$$L^2 S_i{}^j{}_{hk} = -A_t{}^j{}_h A_i{}^t{}_k + A_t^j{}_k A_i{}^t{}_h. \tag{25.16}$$

(24. 21) and (25. 8) imply the equations

$$l^j{}_{,h} = 0, \quad Ll^j{}_{;h} = \delta^j_h - l^j l_h. \tag{25.17}$$

Thus, we obtain the equation

$$\begin{aligned} P_i{}^j{}_{hk}\xi^h &= -\frac{\partial \Gamma^*{}_i{}^j{}_h}{\partial \xi^k}\xi^h + A_i{}^j{}_{k,h} l^h - A_i{}^j{}_t A_h{}^t{}_{k,s} l^h l^s \\ &= A_i{}^j{}_t A_h{}^t{}_k l^h{}_{,s} l^s = 0. \end{aligned}$$

By means of Theorem 15. 4, Γ is h-proper and $\Phi_\Gamma = 1$, hence it must be a-proper by the definition.

§ 26. a-proper metric connections.

In this section, we shall investigate the order of freedom of a-proper metric connections for a regular Finsler manifold $(\mathfrak{X}, L)$. Let Γ be the metric connection of E. Cartan and $\bar{\Gamma}$ be any a-proper metric connection. Let Γ and $\bar{\Gamma}$ be given locally by the Pfaffian forms

$$\omega_i^j = \Gamma_i{}^j{}_h du^h + C_i{}^j{}_h d\xi^h, \quad \bar{\omega}_i^j = \bar{\Gamma}_i{}^j{}_h du^h + \bar{C}_i{}^j{}_h d\xi^h.$$

Then we have

$$\frac{\partial g_{ij}}{\partial u^h} = \bar{\Gamma}_{ijh} + \bar{\Gamma}_{jih}, \qquad \frac{\partial g_{ij}}{\partial \xi^h} = \bar{C}_{ijh} + \bar{C}_{jih}, \tag{26.1}$$

$$\bar{C}_o{}^j{}_i = 0, \qquad \bar{P}_i{}^j{}_{oh} = 0. \tag{26.2}$$

Putting

$$\bar{\Gamma}^*{}_i{}^j{}_h = \bar{\Gamma}_i{}^j{}_h - \bar{C}_i{}^j{}_k \bar{\Gamma}_o{}^k{}_h = \Gamma^*{}_i{}^j{}_h + N_i{}^j{}_h, \tag{26.3}$$

$$\bar{C}_i{}^j{}_h = C_i{}^j{}_h + E_i{}^j{}_h, \tag{26.4}$$

$N_i{}^j{}_h$, $E_i{}^j{}_h$ are the components of tensor fields of the type (1. 2) of $\mathfrak{F}$. We get easily from these equations

$$\bar{\Gamma}_{ijh} = \Gamma^*{}_{ijh} + N_{ijh} + (C_{ijk} + E_{ijk})(\Gamma^*{}_o{}^k{}_h + N_o{}^k{}_h)$$
$$= \Gamma_{ijh} + E_{ijk}\Gamma_o{}^k{}_h + N_{ijh} + (C_{ijk} + E_{ijk}) N_o{}^k{}_h.$$

Accordingly, (26.1) is equivalent to

$$N_{ijh} + N_{jih} + 2C_{ijk} N_o{}^k{}_h = 0, \tag{26.5}$$
$$E_{ijh} + E_{jih} = 0. \tag{26.6}$$

The first part of (26.2) is written as

$$E_i{}^j{}_o = E_o{}^j{}_i = 0. \tag{26.7}$$

By means of (9.5′), the second part of (26.2) is equivalent to

$$\bar{\Gamma}^*{}_i{}^j{}_k = \frac{\partial \bar{\Gamma}^*{}_i{}^j{}_o}{\partial \xi^k} - \left(\frac{\partial \bar{C}_i{}^j{}_k}{\partial u^h} \xi^h - \frac{\partial \bar{C}_i{}^j{}_k}{\partial \xi^h} \bar{\Gamma}^*{}_o{}^h{}_o + \bar{C}_i{}^h{}_k \bar{\Gamma}^*{}_h{}^j{}_o - \bar{C}_h{}^j{}_k \bar{\Gamma}^*{}_i{}^h{}_o\right.$$
$$\left. - \bar{C}_i{}^j{}_h \bar{\Gamma}^*{}_k{}^h{}_o\right) + \bar{C}_i{}^j{}_t \left(\frac{\partial \bar{\Gamma}^*{}_h{}^t{}_o}{\partial \xi^k} \xi^h - \bar{\Gamma}^*{}_o{}^t{}_k\right). \tag{26.8}$$

If we denote the covariant derivatives with respect to Γ by " , " and " ; ", and substitute (26.3), (26.4) into (26.8), then we get

$$\bar{\Gamma}^*{}_i{}^j{}_k = \frac{\partial \Gamma^*{}_i{}^j{}_o}{\partial \xi^k} + \frac{\partial N_i{}^j{}_o}{\partial \xi^k} - \bar{C}_i{}^j{}_{k,o} + \frac{\partial \bar{C}_i{}^j{}_k}{\partial \xi^h} N_o{}^h{}_o - \bar{C}_i{}^h{}_k N_h{}^j{}_o + \bar{C}_h{}^j{}_k N_i{}^h{}_o$$
$$+ \bar{C}_i{}^j{}_h N_k{}^h{}_o + \bar{C}_i{}^j{}_t \left(\frac{\partial \Gamma^*{}_h{}^t{}_o}{\partial \xi^k} \xi^h - \bar{\Gamma}^*{}_o{}^t{}_k\right)$$
$$= \frac{\partial \Gamma^*{}_i{}^j{}_o}{\partial \xi^k} + (N_i{}^j{}_o)_{;k} - (C_i{}^j{}_{k,o} + E_i{}^j{}_{k,o}) + \frac{\partial \bar{C}_i{}^j{}_k}{\partial \xi^h} N_o{}^h{}_o$$
$$- E_i{}^h{}_k N_h{}^j{}_o + E_h{}^j{}_k N_i{}^h{}_o + \bar{C}_i{}^j{}_h N_k{}^h{}_o$$
$$+ \bar{C}_i{}^j{}_t \left(\frac{\partial \Gamma^*{}_h{}^t{}_o}{\partial \xi^k} \xi^h - \Gamma^*{}_o{}^t{}_k + \frac{\partial N_h{}^t{}_o}{\partial \xi^k} \xi^h - N_o{}^t{}_k\right).$$

Furthermore, using Theorem 25.2 and (25.12), we get

$$= \Gamma^*{}_i{}^j{}_k + (N_i{}^j{}_o)_{;k} - E_i{}^j{}_{k,o} + \frac{\partial \bar{C}_i{}^j{}_k}{\partial \xi^h} N_o{}^h{}_o - E_i{}^h{}_k N_h{}^j{}_o$$
$$+ E_h{}^j{}_k N_i{}^h{}_o + \bar{C}_i{}^j{}_h N_k{}^h{}_o + \bar{C}_i{}^j{}_t \left(\frac{\partial N_h{}^t{}_o}{\partial \xi^k} \xi^h - N_o{}^t{}_k\right)$$
$$= \Gamma^*{}_i{}^j{}_k + (N_i{}^j{}_o)_{;k} - E_i{}^j{}_{k,o} + (\bar{C}_i{}^j{}_{k;h} + \bar{C}_t{}^j{}_k C_i{}^t{}_h$$
$$+ \bar{C}_i{}^j{}_t C_k{}^t{}_h - \bar{C}_i{}^t{}_k C_t{}^j{}_h) N_o{}^h{}_o - E_i{}^h{}_k N_h{}^j{}_o + E_h{}^j{}_k N_i{}^h{}_o$$
$$+ \bar{C}_i{}^j{}_h N_k{}^h{}_o + \bar{C}_i{}^j{}_t ((N_h{}^t{}_o)_{;k} \xi^h - N_o{}^h{}_o C_h{}^t{}_k - N_o{}^t{}_k),$$

that is

$$N_i{}^j{}_k = (N_i{}^j{}_o)_{;k} - E_i{}^j{}_{k,o} + (\bar{C}_i{}^j{}_{k;h} + \bar{C}_t{}^j{}_k C^t{}_{ih} - \bar{C}_i{}^t{}_k C_t{}^j{}_h) N_o{}^h{}_o$$
$$- E_{ik}^h N_h{}^j{}_k + E_h{}^j{}_k N_i{}^h{}_o + \bar{C}_i{}^j{}_h (N_o{}^h{}_o)_{;k} - \bar{C}_i{}^j{}_h N_o{}^h{}_k. \tag{26.9}$$

Now, putting

$$N_i{}^j{}_k + C_i{}^j{}_t N_o{}^t{}_k = K_i{}^j{}_k, \tag{26.10}$$

we have

$$K_o{}^j{}_k = N_o{}^j{}_k, \quad K_i{}^j{}_o = N_i{}^j{}_o + C_i{}^j{}_t N_o{}^t{}_o. \tag{26.11}$$

Substituting these into (26.9), we get

$$\begin{aligned} K_i{}^j{}_k = & \; C_i{}^j{}_t K_o{}^t{}_k + (K_i{}^j{}_o - C_i{}^j{}_t K_o{}^t{}_o)_{;k} - E_i{}^j{}_{k,o} \\ & + (\bar{C}_i{}^j{}_{k;h} + \bar{C}_t{}^j{}_k C_i{}^t{}_h - \bar{C}_i{}^t{}_k C_t{}^j{}_h) K_o{}^h{}_o \\ & - E_i{}^h{}_k (K_h{}^j{}_o - C_h{}^j{}_t K_o{}^t{}_o) + E_h{}^j{}_k (K_i{}^h{}_o - C_i{}^h{}_t K_o{}^t{}_o) \\ & + \bar{C}_i{}^j{}_h (K_o{}^h{}_o)_{;k} - \bar{C}_i{}^j{}_h K_o{}^h{}_k \\ = & \; (K_i{}^j{}_o)_{;k} + E_i{}^j{}_h (K_o{}^h{}_o)_{;k} - E_i{}^h{}_k K_h{}^j{}_o + E_h{}^j{}_k K_i{}^h{}_o - E_i{}^j{}_h K_o{}^h{}_k \\ & + (E_i{}^j{}_{k;h} - C_i{}^j{}_{h;k} + C_i{}^j{}_{k;h} + C_t{}^j{}_k C_i{}^t{}_h - C_i{}^t{}_k C_t{}^j{}_h) K_o{}^h{}_o - E_i{}^j{}_{k,o}. \end{aligned}$$

