THEORY OF AFFINE CONNECTIONS OF THE SPACE OF TANGENT DIRECTIONS OF A DIFFERENTIABLE MANIFOLD I, II

TOMINOSUKE ŌTSUKI

It seems to the author that theories of affine connections on the space of tangent directions of a differentiable manifold have been mainly studied by many authors in connection with Finsler manifolds and within the limits of metric connections. But, if we wish to generalize some theorems of Riemann manifolds in Finsler manifolds and try it only utilizing metric connections in the classical sense, we shall encounter various difficulties.

In the present paper, the author will study a general theory of affine connections on the space $\mathfrak{S}(\mathfrak{X})$ of tangent directions of a differentiable manifold $\mathfrak{X}$ based on the theory of connections of vector bundles. Firstly we shall study it from the standpoint of holonomy groups then investigate metric connections in the case Finsler metrics are given on $\mathfrak{S}(\mathfrak{X})$. For a connection Γ given on $\mathfrak{S}(\mathfrak{X})$, we can define holonomy groups in two senses as follow. If we regard $\mathfrak{S}(\mathfrak{X})$ merely as a differentiable manifold and Γ as an affine connection of the vector bundle $\{\mathfrak{Z}, \mathfrak{S}(\mathfrak{X})\}$, we can obtain the homogeneous holonomy group $\widetilde{H}$ and the affine holonomy group $\widetilde{AH}$ by means of parallel displacements alongs curves in $\mathfrak{S}(\mathfrak{X})$. On the other hand, since $\mathfrak{S}(\mathfrak{X})$ is in fact a space as defined above, in order to define the holonomy groups of Γ it must be natural that we utilize only the family of curves in $\mathfrak{S}(\mathfrak{X})$ which are curves piecewise consisting of tangent directions of curves in $\mathfrak{X}$ or curves in fibres of the sphere bundle $\{\mathfrak{S}(\mathfrak{X}), \mathfrak{X}\}$. Thus we obtain the homogeneous holonomy group H and the affine holonomy group AH of Γ as subgroups of $\widetilde{H}$ and $\widetilde{AH}$ respectively. We call $\widetilde{H}$, $\widetilde{AH}$ especially the holonomy groups of Γ in a wide sense.

In Part I, we shall study mainly the relations between H and $\widetilde{H}$. For any regular connection Γ on $\mathfrak{S}(\mathfrak{X})$, we shall reach a concept of derived connection $'\Gamma$ of Γ (§ 15) and shall show that the homogeneous holonomy group H_Γ of Γ coincides with the homogeneous holonomy group $\widetilde{H}_{'\Gamma}$ of $'\Gamma$ in a wide sense (Theorem 15. 1).

In Part II, we shall deal with the analogous problems for affine holonomy groups of Γ. For Γ and its derived connectien $'\Gamma$, in order that their affine holonomy groups AH_Γ and $\widetilde{AH}_{'\Gamma}$ coincide with each others, it must be necessary some condition (Theorem 22. 3). In order to remove the condition, we shall introduce a concept of modified connection ${}^{\triangle}\Gamma$ of Γ for any regular connection on $\mathfrak{S}(\mathfrak{X})$ and show that making use of ${}^{*}\Gamma = {}^{\triangle}({}'\Gamma)$, AH_{Γ} is isomorphic with $\widetilde{AH}_{{}'\Gamma}$ (Theorem 23. 5). In Parts I, II, we shall also prove a theorem on the structure of holonomy groups which is a generalization of a theorem of A. Nijenhuis[1] in affinely connected manifolds in the ordinary sense and shall investigate the properties of the groups which occur under the circumstances that we treat connections on $\mathfrak{S}(\mathfrak{X})$ in place of connections on $\mathfrak{X}$ itself (Theorems 13. 5, 14. 1, 20. 7, 20. 8 and 21. 1).

In Part III, we shall investigate Finsler metrics in connection with the theory of Parts I, II. Contracting the projection map τ of the tangent bundle $\{T(\mathfrak{X}), \mathfrak{X}\}$ on $T(\mathfrak{X}) - \mathfrak{X} = T_0(\mathfrak{X})$ and denoting it by $\bar{\tau}$, we shall investigate connection Γ on $\mathfrak{S}(\mathfrak{X})$ by means of the induced connection $\widetilde{\Gamma}$ of the induced vector bundle $\bar{\tau}^{\diamond}\{T(\mathfrak{X}), \mathfrak{X}\}$ which is induced by the map $T_0(\mathfrak{X}) \rightarrow \mathfrak{C}(\mathfrak{X})$ from Γ. Taking the associated principal bundle $\{\mathfrak{B}, \mathfrak{X}\}$ of $\{T(\mathfrak{X}), \mathfrak{X}\}$ and putting $\bar{\tau}^{\diamond}\{\mathfrak{B}, \mathfrak{X}\} = \{\widetilde{\mathfrak{B}}_0, T_0(\mathfrak{X})\}$, we can define a natural imbedding of $\mathfrak{B}$ into $\widetilde{\mathfrak{B}}_0$[2] and therefore discuss the theory of connections on $\mathfrak{S}(\mathfrak{X})$ by means of $\mathfrak{B}$ only[3]. We shall investigate the relations between this stand point of view and the above mentioned one. Lastly we shall discuss the affine connections in a general sense induced on submanifolds of $\mathfrak{S}(\mathfrak{X})$ from Γ, taking the applications of them in the cases with metrics into consideration.

Part I

1) A. Nijenhuis, On the holonmoy groups of linear connections, II. Properties of general linear connections, Indagationes Mathematicae, Vol. 16, 1954, pp. 17—25.

2) S. S. Chern, Euclidean connection for Finsler spaces, Proc. Nat. Acad. Sci. U. S. A., Vol. 29, 1943, pp. 38—43.

3) L. Auslander, On curvature in Finsler geometry, Trans. Amer. Math. Soc., Vol. 79, 1955, pp. 378—388.

§ 1. **Preliminary (I).**

Let $\mathfrak{X}$ be an n-dimensional differentable manifold and $\mathfrak{Y}$ be an m-dimensional vecor space over the real field. Consider a vector bundle $\mathfrak{F} = \{\mathfrak{Z}, \mathfrak{X}, \pi, \mathfrak{Y}\}$ with $\mathfrak{Z}, \mathfrak{X}, \pi, \mathfrak{Y}$ as its total space, base space, projection map and fibre respectively. In the following, we assume that all spaces and maps have suitable differentiablity. Let a subgroup G of the m-dimensional full linear group $GL(m)$ be the group of bundle of $\mathfrak{F}$, for a coordinate neighborhood system $\{U_\lambda \mid \lambda \in \Lambda\}$, $\varphi_\lambda: U_\lambda \times \mathfrak{Y} \to \pi^{-1}(U_\lambda)$ be the coordinate function of $\mathfrak{F}$ and for $U_\lambda \cap U_\mu \neq \emptyset$

$$g_{\mu\lambda}: U_\lambda \cap U_\mu \to G$$

be the coordinate transformation of $\mathfrak{F}$. By definition, we have for $U_\lambda \cap U_\mu \cap U_\nu \neq \emptyset$

$$g_{\nu\mu}(x) g_{\mu\lambda}(x) = g_{\nu\lambda}(x). \tag{1. 2}$$

We denote the associated principal fibre bundle of $\mathfrak{F}$ by $\hat{\mathfrak{F}} = \{\mathfrak{B}, \mathfrak{X}, \hat{\pi}, G\}$ and the one by $\tilde{\mathfrak{F}} = \{\tilde{\mathfrak{B}}, \mathfrak{X}, \tilde{\pi}, GL(m)\}$ when we replace the group of bundle G of $\mathfrak{F}$ by $GL(m)$. As is well known, if we put the coodinate functions of $\hat{\mathfrak{F}}, \tilde{\mathfrak{F}}$

$$\hat{\varphi}_\lambda : U_\lambda \times G \to \hat{\pi}^{-1}(U_\lambda), \qquad \tilde{\varphi}_\lambda : U_\lambda \times GL(m) \to \tilde{\pi}^{-1}(U_\lambda),$$

then we may put $\hat{\varphi}_\lambda = \tilde{\varphi}_\lambda \mid U_\lambda \times G$ and hence we may consider that $\hat{\mathfrak{F}}$ is a subbundle of $\tilde{\mathfrak{F}}$.

For another differentiable manifold $\mathfrak{X}'$ and a map $f : \mathfrak{X}' \to \mathfrak{X}$, taking $\{f^{-1}(U_\lambda) = U'_\lambda\}$ as a coordinate neighborhood system and putting the coordinate transformation for $U'_\lambda \cap U'_\mu \neq \emptyset$

$$g'_{\mu\lambda} = g_{\mu\lambda} \cdot (f \mid U'_\lambda \cap U'_\mu),$$

we get a vector bundle $\{\mathfrak{Z}', \mathfrak{X}', \pi', \mathfrak{Y}\}$ with its group of bundle G that is called the induced bundle of $\mathfrak{F}$ by f. We denote this by $f^{\diamond}\mathfrak{F}$. It can be easily verifield that the associated principal fibre bundles of $f^{\diamond}\mathfrak{F}$ are $f^{\diamond}\hat{\mathfrak{F}}$ and $f^{\diamond}\tilde{\mathfrak{F}}$.

For two fibre bundles $\mathfrak{F} = \{\mathfrak{Z}, \mathfrak{X}, \pi, \mathfrak{Y}\}$, $\mathfrak{F}' = \{\mathfrak{Z}', \mathfrak{X}, \pi', \mathfrak{Y}'\}$ with the same base space $\mathfrak{X}$, putting their coordinate transformations $g_{\mu\lambda}: U_\lambda \cap U_\mu \to G$ and $g'_{\mu\lambda}: U_\lambda \cap U_\mu \to G'$, we define the map $g''_{\mu\lambda}: U_\lambda \cap U_\mu \to G \times G'$ by

$$g''_{\mu\lambda}(x) = (g_{\mu\lambda}(x), \ g'_{\mu\lambda}(x)) \tag{1.4}$$

and we denote the product fibre bundle which have $g''_{\mu\lambda}$ as its coordinate transformations and $\mathfrak{Y} \times \mathfrak{Y}'$ as its fibre and its total space by $\mathfrak{F} \times \mathfrak{F}'$ and $\mathfrak{Z} \boxtimes \mathfrak{Z}'$ respectively. When $\mathfrak{F}$ and $\mathfrak{F}'$ are vector bundles, furthermore we can define $\mathfrak{F} \otimes \mathfrak{F}'$ as follows. Putting $G \subset GL(m)$, $G' \subset GL(m')$, $m = \dim \mathfrak{Y}$, $m' = \dim \mathfrak{Y}'$, we define the maps $g''_{\mu\lambda}: U_\lambda \cap U_\mu \to G \otimes G' \subset GL(m \times m')$ by

$$g''_{\mu\lambda}(x) = g_{\mu\lambda}(x) \otimes g'_{\mu\lambda}(x): \ \mathfrak{Y} \otimes \mathfrak{Y}' \to \mathfrak{Y} \otimes \mathfrak{Y}' \tag{1.5}$$

then we obtain a vector bundle with $g''_{\mu\lambda}$ as its coordinate transformation and denote it and its total space by $\mathfrak{F} \otimes \mathfrak{F}$ and $\mathfrak{Z} \otimes \mathfrak{Z}'$ respectively.

