A NOTE ON GALOIS THEORY OF DIVISION RINGS

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In their previous paper [3][1], two of the present authors considered Galois theory of division rings under the assumption that the total group is l. f. d. and locally compact, and proved the existence of Galois correspondence between closed regular subgroups of the total group and intermediate subrings.

In this paper, we shall deal with a special class of Galois extensions of which the total groups are l. f. d. but not locally compact and for which there exists still Galois correspondence in the above sense. In § 1, our consideration proceeds somewhat systematically under several assumptions which will be added in order, and § 2 contains an example belonging to the class considered in § 1.

As to terminologies used in this paper we follow [3].

1. Throughout this section, K be a division ring with the center C, and M be a (fixed) maximal (commutative) subfield of K. For any non-empty subset S of K, we shall denote by $V_K(S)$ the centralizer of S in K.

Lemma 1. *Let $\widetilde{M}$ be the group of all inner automorphisms of K generated by non-zero elements of M, and T be an intermediate division subring of K/M.*

(i) *T is Galois over M with $\widetilde{M}_T$ (= the restriction of $\widetilde{M}$ to T) as a Galois group, that is, the fixed subring of $\widetilde{M}_T$ in T is M.*

(ii) *$V_K(T) = V_T(T) = V_{T'}(T)$ for any intermediate subring T' of K/M.*

(iii) *If $[T:M] < \infty$ then $[T:V_T(T)] = [T:M]^2 = [M:V_T(T)]^2 < \infty$.*

Proof. Evidently $M \subset V_T(M) \subset V_K(M) = M$, that is, $V_T(M) = M (= V_T(V_T(M)))$. Further, from $T \supset M$, we obtain $V_K(T) \subset V_K(M) = M$, whence $V_K(T) = V_M(T)$. This proves (ii). Finally, assume $[T:M] < \infty$. Then, as $V_T(M) = M$ and T/M is Galois, we have $[V_T(M): V_T(T)] = [M: V_T(T)] < \infty$. This proves that $[T: V_T(T)] = [T:M] \cdot [M: V_T(T)] < \infty$. Accordingly we have also $[T: M] = [T: V_T(M)] = [M: V_T(T)]$, which proves (iii).

Now we shall set the following assumption.

(a) *K is locally finite over M.*

1) Numbers in brackets refer to the references cited at the end of this paper.

Lemma 2. *Under the assumption* (**a**), *any intermediate* (*division*) *subring*[2] T *of* K/M *is Galois over* M *and the total group* $\mathfrak{G}(T/M)$ *is locally finite dimensional* (*l.f.d.*).

Proof. T/M is Galois and $V_T(M) = M = V_M(M)$ by Lemma 1 (i). Hence $\mathfrak{G}(T/M)$ is l. f. d. by [3, Theorem 3].

We shall add, besides (a), the next assumption:

(**b**) *For any intermediate subfield* M' *of* M/C *with* $[M: M']<\infty$ *and for any intermediate division subring* T *of* K/M *with* $M' \supset V_T(T)$, *there exists a division subring* F *of* T *such that* $[F : M] < \infty$ *and* $M' \supset V_F(F)$.

Remark. If the assumption (**b**) is satisfied then $V_F(M')$ is a central division algebra of finite rank over M', for $[F : V_F(F)] < \infty$ by Lemma 1 (iii), accordingly $M' \supset V_F(F)$ implies $V_F(V_F(M')) = M'$. Thus M' is the center of some division subring F' of T with $[F' : M] < \infty$.

Lemma 3. *Let* M' *be a subfield of* M *with* $[M: M'] < \infty$ *and* T *be an intermediate division subring of* K/M *with* $M' \supset V_T(T)$. *Then under the assumption* (**b**), *for any intemediate division subring* T' *of* K/T, $V_{T'}(M')$ *is a division subring which is finite over* M *and whose center is* M'.