Hence, using the equation $C_i{}^j{}_{h;k} = C_i{}^j{}_{k;h}$, we have

$$\begin{aligned} K_i{}^j{}_k = & \; (K_i{}^j{}_o)_{;k} + E_i{}^j{}_h (K_o{}^h{}_o)_{;k} - E_i{}^h{}_k K_h{}^j{}_o + E_h{}^j{}_k K_i{}^h{}_o - E_i{}^j{}_h K_o{}^h{}_k \\ & + (E_i{}^j{}_{k;h} + C_t{}^j{}_k C_i{}^t{}_h - C_t{}^j{}_h C_i{}^t{}_k) K_o{}^h{}_o - E_i{}^j{}_{k,o}. \end{aligned} \tag{26.12}$$

On the other hand, by means of (26.10), (26.5) is equivalent to

$$K_{ijh} + K_{jih} = 0. \tag{26.13}$$

Furthermore, from (26.12) and (26.7), we get

$$\begin{aligned} K_o{}^j{}_k &= (K_o{}^j{}_o)_{;k} - K_k{}^j{}_o + E_h{}^j{}_k K_o{}^h{}_o + E_i{}^j{}_{k;h} \xi^i K_o{}^h{}_o \\ &= (K_o{}^j{}_o)_{;k} - K_k{}^j{}_o. \end{aligned} \tag{26.14}$$

If we put $K_i{}^j = K_i{}^j{}_h \xi^h$, $K^j = K_i{}^j{}_h \xi^i \xi^h$, then (26.12) is written as

$$\begin{aligned} K_i{}^j{}_k = & \; K_i{}^j{}_{;k} - E_i{}^h{}_k K_h{}^j + E_h{}^j{}_k K_i{}^h + E_i{}^j{}_h K_k^h \\ & + (E_i{}^j{}_{k;h} + C_t{}^j{}_k C_i{}^t{}_h - C_t{}^j{}_h C_{ik}^t) K^h - E_i{}^j{}_{k,h} \xi^h. \end{aligned} \tag{26.15}$$

Hence, $K_i{}^j{}_k$ can be written in terms of $K_i{}^j$, K^j and $E_i{}^j{}_h$.

Conversely, if there exist $E_i{}^j{}_h$ which satisfy (26.6), (26.7) and $\lambda_k{}^* E_i{}^j{}_h = k^{-1} E_i{}^j{}_h (k > 0)$, and $K_i{}^j$, K^j such that

$$K_{ij} + K_{ji} = 0, \qquad K_i{}^j \xi^i = K^j \tag{26.16}$$

and $\lambda_k{}^* K_i{}^j = k K_i{}^j (k > 0)$, then for $K_i{}^j{}_h$ defined by (26.15) we have

$$K_i{}^j{}_k \xi^k = K_i{}^j + E_i{}^j{}_h K_k^h \xi^k + E_i{}^j{}_{k;h} \xi^k K^h,$$

that is

$$K_i{}^j{}_k \xi^k - K_i{}^j = E_i{}^j{}_h (K_k^h \xi^k - K^h) = 0.$$

Furthermore, it is easily seen that $K_i{}^j{}_h$ satisfy (26.13). From (26.15), we get

$$K_i{}^j{}_k\xi^i = (K_i{}^j\xi^i)_{;k} - K_k{}^j.$$

Substituting these into (26. 15), it follows immediately (26. 12). Thus, we obtain the following

Theorem 26.1. *For any a-proper metric connection $\bar{\Gamma}$ with respect to a regular Finsler manifold $(\mathfrak{X}, L)$, the following f-invariant tensor fields*

$$\mathfrak{E} = \frac{1}{2} LE^{ijh}\left(\frac{\partial}{\partial u^i} \wedge \frac{\partial}{\partial u^j}\right) \otimes \frac{\partial}{\partial u^h} \tag{26. 17}$$

of the type $(\mathfrak{F} \wedge \mathfrak{F}) \otimes \mathfrak{F}$ whose inner products with the canonical vector field $\mathfrak{y}$ vanish and

$$\mathfrak{K} = \frac{1}{2L} K^{ij} \frac{\partial}{\partial u^i} \wedge \frac{\partial}{\partial u^j} \tag{26. 18}$$

of the type $\mathfrak{F} \wedge \mathfrak{F}$ are determined by (26. 4) *and* (26. 3), (26. 10) *and* $K_{ij} = g_{jh}K_i{}^h{}_k\xi^k$ *respectively. Conversely, from such $\mathfrak{E}$ and $\mathfrak{K}$, we can uniquely determine an a-proper metric connection $\bar{\Gamma}$ by means of* (26. 4), (26.15), (26.11), (26.10), (26.3) *and the metric connection Γ of E.Cartan.*

§ 27. Metric connections and their holonomy groups.

In this section, we shall classify the a-proper metric connections for a regular Finsler manifold $(\mathfrak{X}, L)$ by means of the tensor fields $\mathfrak{E}$ and $\mathfrak{K}$ in connection with their holonomy groups.

For any a-proper connection, (11. 12) becomes generally $\bar{B}_h = B_h$ $E_h = Y_h$. Therefore, we shall use the notations in Part I, II for the metric connection Γ of E. Cartan with respect to $(\mathfrak{X}, L)$ and the same notations with bars for any a-proper metric connection $\bar{\Gamma}$ without any confusions. Since the basic tangent vector fields B_i, E_i, Q_i^j are locally written by (5. 1), (5. 2), (5. 3) respectively, using the formulas in the last section, we get from (5. 1), (5. 6) the equations

$$\begin{aligned}\bar{B}_i &= B_i - a_i^k N_o{}^h{}_k \frac{\partial}{\partial \xi^h} - N_t{}^k{}_m a_h^t a_i^m \frac{\partial}{\partial a_h^k} \\ &= B_i - \widetilde{N}_o{}^k{}_i(Y_k + \widetilde{C}_h{}^j{}_k Q_j^h) - \widetilde{N}_h{}^j{}_i Q_j^h,\end{aligned}$$

and using (26. 10) we get

$$\bar{B}_i = B_i - \widetilde{K}_o{}^k{}_i Y_k - \widetilde{K}_h{}^k{}_i Q_k^h. \tag{27. 1}$$

Nextly, we get from (5. 2) the equation

$$\bar{E}_i = \bar{Y}_i = Y_i - \widetilde{E}_h{}^k{}_i Q_k^h. \tag{27.2}$$

For the connection Γ, $V_j{}^i{}_{hk}$, $W_j{}^i{}_{hk}$ defined by (13.11) become

$$V_j{}^i{}_{hk} = R_j{}^i{}_{hk}, \quad W_j{}^i{}_{hk} = P_j{}^i{}_{hk}, \tag{27.3}$$

since Γ is a-proper. Accordingly, the Lie algebra $\mathfrak{U}_\Gamma$ over $\widetilde{\mathfrak{B}}_0$ defined in § 13 becomes the subalgebra of $\mathfrak{M}$ which is generated by the elements obtained from $\boldsymbol{R}_{hk}$, $\boldsymbol{P}_{hk}$, $\boldsymbol{S}_{hk}$ operating B_i, Y_i on them. Then, according to the same process in §§ 13—15, we shall obtain from (27.1) and (27.2) the following

Theorem 27.1. *For the metric connection Γ of E. Cartan and any a-proper metric connection $\bar{\Gamma}$ with respect to a regular Finsler manifold $(\mathfrak{X}, L)$, the necessary and sufficient condition in order that*

$$H_\Gamma(y) = H_{\bar{\Gamma}}(y), \quad y \in T_0(\mathfrak{X}),$$

is that $\boldsymbol{E}_h = ((E_j{}^i{}_h))$, $\boldsymbol{K}_h = ((K_j{}^i{}_h)) \in \mathfrak{M}$ belong to $\mathfrak{U}_\Gamma$.

By (27.1), the canonical horizontal tangent vector field of $\bar{\Gamma}$ is given by

$$\bar{B} = y^i \bar{B}_i = B - \widetilde{K}_o{}^k{}_o Y_k - \widetilde{K}_h{}^k{}_o Q_k^h. \tag{27.4}$$

We get from the equation the following

Theorem 27.2. *For any a-proper metric connection $\bar{\Gamma}$ whose canonical horisontal tangent vector field on $\widetilde{\mathfrak{B}}_0$ coincides with the one of the metric connection of E. Cartan, the tensor field $\mathfrak{K}$ of $\mathfrak{F}$ in Theorem 26.1 vanishes everywhere and*

$$K_i{}^j{}_h = -E_i{}^j{}_{h,k}\xi^k \tag{27.5}$$

Furthermore, for any two a-proper metric connection $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$, in order that $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ have the same canonical horizontal tangent bector field, it is necessary and sufficient that $\mathfrak{K}_1 = \mathfrak{K}_2$.

Corollary 27.3. *If an a-proper metric connection $\bar{\Gamma}$ for a regular Finsler manifold $(\mathfrak{X}, L)$ have the same canonical horizontal tangent vector field of the metric connection Γ of E. Cartan and $\bar{C}_i{}^j{}_k = \bar{C}_k{}^j{}_i$, then $\bar{\Gamma} = \Gamma$.*

Proof. From the assumption, we have $E_{ijk} = E_{kji}$. Considering (26.6), it follows that $E_{ijk} = 0$. By virtue of Theorem 27.2, we have $K_i{}^j{}_h = 0$, and hence $\bar{\Gamma} = \Gamma$.