Now, we consider a vector bundle $\mathfrak{F} = \{\mathfrak{Z}, \mathfrak{X}, \pi, \mathfrak{Y}\}$. Any element g of $GL(m)$ is represented by a square matrix (a^β_α) of degree m which may operate on $\mathfrak{Y}$ as follows, taking a fixed base $(y_1, \dots, y_m)$ of $\mathfrak{Y}$

$$g(y) = a^\beta_\alpha(g) v^\alpha y_\beta, \qquad y = v^\alpha y_\alpha, \quad g \in GL(m). \tag{1.6}$$

We regard covariant differentiation of $\mathfrak{F}$ as a linear operator over the real field which transforms the vecor space $\Psi(\mathfrak{F})$ of cross sections of $\mathfrak{F}$ over the algebra of scalar fields on $\mathfrak{X}$ into $\Psi(\mathfrak{F} \otimes T^*(\mathfrak{X}))$

$$D: \ \Psi(\mathfrak{F}) \to \Psi(\mathfrak{F} \otimes T^*(\mathfrak{X})) \tag{1.7}$$

where $T^*(\mathfrak{X})$ is the co-tangent vector bundle of $\mathfrak{X}$ which is dual to $T(\mathfrak{X})$. As is well known, taking the local cross sections $z_{(\lambda)\alpha}(x) = \varphi_\lambda(x, y_\alpha)$, $x \in U_\lambda$, of $\mathfrak{F}$, we have

$$Dz_{(\lambda)\alpha} = z_{(\lambda)\beta} \otimes \omega_{(\lambda)}{}^\beta_\alpha \tag{1.8}$$

where the differential forms $\omega_{(\lambda)}{}^\beta_\alpha$ on U_λ have the properties

(i) $(\omega_{(\lambda)}{}^\beta_\alpha)$ is $L(G)$-valued, (1.9)

(ii) in $U_\lambda \cap U_\mu \neq \emptyset$

$$\omega_{(\lambda)}{}^\beta_\alpha = (a^\beta_\gamma \cdot g_{\lambda\mu}) \left(d\,(a^\gamma_\alpha \cdot g_{\mu\lambda}) + \omega_{(\mu)}{}^\gamma_\rho \,(a^\rho_\alpha \cdot g_{\mu\lambda}) \right) \tag{1.10}$$

where $L(G)$ denote the Lie algebra of G and (i) is required since the group of bundle of $\mathfrak{F}$ is G. Conversely, if we have $\omega_{(\lambda)}{}^\beta_\alpha$ satisfying (1.9) and (1.10), we can define a covariant differetiation D of $\mathfrak{F}$. We say the system $\omega_{(\lambda)}{}^\beta_\alpha$ define a connection Γ of $\mathfrak{F}$. When a differentiabe manifold

$\mathfrak{X}'$ and a map $f: \mathfrak{X}' \rightarrow \mathfrak{X}$ are given, we can define a connection $\Gamma' = f^{\diamond}\Gamma$ of $f^{\diamond}\mathfrak{F}$ which is given by $f^{*}\omega_{(\lambda)}{}_{\alpha}^{\beta}$ on each $U'_{\lambda} = f^{-1}(U_{\lambda})$. $f^{*}\omega_{(\lambda)}{}_{\alpha}^{\beta}$ satisfy clearly (1.9) and (1.10), where we denote by $f^{*}: T^{*}(\mathfrak{X}) \rightarrow T^{*}(\mathfrak{X}')$ in general the dual map of the differential map $f_{*}: T(\mathfrak{X}') \rightarrow T(\mathfrak{X})$ of f.

Now let $\mathfrak{F} = \{\mathfrak{Z}, \mathfrak{X}, \pi, \mathfrak{Y}\}$ be a vector bundle with a connection Γ. Let $\hat{\mathfrak{F}}$ and $\tilde{\mathfrak{F}}$ be the associated principal fibre bundles of $\mathfrak{F}$ above mentioned. For the induced vector bundle $\tilde{\pi}^{\diamond}\mathfrak{F} = \{\mathfrak{V}, \tilde{\mathfrak{B}}, p, \mathfrak{Y}\}$, let $h: \mathfrak{V} \rightarrow \mathfrak{Z}$ be the induced bundle map. On the other hand, since any point $b \in \tilde{\mathfrak{B}}$ is a base of the fibre $\mathfrak{Y}_x$ of $\mathfrak{F}$, $x = \tilde{\pi}(b)$, we denote this by $(z_1(b), \ldots, z_m(b))$ which is called a *frame* at x. Then, we obtain m natural cross sections $\mathfrak{z}_{\alpha}: \tilde{\mathfrak{B}} \rightarrow \mathfrak{V}$ by

$$h(\mathfrak{z}_{\alpha}(b)) = z_{\alpha}(b). \tag{1.11}$$

The induced connection $\tilde{\pi}^{\diamond}\Gamma$ determine m^2 differential froms θ_{α}^{β} on $\tilde{\mathfrak{B}}$ by

$$D\mathfrak{z}_{\alpha} = \mathfrak{z}_{\beta} \otimes \theta_{\alpha}^{\beta} \tag{1.12}$$

and they are written on $\tilde{\pi}^{-1}(U_{\lambda})$ as

$$\tilde{\varphi}_{\lambda}^{*}\ \iota_{\alpha}^{\beta} = b_{\gamma}^{\beta}(d\,a_{\alpha}^{\gamma} + \omega_{(\lambda)}{}_{\rho}^{\gamma}\ a_{\alpha}^{\rho})$$

where $b_{\alpha}^{\beta}(g) = a_{\alpha}^{\beta}(g^{-1})$, $g \in GL(m)$. As is well known, for any right-translation $r_g: \tilde{\mathfrak{B}} \rightarrow \tilde{\mathfrak{B}}$, $g \in GL(m)$,

$$r_g(b) = (z_{\beta}(b)\,a_1^{\beta}(g), \ldots,\ z_{\beta}(b)a_m^{\beta}(g)\,), \tag{1.13}$$

θ_{α}^{β} have the property

$$(r_g)^{*}\ \theta_{\alpha}^{\beta} = b_{\gamma}^{\beta}(g)\theta_{\rho}^{\gamma}\ a_{\alpha}^{\rho}(g). \tag{1.14}$$

It can easily seen that if we restrict ι_{α}^{β} on $\hat{\mathfrak{B}} \subset \tilde{\mathfrak{B}}$, they become $L(G)$-valued.

Then, we take two vector bunde $\mathfrak{F} = \{\mathfrak{Z}, \mathfrak{X}, \pi, \mathfrak{Y}\}$, $\mathfrak{F}' = \{\mathfrak{Z}', \mathfrak{X}', \pi', \mathfrak{Y}\}$ with the same fibre $\mathfrak{Y}$, a map $f: \mathfrak{X}' \rightarrow \mathfrak{X}$ and a bundle map $h: \mathfrak{Z}' \rightarrow \mathfrak{Z}$ such that $\pi \cdot h = f \cdot \pi'$. For any cross section $\xi \in \Psi(\mathfrak{F})$, we can define an element $\xi' = h^{\ominus}\xi$ of $\Psi(\mathfrak{F}')$ by

$$\xi'(x') = (h \mid \mathfrak{Y}_{x'})^{-1}\ (\xi(f(x'))), \quad x' \in \mathfrak{X}'. \tag{1.15}$$

When especially $\mathfrak{F}' = f^{\diamond}\mathfrak{F}$ and h is its induced bundle map, we denote $h^{\ominus}$ by $f^{\ominus}$ in the following. If we put $z'_{(\lambda)\alpha} = h^{\ominus}\ z_{(\lambda)\alpha}$ on $f^{-1}(U_{\lambda})$, then for $U_{\lambda} \cap U_{\mu} \neq \emptyset$ we have

$$z'_{(\lambda)\alpha} = z'_{(\mu)\beta}\ f^{*}(a_{\alpha}^{\beta}\cdot g_{\mu\lambda}).$$

By virtue of the relations and (1.9), (1.10), we can define a connection

Γ' of $\mathfrak{F}'$ from a connection Γ of $\mathfrak{F}$ by

$$Dz'_{(\lambda)\alpha} = z'_{(\lambda)\beta} \otimes f^* \, \omega_{(\lambda);\beta}^{\alpha}. \tag{1.16}$$

We denote this by $\Gamma' = h\# \Gamma$ and call it the *induced connection* from Γ by h. When h is the induced bundle map of f, we have clearly $h\# \Gamma = f^{\diamond} \Gamma$. Now we assume that ξ is written locally as $\xi^\alpha z_{(\lambda)\alpha}$, then we have by (1.15) $\xi' = z'_{(\lambda)\alpha} \, \xi'^\alpha$, $\xi'^\alpha = f^* \xi^\alpha$. Accordingly for $\Gamma' = h\# \Gamma$ we have

$$\begin{aligned} D\xi' &= z'_{(\lambda)\alpha} \otimes D\xi'_\alpha = z'_{(\lambda)\alpha} \otimes \Big(d(f^*\xi^\alpha) + f^* \omega_{(\lambda)\beta}^{\alpha} (f^* \xi^\beta) \Big) \\ &= h\ominus x_{(\lambda)\alpha} \otimes f^* \, D\xi^\alpha, \end{aligned}$$

that is

$$D(h\ominus \xi) = (h\ominus \otimes f^*) \, D\xi, \qquad \xi \in \Psi(\mathfrak{F}). \tag{1.17}$$

This formula may be generalized in the following sense. Denoting the vector bundle over $\mathfrak{X}$ with fibres $T_x^*(\mathfrak{X}) \wedge \cdots \wedge T_x^*(\mathfrak{X})$ which are the exterior products of r $T_x^*(\mathfrak{X})$ by $\{\Lambda^{*r}(\mathfrak{X}), \mathfrak{X}\}$ (or simply $\Lambda^{*r}(\mathfrak{X})$), we extend the operator D as follows :

$$\nabla : \Psi(\mathfrak{F} \otimes \Lambda^{*r}(\mathfrak{X})) \to \Psi(\mathfrak{F} \otimes \Lambda^{*r+1}(\mathfrak{X})), \tag{1.18}$$

$\nabla = D$ when $r = 0$ and

(i) for $\Psi(\mathfrak{F} \otimes \Lambda^{*r}(\mathfrak{X})) \ni \xi_r = \xi \otimes \omega_r$, $\xi \in \Psi(\mathfrak{F})$, $\omega_r \in \Psi(\Lambda^{*r}(\mathfrak{X}))$

$$\nabla \xi_r = D\xi \wedge \omega_r + \xi \otimes d\omega_r \tag{1.19}$$

where the first term of the right hand side is written in the sense $(\xi \otimes \omega_1) \wedge \omega_r = \xi \otimes (\omega_1 \wedge \omega_r)$, and

(ii) for $\Psi(\mathfrak{F} \otimes \Lambda^{*r}(\mathfrak{X})) \ni \xi_r$, any scalar field ϕ of $\mathfrak{X}$

$$\nabla(\phi \xi_r) = \phi \, \nabla \xi_r + (-1)^r \xi_r \wedge d\phi. \tag{1.19'}$$

We can easily prove that ∇ is uniquely determined by (1.19) and (1.19′). Now, denoting $\Lambda^r f^* : \Psi(\Lambda^{*r}(\mathfrak{X})) \to \Psi(\Lambda^{*r}(\mathfrak{X}'))$ by f^*, we can determine the map

$$h\ominus \otimes f^* : \ \Psi(\mathfrak{F} \otimes \Lambda^{*r}(\mathfrak{X})) \to \Psi(\mathfrak{F}' \otimes \Lambda^{*r}(\mathfrak{X}')). \tag{1.20}$$

By virtue of (1.17) and the relation $d \cdot f^* = f^* \cdot d$, we can prove that

$$\nabla((h\ominus \otimes f^*) \xi_r) = (h\ominus \otimes f^*) \nabla \xi_r \tag{1.21}$$

for any integer $r \geqq 0$ and $\xi_r \in (\mathfrak{F} \otimes \Lambda^{*r}(\mathfrak{X}))$.

Now, we say a pair (Γ, ψ) of a connection Γ of $\mathfrak{F}$ and a cross-section ψ of $\mathfrak{F} \otimes T^*(\mathfrak{X})$ define an *affine connection* of $\mathfrak{F}$. Then the affine connection

$$h\# (\Gamma, \psi) \equiv (h\# \Gamma, (h\ominus \otimes f^*) \psi) \tag{1.22}$$

of $\mathfrak{F}'$ is also called the *induce affine connection* from (Γ, ψ_Γ) by h and we denote $h\#$ by $f^\diamond$ when h is the induced bundle map of f.

Lastly, for a vector bundle $\mathfrak{F}$, we define its dual vector bundle $\mathfrak{F}^* = \{\mathfrak{Z}', \mathfrak{X}, \pi', \mathfrak{Y}'\}$, $\mathfrak{Y}' = \mathfrak{Y}^*$ (the dual vector space of $\mathfrak{Y}$), by its coordinate transformation $g'_{\mu\lambda}: U_\lambda \cap U_\mu \rightarrow L(\mathfrak{Y}^*) \approx GL(m)$

$$g'_{\mu\lambda}(x) = (g_{\mu\lambda}(x))^*. \tag{1.23}$$

Its coordinate transformations $\varphi'_\lambda: U_\lambda \times \mathfrak{Y}^* \rightarrow \pi'^{-1}(U_\lambda)$ are given by $\varphi'_{\lambda,x} = (\varphi_{\lambda,x}^{-1})^*: \mathfrak{Y}^* \rightarrow (\mathfrak{Y}_x)^*$, $\varphi'_{\lambda,x}(y') = \varphi'_\lambda(x, y')$. Putting $(y^1, \cdots, y^m)$ the dual base of $(y_1, \cdots, y_m)$, $\varphi'_\lambda(x, y^\alpha) = z_{(\lambda)}{}^\alpha$, then $(z_{(\lambda)}{}^1, \cdots z_{(\lambda)}{}^m)$ is a local field of dual base of $(z_{(\lambda)1}, \cdots z_{(\lambda)m})$. For a connection Γ of $\mathfrak{F}$ given by (1.8) — (1.10), we can define a connection Γ^* of $\mathfrak{F}^*$ by

$$Dz_{(\lambda)}{}^\beta = -z_{(\lambda)}{}^\alpha \otimes \omega_{(\lambda)}{}^\beta_\alpha. \tag{1.24}$$

Then we can naturally define connections by means of Γ and Γ^* for various product bundles of $\mathfrak{F}$ and $\mathfrak{F}^*$. But we denote them by the same symbol Γ.