Proof. By the above remark, there exists a division subring F' of T with $[F' : M] < \infty$ and $V_{F'}(F') = M'$. Now let $T' \supset T$, and we shall prove that $F' = V_{T'}(M')$. Since $F' = V_{F'}(M') \subset V_{T'}(M')$, it follows that $M' = V_{F'}(V_{F'}(M')) = V_{V_{T'}(M')}(V_{F'}(M')) \supset V_{V_{T'}(M')}(V_{T'}(M')) \supset M'$ by Lemma 1 (ii), that is, $V_{V_{T'}(M')}(V_{F'}(M')) = M' = V_{V_{T'}(M')}(V_{T'}(M'))$. Noting that $V_{F'}(M') = F'$ is finite over the center M' (= the center of $V_{T'}(M')$) by Lemma 1 (iii), we obtain $\infty > [F' : M'] = [V_{F'}(M') : M'] = [V_{T'}(M'): V_{V_{T'}(M')}(V_{F'}(M'))] = [V_{T'}(M'): M']$. As $F' \subset V_{T'}(M')$, the equation $[F' : M'] = [V_{T'}(M') : M']$ implies $F' = V_{T'}(M')$, as desired.

Lemma 4. *Under the assumptions* (**a**) *and* (**b**), *there holds* $V_K(V_K(T)) = T$ *for any intermediate* (*division*) *subring* T *of* K/M.

Proof. Set $T_0 = V_K(V_K(T))$. Then evidently $V_K(T_0) = V_K(T)$. Further $V_{T_0}(T_0) = V_K(T_0) = V_K(T) = V_T(T)$ by Lemma 1 (ii). Now let k be an arbitrary element in T_0. Then $[M(k) : M] < \infty$ by (a), and $\infty > [M(k) : M] = [M: V_{M(k)}(M(k))] = [M: V_{T_0}(M(k))]$ by Lemma 1 (iii)

2) As K is locally finite over M, any intermediate subring of K/M is necessarily a division subring.

and (ii). Since $V_{T_0}(M(k)) \supset V_{T_0}(T_0) = V_T(T)$, we obtain $[V_T(V_{T_0}(M(k))): M] < \infty$, $[V_{T_0}(V_{T_0}(M(k))): M] < \infty$ and that $V_{T_0}(M(k))$ is the center of $V_T(V_{T_0}(M(k)))$ as well as of $V_{T_0}(V_{T_0}(M(k)))$ by Lemma 3. Hence, by Lemma 1 (iii), it follows $[V_T(V_{T_0}(M(k))): M] = [M: V_{T_0}(M(k))] = [V_{T_0}(V_{T_0}(M(k))): M]$. As obviously $V_T(V_{T_0}(M(k))) \subset V_{T_0}(V_{T_0}(M(k)))$, the last equation shows $V_T(V_{T_0}(M(k))) = V_{T_0}(V_{T_0}(M(k))) \ni k$, whence k is contained in T. We have proved therefore $T_0 = T$.

Lemma 5. *Let $\{M_\alpha;\ \alpha \in A\}$ be a set of intermediate subfields M_α's of M/C with $[M: M_\alpha] < \infty$, and let $M' = \bigcap_{\alpha \in A} M_\alpha$. If the assumption* (**b**) *is satisfied then $V_K(V_K(M')) = M'$.*

Proof. Since $[M: M_\alpha] < \infty$ and $M_\alpha \supset V_K(K)$, M_α is the center of $V_K(M_\alpha)$ by Lemma 3, which coincides also with $V_K(V_K(M_\alpha))$ by Lemma 1 (ii). Hence $V_K(V_K(M_\alpha)) = M_\alpha$ for all $\alpha \in A$. Accordingly we have $M' \subset V_K(V_K(M')) \subset \bigcap_{\alpha \in A} V_K(V_K(M_\alpha)) = \bigcap_{\alpha \in A} M_\alpha = M'$, whence our assertion $V_K(V_K(M')) = M'$ follows.