Lastly, we shall investigate the analogy with Theorem 27.1 for the group AH. For the metric connection Γ of E. Cartan, (19.10) and (19.11) become clearly

$$\hat{V}^{j}{}_{hk} = w^{i}\widetilde{R}_{i}{}^{j}{}_{hk}, \quad \hat{W}^{j}{}_{hk} = w^{i}\widetilde{P}_{i}{}^{j}{}_{hk} - \widetilde{C}_{h}{}^{j}{}_{k}. \tag{27.6}$$

For the basic horizontal tangent vector fields $\hat{B}_i, E_i$, and $\hat{\bar{B}}_i, \bar{E}_i$ on $\mathfrak{B}$ of the connections Γ_0 and $\bar{\Gamma}_0$ (§18), we get from (27.1), (27.2), (18.1) and (18.2) the equations

$$\begin{aligned} \hat{\bar{B}}_i &= \bar{B}_i - W_i = \hat{B}_i - \widetilde{K}_o{}^k{}_i Y_k - \widetilde{K}_h{}^k{}_i Q^h_k \\ &= \hat{B}_i - \widetilde{K}_o{}^k{}_i E_k - w^h \widetilde{K}_h{}^k{}_i \hat{Q}^o_k - \widetilde{K}_h{}^k{}_i \hat{Q}^h_k, \\ \bar{E}_i &= E_i - w^h \widetilde{E}_h{}^k{}_i \hat{Q}^o_k - \widetilde{E}_h{}^k{}_i \hat{Q}^h_k. \end{aligned}$$

Let $\boldsymbol{K}_h, \boldsymbol{E}_h \in \hat{\mathfrak{M}}$ be the elements defined by

$$\boldsymbol{K}_h = ((w^k \widetilde{K}_k{}^i{}_h, \quad \widetilde{K}_j{}^i{}_h)), \ \boldsymbol{E}_h = ((w^k \widetilde{E}_k{}^i{}_h, \ \widetilde{E}_j{}^i{}_h)).$$

Then, we have the following

Theorem 27.4. *For any a-proper metric connection $\bar{\Gamma}$ and the metric connection Γ of E. Cartan with respect to a regular Finsler manifold $(\mathfrak{X}, L)$, a necessary and sufficient condition in order that*

$$AH_{\Gamma}(y) = AH_{\bar{\Gamma}}(y), \ y \in T_0(\mathfrak{X})$$

is that $\boldsymbol{K}_h$, $\boldsymbol{E}_h$ belong to the Lie algebra $\hat{\mathfrak{U}}_{\Gamma}$ defined in § 20.

§ 28. The cannical imbedding of $\mathfrak{B}$ into $\widetilde{\mathfrak{B}}_0$.

We have regarded $\mathfrak{B}$ as a submanifold of $\widetilde{\mathfrak{B}}$ by a natural imbedding (§2) and put $\widetilde{\mathfrak{B}}_0 = \widetilde{\mathfrak{B}} - \mathfrak{B}$. In this section, we shall define a canonical imbedding[2] $\iota : \mathfrak{B} \to \widetilde{\mathfrak{B}}_0$ by

$$\bar{b} = \iota(b) = (b, \mathrm{e}_n(b)) = (\mathrm{e}_1(b), \cdots, \mathrm{e}_n(b); \mathrm{e}_n(b)). \tag{28.1}$$

For any $y \in T_0(\mathfrak{X})$, $\tilde{\pi}^{-1}(y) \cap \iota(\mathfrak{B})$ is the set of points $\tilde{b} = (b, y)$ such that $\mathrm{e}_n(b) = y$. Hence, $\{\iota(\mathfrak{B}), T_0(\mathfrak{X})\}$ may be considered as a principal fibre bundle whose group of bundle is the subgroup A^*_{n-1} of the elements of $GL(n)$ such that

$$a^1_n = \cdots = a^{n-1}_n = 0, \ a^n_n = 1. \tag{28.2}$$

2) Cf. Footnote 2) in Introduction.

In order to make clear the relation between right translations of $\mathfrak{B}$, $\tilde{\mathfrak{B}}$ and the map ι, we shall prepare some formulas. For any $g \in GL(n)$, let $\psi_{r_g}: \tilde{\mathfrak{B}} \rightarrow \tilde{\mathfrak{B}}$ be the map defined by

$$\begin{aligned} \psi_{r_g}(\tilde{b}) &= (\mathrm{e}_i(bg)\,;\, \mathrm{e}_h(bg)y^h(\tilde{b})) \\ \tilde{b} &= (b, y) = (\mathrm{e}_i(b)\,;\, \mathrm{e}_h(b)y^h(\tilde{b})). \end{aligned} \tag{28.3}$$

We get easily

$$\psi_{r_{g'}} \cdot \psi_{r_g}(\tilde{b}) = (\mathrm{e}_i(bgg')\,;\, \mathrm{e}_h(bgg')y^h(\tilde{b})) = \psi_{r_{gg'}}(\tilde{b}),$$

that is

$$\psi_{r_{gg'}} = \psi_{r_{g'}} \cdot \psi_{r_g}, \quad g, g' \in GL(n). \tag{28.4}$$

Then, we have

$$\begin{aligned} r_{g'}(\psi_{r_g}(\tilde{b})) &= (\mathrm{e}_i(bgg')\,;\, \mathrm{e}_h(bg)y^h(\tilde{b})) \\ &= (\mathrm{e}_i(bg'g'^{-1}gg')\,;\, \mathrm{e}_h(bg'g'^{-1}gg')y^h(\tilde{b}g')) \\ &= \psi_{r_{g'^{-1}gg'}}(\mathrm{e}_i(bg')\,;\, \mathrm{e}_h(bg')y^h(\tilde{b}g')) \\ &= \psi_{r_{g'^{-1}gg'}}(\mathrm{e}_i(bg')\,;\, \mathrm{e}_h(b)y^h(\tilde{b})) = \psi_{r_{g'^{-1}gg'}} \cdot r_{g'}(\tilde{b}), \end{aligned}$$

that is

$$r_{g'} \cdot \psi_{r_g} = \psi_{r_{g'^{-1}gg'}} \cdot r_{g'}, \quad g, g' \in GL(n). \tag{28.5}$$

Lemma 28.1. *For any $g \in GL(n)$, the two maps r_g and ψ_{r_g} are identical only on the uion of $\mathfrak{B}$ ($\subset \tilde{\mathfrak{B}}$) and the subset*

$$\tilde{\mathfrak{B}}_g = \{\tilde{b} \mid \tilde{b} \in \tilde{\mathfrak{B}}_0, \quad a_i^j(g)y^i(\tilde{b}) = y^j(\tilde{b})\}. \tag{28.6}$$

Proof. By (28.3), $\psi_{r_g}(\tilde{b}) = r_g(\tilde{b})$ is equivalent to $\mathrm{e}_h(bg)y^h(\tilde{b}) = \mathrm{e}_h(b) a_k^h(g)y^k(\tilde{b}) = \mathrm{e}_h(b)y^h(\tilde{b})$, that is $a_k^h(g)y^k(\tilde{b}) = y^h(\tilde{b})$.

Now, for any point $c = (c^1, \cdots, c^n)$ of R^n, we put

$$\mathfrak{B}_c = \{\tilde{b} \mid \tilde{b} \in \tilde{\mathfrak{B}}, \ y^i(\tilde{b}) = c^i\}, \tag{28.7}$$

$$G_c = \{g \mid g \in GL(n), \ a_j^i(g)c^j = c^i\}. \tag{28.8}$$

$\mathfrak{B}_c$ is the locus of the points of $\tilde{\mathfrak{B}}$ whose second component is c, when we regard $\tilde{\mathfrak{B}}$ as $\tilde{\mathfrak{B}} = \mathfrak{B} \times R^n$. We have easily the following lemma.

Lemma 28.2. *The set of $g \in GL(n)$ such that $r_g \mid \mathfrak{B}_c = \psi_{r_g} \mid \mathfrak{B}_c$ is G_c and we have*

$$\begin{cases} \mathfrak{B}_{(0,\cdots,0)} = \mathfrak{B}, & G_{(0,\cdots,0)} = GL(n) \\ \mathfrak{B}_{(0,\cdots,0,1)} = \iota(\mathfrak{B}), & G_{(0,\cdots,0,1)} = A^*_{n-1} \\ G_{kc} = G_c, & k \neq 0, \\ \widetilde{\mathfrak{B}}_0 = \bigcup_{g \in G_c, c \neq (0,\cdots,0)} \mathfrak{B}_c. \end{cases} \tag{28.9}$$

Now, from the definitions of ψ_g and ι, we get

$$\iota \cdot r_g = \psi_g \cdot \iota, \qquad g \in GL(n). \tag{28.10}$$

Hence, from Lemma 28. 2 we have easily the following

Lemma 28. 3. *We have the relation*

$$\iota \cdot r_g = r_g \cdot \iota \tag{28.11}$$

only for $g \in A^*_{n-1}$.

Now, for a connection $\widetilde{\Gamma} = \rho^{\diamond}\Gamma$ of $\mathfrak{F}$, since we have

$$\iota^* Dy^j = \iota^*(dy^j + \theta^j_k y^k) = \iota^* \theta^j_n, \tag{28.12}$$

only $\iota^*\theta^j$, $\iota^*\theta^j_i$ are linearly independent each others at each point of $\iota(\mathfrak{B})$. For any tangent vector $V \in T_b(\mathfrak{B})$, if we put $\iota(b) = \bar{b}$ and

$$\iota_* V = x^i B_i + v^i E_i + z^i_j Q^j_i,$$

then we have $x^i = \langle \iota_* V, \theta^i \rangle$, $z^i_j = \langle \iota_* V, \theta^i_j \rangle$ and $v^i = \langle \iota_* V, Dy^i \rangle = \langle V, \iota^*(Dy^i) \rangle = \langle V, \iota^* \theta^i_n \rangle = z^i_n$ and hence

$$\iota^* V = x^i B_i + z^i_j (Q^j_i + \delta^j_n E_i). \tag{28.13}$$

Accordingly, if we take a system of differential forms and its dual system of tangent vector fields on $\widetilde{\mathfrak{B}}_0$ such that

$$\{\theta^j,\ Dy^j - \theta^j_n,\ \theta^j_i\} \text{ and } \{B_i,\ E_i,\ Q^j_i + \delta^j_n E_i\}, \tag{28.14}$$

then we have $Dy^j - \theta^j_n = 0$ on $\iota(\mathfrak{B})$ and that B_i, $Q^j_i + \delta^j_n E_i$ are tangent to $\iota(\mathfrak{B})$ at each point of $\iota(\mathfrak{B})$. Hence, we can write the torsion forms Θ^j and Θ^j_i on $\iota(\mathfrak{B})$ as follows

$$\iota^* \Theta^j = -\frac{1}{2} \widetilde{T}_i{}^j{}_k \theta^i \wedge \theta^k - \widetilde{C}_i{}^j{}_h \widetilde{M}^h_k \theta^h \wedge \theta^k_n, \tag{28.15}$$

$$\iota^* \Theta^j_i = \frac{1}{2} \widetilde{R}_i{}^j{}_{hk} \theta^h \wedge \theta^k + \widetilde{P}_i{}^j{}_{ht} \widetilde{M}^t_k \theta^h \wedge \theta^k_n + \frac{1}{2} \widetilde{S}_i{}^j{}_{ts} \widetilde{M}^t_h \widetilde{M}^s_k \theta^h_n \wedge \theta^k_n. \tag{28.16}$$

§ 29. The canonical imbedding and connections.