§ 2. **Preliminary (II).**

Let $\mathfrak{X}$ be an n-dimensional differentiable manifold and consider its tangent bundle $\{T(\mathfrak{X}), \mathfrak{X}, \tau, R^n\}$, simply denote this by $T(\mathfrak{X})$, where R^n is the n dimensional coordinate space and is regarded as a vector space. For any coordinate neighborhood (U, u) of $\mathfrak{X}$ where $u: U \rightarrow R^n$, $u(x) = (u^1(x), \cdots, u^n(x))$, the coordinate function $\varphi_U: U \times R^n \rightarrow \tau^{-1}(U)$ of $T(\mathfrak{X})$ is given by

$$\varphi_U(x, \tilde{e}_i) = \frac{\partial}{\partial u^i}(x), \tag{2.1}$$

where $\tilde{e}_i = (\delta^1_i, \cdots, \delta^n_i)$. Taking another coordinate neighborhood (V, v), $U \cap V \neq \emptyset$, we have the coordinate transformation of $T(\mathfrak{X})$ given by

$$g_{VU} = \varphi_V^{-1} \cdot \varphi_U = \left(\frac{\partial v^j}{\partial u^i}\right): \quad U \cap V \rightarrow GL(n). \tag{2.2}$$

Analogously, the dual vector bundle $\{T^*(\mathfrak{X}), \mathfrak{X}\}$ of $\{T(\mathfrak{X}), \mathfrak{X}\}$ has its coordinate function for (U, u) defined by

$$\varphi_U(x, \tilde{e}^i) = du^i(x) \tag{2.1'}$$

where $\{\tilde{e}^i\}$ is the dual base of $\{\tilde{e}_i\}$. We denote this vector bundle simply by $T^*(\mathfrak{X})$.

Let $\{\mathfrak{B}, \mathfrak{X}, \pi, GL(n)\}$ be the associated principal fibre bundle of $T(\mathfrak{X})$, then $\mathfrak{B}$ is, as is well known, the space of all admissible maps of

$T(\mathfrak{X})$. For (U, u), the coordinate function $\tilde{\varphi}_U$ of the fibre bundle is given by

$$\tilde{\varphi}_U(x, g) = \left(a_1^j(g) \frac{\partial}{\partial u^j}(x), \cdots, a_n^j(g) \frac{\partial}{\partial u^j}(x)\right). \tag{2.3}$$

According to § 1, for any $b \in \mathfrak{B}$, $b = \tilde{\varphi}_U(x, g)$, we have

$$\mathrm{e}_i(b) = a_i^j(g) \frac{\partial}{\partial u^j}(x) \tag{2.4}$$

which become a base (frame) of $T(\mathfrak{X})$ at $\pi(b)$. Let $\pi^{\diamond}\{T(\mathfrak{X}), \mathfrak{X}, \tau, R^n\} = \{\mathfrak{B}_n, \mathfrak{B}, \chi, R^n\}$ and $\nu : \mathfrak{B}_n \to T(\mathfrak{X})$ be its induced bundle map from π. Then, by means of (1.11), we determine the natural cross sections $\mathfrak{e}_i : \mathfrak{B} \to \mathfrak{B}_n$, $\chi \cdot \mathfrak{e}_i = 1$ by the equations

$$\nu(\mathfrak{e}_i(b)) = \mathrm{e}_i(b). \tag{2.5}$$

Making use of these concepts, we shall consider the following fibre bundles. Firstly, putting $\{T(\mathfrak{X}) \boxtimes T(\mathfrak{X}), \mathfrak{X}\} = \{T(\mathfrak{X}), \mathfrak{X}\} \times \{T(\mathfrak{X}), \mathfrak{X}\}$, define the maps

$$\tilde{\tau}, \ \tau_1 : \ T(\mathfrak{X}) \boxtimes T(\mathfrak{X}) \to T(\mathfrak{X}) \tag{2.6}$$

by $\tilde{\tau}(w, y) = y$, $\tau_1(w, y) = w$, $w, y \in T_x(\mathfrak{X})$, then we see easily that $\{T(\mathfrak{X}) \boxtimes T(\mathfrak{X}), T(\mathfrak{X}), \tilde{\tau}, R^n\} = \tau^{\diamond}\{T(\mathfrak{X}), \mathfrak{X}, \tau, R^n\}$ and τ_1 is its induced bundle map from τ. As is stated in § 1, the associated principal fibre bundle $\{\widetilde{\mathfrak{B}}, T(\mathfrak{X}), \tilde{\pi}, GL(n)\}$ of $\{T(\mathfrak{X}) \boxtimes T(\mathfrak{X}), T(\mathfrak{X})\}$ is the induced bundle $\tau^{\diamond}\{\mathfrak{B}, \mathfrak{X}, \pi, GL(n)\}$. Let its induced bundle map be

$$\tau_p : \widetilde{\mathfrak{B}} \to \mathfrak{B}. \tag{2.7}$$

If for any $\tilde{b} \in \widetilde{\mathfrak{B}}$, we put $\tau_p(\tilde{b}) = b$, $\tilde{\pi}(\tilde{b}) = y$, we may consider $\tilde{b} = (b, y)$. Hence, we can easily prove that

$$\{\widetilde{\mathfrak{B}}, \mathfrak{X}, \tau \cdot \tilde{\pi}, GL(n) \times R^n\} = \{\mathfrak{B}, \mathfrak{X}, \pi, GL(n)\} \times \{T(\mathfrak{X}), \mathfrak{X}, \tau, R_n\}. \tag{2.8}$$

On the other hand, for any $\tilde{b} \in \widetilde{\mathfrak{B}}$, y is written as

$$y = \tilde{\pi}(\tilde{b}) = y^i \mathrm{e}_i(b),$$

from which we obtain n natural functions

$$y^i : \widetilde{\mathfrak{B}} \to R. \tag{2.9}$$

Making use of the n functions, we have the relation $\widetilde{\mathfrak{B}} = \mathfrak{B} \times R^n$. Let $\tilde{\pi}^{\diamond}\{T(\mathfrak{X}) \boxtimes T(\mathfrak{X}), T(\mathfrak{X}), \tilde{\tau}, R^n\} = \{\widetilde{\mathfrak{B}}_n, \widetilde{\mathfrak{B}}, \widetilde{\chi}, R^n\}$, then its natural cross sections $\mathfrak{z}_i : \widetilde{\mathfrak{B}} \to \widetilde{\mathfrak{B}}_n$, $\widetilde{\chi} \cdot \mathfrak{z}_i = 1$ are given by

$$\tilde{\nu}(\mathfrak{z}_i(\tilde{b})) = (\mathrm{e}_i(\tau_p(\tilde{b})), \ \tilde{\pi}(\tilde{b})) \tag{2.10}$$

where $\tilde{\nu}$ is the induced bundle map from π of $\tilde{\mathfrak{B}}_n$ into $T(\mathfrak{X}) \boxtimes T(\mathfrak{X})$. As is easily seen, $\{\tilde{\mathfrak{B}}_n, \tilde{\mathfrak{B}}, \tilde{\chi}, R_n\}$ is the induced vector bundle of $\{\mathfrak{B}_n, \mathfrak{B}, \chi, R^n\}$ from the map $\tau_p: \tilde{\mathfrak{B}} \to \mathfrak{B}$, its induced bundle map $\tau_n: \tilde{\mathfrak{B}}_n \to \mathfrak{B}_n$ is given by

$$\tau_n(\mathfrak{z}_i(\tilde{b})) = \mathfrak{e}_i(\tau_p(\tilde{b})). \tag{2.11}$$

By (2.5), (2.10) and (2.11), we have

$$(\nu \cdot \tau_n \cdot \mathfrak{z}_i)(\tilde{b}) = \nu(\mathfrak{e}_i(\tau_p(\tilde{b}))) = \mathrm{e}_i(\tau_p(b)) = (\tau_1 \cdot \tilde{\nu} \cdot \mathfrak{z}_i)(\tilde{b})$$

hence

$$\nu \cdot \tau_n = \tau_1 \cdot \tilde{\nu} \tag{2.12}$$

Thus we obtain the following commutative diagram

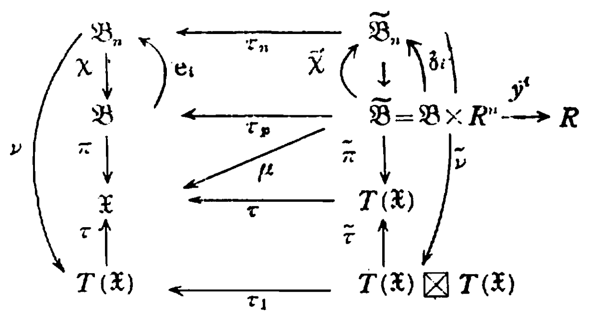

Diagram 1.

Let R^+ be the group of positive real numbers with respect to multiplication. For any $k \in R$, we define an operator λ_k on $T(\mathfrak{X})$, $T(\mathfrak{X}) \boxtimes T(\mathfrak{X})$, $\tilde{\mathfrak{B}}$ such that

$$\begin{aligned} &\lambda_k(y) = ky, && y \in T_x(\mathfrak{X}) \\ &\lambda_k(w, y) = (w, ky), && (w, y) \in T_x(\mathfrak{X}) \times T_x(\mathfrak{X}) \\ &\lambda_k(\tilde{b}) = \lambda_k(\tau_p(\tilde{b}), \tilde{\pi}(\tilde{b})) = (\tau_p(\tilde{b}), k\tilde{\pi}(\tilde{b})) && \end{aligned} \tag{2.13}$$

and in some place we denote this simply by k. Clearly, R^+ is regarded as a transformation group opreating on $T(\mathfrak{X})$, $T(\mathfrak{X}) \boxtimes T(\mathfrak{X})$, $\tilde{\mathfrak{B}}$. We may identify $\mathfrak{X}$, $T(\mathfrak{X})$, $\mathfrak{B}$ with the images of $T(\mathfrak{X})$, $T(\mathfrak{X}) \boxtimes T(\mathfrak{X})$, $\tilde{\mathfrak{B}}$ under the map λ_0 respectively. Classifying the spaces $T_0(\mathfrak{X}) = T(\mathfrak{X}) - \mathfrak{X}$, $T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X})$, $\tilde{\mathfrak{B}}_0 = \tilde{\mathfrak{B}} - \mathfrak{B}$ by the group R^+, we obtain the spaces $\mathfrak{S} = \mathfrak{S}(\mathfrak{X})$, $\mathfrak{Z}$, $\mathfrak{P}$ which are differentiable manifolds. Let ρ, ρ_1, ρ_p be the natural map

$$\rho: T_0(\mathfrak{X}) \to \mathfrak{S}, \quad \rho_1: T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X}) \to \mathfrak{Z}, \quad \rho_p: \mathfrak{B} \to \mathfrak{P}. \tag{2.14}$$

Clearly $\mathfrak{S}$ is the space of of tangent directions of $\mathfrak{X}$. Since

$$\lambda_k \cdot \tilde{\pi} = \tilde{\pi} \cdot \lambda_k, \quad \lambda_k \cdot \tilde{\tau} = \tilde{\tau} \cdot \lambda_k \tag{2.15}$$

we obtain naturally two vector bundles $\{\mathfrak{Z}, \mathfrak{S}, \tau_0, R^n\}$ and its associated principal fibre bundle $\{\mathfrak{P}, \mathfrak{Z}, \pi_0, GL(n)\}$ by virtue of ρ, ρ_1, ρ_p. Furthermore, let λ_k operate also on $\widetilde{\mathfrak{B}}_n$ by

$$\lambda_k(\mathfrak{z}_i(\tilde{b})) = \mathfrak{z}_i(\lambda_k(\tilde{b})), \tag{2.16}$$

then since we have $\lambda_k \cdot \tilde{\chi} = \tilde{\chi} \cdot \lambda_k$, we can classify $\widetilde{\mathfrak{B}}_n - \mathfrak{B}_n$ by the group R^+ and put $\mathfrak{P}_n = (\widetilde{\mathfrak{B}}_n - \mathfrak{B}_n)/R^+$, regarding $\mathfrak{B}_n$ as a submanifold of $\widetilde{\mathfrak{B}}_n$ by $\mathfrak{B}_n \approx \tilde{\chi}^{-1}(\mathfrak{B})$. Let be the natural map

$$\rho_n : \widetilde{\mathfrak{B}}_n - \mathfrak{B}_n \to \mathfrak{P}_n, \tag{2.17}$$

then by (2.15), (2.10) and (2.13), we have

$$\begin{aligned} \tilde{\nu}(\lambda_k(\mathfrak{z}_i(\tilde{b}))) &= \tilde{\nu}(\mathfrak{z}_i(\lambda_k(\tilde{b}))) \\ &= (\mathrm{e}_i(\tau_p(\lambda_k(\tilde{b}))), \tilde{\pi}(\lambda_k(\tilde{b}))) \\ &= (\mathrm{e}_i(\tau_p(\tilde{b})), \lambda_k(\tilde{\pi}(\tilde{b})) = \lambda_k(\tilde{\nu}(\mathfrak{z}_i(\tilde{b}))), \end{aligned}$$

that is

$$\tilde{\nu} \cdot \lambda_k = \lambda_k \cdot \tilde{\nu}. \tag{2.18}$$

Hence the vector bundle $\{\mathfrak{P}_n, \mathfrak{P}, \chi_0, R^n\}$ derived from $\{\widetilde{\mathfrak{B}}_n, \widetilde{\mathfrak{B}}, \tilde{\chi}, R^n\}$ by $/R^+$ is the induced bundle of $\{\mathfrak{Z}, \mathfrak{S}, \tau_0, R^n\}$ by the map $\pi_0 : \mathfrak{P} \to \mathfrak{S}$ whose induced bundle map $\nu_0 : \mathfrak{P}_n \to \mathfrak{Z}$ is obtained from $\tilde{\nu}$. We can easily see from (2.16) that the natural cross sections $\mathfrak{w}_i$ of $\{\mathfrak{P}_n, \mathfrak{P}\}$ are defined by

$$\mathfrak{w}_i(\rho_p(\tilde{b})) = \rho_n(\mathfrak{z}_i(\tilde{b})), \quad \tilde{b} \in \widetilde{\mathfrak{B}}_0. \tag{2.19}$$

