A NOTE ON CONJUGATES

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Recently in his paper [1] I. N. Herstein proved the following : *If in a division ring D an element $d \in D$ has only a finite number of conjugates then it has only one, that is, d is in the center of D.* On the other hand, relating to Galois theory of infinite degree, F. Kasch has obtained the following ([2, Satz 4]) : *Let U be an arbitrary ring, and D be an infinite division subring of U. If t is an element of U not contained in $V_U(D)$ then it has an infinite number of conjugates by non-zero elements in D.* Noting that any finite division ring is commutative by a well-known theorem of Wedderburn, one will readily see that Herstein's theorem is an easy consequence of Kasch's.

In what follows, we shall prove a theorem which is a sharpening as well as a generalization of Kasch's theorem. And one will see also that Herstein's theorem can be shown without making use of Cartan's theorem.

We use the following conventions throughout : By a ring we mean a ring with an identity, and by a subring we mean one which contains this identity. By a simple ring we shall mean a two-sided simple ring with minimum condition for one-sided ideals. For any non-empty subset B in a ring A, $V_A(B)$ will denote the centralizer of B in A. If K is a division ring then K^* will be the group of its non-zero elements under the multiplication of K. And for any set S, $\overline{\overline{S}}$ will signify the cardinal number of S.

Now we shall begin our course with the following sharpening of Kasch's theorem.

Lemma. *Let D be an infinite division subring of a ring U, and T be the set of conjugates of an element $t \in U$ by all non-zero elements in D. Then $\overline{\overline{T}} = \overline{\overline{D}}$ or 1.*

Proof. Clearly $\overline{\overline{T}}$ coincides with the index of $V_D(t)^*$ in D^*. Hence we have $\overline{\overline{D}} = \overline{\overline{T}} \cdot \overline{\overline{V_D(t)}}$. Now we assume $V_D(t) \subsetneqq D$. Then there exists some $d \in D$ with $dt \neq td$. And for any different v, v' in $V_D(t)$, there holds $(d+v)\,t\,(d+v)^{-1} \neq (d+v')\,t\,(d+v')^{-1}$. For, if not, $(d+v)\,t\,(d+v)^{-1} = t' = (d+v')\,t\,(d+v')^{-1}$ implies $t = t'$, from which we can readily obtain a contradiction $dt = td$. The last fact shows evidently $\overline{\overline{V_D(t)}} \leq \overline{\overline{T}}$, accordingly we have $\overline{\overline{D}} = \overline{\overline{T}} \cdot \overline{\overline{V_D(t)}} = \overline{\overline{T}}$, as desired.

Our principal theorem is stated as follows.

Theorem. *Let R be an infinite simple subring of a ring U, and T be the set of conjugates of an element $t \in U$ by all regular elements in R. Then $\overline{\overline{T}} = \overline{\overline{R}}$ or 1.*

Proof. We set, throughout the proof, $R = \sum_{i,j=1}^{n} De_{ij}$ where e_{ij}'s are matric units and $D = V_R(\{e_{ij}\})$ is a division ring. Then, as is well-known, $U = \sum_{i,j=1}^{n} Ve_{ij}$, where $V = V_U(\{e_{ij}\})$. Since our assertion for $n = 1$ is the above lemma itself, we shall assume $n > 1$, and set $t = \sum_{i,j=1}^{n} c_{ij}e_{ij}$ with $c_{ij} \in V$. Now we shall prove that if $\overline{\overline{T}} < \overline{\overline{R}}$ then t is in $V_U(R)$. By assumption $\overline{\overline{\{dtd^{-1};\ d \in D^*\}}} \leqq \overline{\overline{T}} < \overline{\overline{R}} = \overline{\overline{D}}$, *a fortiori*, $\overline{\overline{\{dc_{ij}d^{-1};\ d \in D^*\}}} < \overline{\overline{D}}$ for any i, j. Hence all c_{ij}'s are contained in $V_V(D) = V_U(R)$ by the above lemma. If $c_{pq} \neq 0$ for some $p \neq q$ then there holds

$$(1 - de_{qp})\, t\, (1 - de_{qp})^{-1} = (1 - de_{qp})\, t\, (1 + de_{qp})$$
$$= \sum_{i,j=1}^{n} c_{ij}e_{ij} - \sum_{j=1}^{n} dc_{pj}e_{qj} + \sum_{i=1}^{n} dc_{iq}e_{ip} - d^2c_{pq}e_{qp}$$

for any $d \in D$. Noting that the coefficient of e_{pp} in the last equation is $c_{pp} + dc_{pq}$, we have a contradiction $\overline{\overline{T}} \geqq \overline{\overline{D}} = \overline{\overline{R}}$. Hence $t = \sum_{i=1}^{n} c_{ii}e_{ii}$. Again for any $p \neq q$ and $d \in D$, there holds

$$(1 - de_{qp})\, t\, (1 - de_{qp})^{-1} = \sum_{i=1}^{n} c_{ii}e_{ii} - dc_{pp}e_{qp} + dc_{qq}e_{qp}.$$

As the coefficient of e_{qp} is $d(c_{qq} - c_{pp})$, we must have $c_{qq} - c_{pp} = 0$, that is, $t = c_{pp} \in V_U(R)$. This completes the proof.

The next is only a restatement of our theorem.

Corollary. *Let R be an infinite simple subring of a ring U, and T be a subset of U which is transformed into itself by all regular elements in R. If $\overline{\overline{T}} < \overline{\overline{R}}$ then $T \subset V_U(R)$.*

REFERENCES

[1] I. N. HERSTEIN, Conjugates in division rings, Proc. Amer. Math. Soc., 7 (1956) 1021—1022.

[2] F. KASCH, Eine Bemerkung über innere Automorphismen, Math. J. Okayama Univ., 6 (1957) 131—133.

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