The differential forms θ_i^j restricted on the bundle space $\iota(\mathfrak{B})$ of the principal fibre bundle $\{\mathfrak{B},\ T_0(\mathfrak{X}),\ \bar{\pi}\cdot\iota,\ A_{n-1}^*\} \cong \{\iota(\mathfrak{B}),\ T_0(\mathfrak{X})\}$ satisfy the equation

$$\text{(i)}\quad r_g^*\,\theta_i^j = b_k^j(g)\theta_h^k\,a_i^h(g),\quad g \in A_{n-1}^* \tag{29.1}$$

by (1.14) and (28.11). For any coordinate neighborhood $(U,\ u)$ and $y = \xi^j(y)\dfrac{\partial}{\partial u^j} \in T_0(\mathfrak{X})$, we put

$$G_{U,y} = \{g \mid a_n^j(g) = \xi^j(y),\ g \in GL(n)\}, \tag{29.2}$$

which is a left coset of $GL(n)$ by A_{n-1}^*. Furthermore, we define a manifold G_U by

$$G_U = \bigcup_{\tau(y)\in U} y \times G_{U,y} \subset \tau^{-1}(U) \times GL(n) \tag{29.3}$$

and a map $\bar{\varphi}_U:\ G_U \to \pi^{-1}(U) \subset \mathfrak{B}$ by

$$\bar{\varphi}_U(y,\ g) = \bar{\varphi}_{U,y}(g) = (a_i^k(g)\frac{\partial}{\partial u^k}(x)). \tag{29.4}$$

In the following, we assume that $\alpha,\ \beta,\ \gamma, \cdots$ run over $1, 2, \cdots,\ n-1$. For another coordinate neighborhood $(\bar{U},\ \bar{u})$ such that $U \cap \bar{U} \neq \emptyset$, since we have

$$\bar{\xi}^j(y) = a_i^j(g_{\bar{U}U}(x))\xi^i(y),\quad \tau(y) = x, \tag{29.5}$$

we get from (29.2) the equation

$$G_{\bar{U},y} = g_{\bar{U}U}(x)G_{U,y}. \tag{29.6}$$

Since we have furthermore

$$\begin{aligned}\bar{\varphi}_U(y,\ g) &= \left(a_\alpha^k(g)\frac{\partial \bar{u}^h}{\partial u^k}\frac{\partial}{\partial \bar{u}^h}(x),\ \xi^k(y)\frac{\partial \bar{u}^h}{\partial u^k}\frac{\partial}{\partial \bar{u}^h}(x)\right)\\ &= \left(a_\alpha^k(g_{\bar{U}U}(x)g)\frac{\partial}{\partial \bar{u}^k}(x),\ \bar{\xi}^k(y)\frac{\partial}{\partial \bar{u}^k}(x)\right),\end{aligned}$$

we get

$$\bar{\varphi}_U(y,\ g) = \bar{\varphi}_{\bar{U}}(y,\ \bar{g}),\ \bar{g} = g_{\bar{U}U}(x)g,\ g \in G_{U,y}. \tag{29.7}$$

Now, if we denote the coordinate functions of $\{\mathfrak{B},\ \mathfrak{X}\}$ by $\bar{\varphi}_U$ according to (2.3), we get

$$\bar{\varphi}_{U,y} = \bar{\varphi}_U \mid G_{U,y} \tag{29.8}$$

and hence

$$\text{(ii)}\qquad (\bar{\varphi}_{U,y})^*\,\theta_i^j = b_h^j da_i^h \text{ on } G_{U,y}. \tag{29.9}$$

Theorem 29.1. *If for the principal fibre bundle* $\{\mathfrak{B}, T_0(\mathfrak{X}), \tilde{\pi}\cdot\iota, A^*_{n-1}\}$, *differential forms* θ^j_i *on* $\mathfrak{B}$ *satisfy the conditions* (i), (ii), *then the system determine a connection* $\widetilde{\Gamma}$ *of* $\mathfrak{F}$ *and it define the connection* $(\tilde{\pi}\cdot\iota)^{\diamond}\widetilde{\Gamma}$.

Proof. By means of (ii), for the coordinate neighborhood (U, u), θ^j_h are written as

$$\theta^j_i = b^j_h(g)\,\{d\,a^h_i(g) + \omega^h_k\,a^k_i(g)\},$$
$$w^j_i = \omega^j_i(u,\ \xi,\ g\ ;\ du,\ d\xi).$$

Hence, for any $g' \in A^*_{n-1}$, we have

$$r^*_{g'}\,\theta^j_i = b^j_t(g')b^t_h(g)\,\{da^h_k(g) + r^*_{g'}\,\omega^h_m\,a^m_k(g)\}a^k_i(g').$$

Furthermore, supposing (ii), we get

$$r^*_{g'}\,\omega^j_i = \omega^j_i(u,\ \xi,\ gg'\,;du,\ d\xi) = \omega^j_i(u,\ \xi,\ g\ ;\ du,\ d\xi)$$
$$g \in G_{U,y},\ g' \in A^*_{n-1}.$$

Accordingly, $\omega^j_i(u,\ \xi,\ g\ ;\ du,\ d\xi)$, $g \in G_{U,y}$, depends only on $G_{U,y}$ and hence it must be of the form $\omega^j_i(u,\ \xi\ ;\ du,\ d\xi)$. For any coordinate neighborhood $(\bar{U},\ \bar{u})$ such that $U \cap \bar{U} \neq \emptyset$, we get by (29.7)

$$\begin{aligned}\theta^j_i &= b^j_h(\bar{g})\,\{da^h_i(\bar{g}) + \bar{\omega}^h_k\,a^k_i(\bar{g})\}\\ &= b^j_t(g)b^t_h(g_{\bar{U}U})\{a^h_k(g_{\bar{U}U})da^k_i(g) + da^h_k(g_{\bar{U}U})a^k_i(g)\\ &\qquad + \bar{\omega}^h_k\,a^k_m(g_{\bar{U}U})a^m_i\,(g)\}\end{aligned}$$

and hence

$$\begin{aligned}\omega^j_i &= b^j_h(g_{\bar{U}U})\,\{da^h_i(g_{\bar{U}U}) + \bar{\omega}^h_k\,a^k_i(g_{\bar{U}U})\}\\ &= \frac{\partial u^j}{\partial \bar{u}^h}\left(d\,\frac{\partial \bar{u}^h}{\partial u^i} + \bar{\omega}^h_k\,\frac{\partial \bar{u}^k}{\partial u^i}\right).\end{aligned}$$

Accordingly, the system of ω^j_i define a connection $\widetilde{\Gamma}$ of $\mathfrak{F} = \{T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X}),\ T_0(\mathfrak{X})\}$. It is clear that the images of the differential forms on $\widetilde{\mathfrak{B}}_0$ for $\widetilde{\Gamma}$ under ι^* may become θ^j_i. q. e. d.

Theorem 29.2. *If a connection* Γ' *of the vector bundle* $\{\widetilde{\mathfrak{B}}_n - \mathfrak{B}_n, \widetilde{\mathfrak{B}}_0\}$ *is given by* $D\mathfrak{z}_i = \mathfrak{z}_j \otimes \theta^j_i$ *and* θ^j_i *satisfy*

(i) $r^*_g\,\theta^j_i = b^j_k(g)\theta^k_h\,a^h_i(g)$, $\quad g \in GL(n)$,

(ii) $(\tilde{\varphi}_{U,y})^*\,\theta^j_i = b^j_h\,da^h_i$,

then Γ' *is derived from a connection* $\widetilde{\Gamma}$ *of* $\mathfrak{F}$ *by the projection map* $\tilde{\pi}_0$

of $\{\widetilde{\mathfrak{B}}_0, \widetilde{T}_0(\mathfrak{X})\}$.

We can prove this theorem by an analogous method to the verifica tion of Theorem 29. 1.

We have also the following theorem in connection with the equation of structure analogous to the theorem in the ordinary cases.

Let $\bar{\mathfrak{F}}$ be the induced vector bundle of $\mathfrak{F}$ by the natural map ψ : $\overline{T}_0(\mathfrak{X}) = \mathfrak{B}/A^{*+}_{n-1} \to T_0(\mathfrak{X}) = \mathfrak{B}/A^{*}_{n-1}$ and $\eta : \mathfrak{B} \to \overline{T}_0(\mathfrak{X})$ be the natural projection, where $A^{*+}_{n-1} = \{g \mid |a^j_i(g)| > 0,\ g \in A^{*}_{n-1}\}$.

Theorem 29. 3.[3] *If an affine connection* $(\Gamma', d\mathfrak{p})$ (§ § 2, 3) *of the vector bundle* $\{\mathfrak{B}_n, \mathfrak{B}\}$ *is given by* $D\mathfrak{e}_i = \mathfrak{e}_j \otimes \theta^j_i$, $d\mathfrak{p} = \mathfrak{e}_j \otimes \theta^j (= (\pi \circ \otimes \pi^*) dp$, $dp = \frac{\partial}{\partial u^i} \otimes du^i)$ *and its torsion forms and curvature forms*

$$\Theta^j = D\theta^j = d\theta^j + \theta^j_i \wedge \theta^i, \quad \Theta^j_i = d\theta^j_i + \theta^j_k \wedge \theta^k_i$$

can be written only in terms of θ^j *and* θ^j_n, *then it is induced from an affine connection of* $\bar{\mathfrak{F}}$ *by the map* η.