We have also the relation on $\widetilde{\mathfrak{B}}_n - \mathfrak{B}_n$

$$\nu_0 \cdot \rho_n = \rho_1 \cdot \tilde{\nu}_0, \tag{2.20}$$

where $\tilde{\nu}_0 = \tilde{\nu} \mid \widetilde{\mathfrak{B}}_n - \mathfrak{B}_n$. In the following we denote by $\bar{\tau}, \bar{\tau}_1, \bar{\tau}_p, \bar{\tau}_n$ the maps $\tau, \tau_1, \tau_p, \tau_n$ contracting on $T_0(\mathfrak{X})$, $T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X})$, $\widetilde{\mathfrak{B}}_0$, $\widetilde{\mathfrak{B}}_n - \mathfrak{B}_n$ respectively.

From these relations, we can naturally define the maps $\sigma : \mathfrak{S} \to \mathfrak{X}$, $\sigma_1 : \mathfrak{Z} \to T(\mathfrak{X})$, $\sigma_p : \mathfrak{P} \to \mathfrak{B}$ and $\sigma_n : \mathfrak{P}_n \to \mathfrak{B}_n$ such that

$$\bar{\tau} = \sigma \cdot \rho, \quad \bar{\tau}_1 = \sigma_1 \cdot \rho_1, \quad \bar{\tau}_p = \sigma_p \cdot \rho_p, \quad \bar{\tau}_n = \sigma_n \cdot \rho_n \tag{2.21}$$

We see easily that $\{\mathfrak{Z}, \mathfrak{S}\} = \sigma^{\diamond}\{T(\mathfrak{X}), \mathfrak{X}\}$, $\{\mathfrak{P}, \mathfrak{S}\} = \sigma^{\diamond}\{\mathfrak{B}, \mathfrak{X}\}$, $\{\mathfrak{P}_n,$

$\mathfrak{P}\} = \sigma_p^{\diamond}\{\mathfrak{B}_n, \mathfrak{B}\}$ and their induced bundle maps are σ_1, σ_p, σ_n and the relations

$$\tau\cdot\sigma_1=\sigma\cdot\tau_0,\quad \pi\cdot\sigma_p=\sigma\cdot\pi_0,\quad \chi\cdot\sigma_n=\sigma_p\cdot\chi_0 \tag{2. 22}$$

hold. Furthermore, since by (2. 18) we have

$$\sigma_n(\mathfrak{w}_i(\rho_p(\tilde b)))=(\sigma_n\cdot\rho_n)(\mathfrak{z}_i(\tilde b))=\bar\tau_n(\mathfrak{z}_i(\tilde b))=\mathfrak{e}_i(\bar\tau_p(\tilde b))$$
$$=\mathfrak{e}_i(\sigma_p(\rho_p(\tilde b))),$$

we get

$$\sigma_n\cdot\mathfrak{w}_i=\mathfrak{e}_i\cdot\sigma_p \tag{2. 23}$$

Lastly, by (2. 19), (2. 21), (2. 20) we have

$$\nu\cdot\sigma_n\cdot\mathfrak{w}_i(\rho_p(\tilde b))=\nu\cdot\sigma_n\cdot\rho_n(\mathfrak{z}_i(\tilde b))=\nu\cdot\bar\tau_n(\mathfrak{z}_i(\tilde b))$$
$$=\bar\tau_1\cdot\tilde\nu_0(\mathfrak{z}_i(\tilde b))=\sigma_1\cdot\rho_1\cdot\tilde\nu^0(\mathfrak{z}_i(\tilde b))$$
$$=\sigma_1\cdot\nu_0\cdot\rho_n(\mathfrak{z}_i(\tilde b))=\sigma_1\cdot\nu_0\cdot\mathfrak{w}_i(\rho_p(\tilde b)),$$

hence

$$\nu\cdot\sigma_n=\sigma_1\cdot\nu_0 \tag{2. 24}$$

Thus we obtain the following commutative diagram

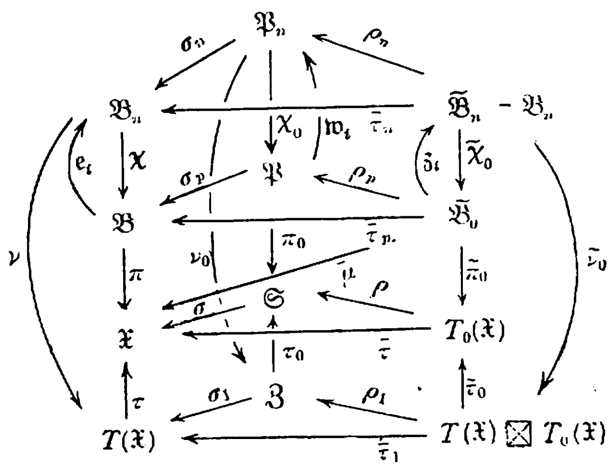

Diagram 2

where we put $\bar\mu=\mu\mid\tilde{\mathfrak{B}}_0$. In the following, we shall use the notations τ, τ_1, τ_p, τ_n, μ, $\tilde\tau$, $\tilde\pi$, $\tilde\chi$, $\tilde\nu$ in place of $\bar\tau$, $\bar\tau_1$, $\bar\tau_p$, $\bar\tau_n$, $\bar\mu$, $\tilde\tau_0$, $\tilde\pi_0$, $\tilde\chi_0$, $\tilde\nu_0$ respectively in cases that there will be no confusion.

§ 3. Connection Γ on the vector bundle $\{\mathfrak{Z}, \mathfrak{S}\}$.

For a given connection Γ of the vector bundle $\{\mathfrak{Z}, \mathfrak{S}\}$, we investigate the induced connection $\rho^{\diamond}\Gamma$ of the vector bundle $\mathfrak{F} = \{T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X}), T_0(\mathfrak{X})\}$. For a coordinate neighborhood (U, u), we define a natural local coordinate system (u^i, ξ^i) on $\tau^{-1}(U)$ such that for $y \in \tau^{-1}(U)$

$$u^i(x), \qquad y = \xi^i \frac{\partial}{\partial u_i}(x), \qquad x = \tau(y) \tag{3.1}$$

and we call them the *canonical local coordinates* of $T(\mathfrak{X})$ for (U, u). Accordingly, the differential forms $\omega_i^j = \omega_{U i}^{\ j}$ on $\tau^{-1}(U)$ of $\widetilde{\Gamma} = \rho^{\diamond}\Gamma$ are written as

$$\omega_i^j = \Gamma_{ik}^j(u, \xi) du^k + C_{ik}^j(u, \xi)\, d\xi^k \tag{3.2}$$

and in $U \cap V \neq \emptyset$ we have by (1.10) and (2.1)

$$\omega_{U i}^{\ j} = \frac{\partial u^j}{\partial v^k}\left(d\frac{\partial v^k}{\partial u^i} + \omega_{V h}^{\ k}\frac{\partial v^h}{\partial u^i}\right) \tag{3.3}$$

Since for another (V, v) we have

$$y = \eta^i \frac{\partial}{\partial v^i}(x) = \eta^i\left(\frac{\partial u^k}{\partial v^i}\frac{\partial}{\partial u^k}\right)(x),$$

hence for $U \cap V \neq \emptyset$ we get

$$\xi^k = \frac{\partial u^k}{\partial v^h}\eta^h, \quad \text{or} \quad \eta^h = \frac{\partial v^h}{\partial u^k}\xi^k. \tag{3.4}$$

Accordingly, the equations (3.3) are rewritten as

$$\Gamma_{U ik}^{\ j} = \frac{\partial u^j}{\partial v^m}\left\{\frac{\partial^2 v^m}{\partial u^k \partial u^i} + \Gamma_{V ts}^{\ m}\frac{\partial v^t}{\partial u^i}\frac{\partial v^s}{\partial u^k} + C_{V ts}^{\ m}\frac{\partial v^t}{\partial u^i}\frac{\partial^2 v^s}{\partial u^k \partial u^h}\xi^h\right\} \tag{3.5}$$

$$C_{U ik}^{\ j} = \frac{\partial u^j}{\partial v^m} C_{V ts}^{\ m}\frac{\partial v^t}{\partial u^i}\frac{\partial v^s}{\partial u^k} \tag{3.6}$$

According to § 1, the condition that $\widetilde{\Gamma}$ is a induced connection from a connection of $\{\mathfrak{Z}, \mathfrak{S}\}$ by ρ is that ω_i^j are induced from differential forms on $\sigma^{-1}(U)$. We may assume $\xi^n \neq 0$ on $\tau^{-1}(U)$, hence putting $\xi^i = t^i \xi^n$ we get

$$\omega_i^j = \Gamma_{ik}^j(u, t\xi^n)\, du^k + C_{ik}^j(u, t\xi^n)\xi^n\, dt^k + C_{ik}^j(u, t\xi^n)t^k d\xi^n.$$

Since $(u^1, \cdots, u^n, t^1, \cdots, t^{n-1})$ are regarded as coordinates on $\sigma^{-1}(U)$, in order that the forms depend only the coordinates, it is necessary and sufficient that

$$\Gamma_{ik}^j(u, t\xi) = \Gamma_{ik}^j(u, \xi), \quad C_{ik}^j(u, t\xi) = t^{-1}C_{ik}^j(u, \xi), \quad t > 0 \tag{3.7}$$

$$C_{ik}^{j}(u, \xi)\xi^{k} = 0 \tag{3.8}$$

considering other cases.

Now, the differential forms θ_i^j on $\widetilde{\mathfrak{B}}_0$ for the connection $\widetilde{\Gamma}$ are written on $\tilde{\mu}^{-1}(U)$ by means of (1.13) as

$$\theta_i^j = b_k^j(da_i^k + \omega_h^k a_i^h) = b_k^j(da_i^k + \Gamma_{hl}^{k} a_i^h du^l + C_{hl}^k a_i^h d\xi^l) \tag{3.9}$$

regarding u^j, ξ^j and a_i^j as local coordinates of $\widetilde{\mathfrak{B}}$. As is well known, $\mathfrak{B}$ has particular differential forms θ^j which is written in $\pi^{-1}(U)$

$$\theta^j = b_i^j du^i \tag{3.10}$$

We shall denote also $(\tau_p)^* \theta^j$ by θ^j. The tensor field dp of type (1, 1) which represents the identical transformation of $T(\mathfrak{X})$ is written locally as

$$dp = \frac{\partial}{\partial u^i} \otimes du^i, \tag{3.11}$$

from which we get the tensor field $\pi^\circ\, dp\, (\equiv (\pi^\circ \otimes \pi^*)\, dp) = d\,\mathfrak{p}$ of the vector bundle $\{\mathfrak{B}^n, \mathfrak{B}\}$ and

$$(\tau_p)^\circ \cdot \pi^\circ(dp) = \tilde{\pi}^\circ \cdot \tau^\circ(dp) = \mathfrak{z}_i \otimes \theta^i. \tag{3.12}$$

We denote also $\tau^\circ(dp)$, $\tau_p^\circ(d\mathfrak{p})$ by dp, $d\mathfrak{p}$ respectively. Furthermore, using the functions $y^i : \widetilde{\mathfrak{B}} \to R$, we define a perticular vector field $\mathfrak{z}$ of $\{\widetilde{\mathfrak{B}}_n, \widetilde{\mathfrak{B}}_p\}$ by

$$\mathfrak{z} = y^j\, \mathfrak{z}_j.$$

On the other hand, we define a particular vector field $\mathfrak{y}$ of $\mathfrak{F}$ by $\mathfrak{y}(y) = (y, y)$. Then we get easily

$$\mathfrak{z} = \tilde{\pi}^\circ\, \mathfrak{y}. \tag{3.13}$$

For $\mathfrak{y}$ and $\mathfrak{z}$, we have the relations analogous to (3.10)

$$y^j = b_i^j \xi^i, \quad \tilde{\pi}(\tilde{b}) = y = \xi^i\, \partial/\partial u_i. \tag{3.14}$$