Here, besides (**a**) and (**b**), we shall add the last assumption:

(**c**) *Given any intermediate subfield M' of M/C, there exists a set $\{M_\alpha;\ \alpha \in A\}$ of intermediate subfields M_α's of M/C such that $M' = \bigcap_{\alpha \in A} M_\alpha$ and $[M: M_\alpha] < \infty$ for all $\alpha \in A$.*

From Lemmas 4 and 5, we obtain the following:

Theorem 1. *If the assumptions* (**a**), (**b**) *and* (**c**) *are satisfied then, between intermediate subfields M' of M/C and intermediate (division) subrings T of K/M, there exist the following mutally inverse one-to-one correspondences:*

$$T \longrightarrow V_K(T), \qquad M' \longrightarrow V_K(M').$$

Lemma 6. *Suppose that the assumptions* (**a**), (**b**) *and* (**c**) *are satisfied. If $[M': M' \cap M_0] < \infty$ for any intermediate subfields M', M_0 of M/C with $[M: M_0] < \infty$ then K is totally locally finite over M, that is, K is locally finite over any intermediate division subring of K/M.*

Proof. Let T be an arbitrary intermediate (division) subring of K/M, and S be a finite subset of K. If we denote by M' and M_0 the centers of T and $M(S)$ respectively, we have $V_K(T) = M'$ and $V_K(M(S)) = M_0$ by Lemma 1 (ii), and further $[M: M_0] < \infty$ by Lemma 1 (iii). Hence, by assumption, $[M': M' \cap M_0] < \infty$. Noting that $V_{V_K(M' \cap M_0)}(V_K(M' \cap M_0)) = V_K(V_K(M' \cap M_0)) = M' \cap M_0$ by Lemma 1 (ii) and

Lemma 5, we obtain $[V_K(M' \cap M_0) : V_{V_K(M' \cap M_0)}(M')] = [M' : M' \cap M_0]$. On the other hand, noting that $V_K(M' \cap M_0) \supset V_K(M') = V_K(V_K(T)) = T$ by Lemma 4, we have $T = V_K(M') = V_{V_K(M' \cap M_0)}(M')$. It follows therefore $[V_K(M' \cap M_0) : T] = [M' : M' \cap M_0] < \infty$. Clearly $V_K(M' \cap M_0) \supset V_K(M_0) = M(S)$ by Lemma 4, whence $[T(S) : T] < [M' : M' \cap M_0] < \infty$. This proves that K is totally locally finite over M.

In case the total group $\mathfrak{G}(K/M)$ is l. f. d., we can introduce into $\mathfrak{G}(K/M)$ the Krull's topology [2]. Further, if K is totally locally finite over M then we can prove, by making use of the standard method as in the proof of [4, Theorem 7], that any closed regular subgroup of $\mathfrak{G}(K/M)$ is a total subgroup. Thus, the following theorem is an easy consequence of Lemmas 2, 6 and Theorem 1.

Theorem 2. *Under the same assumptions as in Lemma* 6, *there exists a one-to-one dual correspondence between closed regular subgroups of* $\mathfrak{G}(K/M)$ *and intermediate subrings of* K/M, *in the usual sense of Galois theory.*