Proof. By the assumption, the system of Pfaffian equations

$$\theta^j = 0, \qquad \theta^j_n = 0$$

is completely integrable. Since locally $\theta^j = b^j_i\, du^i$ and

$$\Theta^j = db^j_i \wedge du^i + \theta^j_i \wedge \theta^i = (\theta^j_i - b^j_h da^h_i) \wedge \theta^i,$$

$\theta^j_i - b^j_h\, da^h_i$, especially $b^j_h\, da^h_n$ must be linear combinations of θ^i and θ^i_n. Accordingly we may regard $a^j_n = \xi^j$ and u^j as integrals of the above mentioned Pfaffian equations. Nextly, if we write locally θ^j_i as $\theta^j_i = b^j_k(da^k_i + \omega^k_h\, a^h_i)$, then we have $\Theta^j = (\theta^j_i - b^j_h\, da^h_i) \wedge \theta^i = b^j_k\, \omega^k_h \wedge du^h$. Hence, ω^j_i must be linear combinations of du^h, $d\xi^h$. Furthermore, we have $\Theta^j_i = b^j_k(d\omega^k_h + \omega^k_m \wedge \omega^m_h)\, a^h_i$, and hence $d\omega^k_h + \omega^k_m \wedge \omega^m_h$ must be written in terms of du^i, $d\xi^i$ only. Accordingly, $d\omega^k_h$ must be so. If we put

$$\omega^j_i = \Gamma_i{}^j{}_k\, du^k + C_i{}^j{}_k\, d\xi^k, \quad g \in G_{U,y}, \ y = \xi^i \frac{\partial}{\partial u^i}$$

then $\Gamma_i{}^j{}_k$, $C_i{}^j{}_k$ depend only on the connected components of $G_{U,y}$ for a fixed y. On the other hand, the connected components of A^{*}_{n-1} are A^{*+}_{n-1} and $A^{*-}_{n-1} = \{g \mid |a^j_i(g)| < 0,\ g \in A^{*}_{n-1}\}$. $\Gamma_i{}^j{}_k$ and $C_i{}^j{}_k$ must be local func-

[3] This theorem is a generalization of a theorem in p. 102 of S. S. Chern's book, Lecture note on differential geometry, Chicago (1952), which must be slightly corrected.

tions on $\overline{T}_0(\mathfrak{X}) = \mathfrak{B}/A^{*+}_{n-1}$. Thus we see easily that ω^j_i define a connection $\overline{\Gamma}$ of $\overline{\mathfrak{F}} = \psi^{\diamond}\mathfrak{F}$. Accordingly, we can easily prove that $(\Gamma', d\mathfrak{p})$ is the induced affine connection from $(\overline{\Gamma}, (\psi^\circ \otimes \psi^*)dp)$ of the vector bundle $\overline{\mathfrak{F}} = \psi^{\diamond}\mathfrak{F}$ *by* η.

Theorem 29.4. *Let an affine connection* $(\Gamma', d\mathfrak{p})$, $d\mathfrak{p} = (\overline{\mu}^\circ \otimes \overline{\mu}^*) dp$, *of the vector bundle* $\{\widetilde{\mathfrak{B}}_n - \mathfrak{B}_n, \widetilde{\mathfrak{B}}_0\}$ *be given by* $D\mathfrak{z}_i = \mathfrak{z}_j \otimes \theta^j_i$ *and assume that the components* Dy^j *of* $D\mathfrak{z} = \mathfrak{z}_j \otimes Dy^j$ *are every where independent of each other with* θ^i. *If its torsion forms* Θ^j *and curvature forms* Θ^j_i *are written in terms of* θ^j, Dy^j *only and* $\Theta^j \equiv 0 \pmod{\theta^i}$, *then* $(\Gamma', d\mathfrak{p})$ *can be induced from an affine connection of the vector bundle* $\overline{\mathfrak{F}}$ *by the natural map* $\tilde{\eta} : \widetilde{\mathfrak{B}}_0 \to \widetilde{\mathfrak{B}}_0/GL^+(n) \approx \overline{T}_0(\mathfrak{X}) = \mathfrak{B}/A^{*+}_{n-1}$.

Proof. Making use of local canonical coordinates (u^j, ξ^j, a^j_i), if we put $\theta^j_i = b^j_k(da^k_i + \omega^k_h a^h_i)$, then we can set $Dy^j = dy^j + \theta^j_i y^i = b^j_i D\xi^i$, $D\xi^i = d\xi^i + \omega^i_h \xi^h$. Accordingly, Θ^j, Θ^j_i can be written locally in terms of du^h, $D\xi^h$ only and $\Theta^j \equiv 0 \pmod{du^i}$. Since $\Theta^j = b^j_k\, \omega^k_h \wedge du^h$, ω^j_i are of the forms

$$\omega^j_i = \Gamma^{*j}_{i\ k}\, du^k + C_i{}^j{}_k\, D\xi^k,$$

where $\Gamma^{*j}_{i\ k}$ and $C_i{}^j{}_k$ are functions on $\overline{\mu}^{-1}(U)$ and are uniquely determined by virtue of the linear independency of θ^j, Dy^j. Since we have

$$\Theta^j_i = b^j_k(d\omega^k_h + \omega^k_t \wedge \omega^t_h)a^h_i,$$

$d\omega^j_i$ must be differential forms of degree 2 in du^k, $D\xi^k$. Furthermore, since we have

$$dD\xi^k + \omega^k_h \wedge D\xi^h = \xi^i(d\omega^k_i + \omega^k_h \wedge \omega^h_i),$$

$dD\xi^j$ are also differential forms of degree 2 in du^k, $D\xi^k$. Now, we have

$$\begin{aligned} d\omega^j_i &\equiv d\Gamma^{*j}_{i\ k} \wedge du^k + dC_i{}^j{}_k \wedge D\xi^k \\ &\equiv \frac{\partial \Gamma^{*j}_{i\ k}}{\partial a^h_t}\, da^h_t \wedge du^k + \frac{\partial C_i{}^j{}_k}{\partial a^h_t}\, da^h_t \wedge D\xi^k \\ &\qquad (\text{mod forms of degree 2 in } du^h \text{ and } d\xi^h), \end{aligned}$$

hence we have

$$\frac{\partial \Gamma^{*j}_{i\ k}}{\partial a^h_t} = \frac{\partial C_i{}^j{}_k}{\partial a^h_t} = 0.$$

Accordingly, with respect to g, $\Gamma^{*j}_{i\ k}$ and $C_i{}^j{}_k$ depend only on the connected components of $GL(n)$, that is they are locally regarded as functions on $\widetilde{\mathfrak{B}}_0/GL^+(n)$. We can easily prove that Γ' must be induced from a connection of the induced vector bundle, which is induced from $\mathfrak{F}$ by the map $\widetilde{\mathfrak{B}}_0/GL^+(n) \to \widetilde{\mathfrak{B}}_0/GL(n) = T_0(\mathfrak{X})$, by the natural map $\widetilde{\mathfrak{B}}_0 \to \widetilde{\mathfrak{B}}_0/GL^+(n)$. Now, since $A^{*+}_{n-1} = GL^+(n) \cap A^*_{n-1}$, $GL^+(n)/A^{*+}_{n-1} = R^n - 0$ and $\iota(\mathfrak{B})GL^+(n) = \widetilde{\mathfrak{B}}_0$ we have

$$\widetilde{\mathfrak{B}}_0/GL^+(n) = \iota(\mathfrak{B})/A^{*+}_{n-1} \approx \mathfrak{B}/A^{*+}_{n-1} = T_0(\mathfrak{X}).$$

Thus, the theorem has been proved.

Corollary 29. 5. *In this theorem, in order that $(\Gamma', d\mathfrak{p})$ can be induced from an affine connection $(\widetilde{\Gamma}, dp)$ of $\mathfrak{F}$, it is sufficient that for some $g \in GL^-(n)$, we have*

$$r_g^* \theta_i^j = b_k^j(g)\,\theta_h^k\, a_i^h(g).$$

§ 30. **Geodesics.**

In § 6, we have shown that the condition in order that the supporting curve C' of a development of a curve $\bar{C}$ in $T_0(\mathfrak{X})$ be a straight line is that for $\bar{C}$ there exists a function ψ of t such that in local coordinates

$$\frac{d^2}{dt^2}u^j + \frac{\omega_i^j}{dt}\frac{du^i}{dt} = \psi\frac{du^j}{dt}. \tag{30. 1}$$

Let $\bar{C}$ be given by $\bar{f}\colon I \to T_0(\mathfrak{X})$ and put $f = \tau \cdot \bar{f}$. If $\bar{C}$ is a proper a-curve, we have a function χ of t such that $\xi^j \cdot \bar{f} = \chi \frac{d}{dt}(u^j \cdot f)$, $\chi \neq 0$. Considering to take the parameter in the opposite direction, we may put $\chi > 0$. The left hand side of the above equation become by means of (3.7), (3.8), (4.2) and (4.11)

$$\begin{aligned}\frac{1}{dt^2}\bar{f}^*(d^2u^j + \omega_i^j du^i) &= \frac{1}{dt^2}\left(\frac{df}{dt}\right)^*(d^2u^j + \omega_i^j du^i)\\ &= \frac{1}{dt^2}\left(\frac{df}{dt}\right)^*\{\Phi_k^j(d^2u^k + \Gamma^{*k}_{i\,h}\,du^i\,du^h)\} = \psi\frac{d}{dt}(u^j \cdot f).\end{aligned}$$

Hence $C = \tau\,\bar{C}$ is locally given by the differential equations

$$\frac{d^2u^j}{dt^2} + \Gamma^{*j}_{i\,k}\left(u^h, \frac{du^h}{dt}\right)\frac{du^i}{dt}\frac{du^k}{dt} = \psi\frac{du^j}{dt}. \tag{30. 2}$$

Accordingly, for the lift $\bar{C}$ in $T_0(\mathfrak{X})$ of C, we have

$$\bar{f}^*\xi^j = f^*\,du^j/dt,\ \bar{f}^*\,D\xi^j/dt = \psi\,\bar{f}^*\,\xi^j. \tag{30. 3}$$

Therefore, the submanifold $\tilde{\pi}^{-1}(\bar{C})$ in $\widetilde{\mathfrak{B}}_0$ is a solution of the system of Pfaffian equations (in $\widetilde{\mathfrak{B}}_0 \times R$)

$$\theta^j = y^j\,dt, \qquad Dy^j = \psi\, y^j\,dt. \tag{39.4}$$

This solution clearly satisfies also the equations

$$y^j\,\theta^k - y^k\,\theta^j = 0, \qquad y^j\,Dy^k - y^k\,Dy^j = 0. \tag{30.5}$$

Proposition 30.1. *The system of Pfaffian equations* (30.5) *in* $\widetilde{\mathfrak{B}}_0$ *is completely integrable and its integral manifolds are the connected components of the inverse images of the lifts of geodesics in* $\mathfrak{X}$ *with respect to* $\widetilde{\Gamma}$ *under the map* $\tilde{\pi}: \widetilde{\mathfrak{B}}_0 \to T_0(\mathfrak{X})$.