(3.6) shows that C_{ik}^j are the components of a tensor field $\mathfrak{C}$ of the type (1, 2) with respect to (u^j, ξ^j). Let us put

$$\tilde{\pi}^\circ \mathfrak{C} = \widetilde{C}_{ik}^j\, \mathfrak{z}^i \otimes \mathfrak{z}_j \otimes \mathfrak{z}^k \tag{3.15.}$$

where $\mathfrak{z}^i$ are the dual vector fields to $\mathfrak{z}_i$. Then, the coefficients $\widetilde{C}_{ik}^j$ are written on $\tilde{\mu}^{-1}(U)$ as

$$\widetilde{C}_{ik}^j = b_l^j\, C_{lm}^l\, a_i^l\, a_k^m. \tag{3.16}$$

The differential forms $\{\theta^j, \theta_i^j\}$ define an *affine connection* on the vector bundle $\tilde{\pi}^{\diamond}\mathfrak{F}$ which will be investigated in the following. The connection Γ of $\{\mathfrak{Z}, \mathfrak{S}\}$ determines an affine connection in the sense, that is, for any curve $\bar{C}$ of $T_0(\mathfrak{X})$ given by a map $\bar{f}: I \to T_0(\mathfrak{X})$ which is locally represented by means of u^i, ξ^i, we consider the system of differential equations on the space of affine frames $R^n \times GL(n)$

$$\frac{dp'}{dt} = e'_i \frac{d}{dt}(u^i \cdot \bar{f}), \quad \frac{de'_i}{dt} = e'_j \frac{f^*\omega_i^j}{dt}$$

in connection with (3.11) and (1.8) and look for the solutions of the equations. As easily seen, the solution is uniquely determined save for affine motions. Then we call the figure $(p'(t),\ e'_i(t)\ u^i(f(t)))$ in R^n a developement of $\bar{C}$ and $p'(t)$ a development of the curve $f = \tau \cdot \bar{f}: I \to \mathfrak{X}$ with respect to $\bar{C}$. The appropriateness of this concept will be showed in the following paragraphs.

§ 4. The homorphism Φ derived from Γ.

The covariant differentiation of $\mathfrak{y}$ with respect to $\tilde{\Gamma}$ is given for (U, u) by

$$D\mathfrak{y} = \frac{\partial}{\partial u^j} \otimes D\xi^j, \quad D\xi^j = d\xi^j + \omega_i^j \xi^i \tag{4.1}$$

and we have

$$D\xi^j = (\delta_k^j + C_{ik}^j \xi^i)\, d\xi^k + \Gamma_{ik}^j \xi^i du^k.$$

From the right hand side, we can define a tensor field Φ of $\mathfrak{F}$ of the type (1.1) whose component with respect $\partial/\partial u^i$ are given by

$$\Phi_i^j = \delta_i^j + C_{ki}^j \xi^k \tag{4.2}$$

and which have the property as a homomorphism of $\mathfrak{F}$ that $\tilde{\tau}_0 \cdot \Phi = 1 \cdot \tilde{\tau}_0$ Furthermore, we put $\tilde{\Phi} = \tilde{\pi}^{\circ}\Phi = \tilde{\Phi}_i^j \mathfrak{z}^i \otimes \mathfrak{z}_j$, then we have

$$\tilde{\Phi}_i^j = b_k^j \Phi_h^k a_i^h \tag{4.3}$$

and the homomorphism of $\tilde{\pi}^{\diamond}\mathfrak{F}$ corresponding to this have the analogous relation

$$\tilde{\chi}_0 \cdot \tilde{\Phi} = 1 \cdot \tilde{\chi}_0 \tag{4.4}$$

We call $\Phi = \Phi_\Gamma$, $\tilde{\Phi} = \tilde{\Phi}_\Gamma$, the *derived homomorphisms from* $\tilde{\Gamma}$. When

Φ is an isomorphism, we say that $\widetilde{\Gamma}$ is *regular*. Since $D\xi^j$ is written as

$$D\xi^j = \Phi^j_k d\xi^k + \Gamma^j_{ik}\xi^i du^k,$$

we can make use of the $2n$ forms du^j, $D\xi^j$ in place of du^j, $d\xi^j$ for a regular connection $\widetilde{\Gamma}$.

In the following, we assume that $\widetilde{\Gamma}$ is regular and let M^j_i, $\widetilde{M}^j_i$ be the components of Φ^{-1}, $\widetilde{\Phi}^{-1}$ with respect to $\partial/\partial u^i$, $\mathfrak{z}_i$ respectively. Then we have

$$M^j_k\Phi^k_i = \Phi^j_k M^k_i = \delta^j_i, \quad \widetilde{M}^j_k\widetilde{\Phi}^k_i = \widetilde{\Phi}^j_k \widetilde{M}^k_i = \delta^j_i, \tag{4. 5}$$

it follows that by (4. 2)

$$M^j_i + M^j_k C^k_{li}\xi^l = \delta^j_i, \quad M^j_i + C^j_{lk}\xi^l M^k_i = \delta^j_i \tag{4. 6}$$

and the equations for $\widetilde{M}^j_i$ similar to (4. 6). Making use of these equations, we have

$$d\xi^j = M^j_i(D\xi^i - \Gamma^i_{lk}\xi^l du^k),$$

hence by (3. 2) we get on $\bar{\tau}^{-1}(U)$

$$\begin{aligned}\omega^j_i &= \Gamma^j_{ik} du^k + C^j_{ik} M^k_h (D\xi^h - \Gamma^h_{lm}\xi^l du^m)\\ &= (\Gamma^j_{ik} - C^j_{im} M^m_h \Gamma^h_{lk}\xi^l)\, du^k + C^j_{ih} M^h_k D\xi^k.\end{aligned}$$

If we put

$$\Gamma^{*j}_{ik} = \Gamma^j_{ik} - C^j_{im} M^m_h \Gamma^h_{lk}\xi^l, \tag{4. 7}$$

then ω^j_i are written as

$$\omega^j_i = \Gamma^{*j}_{ik} du^k + C^j_{ih} M^h_k D\xi^k. \tag{4. 8}$$

The equations above clearly hold good for any regular connection $\widetilde{\Gamma}$ of $\mathfrak{F}$. If $\widetilde{\Gamma} = \rho^{\diamond}\Gamma$, we obtain firstly from (3. 8) the equations

$$\Phi^j_i\xi^i = \xi^j + C^j_{ki}\xi^k\xi^i = \xi^j, \quad \widetilde{\Phi}^j_i y^i = y^j \tag{4. 9}$$

which show that $\Phi(\mathfrak{z}) = \mathfrak{z}$, $\widetilde{\Phi}(\mathfrak{z}) = \mathfrak{z}$. We have easily

$$M^j_i\xi^i = \xi^j, \quad \widetilde{M}^j_i y^i = y^j, \tag{4. 10}$$

hence from (4. 6) and (4. 2) it follows that

$$\begin{aligned}\Gamma^{*j}_{ik}\xi^i &= \Gamma^j_{ik}\xi^i - C^j_{im}\xi^i M^m_h \Gamma^h_{lk}\xi^l\\ &= \Gamma^j_{ik}\xi^i + (M^j_h - \delta^j_h)\,\Gamma^h_{lk}\xi^l = M^j_h \Gamma^h_{lk}\xi^l,\end{aligned}$$

that is

$$\Gamma^{*j}_{ik}\xi^i = M^j_h \Gamma^h_{ik}\xi^i. \tag{4.11}$$

Using this equations, (4. 7) and $D\xi^j$ are also written as

$$\Gamma^j_{ik} = \Gamma^{*j}_{ik} + C^j_{ih}\Gamma^{*h}_{lk}\xi^l, \tag{4.12}$$

$$D\xi^j = \Phi^j_k(d\xi^k + \Gamma^{*k}_{ih}\xi^i du^h) \tag{4.13}$$

On the other hand, the covariant differentiation of the field $\mathfrak{z}$ with respect to $\tilde{\pi}^\diamond\tilde{\Gamma}$ is given by

$$D\mathfrak{z} = \mathfrak{z}_j \otimes Dy^j, \quad Dy^j = dy^j + \theta^j_i y^i. \tag{4.14}$$

We may regard u^j, ξ^j, a^j_i as local coordinates on $\mu^{-1}(U)$ and call them the canonical local coordinates for (U, u). By means of (3. 14), we may also regard u^j, y^j, a^j_i as local coordinates, of which y^j are defined on all the space $\tilde{\mathfrak{B}}$. As it is easily seen from (3. 13), (3. 14), we have

$$Dy^j = b^j_i D\xi^i. \tag{4.15}$$

This can be represented on $\tilde{\mathfrak{B}}_0$ by the equation

$$D\mathfrak{z} = (\tilde{\pi}^\circ \otimes \tilde{\pi}^*)\, D\mathfrak{y}. \tag{4.16}$$

Now, it is clear that θ^j, $d\xi^j$, θ^j_i are linearly indepent and become a base of $T^*(\bar{\mu}^{-1}(U))$ at each point of $\bar{\mu}^{-1}(U)$. If Φ is isomorphism, the same fact holds good for θ^j, $D\xi^j$, θ^j_i. Hence by means of (4. 15), θ^j, Dy^j, θ^j_i form a base of $T^*(\bar{\mu}^{-1}(U))$. Thus we obtain

Proposition 4. 1. *If Γ is regular, the differential forms θ^j, Dy^j, θ^j_i on $\tilde{\mathfrak{B}}_0$ form a base of $T^*(\tilde{\mathfrak{B}}^0)$ at each point.*

§ 5 The basic horizontal tangent vector fields and basic vertical tangent vector fields.

We say any tangen vector X of $\tilde{\mathfrak{B}}_0$ is *horizontal* if $\langle X, \theta^i_j \rangle = 0$ and *vertical* if it is tangent to the fibre of $\{\tilde{\mathfrak{B}}^0, T_0(\mathfrak{X})\}$ through the origin of X. By means of (3. 10), (4. 15), the condition that X is vertical is that $\langle X, \theta^j \rangle = \langle X, Dy^j \rangle = 0$ when the connection is regular. Any tangent vector field X of $\tilde{\mathfrak{B}}_0$ which is every where horizontal or vertical is called a *basic horizontal* or *vertical tangent vector field* if its inner poducts with θ^j, Dy^j, θ^j_i are constant.

Let B_i, E_i, Q_i^j be the dual tangent vector fields of θ^j, Dy^j, θ_i^j, then B_i, E_i are basic horizontal and Q_i^j are basic vertical from the above definition. Let us write them by means of canonical local coordinates for (U, u). By (3.10), (4.11), (4.15), (3.9), (4.8), we have

$$\begin{aligned}\theta^j &= b_k^j du^k,\\ Dy^j &= b_l^j \Phi_m^l \Gamma^{*m}_{ik} \xi^i du^k + b_l^j \Phi_k^l d\xi^k,\\ \theta_i^j &= b_l^j \Gamma^l_{mk} a_i^m du^k + b_l^j C^l_{mk} a_i^m d\xi^k + b_k^j \delta_i^h da_h^k.\end{aligned}$$

Since $\partial/\partial u^i$, $\partial/\partial \xi^i$, $\partial/\partial a_j^i$ are locally dual to du^j, $d\xi_i^j$, da_i^j, making use of the inverse of the matrix of order (n^2+2n) consisting of the coefficients of the above equations and (4.12), we get

$$\begin{aligned}B_i &= a_i^k \partial/\partial u^k - \Gamma^{*k}_{lm} \xi^l a_i^m \partial/\partial \xi^k - \Gamma^{*k}_{lm} a_h^l a_i^m \partial/\partial a_h^k,\\ E_i &= \qquad\qquad M_i^k a_i^l \partial/\partial \xi^k - C^k_{lm} a_h^l M_s^m a_i^s \partial/\partial a_h^k,\\ Q_i^j &= \qquad\qquad\qquad\qquad \delta_h^j a_i^k \partial/\partial a_h^k\end{aligned}$$

that is

$$B_i = a_i^k\left(\frac{\partial}{\partial u^k} - \Gamma^{*h}_{lk}\xi^l \frac{\partial}{\partial \xi^h}\right) - \Gamma^{*k}_{lm} a_h^l a_i^m \frac{\partial}{\partial a_h^k}, \tag{5.1}$$

$$E_i = a_i^l M_l^k\left(\frac{\partial}{\partial \xi^k} - C^h_{sk} a_m^s \frac{\partial}{\partial a_m^h}\right), \tag{5.2}$$

$$Q_i^j = a_i^k \frac{\partial}{\partial a_j^k}. \tag{5.3}$$

We shall give a geometrical significance of the above equations for the connection $\tilde{\Gamma}$. For any point $\tilde{b} \in \tilde{\mu}^{-1}(U)$, putting $x = \mu(\tilde{b})$, $b = \tau_p(\tilde{b})$, $\tilde{\pi}(b) = y$, we get by (2.4)

$$\mathrm{e}_i(b) = a_i^j \frac{\partial}{\partial u^j}(x), \qquad y = \xi_0^j \frac{\partial}{\partial u^j}(x).$$

Let C be a curve in $\mathfrak{X}$ given by $f : I \to \mathfrak{X}$, $f(0) = x$ whose tangent vector at x is $\mathrm{e}_i(b)$, where I denotes an open interval containg 0. If we parallely displace the tangent vector y of $\mathfrak{X}$ at x with respect to $\tilde{\Gamma}$, this is done by solving the differential equations

$$\frac{dv^j}{dt} + \Gamma^j_{hk}(f(t), v) v^h \frac{d(u^k \cdot f)}{dt} + C^j_{hk}(f(t), v) v^h \frac{dv^k}{dt} = 0$$

under the initial condition $v^j(0) = \xi^j$. By (4.13), we have

$$\frac{dv^j}{dt} + \Gamma^{*j}_{hk}(f(t), v) v^h \frac{d(u^k \cdot f)}{dt} = 0.$$

Accordingly, the curve $\bar{C}$ in $T(\mathfrak{X})$ given by $\bar{f}: I \to T(\mathfrak{X})$ such that $\tau \cdot \bar{f} = f$, $\xi^j \cdot \bar{f} = v^j$ have the tangent vector at y

$$a_i^k \frac{\partial}{\partial u^k} - \Gamma^{*k}_{ih}(x, y)\, \xi^l a_i^h \frac{\partial}{\partial \xi^k}$$

since $\left[\dfrac{d(u^k \cdot f)}{dt}\right]_{t=0} = a_i^k$.