2. The object with which we shall deal in this section is a special type of division rings which has been considered in [1] and [3, § 4]. In pp. 23—24 of [1], G. Köthe proved that there exists a (countably) infinite number of central division algebras over the rational number field C: $\{D_1, D_2, \cdots\}$ such that $([D_i : C], [D_j : C]) = 1$ for $i \neq j$. Since each D_i is a cyclic division algebra over C, D_i contains a maximal subfield M_i which is cyclic over C. Clearly $D^{(i)} = D_1 \times_C D_2 \times_C \cdots \times_C D_i$ is a central division algebra over C. If $i < j$, by the canonical isomorphism, $D^{(i)}$ may be considered as a division subalgebra of $D^{(j)}$, and $K = \bigcup_{i=1}^{\infty} D^{(i)}$ may be considered. Throughout this section, K will signify this division ring $\bigcup_{i=1}^{\infty} D^{(i)}$ and M will mean the maximal commutative subfield $\bigcup_{i=1}^{\infty} M^{(i)}$ of K where $M^{(i)} = M_1 \times_C M_2 \times_C \cdots \times_C M_i$. Since $M_1, M_2, \cdots, M_n, \cdots$ are independent over C as subfields of M, we shall set $M = M_1 \times M_2 \times \cdots = \prod_{i=1}^{\infty} M_i$[3]. Then $M^{(i)} = M_1 \times M_2 \times \cdots \times M_i = \prod_{j=1}^{i} M_j$.

In what follows, we shall show that the assumptions (**a**), (**b**) and (**c**) in §1 are fulfilled with respect to this K and M.

3) This means that the intersection of any M_i and the composite of all M_j's except M_i is C and the composite of all M_i's is M.

A. Let S be an arbitrary finite subset of K. Then, there exists some $D^{(i)}$ containing S. Consider the division subring $D^{(i)} \times_C R^{(i)}$ of K where $R^{(i)} = \prod_{\nu=i+1}^{\infty} M_\nu$. Clearly $D^{(i)} \times_C R^{(i)} \supset M(S)$ and $[D^{(i)} \times_C R^{(i)} : M] = [M^{(i)}: C]$, and so $[M(S) : M] < \infty$. Thus, the assumption (**a**) is fulfilled.

B. The field $M^{(s)} = M_1 \times M_2 \times \cdots \times M_s$ is cyclic over C and the Galois group (the total group) $\mathfrak{G}$ of $M^{(s)}/C$ can be represented as a direct product of subgroups $\mathfrak{G}_1, \cdots, \mathfrak{G}_s$: $\mathfrak{G} = \mathfrak{G}_1 \times \cdots \times \mathfrak{G}_s$ where $\mathfrak{G}_i$ is isomorphic to the Galois group of M_i/C and $\mathfrak{G}^{(i)} = \mathfrak{G}_1 \times \cdots \times \mathfrak{G}_{i-1} \times \mathfrak{G}_{i+1} \times \cdots \times \mathfrak{G}_s$ is the Galois group of $M^{(s)}/M_i$. Let H be an intermediate subfield of $M^{(s)}/C$ and $\mathfrak{H}$ be the Galois group of $M^{(s)}/H$. Then the Galois group $\mathfrak{H}\mathfrak{G}^{(i)}$ of $M^{(s)}/H \cap M_i$ is equal to $\mathfrak{H}_i \times \mathfrak{G}^{(i)}$ where $\mathfrak{H}_i = \mathfrak{H} \cap \mathfrak{G}_i$. As evidently $\mathfrak{H} = \bigcap_{i=1}^{s} (\mathfrak{H}_i \times \mathfrak{G}^{(i)})$, we obtain $H = (H \cap M_1) \times \cdots \times (H \cap M_s)$.

Now let W be any intermediate subfield of M/C. Then $W \subset \prod_{i=1}^{\infty} (W \cap M_i)$ evidently. If w is an arbitrary element in W then there exists an integer s such that $w \in M^{(s)}$. Hence, by the above remark, $C(W) = (C(w) \cap M_1) \times \cdots \times (C(w) \cap M_s) \subset (W \cap M_1) \times \cdots \times (W \cap M_s) \subset \prod_{i=1}^{\infty} (W \cap M_i)$, whence it follows that $W = \prod_{i=1}^{\infty} (W \cap M_i)$. Further, if $[M: W] < \infty$ then we can easily see that $W \supset M_i$ for almost all i. These remarks will be used very often in the sequel.