Proof. We denote by "$\equiv$" the equality mod the ideal of the algebra $\Psi(\Lambda^*(\widetilde{\mathfrak{B}}_0))$ of differential forms on $\widetilde{\mathfrak{B}}_0$ generated by the left hand sides of (30.5) with coefficients in $\mathfrak{A}$ (= the algebra of scalar fields on $\widetilde{\mathfrak{B}}_0$). Since we have

$$y^j\theta^k \equiv y^k\,\theta^j, \quad y^j\,Dy^k \equiv y^k\,Dy^j,$$

there exist differential forms θ, ω such that

$$\theta^j \equiv y^j\,\theta, \quad Dy^j \equiv y^j\,\omega. \tag{30.6}$$

Accordingly, we get from (4.10), (6.10), (6.11), (9.13) and (30.6)

$$\Theta^j \equiv 0, \qquad \Theta^j_i \equiv 0.$$

Hence, we have

$$d(y^j\,\theta^k - y^k\,\theta^j) = Dy^j \wedge \theta^k - Dy^k \wedge \theta^j + y^j\,\Theta^k - y^k\,\Theta^j \equiv 0,$$
$$d(y^jDy^k - y^kDy^j) = Dy^j \wedge Dy^k - Dy^k \wedge Dy^j + y^jy^h\Theta^k_h - y^ky^h\Theta^j_h \equiv 0.$$

According to a theorem of Frobenius, the equation (30.5) is completely integrable. Now, for any $g \in GL(n)$, $r_g^*\,y^j = b^j_i(g)y^i$, $r_g^*\,\theta^j = b^j_i(g)\theta^i$, $r_g^*\,Dy^j = b^j_i(g)Dy^i$, hence (30.5) is invariant under any right translation of $\widetilde{\mathfrak{B}}_0$. Accordingly, its integral manifolds are transformed each others by the right translations. On the other hand, θ^j and Dy^j vanish on the fibres of $\{\widetilde{\mathfrak{B}}_0, T_0(\mathfrak{X})\}$. Accordingly, any integral manifold P of (30.5) may be considered as a locus of fibres of $\{\widetilde{\mathfrak{B}}_0, T_0(\mathfrak{X})\}$. Let $\tilde{\pi}(P) = \bar{C}$ then $\bar{C}$ satisfies locally

$$\xi^j\,du^k - \xi^k\,du^j = 0, \qquad \xi^j\,D\xi^k - \xi^k\,D\xi^j = 0.$$

Since $\widetilde{\Gamma}$ is regular, $\bar{C}$ must be a curve in $T_0(\mathfrak{X})$.

Proposition 30.2. *For the principal bundle* $\{\iota(\mathfrak{B}), T_0(\mathfrak{X})\}$, *the equations characterizing the geodesics are*

$$\theta^\alpha = 0, \qquad \theta^\alpha_n = 0. \tag{30.7}$$

Proof. Covering ι^* on (30. 5), we get (30. 7) by means of (28. 12). Since $\iota(\mathfrak{B})$ is a submanifold of $\widetilde{\mathfrak{B}}_0$, (30. 7) is also completely integrable. Accordingly, the integral manifolds of (30. 7) are the intersection of the integral manifolds of (30. 5) and $\iota(\mathfrak{B})$.

§ 31. Submanifolds of $T_0(\mathfrak{X})$ and the induced connections.

Let us consider an n-dimensional submanifold $\mathfrak{X}_1$ of $T_0(\mathfrak{X})$, such that $\rho \mid \mathfrak{X}_1$ is a one-one map. Putting $\mathfrak{Z}_1 = \bar{\tau}^{-1}(\mathfrak{X}_1)$, $\mathfrak{B}_1 = \tilde{\pi}^{-1}(\mathfrak{X}_1)$, we denote by the same symbol ι_1 the imbedding of $\mathfrak{X}_1, \mathfrak{Z}_1, \mathfrak{B}_1$ into $T_0(\mathfrak{X}), \mathfrak{Z}, \widetilde{\mathfrak{B}}_0$. Then, we get from an affine connection $(\Gamma, (\bar{\tau}^{\circ} \otimes \bar{\tau}^*)dp)$ of $\mathfrak{F}$ the induced affine connection $(\Gamma_1, \psi_{\Gamma_1})$ of the induced vector bundle $\mathfrak{F}_1 = \{\mathfrak{Z}_1, \mathfrak{X}_1\} = \iota_1{}^{\diamond} \mathfrak{F}$ by the imbedding ι_1, that is $\Gamma_1 = \iota_1{}^{\diamond} \widetilde{\Gamma}$, $\psi_{\Gamma_1} = ((\tau \cdot \iota_1)^{\circ} \otimes (\tau \cdot \iota_1)^*)dp \in \Psi(\mathfrak{F}_1 \otimes T^*(\mathfrak{X}_1))$. With respect to the local coordinate neighborhood (U, u), we have

$$((\tau \cdot \iota_1)^{\circ} \otimes (\tau \cdot \iota_1)^*)dp = (\tau \cdot \iota_1)^{\circ} \frac{\partial}{\partial u^i} \otimes d(u_i \cdot \tau \cdot \iota_1). \tag{31. 1}$$

Now, for any point $y \in \mathfrak{X}_1$, we define the order of singularity at y by

$$m(y) = n - \dim \tau_*(T_y(\mathfrak{X}_1)). \tag{31. 2}$$

We say especially y is an ordinary point when $m(y) = 0$. Let (v^j) be local coordinates of $\mathfrak{X}_1$ near y and put $u^j = u^j(v^i)$, $\xi^j = \xi^j(v^i)$, then we have at y

$$n - m(y) = \operatorname{rank}\left(\frac{\partial u^j}{\partial v^i}\right).$$

Example. Let X be a Euclidean plane and let x, y be its Descartes coordinates, then we can represent any point (vector) of $T(\mathfrak{X})$ by $(x, y; v, \varphi)$ where v is the length of this vector and φ is its argument. Let $\mathfrak{X}_1$ be the submanifold defined by

$$y - x \tan \varphi = 0, \quad -\frac{\pi}{2} < \varphi < \frac{\pi}{2}, \quad v = 1.$$

$\mathfrak{X}_1$ is clearly a regular submanifold of dimension 2. Save for the points such that $x = 0$, the points of $\mathfrak{X}_1$ and $\mathfrak{X}$ are in one-one correspondence under the projection.

Now, between the tangent bundle $T(\mathfrak{X}_1)$ and the vector bundle $\mathfrak{F}_1$, we define a homomorphism $h: T(\mathfrak{X}_1) \rightarrow \mathfrak{Z}_1$ which covers the identity transformation of $\mathfrak{X}_1$ by

$$h \frac{\partial}{\partial v^i} = \frac{\partial u^j}{\partial v^i} (\tau \cdot \iota_1)^{\circ} \frac{\partial}{\partial u^j}. \tag{31. 3}$$

Clearly, this definition is independent of the choice of local coordinates.

h is a generalized bundle map of vector bundles. The dual map $h^{\times} : \mathfrak{Z}_1^* \to T^*(\mathfrak{X}_1)$ of h is given by

$$h^{\times}((\tau \cdot \iota_1)^{\circ}\, du^j) = \frac{\partial u^j}{\partial v^i}\, dv^i = (\tau \cdot \iota_1)^*\, du^j. \tag{31.4}$$

The set of points such that $m(y) = 0$ is an open subset of $\mathfrak{X}_1$. Since h is an isomorphism in this set, we may identity $\mathfrak{F}_1$ and $T(\mathfrak{X}_1)$ through h. Then $(\tau \cdot \iota_1)^{\circ} du^j$ of $\mathfrak{F}_1^*$ may be identified to $(\tau \cdot \iota_1)^* du^j$ of $T^*(\mathfrak{X}_1)$ by means of (31.4). In this part of $\mathfrak{X}_1$, we have

$$dp_1 \equiv (h\Theta \otimes 1)\, \psi_{\Gamma_1} = \frac{\partial}{\partial v^i} \otimes dv^i.$$

Accordingly, the affine connection $(\Gamma_1, \psi_{\Gamma_1})$ of $\mathfrak{F}_1$ can determine an affine connection $h\#(\Gamma_1, \psi_{\Gamma_1})$ for $\mathfrak{X}_1$ in the ordinary sense.

Now, from any tensor field $\mathfrak{K}_1$ of $\mathfrak{F}_1$ of the type $(0, q)$, we can induce uniquely and naturally a tensor field $\bar{\mathfrak{K}}$ of $T(\mathfrak{X}_1)$ of the same type by $h^{\times}$.

Let us consider a Finsler manifold $(\mathfrak{X}, L)$, let $\tilde{\Gamma}$ be a metric connection of $(\mathfrak{X}, L)$ and let $\mathfrak{g}$ be its metric tensor. We put $\mathfrak{g}_1 = \iota_1^{\circ}\, \mathfrak{g}$.

Proposition 31.1. *The induced affine connection $(\Gamma_1, \psi_{\Gamma_1})$ of $\mathfrak{F}_1$ from $\tilde{\Gamma}$ by ι_1 is a metric connection with respect to $\mathfrak{g}_1$.*

Proof. By means of (1.17), we have

$$D\mathfrak{g}_1 = D(\iota_1^{\circ}\, \mathfrak{g}) = (\iota_1^{\circ} \otimes \iota_1^*) D\mathfrak{g} = 0.$$

Since $\mathfrak{g}_1$ is of the type $(0, 2)$, $h^{\times}\mathfrak{g}_1 = \bar{\mathfrak{g}}$ is a symmetric, covariant tensor field of $T(\mathfrak{X}_1)$ of degree 2. The following proposition is evident.