Nextly, let $\bar{C}$ be a curve in $T_x(\mathfrak{X})$ given by $\bar{f}: I \to T_x(\mathfrak{X})$, $\bar{f}(0) = y$ and

$$\left[\frac{D(\xi^j \cdot \bar{f})}{dt}\right]_{t=0} = a_i^j,$$

then the tangent vector at y is

$$a_i^h M_h^k \frac{\partial}{\partial \xi^k}$$

since we have analogously

$$\left[\frac{d(\xi^j \cdot \bar{f})}{dt}\right]_{t=0} = \left[M_k^j \frac{D(\xi^k \cdot \bar{f})}{dt}\right]_{t=0} = a_i^k M_k^j.$$

Lastly, for (5. 3) it is clear by means of its form that Q_i^j is the image of the tangent vector $\partial/\partial a_j^i$ at the unit element e of $GL(n)$ by $\tilde{b}$ as an admissible map $\tilde{b}: GL(n) \to \tilde{\pi}^{-1}(y)$ of $\{\tilde{\mathfrak{B}}_0, T_0(\mathfrak{X})\}$. Thus, we obtain

Proposition 5.1. *The basic tangent vector fields B_i, E_i, Q_i^j on $\tilde{\mathfrak{B}}_0$ for a regular connection $\tilde{\Gamma}$ have the following geometrical significances as follow. Let $\tilde{b} \in \tilde{\mathfrak{B}}_0$, $b = \tau_p(\tilde{b})$, $y = \tilde{\pi}(\tilde{b})$, $x = \mu(\tilde{b})$.* i) *If we take any curve through x with its tangent vector $e_i(b)$ at x and define a curve in $T_0(\mathfrak{X})$ by parallel displaing y along the curve with respect to $\tilde{\Gamma}$, then the tangent vector of the lifted curve at y is $\tilde{\pi}_*(B_i(\tilde{b}))$.* ii) *If we take a curve in $T_x(\mathfrak{X})$ through y such that the tangent vector at the beginning point of the development of the curve is the image of $e_i(b)$, then the tangent vector of the curve at y is $\tilde{\pi}_*(E_i(\tilde{b}))$.* iii) *$Q_i^j(b)$ is the image of $\partial/\partial a_j^i$ at e by the admissible map $\tilde{b}: GL(n) \to \tilde{\pi}^{-1}(y)$.*

Lastly, we define other tangent vector fields which will be frequently utilized in the following. By means of $y^1, \cdots, y^n$, we have the n tangent vector $\partial/\partial y^i$ which are independent of connections. Since we have $y^j = b_i^j \xi^i$ or $\xi^i = a_j^i y^j$ by means of canonical local coordinates, we get

$$\frac{\partial}{\partial y^i} = a_i^k \frac{\partial}{\partial \xi^k}. \tag{5.4}$$

We define horizontal tangent vector field Y_i on $\widetilde{\mathfrak{B}}_0$ by

$$\tilde{\pi}_*(Y_i) = \tilde{\pi}_*\left(\frac{\partial}{\partial y^i}\right)$$

Since we have locally

$$\tilde{\pi}_* E_i = a_i^l M_l^k \frac{\partial}{\partial \xi^k} = a_i^l M_l^h b_h^m a_m^k \frac{\partial}{\partial \xi^k} = \widetilde{M}_i^m \tilde{\pi}_*(Y_m),$$

it follows that

$$Y_i = \widetilde{\Phi}_i^k E_k \quad \text{or} \quad E_k = \widetilde{M}_k^i Y_i, \tag{5.5}$$

hence by (5.2), (4.5)

$$Y_i = a_i^k\left(\frac{\partial}{\partial \xi^k} - C_{sk}^h a_m^s \frac{\partial}{\partial a_m^h}\right) = \frac{\partial}{\partial y^i} - \widetilde{C}\,{}^k_{hi} Q_k^h. \tag{5.6}$$

§ 6. **Torsion forms, curvature forms and developments of curves.** By means of (1.18), the covariant differentiations of the cross section dp of $\mathfrak{F} \otimes T^*(\mathfrak{X})$ is written on $\tilde{\tau}^{-1}(U)$ as

$$\nabla\,(dp) = \frac{\partial}{\partial u^j} \otimes \Omega^j, \tag{6.1}$$

$$\nabla\left(\nabla\left(\frac{\partial}{\partial u^i}\right)\right) = \nabla^2\left(\frac{\partial}{\partial u^i}\right) = \frac{\partial}{\partial u^j} \otimes \Omega_i^j \tag{6.2}$$

where

$$\Omega^j = d(du^j) + \omega_i^j \wedge du^i = \omega_i^j \wedge du^i, \tag{6.3}$$

$$\Omega_i^j = d\omega_i^j + \omega_k^j \wedge \omega_i^k. \tag{6.4}$$

Ω^j and Ω_i^j are called the *torsion forms* and the *curvature forms* of $\widetilde{\Gamma}$ for the canonical local coordinates respectively. Since $\widetilde{\Gamma}$ is regular, Ω^j and Ω_i^j are written in terms of du^k, $D\xi^k$ only, hence putting

$$\gamma^j = M_i^j D\xi^i \tag{6.5}$$

we can write them as

$$\Omega^j = -\Gamma^{*j}_{ik}\, du^i \wedge du^k - C_{ik}^j\, du^i \wedge \gamma^k, \tag{6.6}$$

$$\Omega_i^j = \frac{1}{2} R^j_{ihk}\, du^h \wedge du^k + P^j_{ihk}\, du^h \wedge \gamma^k + \frac{1}{2} S^j_{ihk}\, \gamma^h \wedge \gamma^k, \tag{6.7}$$

$$R^j_{ihk} = -R^j_{ikh}, \quad S^j_{ihk} = -S^j_{ikh}.$$

On the other hand, for the natural cross section $\mathfrak{y}$ of $\mathfrak{F}$ we have easily

$$\nabla \mathfrak{y} = \frac{\partial}{\partial u^j} \otimes D\xi^j, \quad \nabla^2 \mathfrak{y} = \frac{\partial}{\partial u^j} \otimes \Omega_i^j \xi^i, \tag{6.8}$$

that is

$$d(D\xi^j) + \omega_i^j \wedge D\xi^i = \Omega_i^j \xi^i. \tag{6.9}$$

For $\tilde{\pi}^\diamond \mathfrak{F}$, we have analogously the following equations. Since $\mathfrak{z}_i = a_i^j \tilde{\pi}^\circ (\partial/\partial u^i)$ on $\mu^{-1}(U)$, we obtain by means of (1.21), (3.12), (3.13)

$$\begin{aligned}\nabla (d\mathfrak{p}) &= \tilde{\pi}^\circ(\partial/\partial u^j) \otimes \tilde{\pi}^* \Omega^j = \mathfrak{z}_j \otimes b_i^j \tilde{\pi}^* \Omega^i \\ &= \mathfrak{z}_j \otimes (d\theta^j + \theta_i^j \wedge \theta^i)\end{aligned} \tag{6.1'}$$

and

$$\begin{aligned}\nabla^2 \mathfrak{z}_i &= a_i^h \nabla^2(\tilde{\pi}^\circ(\partial/\partial u^h)) = a_i^h \tilde{\pi}^\circ(\partial/\partial u^k) \otimes \tilde{\pi}^* \Omega_h^k = \mathfrak{z}_j \otimes b_k^j (\tilde{\pi}^* \Omega_h^k)\, a_i^h \\ &= \mathfrak{z}_j \otimes (d\theta_i^j + \theta_k^j \wedge \theta_i^k).\end{aligned} \tag{6.2'}$$

Hence the torsion forms Θ^j and the curvature forms Θ_i^j of $\tilde{\pi}^\diamond \widetilde{\Gamma}$ are written on $\bar{\mu}^{-1}(U)$ as

$$\begin{aligned}\Theta^j &\equiv d\theta^j + \theta_i^j \wedge \theta^i = b_i^j \tilde{\pi}^* \Omega^i \\ &= -\frac{1}{2} \widetilde{T}^j_{ik}\, \theta^i \wedge \theta^k - \widetilde{C}^j_{ik}\, \theta^i \wedge \tilde{\gamma}^k, \\ \Theta_i^j &\equiv d\theta_i^j + \theta_k^j \wedge \theta_i^k = b_k^j (\tilde{\pi}^* \Omega_h^k)\, a_i^h \\ &= \frac{1}{2} \widetilde{R}^j_{ihk}\, \theta^h \wedge \theta^k + \widetilde{P}^j_{ihk}\, \theta^h \wedge \tilde{\gamma}^k + \frac{1}{2} \widetilde{S}^j_{ihk}\, \tilde{\gamma}^h \wedge \tilde{\gamma}^k\end{aligned} \tag{6.10}$$

where $\widetilde{T}^j_{ik}$, $\widetilde{C}^j_{ik}$, $\widetilde{R}^j_{ihk}$, $\widetilde{P}^j_{ihk}$, $\widetilde{S}^j_{ihk}$ are the components of the images of the tensor fields of $\mathfrak{F}$ under $\tilde{\pi}^\circ$ with respect to the natural base $\{\mathfrak{z}_i\}$ which have locally components $\Gamma^{*j}_{ik} - \Gamma^{*j}_{ki}$, C^j_{ik}, R^j_{ihk}, S^j_{ihk} respectively. Clearly we have

$$\widetilde{R}^j_{ihk} = b^j_{j'} a^{i'}_i a^{h'}_h a^{k'}_k R^{j'}_{i'h'k'} \quad \text{etc.},$$

where we put by (4.15)

$$\tilde{\gamma}^h = \widetilde{M}^h_k Dy^k. \tag{6.12}$$

For $\mathfrak{z}$, we have

$$\nabla^2 \mathfrak{z} = \nabla (\mathfrak{z}_j \otimes Dy^j) = \mathfrak{z}_j \otimes (d\, Dy^j + \theta^j_i \wedge Dy^i)$$

hence

$$dDy^j + \theta^j_i \wedge Dy^i = y^i\, \Theta^j_i. \tag{6.13}$$

Now, for a curve $\bar{C}$ in $T_0(\mathfrak{X})$ given by $\bar{f}: \ I \to T_0(\mathfrak{X})$, let $\iota : \tilde{\pi}^{-1}(\bar{C}) \to \widetilde{\mathfrak{B}}_0$ be the imbedding map, then we get by (6.10) — (6.13) the equations

$$\iota^* \Theta^j = \iota^* \Theta^j_i = 0.$$

Hence, the equations of Pfaffian forms in $R^n \times GL(n) \times \tilde{\pi}^{-1}(\bar{C})$

$$dp' = \mathrm{e}'_j\, \iota^* \theta^j, \qquad d\mathrm{e}'_i = \mathrm{e}'_i\, \iota^* \theta^j_i \tag{6.14}$$

are completely integrable by a Theorem of Frobenious. Furthermore p' depends only on the parameter t of $\bar{C}$, since $\iota^* \theta^j$ depend only on dt. Accordingly, p' determine a curve C' in R^n as an affine space. For a solution p', e'_i, we get in $\mu^{-1}(U) \cap \tilde{\pi}^{-1}(\bar{C})$

$$d(\mathrm{e}'_i y^i) = \mathrm{e}'_j (dy^j + \iota^* \theta^j_i\, y^i) = \mathrm{e}'_j (\iota^* Dy^j) = \mathrm{e}'_j\, b^j_i\, \bar{f}^* (D\xi^i)$$

this shows that $\mathrm{e}'_j\, y^j$ depends only t. Let us consider the curve C in $\mathfrak{X}$ given by $f = \tau \cdot \bar{f}$. From the above circumstances, we may call the figure composed of the curve C' and the vector field $\mathrm{e}'_j y^j$ along C' a *development of $\bar{C}$ with respect to $\tilde{\Gamma}$*. We call C' a *supporting curve* of the development of $\bar{C}$.