Now let T be an intermediate subring of K/M, and M' be an intermediate subfield of M/C with $[M: M'] < \infty$ and $M' \supset V_T(T)$. Then, by the above remark, there exists an integer j such that $M' \supset R^{(j)} = \prod_{\nu=j+1}^{\infty} M_\nu$. If we set $T_\nu = (D^{(\nu)} \times_C R^{(\nu)}) \cap T$, we readily see that $T = \bigcup_{\nu=1}^{\infty} T_\nu$. And so $V_T(T) = V_T(\bigcup_{\nu=1}^{\infty} T_\nu) = \bigcap_{\nu=1}^{\infty} V_T(T_\nu) = \bigcup_{\nu=1}^{\infty} V_{T_\nu}(T_\nu)$ by Lemma 1 (ii). By the above remark, $V_{T_\nu}(T_\nu) = \prod_{i=1}^{\infty} M_{\nu,i}$, where $M_{\nu,i} = V_{T_\nu}(T_\nu) \cap M_i$. As $V_{T_\nu}(T_\nu) = V_T(T_\nu) \supset V_T(T_{\nu+1}) = V_{T_{\nu+1}}(T_{\nu+1})$ for all ν, we obtain the descending chain $M_i \supset M_{1,i} \supset \cdots \supset M_{\nu,i} \supset \cdots \supset C$. Noting that $[M_i : C] < \infty$ and $\bigcap_{\nu=1}^{\infty} M_{\nu,i} = (\bigcap_{\nu=1}^{\infty} V_{T_\nu}(T_\nu)) \cap M_i = V_T(T) \cap M_i$, we can find an integer ν_i such that $M_{\nu',i} = V_T(T) \cap M_i$ for all $\nu' > \nu_i$. Hence, if $\mu > \max(\nu_1, \cdots, \nu_j)$ then $V_{T_\mu}(T_\mu) = \prod_{i=1}^{\infty} (M_{\mu,i}) \subset (V_T(T) \cap M_1) \times \cdots \times (V_T(T) \cap M_j) \times R^{(j)} \subset M'$ by the assumption $M' \supset V_T(T)$. This fact together with $[T_\mu : M] < \infty$ shows that the assumption (**b**) is satisfied.

C. If W is an intermediate subfield of M/C then $W=\prod_{\rho=1}^{\infty}(W\cap M_\rho)$ by the remark in **B**. Consider the set of subfields $\{W_i=(\prod_{\rho=1}^{i}(W\cap M_\rho)\times R^{(i)};\ i=1,\ 2,\cdots\}$. Then $[M:W_i]<\infty$ for all i and $\bigcap_{i=1}^{\infty}W_i=W$. The assumption (**c**) is therefore fulfilled.

D. Let W and M' be intermediate subfields of M/C with $[M:M']<\infty$. Then, again by the remark in **B**, $W\cap M'=\prod_{i=1}^{\infty}(W\cap M'\cap M_i)$. Since $[M:M']<\infty$, there exists an integer j such that $M'\cap M_i=M_i$ for all $i>j$, whence $W\cap M'\cap M_i=W\cap M_i$. We obtain therefore $W\cap M'=\prod_{i=1}^{j}(W\cap M'\cap M_i)\times\prod_{i=j+1}^{\infty}(W\cap M_i)$, over which $W=\prod_{i=1}^{\infty}(W\cap M_i)$ is finite. This proves that Theorem 2 holds for K and M con sidered in this section.

REFERENCES

[1] G. KÖTHE, Schiefkörper unendlichen Ranges über dem Zentrum, Math. Ann., 105 (1931) 14—39.

[2] W. KRULL, Galoissche Theorie der unendlichen algebraischen Erweiterungen, Math. Ann., 100 (1928) 687—698.

[3] T. NAGAHARA and H. TOMINAGA, On Galois theory of division rings, Math. J. Okayama Univ., 6 (1956) 1—21.

[4] N. NOBUSAWA, An extension of Krull's Galois theory to division rings, Osaka Math. J., 7 (1955) 1—6.

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