Proposition 31.2. *If the torsion forms of Γ_1 vanish in the open subset of the ordinary points of $\mathfrak{X}_1$, then Γ_1 can be regarded there as the connection of Levi-Civita with respect to the metric $\bar{\mathfrak{g}}$.*

A development of a curve C_1 in $\mathfrak{X}_1$ with respect to $(\Gamma_1, \psi_{\Gamma_1})$ is given by a solution of the equations

$$dp' = \mathrm{e}'_j\, \iota_1^*\, \theta^j, \qquad d\mathrm{e}'_i = \mathrm{e}'_j\, \iota_1^*\, \theta_i^j,$$

and it has a Euclidean length since Γ_1 is metric with respect to $\mathfrak{g}_1$. Since, along C_1, we have

$$\begin{aligned} ds'^2 &= dp' \cdot dp' = (\mathrm{e}'_i \cdot \mathrm{e}'_j)\theta^i \theta^j \\ &= g_{ij}(u(v),\ \xi(v))\, du^i du^j = \bar{g}_{ij}(v) dv^i dv^j, \\ \bar{\mathfrak{g}} &= \bar{g}_{ij}\, dv^i \otimes dv^i, \end{aligned}$$

the length of C_1 with respect to $(\Gamma_1, \psi_{\Gamma_1})$ is the length with respect to the non-negative metric tensor $\bar{\mathfrak{g}}$ of $T(\mathfrak{X}_1)$. Accordingly, the metrical theory of $\mathfrak{X}_1$ with the metric tensor $\bar{\mathfrak{g}}$ may be done in the vector bundle $\mathfrak{F}_1$.

Theorem 31. 3. *Let C be a geodesic of a Finsler manifold $(\mathfrak{X}, L)$ and $\mathfrak{X}_1$ be an n-dimensional submanifold of $T_0(\mathfrak{X})$ containing the lift $\bar{C}$ of C such that $\tau \mid \mathfrak{X}_1$ is one-one. Then, C is a geodesic with respect to the induced affine connection $(\Gamma_1, \psi_{\Gamma_1})$ of $\mathfrak{F}_1$.*

Proof. By means of Proposition (30. 5), it must be on $\tilde{\pi}^{-1}(\bar{C})$

$$y^j \theta^k - y^k \theta^j = 0, \quad y^j Dy^k - y^k Dy^j = 0.$$

By the assumption of this theorem, $\tilde{\pi}^{-1}(\bar{C})$ is also a submanifold of $\mathfrak{B}_1$. According to the definition of $(\Gamma_1, \psi_{\Gamma_1})$ and $\mathfrak{F}_1$, the equations above must hold good for $(\mathfrak{B}_1, \mathfrak{X}_1)$. We can prove also this theorem by the following method. On $\iota(\mathfrak{B})$, the above equations of Pfaffian forms become

$$\theta^\alpha = 0, \quad \theta^\alpha_n = 0$$

by (28. 12). If we consider these equations in $\mathfrak{X}_1$, then they are the condition that the developements of $\bar{C}$ with respect to the affine connection $(\Gamma_1, \psi_{\Gamma_1})$ are straight lines.

§ 32. Geodesics and the first variations.

In the following sections, we shall investigate geodesics in a positive regular Finsler manifold $(\mathfrak{X}, L)$ from the stand point of the vector bundle $\mathfrak{F}$.

Let $\bar{C}$ be a curve in $T_0(\mathfrak{X})$ given by $\bar{f} : I \to T_0(\mathfrak{X})$, $I = [0, a]$. Its length for the interval $[0, t]$ may be defined by

$$s = \int_0^t (\tilde{g}_{ij}\, \theta^i \theta^j)^{\frac{1}{2}}. \tag{32. 1}$$

For simplicity, we will take only the points (frames) of $\widetilde{\mathfrak{B}}_0$ which are orthonormal. Then, we have in the submanifold of these points

$$\tilde{g}_{ij} = \delta_{ij}. \tag{32. 2}$$

Since for the metric connection $\widetilde{\Gamma}$ of E. Cartan we have $D\mathfrak{g} = 0$, we get by means of (32. 2) the equation

$$\theta_{ij} + \theta_{ji} = 0, \tag{32. 3}$$

where we put $\theta_{ij} = \tilde{g}_{jk}\, \theta^k_i$ on $\widetilde{\mathfrak{B}}_0$. Now, we calculate the first variation of the lengths of a family of curves $\bar{C}_\varepsilon$ in $T_0(\mathfrak{X})$ such that $\bar{C}_0 = \bar{C}$ at $\varepsilon = 0$. We get easily

$$\delta s = \int_0^a \delta(\textstyle\sum_i \theta^i \theta^i)^{\frac{1}{2}}$$
$$= \int_0^a (\textstyle\sum_i \theta^i \theta^i)^{-\frac{1}{2}} \sum_j \theta^j \delta\theta^j.$$

Now, we shall denote the quantities corresponding to the variation $\delta\varepsilon$ of ε by the symbols with bars, that is

$$\theta^j(\delta) = \bar{\theta}^j, \quad \theta^j_i(\delta) = \bar{\theta}^j_i, \quad Dy^j(\delta) = \overline{Dy^j}, \text{ etc.}, \tag{32. 4}$$

For the connection $\widetilde{\Gamma}$, (6. 10) is written as

$$d\theta^j + \theta^j_i \wedge \theta^i = \Theta^j = -\widetilde{C}_i{}^j{}_k \theta^i \wedge Dy^k. \tag{32. 5}$$

Hence, we have

$$\delta\theta^j - d\bar{\theta}^j = \theta^j_k \bar{\theta}^k - \bar{\theta}^j_k \theta^k - \widetilde{C}_i{}^j{}_k(\bar{\theta}^i Dy^k - \theta^i \overline{Dy^k}).$$

Accordingly, making use of (32. 3), we get

$$\sum_j \theta^j \delta\theta^j = \sum_j \theta^j d\bar{\theta}^j + \sum_j \theta^j \bar{\theta}^k \theta^j_k - \sum_j \widetilde{C}_i{}^j{}_k \theta^j(\bar{\theta}^i Dy^k - \theta^i \overline{Dy^k}).$$

Substituting these into the right hand side of δs, we get

$$\delta s = \int_0^a \sum_j \left\{ \frac{\theta^j}{ds} d\bar{\theta}^j + \frac{\theta^j}{ds} \bar{\theta}^k \theta^j_k - \widetilde{C}_i{}^j{}_k \frac{\theta^j}{ds} \bar{\theta}^i Dy^k + \widetilde{C}_i{}^j{}_k \frac{\theta^i}{ds} \theta^j \overline{Dy^k} \right\},$$

that is

$$\delta s = \left[\sum_j \bar{\theta}^j \frac{\theta^j}{ds} \right]_0^a - \int_0^a \sum_j \left\{ d \frac{\theta^j}{ds} + \frac{\theta^k}{ds} \theta^j_k + \widetilde{C}^j{}_{ik} \frac{\theta^i}{ds} Dy^k \right\} \bar{\theta}^j$$
$$+ \int_0^a \widetilde{C}_{ijk} \frac{\theta^i}{ds} \frac{\theta^j}{ds} \overline{Dy^k}\, ds, \tag{32. 6}$$

where we put $\widetilde{C}^j{}_{ik} = \tilde{g}^{jh} \tilde{g}_{im} \widetilde{C}_h{}^m{}_k$, $\widetilde{C}_{ijh} = \tilde{g}_{jk} \widetilde{C}_i{}^k{}_h$. The general formula (32. 6) follows immediately

Theorem 32. 1. *For a regular Finsler manifold* $(\mathfrak{X}, L)$, *in order that a curve* $\bar{C}$ *in* $T_0(\mathfrak{X})$ *have a relatively minimum length and furthermore it is so in a family of curves* $\bar{C}_\varepsilon$, $\bar{C}_0 = \bar{C}$, *it is necessary that we have along* $\bar{C}$

$$d \frac{\theta^j}{ds} + \theta^j_k \frac{\theta^k}{ds} + \widetilde{C}^j{}_{ik} \frac{\theta^i}{ds} Dy^k = 0, \tag{32. 7}$$

$$\widetilde{C}_{ijk} \frac{\theta^i}{ds} \frac{\theta^j}{ds} = 0, \tag{32. 8}$$

and

$$\left[\sum_j \bar{\theta}^j \frac{\theta^j}{ds} \right]_0^l = 0, \tag{32. 9}$$

where l *is the length of* $\bar{C}$.

If wo use the local coordinates (u^j), the above conditions become

$$\frac{d^2u^j}{ds^2} + \Gamma_{k}{}^{j}{}_{i}(u(s),\ \xi(s))\frac{du^k}{ds}\frac{du^i}{ds} + C_{k}{}^{j}{}_{i}(u(s),\ \xi(s))\frac{du^k}{ds}\frac{d\xi^i}{ds}$$
$$+ C^j{}_{ik}(u(s),\ \xi(s))\frac{du^i}{ds}\frac{D\xi^k}{ds} = 0, \tag{32.7'}$$

$$C_{ijk}(u(s),\ \xi(s))\frac{du^i}{ds}\frac{du^j}{ds} = 0, \tag{32.8'}$$

$$\left[g_{ij}(u(s),\ \xi(s))\frac{du^i}{ds}\delta u^j\right]_0^l = 0. \tag{32.9'}$$

Especially, if $\bar{C}$ is a proper α-curve, we may consider that along $\bar{C}$ we have generally

$$\theta^j = y^j\, ds. \tag{32.10}$$

Accordingly, (32.9) will be automatically satisfied by (26.2) and the third parts in the left hand side of (32.8) vanish by (25.10) since $\widetilde{\Gamma}$ is the metric connection of E. Cartan.

Theorem 32. 2. *For a positive regular Finsler manifold $(\mathfrak{X}, L)$, in order that a proper α-curve $\bar{C}$ in $T_0(\mathfrak{X})$ have a relatively minimum length and furthermore it is so in a family of curves $\bar{C}_\varepsilon$, it is necessary that*

$$d\frac{\theta^j}{ds} + \theta^j_k\frac{\theta^k}{ds} = 0, \tag{32.11}$$

and

$$[\textstyle\sum \bar{\theta}^j y^j]_0^l = 0. \tag{32.12}$$

(32.11) is locally written as

$$\frac{d^2u^j}{ds^2} + \Gamma_i{}^j{}_k\frac{du^i}{ds}\frac{du^k}{ds} = 0 \tag{32.13}$$

which is essentially equivalent with (30.2). $\bar{C}$ may be regarded as the lift in $T_0(\mathfrak{X})$ of a geodesic of the Finsler manifold $(\mathfrak{X},\ L)$.

§ 33. Geodesics and the second variations.