Let C *be* a curve of class C^r $(r \geqq 2)$ in $\mathfrak{X}$ given by $f: \ I \to \mathfrak{X}$ and without stationary points with respect to t, that is $f_* \, \partial/\partial t \neq 0$ everywhere. Then, the map $\bar{f} = f_* \, \partial/\partial t$ clearly define a curve $\bar{C}$ in $T_0(\mathfrak{X})$. We say that $\bar{C}$ is the *lift of C in $T_0(\mathfrak{X})$*. Let us consider a development of $\bar{C}$. Since we have locally

$$f_* \frac{\partial}{\partial t} = \left(u^j \cdot f, \ \frac{d}{dt}(u^j \cdot f)\right),$$

we have

$$\frac{dp'}{dt} = \mathrm{e}'_j\, b^j_i \frac{d(u^i \cdot f)}{dt}, \qquad \mathrm{e}'_j y^j = \mathrm{e}'_j\, b^j_i\, \xi^i = \mathrm{e}'_j b^j_i \frac{d(u^j \cdot f)}{dt}$$

that is

$$\frac{dp'}{dt} = \mathrm{e}'_j\, y^j.$$

Thus we obtain

Proposition 6. 1. *For any curve C in $\mathfrak{X}$ without stationary points, a development of the lift in $T_0(\mathfrak{X})$ of C is the supporting curve C' of the development and the field of tangent vectors of C'.*

Lastly, we induce the condition that the supporting curve of a development of a curve in $T_0(\mathfrak{X})$ is an affine straight line. From the equations above, it is necessary and sufficient that there exists a function ψ of t such that

$$\frac{d^2p'}{dt^2} = \psi \frac{dp'}{dt}.$$

By a simple calculation, this equation is locally written as

$$d\left(\frac{d}{dt}(u^j \cdot f)\right) + a^j_k \iota^* \theta^k_h b^h_i \frac{d}{dt}(u^i \cdot f) + a^j_k db^k_i \frac{d}{dt}(u^i \cdot f) = \psi d(u^j \cdot f),$$

making use of (3. 9) we have

$$\frac{d^2}{dt^2}(u^j \cdot f) + \frac{\bar{f}^* \omega^j_i}{dt} \frac{d(u^i \cdot f)}{dt} = \psi \frac{d(u^j \cdot f)}{dt}. \tag{6. 15}$$

When $\bar{C}$ is the lift of a curve C in $\mathfrak{X}$, since $\xi^j = \frac{d}{dt}(u^j \cdot f)$ along $\bar{C}$, (6. 15) is written by means of (4. 2), (4. 11) as

$$\frac{d^2u^j}{dt^2} + \Gamma^j_{ik}\left(u, \frac{du}{dt}\right)\frac{du^i}{dt}\frac{du^k}{dt} + C^j_{ik}\left(u, \frac{du}{dt}\right)\frac{du^i}{dt}\frac{d^2u^k}{dt^2}$$

$$= \Phi^j_k\left(u, \frac{du}{dt}\right)\left(\frac{d^2u^k}{dt^2} + \Gamma^{*k}_{ih}\frac{du^i}{dt}\frac{du^h}{dt}\right) = \psi \frac{du^j}{dt},$$

hence

$$\frac{d^2u^j}{dt^2} + \Gamma^{*j}_{ik}\left(u, \frac{du}{dt}\right)\frac{du^i}{dt}\frac{du^k}{dt} = \psi \frac{du^j}{dt} \tag{6. 16}$$

§ 7. α-curves and horizontal curves.

For a given curve C of class C^1 in $T_0(\mathfrak{X})$ given by $\bar{f}: I \to T_0(\mathfrak{X})$, we can prove that there exist curves $\tilde{C}$ in $\mathfrak{B}_0$ which are given by maps $\tilde{f}: I \to \tilde{\mathfrak{B}}_0$ such that $\bar{\pi} \cdot \tilde{f} = \bar{f}$ and $\tilde{f}_* \partial/\partial t$ is horizontal and that these curves $\tilde{C}$ are transformed each others by right translations of $\{\tilde{\mathfrak{B}}_0, T_0(\mathfrak{X})\}$. These curves are called *horizontal lifts* in $\tilde{\mathfrak{B}}_0$ of $\bar{C}$.

A tangent vector of $\bar{C}$ is locally written as

$$X=\bar{f}_*\frac{\partial}{\partial t}=\frac{d(u^j\cdot f)}{dt}\frac{\partial}{\partial u^j}+\frac{d(\xi^j\cdot\bar{f})}{dt}\frac{\partial}{\partial\xi^j}. \tag{7.1}$$

Assume $\bar{f}(t)\in\tau^{-1}(U)$ and take a point $\tilde{b}\in\widetilde{\mathfrak{B}}_0$ over $\bar{f}(t)$, then we get from (5.1) and (5.2)

$$b_i^j\tilde{\pi}_*B_j=\frac{\partial}{\partial u_i}-\Gamma^{*h}_{il}\xi^l\frac{\partial}{\partial\xi^h},\ \Phi_i^k b_k^j\tilde{\pi}_*E_j=\frac{\partial}{\partial\xi^i}, \tag{7.2}$$

hence

$$\frac{\partial}{\partial u^i}=b_i^j\tilde{\pi}_*B_j+\Gamma^{*h}_{il}\xi^l\Phi_h^k b_k^j\tilde{\pi}_*E_j. \tag{7.3}$$

By means of (7.1) and (4.13), we get therefore

$$X=\frac{d(u^i\cdot f)}{dt}\{b_i^j\tilde{\pi}_*B_j+\Gamma^{*h}_{il}\xi^l\Phi_h^k b_k^j\tilde{\pi}_*E_j\}+\frac{d(\xi^i\cdot\bar{f})}{dt}\Phi_i^k b_k^j\tilde{\pi}_*E_j$$

$$=b_i^j\frac{d(u^i\cdot f)}{dt}\tilde{\pi}_*B_j+\left\{\frac{d(\xi^i\cdot\bar{f})}{dt}+\Gamma^{*i}_{hl}(\xi^h\cdot\bar{f})\frac{d(u^l\cdot f)}{dt}\right\}\Phi_i^k b_k^j\tilde{\pi}_*E_j,$$

that is

$$X=\bar{f}_*\frac{\partial}{\partial t}=b_i^j\left\{\frac{d(u^i\cdot f)}{dt}\tilde{\pi}_*B_j+\frac{\bar{f}^*D\xi^i}{dt}\tilde{\pi}_*E_j.\right\}. \tag{7.4}$$

From this equation, we see that the horigontal tangent vector $\widetilde{X}$ at $\tilde{b}$ such that $\tilde{\pi}_*\widetilde{X}=X$ is written simply as

$$\widetilde{X}=\frac{\theta^j}{dt}B_j+\frac{Dy^j}{dt}E_j. \tag{7.5}$$

When $\bar{C}$ is the lift of C in $\mathfrak{X}$, since we have locally

$$\bar{f}=\left(u^j\cdot f,\ \frac{d}{dt}(u^j\cdot f)\right)\quad\text{or}\quad \xi^j\cdot\bar{f}=\frac{d}{dt}(u^j\cdot f),$$

$\widetilde{X}$ may be written as simply

$$\widetilde{X}=y^jB_j+\frac{Dy^j}{dt}E_j. \tag{7.6}$$

Conversely, if $\bar{C}$ satisfies (7.6), then it must be $\frac{d}{dt}(u^j\cdot f)=\xi^j\cdot f$, hence $\bar{C}$ is the lift of $\tau(\bar{C})=C$.

Here, we introduce some definitions. We call the horizontal tangent vector field $B=y^jB_j$ over $\widetilde{\mathfrak{B}}_0$ the *canonical horizontal vector field* of the connection $\widetilde{\Gamma}$. We define the fields Ξ, Π, Π^+, Π^- of the subspaces $\Xi_{\tilde{b}}$, $\Pi_{\tilde{b}}$ and subsets $\Pi^+_{\tilde{b}}$, $\Pi^-_{\tilde{b}}$ of the tangent space $T_{\tilde{b}}(\widetilde{\mathfrak{B}}_0)$ such that

$$\Xi_{\tilde{b}} = \sum RE_i(\tilde{b}),\ \Pi_{\tilde{b}} = RB(\tilde{b}) + \Xi_{\tilde{b}},\ \Pi^{\pm}_{\tilde{b}} = R^{\pm}B(b) + \Xi_{\tilde{b}}$$

respectively, where R, R^+, R^- denote the real field and its subsets of all the positive numbers and negative numbers respectively.

Then, the above calculation proved the following theorem.

Theorem 7. 1. *A necessary and sufficint condition that a curve $\bar{C}$ in $T_0(\mathfrak{X})$ is the lift of a curve in $\mathfrak{X}$ is that the tangent vectors of a horizontal lift of $\bar{C}$ in $\widetilde{\mathfrak{B}}_0$ belong always to the field $B + \Xi$.*

Nextly, for a curve in $\mathfrak{X}$ given by $f:\ I \to \mathfrak{X}$, we call the curve in $\mathfrak{S}(\mathfrak{X})$ given by the map $\varphi = \rho \cdot (f_*\partial/\partial t)$ the *lift of C in $\mathfrak{S}(\mathfrak{X})$*. By (2. 21), we have $\sigma \cdot \varphi = f$. Then, we shall prove a theorem analogous to Theorem 7. 1 regarding the lift of curve in $\mathfrak{S}(\mathfrak{X})$.

Theorem 7. 2. *A necessary and sufficient condition that a curve $\bar{C}$ in $T_0(\mathfrak{X})$ has the property such that $\rho(\bar{C})$ or $\rho(\varepsilon\bar{C})$ is the lift in $\mathfrak{S}(\mathfrak{X})$ of the curve $\tau(\bar{C})$ is that the tangent vectors of a horizontal lift of $\bar{C}$ in $\widetilde{\mathfrak{B}}_0$ belong always to the field Π^+ or Π^- respectively, where ε means the symmetric transformation of vector bundles.*

Proof. Let $\bar{f}_0:\ I \to T_0(\mathfrak{X})$ be the lift of $f = \tau \cdot \bar{f}$, that is $\bar{f}_0 = f_*\partial/\partial t$. If $\rho(\bar{C})$ or $\rho(\varepsilon\bar{C})$ is the lift of the curve C given by f, then it must be $\xi^i \cdot \bar{f} = \psi \dfrac{d}{dt}(u^i \cdot f) = \psi(\xi^i \cdot \bar{f}_0)$ for some positive or negative function ψ, so that the tangent vectors of a horizontal lift of $\bar{C}$ in $\widetilde{\mathfrak{B}}_0$ belong by means of (7. 5) to Π^+ or Π^- respectively. The converse can be easily proved.

Now, for a curve $\bar{C}$ in $T_x(\mathfrak{X}) - x$ given by a map $\bar{f}$, the equation (7. 4) and(7. 5) become clearly

$$X = b^j_i \frac{D}{dt}(\xi^i \cdot \bar{f})\, \tilde{\pi}_* E_j, \quad \widetilde{X} = \frac{Dy^j}{dt} E_j.$$

Conforming to these circumstances, a curve $\bar{C}$ in $T_0(\mathfrak{X})$ is called a *proper α-curve* (regarding as a parameter curve), if it is of class C^1 and its image under ρ or $\rho \cdot \varepsilon$ is the lift in $\mathfrak{S}(\mathfrak{X})$ of a curve in $\mathfrak{X}$ and $\bar{C}$ is called an *α-curve* if it is arc-wise proper α-curves or curves of class C^1 in fibres of $\{\mathfrak{S}(\mathfrak{X}),\ \mathfrak{X}\}$.

Then, we obtain immediately the following theorem which will be fundamental for our considerations.

Theorem 7. 3. *For any α-curve $\bar{C}$, the tangent vectors of a horizontal lift $\widetilde{C}$ of $\bar{C}$ in $\mathfrak{B}_0$ belong always to the field H, and the converse is also true.*

§ 8. The homogeneous holonomy group of Γ.