We shall calculate the second variation of (32.1) when $\bar{C}$ is an α-curve and $C = \tau\bar{C}$ is a geodesic of the Finsler manifold $(\mathfrak{X},\ L)$.

In order to simplify the calculations, we shall utilize only the orthonormal frames in the image of $\mathfrak{B}$ in $\widetilde{\mathfrak{B}}_0$ under the canonical imbedding (§ 28). We have generally on $\iota(\mathfrak{B})$

$$Dy^j = \theta^j_n, \qquad y^j = \delta^j_n, \tag{33.1}$$

and hence we have for $\bar{C}$ which is the lift of a geodesic

$$\theta^{\alpha} = 0, \qquad \theta^{n} = ds, \qquad \theta^{\alpha}_{n} = 0 \tag{33.2}$$

by Proposition 30.2 and (32.1). On the other hand, for any $\bar{C}$ in $T_0(\mathfrak{X})$, (32.6) is written as

$$\delta s = \left[\sum_j \bar{\theta}^j \frac{\theta^j}{ds}\right]_0^a - \int_0^a \sum_j \left(d \frac{\theta^j}{ds} + \theta^j_k \frac{\theta^k}{ds} + \widetilde{C}^j{}_{ik} \frac{\theta^i}{ds} \theta^k_n\right)\bar{\theta}^j + \int_0^a \widetilde{C}_{ijk} \frac{\theta^i}{ds} \frac{\theta^j}{ds} \bar{\theta}^k_n \, ds \tag{33.3}$$

using only such frames. From the our assumption for $\bar{C}$ and by (33.2) we get

$$\delta^2 s = \left[\delta\bar{\theta}^n + \sum_j \bar{\theta}^j \delta \frac{\theta^j}{ds}\right]_0^a - \int_0^a \sum_j \delta\left(d \frac{\theta^j}{ds} + \theta^j_k \frac{\theta^k}{ds} + \widetilde{C}^j{}_{ik} \frac{\theta^i}{ds} \theta^k_n\right)\bar{\theta}^j + \int_0^a \delta \widetilde{C}_{ijk} \frac{\theta^i}{ds} \frac{\theta^j}{ds} \bar{\theta}^k_n \, ds. \tag{33.4}$$

We shall deform the equation into an equation which will have geometrical significances in our standpoint. Firstly, we have

$$\begin{aligned}\delta \widetilde{C}_{ijk} \frac{\theta^i}{ds} \frac{\theta^j}{ds} &= (\widetilde{C}_{ijk,h}\, \bar{\theta}^h + \widetilde{C}_{ijk;h}\, \bar{\theta}^h_n + \widetilde{C}_{ljk}\, \bar{\theta}^l_i + \widetilde{C}_{ilk}\, \bar{\theta}^l_j + \widetilde{C}_{ijl}\bar{\theta}^l_k)\, y^i y^j \\ &= \widetilde{C}_{ijk,h}\, y^i y^j \bar{\theta}^h + \widetilde{C}_{ijk;h}\, y^i y^j \bar{\theta}^h_n \\ &= (\widetilde{C}_{ijk} y^i)_{,h}\, y^j \bar{\theta}^h + \{(\widetilde{C}_{ijk}\, y^i)_{;h} - \widetilde{C}_{hjk}\}\, y^j \bar{\theta}^h_n.\end{aligned}$$

Accordingly, (33.4) is written as

$$\delta^2 s = \left[\delta\bar{\theta}^n + \sum_j \bar{\theta}^j \delta \frac{\theta^j}{ds}\right]_0^a - \int_0^a \sum_j \left(d\delta \frac{\theta^j}{ds} + \delta\theta^j_n + \theta^j_k \delta \frac{\theta^k}{ds}\right)\bar{\theta}^j.$$

Making use of (32.3), this is equivalent to

$$\delta^2 s = [\delta\bar{\theta}^n]_0^a + \int_0^a \sum_j \left\{\delta \frac{\theta^j}{ds}(d\bar{\theta}^j + \theta^j_k \bar{\theta}^k) - \delta\theta^j_n \bar{\theta}^j\right\}. \tag{33.5}$$

On the other hand, along $\bar{C}$ we have easily

$$\frac{\theta^j}{ds} = \left(\sum \theta^k \theta^k\right)^{-\frac{1}{2}} \theta^j,$$

$$\delta \frac{\theta^j}{ds} = \frac{\delta\theta^j}{ds} - \theta^j \frac{\delta ds}{ds^2} = \frac{\delta\theta^j}{ds} - \frac{\theta^j \sum_k \theta^k \delta\theta^k}{ds^3} = \frac{\delta\theta^j - \delta^j_n \delta\theta^n}{ds}$$

and

$$\delta\theta_n^j = d\bar\theta_n^j - \bar\theta_k^j\theta_n^k + \theta_k^j\bar\theta_n^k + \frac{1}{2}\widetilde{R}_n{}^j{}_{ik}(\bar\theta^i\theta^k - \theta^i\bar\theta^k)$$

$$+ \widetilde{P}_n{}^j{}_{ik}(\bar\theta^i\theta_n^k - \theta^i\bar\theta_n^k) + \frac{1}{2}\widetilde{S}_n{}^j{}_{ik}(\bar\theta_n^i\theta_n^k - \theta_n^i\bar\theta_n^k)$$

$$= d\bar\theta_n^j + \theta_\beta^j\bar\theta_n^\beta + \widetilde{R}_n{}^j{}_{\beta n}\bar\theta^\beta\, ds - \widetilde{P}_n{}^j{}_{n\beta}\bar\theta_n^\beta\, ds.$$

Since $\widetilde{\Gamma}$ is a-proper, $\widetilde{P}_i{}^j{}_{nk} = \widetilde{P}_i{}^j{}_{hk} y^h = 0$. Substituting these into (33. 5), we get

$$\delta^2 s = [\delta\bar\theta^n]_0^a + \int_0^a \left\{\sum_\alpha \frac{\delta\theta^\alpha}{ds} D\bar\theta^\alpha - \sum_j d\bar\theta_n^j\bar\theta^j - \sum_j \theta_\beta^j\bar\theta_n^\beta\bar\theta^j + \widetilde{R}_{njn\beta}\bar\theta^j\bar\theta^\beta\, ds\right\}$$

$$= [\delta\bar\theta^n + \bar\theta_j^n\bar\theta^j]_0^a + \int_0^a \left\{\sum_\alpha \frac{\delta\theta^\alpha}{ds} D\bar\theta^\alpha - \bar\theta_j^n\, d\bar\theta^j - \sum_j \theta_\beta^j\bar\theta_n^\beta\bar\theta^j\right.$$

$$\left. + \widetilde{R}_{njn\beta}\bar\theta^j\bar\theta^\beta\, ds\right\}$$

Along $\bar C$, we have easily also

$$\delta\theta^\alpha = d\bar\theta^\alpha + \theta_k^\alpha\bar\theta^k - \bar\theta_k^\alpha\theta^k - \widetilde{C}_i{}^\alpha{}_k\bar\theta^i\theta_n^k + \widetilde{C}_n{}^\alpha{}_k\bar\theta_n^k\, ds$$

$$= D\bar\theta^\alpha - \bar\theta_n^\alpha\, ds$$

and hence

$$\sum_\alpha \frac{\delta\theta^\alpha}{ds} D\bar\theta^\alpha - \bar\theta_j^n\, d\bar\theta^j - \sum_j \theta_\beta^j\bar\theta_n^\beta\bar\theta^j$$

$$= \sum_\alpha \frac{D\bar\theta^\alpha - \bar\theta_n^\alpha\, ds}{ds} D\bar\theta^\alpha - \bar\theta_j^n(d\bar\theta^j + \theta_k^j\bar\theta^k)$$

$$= \sum_\alpha \frac{D\bar\theta^\alpha}{ds}\frac{D\bar\theta^\alpha}{ds}\, ds.$$

Thus, the second variation along $\bar C$ is written as

$$\delta^2 s = [\delta\bar\theta^n + \bar\theta_j^n\bar\theta^j]_0^l + \int_0^l \left(\sum_\alpha \frac{D\bar\theta^\alpha}{ds}\frac{D\bar\theta^\alpha}{ds} + \widetilde{R}_{ninj}\bar\theta^i\bar\theta^j\right), \qquad (33.\ 6)$$

where l is the length of $\bar C$.

When the curves of the family $\bar C_\varepsilon$ are all a-curves, we may put $\theta^j = y^j\, ds$ for each curve. Using the canonical imbedding of $\mathfrak{B}$ and only the orthonormal frames, we may put $\theta^\alpha = 0$ along each $\bar C_\varepsilon$ and

$$D\bar\theta^\alpha = d\bar\theta^\alpha + \theta_j^\alpha\bar\theta^j = \delta\theta^\alpha + \bar\theta_j^\alpha\theta^j = \bar\theta_n^\alpha\, ds$$

along $\bar{C}$. Thus, in this case, the formala (33. 6) is written as

$$\delta^2 s = [\delta\bar{\theta}^n + \bar{\theta}^n_i \bar{\theta}^i]_0^l + \int_0^l (\sum_\alpha \bar{\theta}^\alpha_n \bar{\theta}^\alpha_n + \widetilde{\mathrm{R}}_{,inj} \bar{\theta}^i \bar{\theta}^j). \tag{33.7}$$

Remark. In his paper[4], L. Auslander investigated the geodesics in Finsler manifolds and tried generalizations of a theorem of S.B. Myers and some other grobal theorems in Riemann manifolds. The basis of his arguments was that through the medium of the induced Riemann manifolds on local cross sections in the tangent sphere bundle $\mathfrak{S}(\mathfrak{X})$, the grobal properties of the Riemann manifolds may be carried into the Finsler manifold $(\mathfrak{X}, L)$ and that Theorem 4. 2 and Theorem 4. 3 in his paper are of importance. But, it is clear that his Theorem 4. 3 does not necessary hold good because the range of the families of curves in $T_0(\mathfrak{X})$ containing the lift of a geodesic in Theorem 4. 3 different from the ones in Theorem 4. 2 and his Theorem 4. 2 and Theorem 4. 3 may be considered as interpretations of the formulas (33. 7) and (33. 6) respectively.

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4) Cf. L. Auslander, On curvature in Finsler geometry, Trans. Amer. Math. Soc., Vol. 79 (1955), 378—388, Theorems 4. 2 and 4. 3.