According to the theory of connections of vector bundles, we can define the holonomy groups of the connection $\widetilde{\Gamma} = \rho^{\diamond}\Gamma$ of the vector bundle $\mathfrak{F} = \{T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X}),\ T_0(\mathfrak{X})\}$ which may be regarded as the holonomy groups of the connection Γ of the vector bundle $\{\mathfrak{Z}, \mathfrak{S}\}$. For any two points $y_0,\ y_1 \in T_0(\mathfrak{X})$, take a curve $\bar{C}$ arc-wise of class C[1] which joins from y_0 to y_1. For any frame $\tilde{b}_0$ of $\mathfrak{F}$ at y_0, let $\tilde{b}_1$ be the frame of $\mathfrak{F}$ at y_1 which is obtained by prallel displacing $\tilde{b}_0$ along the curve $\bar{C}$. As is well known, the point $\tilde{b}_1$ is the end point of the horizontal lift $\widetilde{C}$ of $\bar{C}$ with the beginning point $\tilde{b}_0$ and the holizontal lifts of the curve $\bar{C}$ are transformed each other by right translations (1. 14) of the principal fibre bundle $\{\widetilde{\mathfrak{B}}_0, T_0(\mathfrak{X})\}$. Thus corresponding $\tilde{b}_1$ to $\tilde{b}_0$, we obtain an isomorphism

$$h_{\bar{C}} \ :\ G_{y_1} = L(V_{y_1}) \rightarrow G_{y_0} = L(V_{y_0}) \tag{8. 1}$$

which is commutative with the right translations, where V_y and G_y means the fibre of $\mathfrak{F}$ and $\{\widetilde{\mathfrak{B}}_0,\ T_0(\mathfrak{X})\}$ at the point y and $L(V_y)$ means the group of linear automorphisms of the vector space V_y. By virtue of these properties, we may regard $h_{\bar{C}}$ as an isomorphism of V_{y_1} onto V_{y_0}. Furthermore, let $\bar{C}_1$ be a curve joining y_1 to y_2, then, denoting the curve connecting $\bar{C}$ and $\bar{C}_1$ at y_1 by $\bar{C}\bar{C}_1$, we obtain easily the equation

$$h_{\bar{C}\bar{C}_1} = h_{\bar{C}} \cdot h_{\bar{C}_1} \tag{8. 2}$$

For a fixed point y, putting $y_0 = y = y_1$. All the maps $h_{\bar{C}}$ make up, as is well known, a subgroup $\widetilde{H}(y)$ of $L(V_y) \approx GL(n)$ which is called the *(homogeneous) holonomy group at y in a wide sense* of the connection $\widetilde{\Gamma}$. When we take only the curves $\bar{C}$ homotopic to zero, the coresponding subgroup $\widetilde{H}^0(y)$ of $\widetilde{H}(y)$ is called the *restricted (homogeneous) holonomy group at y in a wide sense.* If $\mathfrak{X}$ is separable, $\widetilde{H}^0(y)$ is a Lie group.

We have stated above the holonomy groups of $\widetilde{\Gamma}$ in a general sense regarding it as a connection of the vector bundle $\mathfrak{F}$ and merely $T_0(\mathfrak{X})$ as a differentiable manifold. But, we should naturally restrict our consideration on α-curves only to define the holonomy groups of $\widetilde{\Gamma}$, since $\widetilde{\Gamma}$ is an affine connection of the special vector bundle $\mathfrak{F} = \{T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X}),\ T_0(\mathfrak{X})\}$. And so we call the group $H(y)$ which is defined by taking only α-curves in the above definition of $\widetilde{H}(y)$ the *(homogeneous) holonomy group at y of* $\widetilde{\Gamma}$. We also call the group $H^0(y)$ which is defined by taking only α-curves homotopic to zero through only α-curves in the definition of $\widetilde{H}^0(y)$ the *restricted (homogeneous) holonomy group at y*. $H^0(y)$ is an invariant Lie subgroup of $H(y)$. For two α-curves $\bar{C}_0$, $\bar{C}_1$, if we can deform $\bar{C}_0$ to $\bar{C}_1$ through only α-curves, we say that $\bar{C}_0$ is *α-homotopic to* $\bar{C}_1$. Needless to say, if $\bar{C}_0$ is α-homotopic to $\bar{C}_1$, $\bar{C}_0$ is homotopic to $\bar{C}_1$. On the converse, we shall investigate in future.

Now, for any point $\tilde{b} \in \widetilde{\mathfrak{B}}_0$, we consider all the curves through $\tilde{b}$ which are horizontal curves tangent to the field Π at each point. And, let $P(\tilde{b})$ be the locus of the end points of these curves. We have imediately

Proposition 8. 1. *If* $P(\tilde{b}) \ni \tilde{b}_1$, *then* $P(\tilde{b}) = P(\tilde{b}_1)$.

Proposition 8. 2. *For any* $g \in GL(n)$, $\tilde{b} \in \widetilde{\mathfrak{B}}_0$, *we have* $r_g(P(\tilde{b})) = P(r_g(\tilde{b}))$.

Proof. Since horizontal curves are transformed each others by right translations of $\{\mathfrak{B}_0,\ T_0(\mathfrak{X})\}$, it is sufficient to prove the proposition that

$$(r_g)_* \Pi = \Pi, \quad (r_g)_* \Xi = \Xi \tag{8. 3}$$

From (1. 15), (3. 10) and (4. 15), we have for θ^j, Dy^j, θ^j_i

$$(r_g)^*\theta^j = b^j_i(g)\theta^i, \quad (r_g)^*Dy^j = b^j_i(g)Dy^i, \quad (r_g)^*\theta^j_i = b^j_k(g)\,\theta^k_h a^h_i(g), \tag{8. 4}$$

hence for the dual tangent vector fields B_i, E_i, Q^j_i we have

$$(r_g)_* B_i = b^j_i(g)\, B_j, \quad (r_g)_* E_i = b^j_i(g)\, E_j, \quad (r_g)_* Q^j_i = a^j_k(g) Q^k_h\, b^h_i(g). \tag{8. 5}$$

Furthermore, from (3. 14) we have

$$(r_g)^* y^i = b^j_i(g) y^i.$$

Accordingly, for the canonical horizontal vector field B we have

$$\begin{aligned}(r_g)_*\,(y^i(\tilde{b})\,B_i(\tilde{b})) &= y^i(\tilde{b})\,(r_g)_*\,B_i(\tilde{b}) = y^i(\tilde{b})\,b^j_i(g)\,B_j\,(\tilde{b}g)\\ &= y^j(\tilde{b}g)B_j(\tilde{b}g),\end{aligned}$$

that is

$$(r_g)_* B = B. \tag{8. 6}$$

From (8. 6) and (8. 5), we get immediately (8. 3).

By virtue of the definition of $H(y)$ and Theorem 7. 3, we have the relation

$$H(y)(\tilde{b}) = P(\tilde{b}) \cap \tilde{\pi}^{-1}(y), \quad y = \tilde{\pi}(\tilde{b}). \tag{8. 7}$$

Now, in the set of all the curves arc-wise of C^1 in $T_0(\mathfrak{X})$ which start from and return to a point y, we define a natural equivalence relation such that

$$C_1 C_2 \cdots C_i C_i^{-1} C_{i+2} \cdots C_m \sim C_1 C_2 \cdots C_{i-1} C_{i+2} \cdots C_m,$$

then the set of equivalence classes make up, as is well known, a group $\tilde{\Omega}(y)$. We shall also represent the class itself containing a curve C of the above mentioned set by the same symbol C. h in (8. 1) is a homomorphism $\tilde{\Omega}(y) \to L(V_y)$. Let $\Omega(y)$ be the subset of $\tilde{\Omega}(y)$ consisting of α-curves only, then $H(y) = h(\Omega(y))$. Furthermore, let $\Omega^0(y)$ be the subset of $\Omega(y)$ consisting of α-curves α-homotopic to zero. Then, $\Omega^0(y)$ is clearly an invariant subgroup of $\Omega(y)$. Let $\tilde{\Omega}^0(y)$ be the subset of $\tilde{\Omega}(y)$ consisting of curves homotoptic to zero. Since

$$\Omega^0(y) \subset \Omega(y) \cap \tilde{\Omega}^0(y),$$

we have the natural homomorphisms

$$\Omega(y)/\Omega^0(y) \to H(y)/H^0(y) \to F(T_0(\mathfrak{X})),$$

where $F(T_0(\mathfrak{X}))$ denotes the fundamental group of the space $T_0(\mathfrak{X})$.
If $\mathfrak{X}$ is separable, $T_0(\mathfrak{X})$ is also separable. Thus, we get the following

Proposition 8. 3. *$H^0(y)$ is isomorphic to a Lie subgroup of $GL(n)$ and the connected component of the group $H(y)$.*

From this proposition and (8. 7), we can prove the theorem.

Theorem 8. 4. *For any $\tilde{b} \in \tilde{\mathfrak{B}}_0$, $y = \tilde{\pi}(\tilde{b})$, $H^0(y)(\tilde{b})$ is the connected component containg $\tilde{b}$ of the intersection of the fibre $\tilde{\pi}^{-1}(y)$ and $P(\tilde{b})$ which is an integral manifold of the field Π.*

§ 9. Torsion tensors and curvature tensors.

For the future discussion, we shall calculate the torsion tensor and the curvature tensors of $\widetilde{\Gamma}$ and show some properties. We had locally by (3. 2), (4. 8), (4. 2), (4. 13)

$$\omega_i^j = \Gamma_{ik}^j\, du^k + C_{ik}^j\, d\xi^k = \Gamma^{*j}_{ik}\, du^k + C_{ik}^j \gamma^k, \tag{9.1}$$

$$D\xi^j = \Phi_k^j\, d\xi^k + \Gamma_{lk}^j\, \xi^l\, du^k = \Phi_k^j (d\xi^k + \Gamma^{*k}_{lh}\, \xi^l\, du^h). \tag{9.2}$$

Hence, with respect to the canonical coordinates $(u^j,\ \xi^j)$ for $(U,\ u)$, the torsion forms of $\widetilde{\Gamma}$ are lacally written by (6. 6) as

$$\Omega^j = -\frac{1}{2}\, T_{ik}^j\, du^i \wedge du^k - C_{ik}^j\, du^i \wedge \gamma^k, \quad \gamma^k = M_h^k\, D\xi^h,$$

$$T_{ik}^j = -T_{ki}^j = \Gamma^{*j}_{ik} - \Gamma^{*j}_{ki}. \tag{9.3}$$

The tensor fields of $\mathfrak{F}$ of the type (1. 2) with the components locally T_{ik}^j, C_{ik}^j are called the torsion tensors of the first and second kinds respectively. $\widetilde{T}_{ik}^j$, $\widetilde{C}_{ik}^j$ in (6. 10) are the components of the image tensors of the tensors under $\tilde{\pi}^\circ$.

The curvature forms Ω_i^j was of the form (6. 7), hence we have by (6. 4), (9. 1)

$$\Omega_i^j = \frac{\partial \Gamma_{ik}^j}{\partial u^h}\, du^h \wedge du^k - \frac{\partial \Gamma_{ih}^j}{\partial \xi^k} du^h \wedge d\xi^k + \frac{\partial C_{ik}^j}{\partial u^h}\, du^h \wedge d\xi^k$$

$$+ \frac{\partial C_{ik}^j}{\partial \xi^h}\, d\xi^h \wedge d\xi^k + (\Gamma^{*j}_{lh}\, du^h + C_{lh}^j\, \gamma^h)\, (\Gamma^{*l}_{ik}\, du^k + C_{ik}^l\, \gamma^k)$$

in which substitute (9. 2), we get

$$= \frac{\partial \Gamma_{ik}^j}{\partial u^h}\, du^h \wedge du^k - \frac{\partial \Gamma_{ih}^j}{\partial \xi^t}\, du^h \wedge (M_k^t\, D\xi^k - \Gamma^{*t}_{lk}\, \xi^l\, du^k)$$

$$+ \frac{\partial C_{it}^j}{\partial u^h}\, du^h \wedge (M_k^t\, D\xi^k - \Gamma^{*t}_{lk}\, \xi^l\, du^k)$$

$$+ \frac{\partial C_{is}^j}{\partial \xi^t}\, (M_h^t\, D\xi^h - \Gamma^{*t}_{lh}\, \xi^l\, du^h) \wedge (M_k^s\, D\xi^k - \Gamma^{*s}_{mk}\, \xi^m\, du^k)$$

$$+ (\Gamma^{*j}_{lh}\, du^h + C_{lh}^j\, \gamma^h) \wedge (\Gamma^{*l}_{ik}\, du^k + C_{ik}^l\, \gamma^k)$$

$$= \left\{ \frac{\partial \Gamma_{ik}^j}{\partial u^h} + \frac{\partial \Gamma_{ih}^j}{\partial \xi^t}\, \Gamma^{*t}_{lk} \xi^l - \frac{\partial C_{it}^j}{\partial u^h}\, \Gamma^{*t}_{lk}\, \xi^l + \frac{\partial C_{is}^j}{\partial \xi^t}\, \Gamma^{*t}_{lh}\, \xi^l\, \Gamma^{*s}_{mk}\, \xi^m \right.$$

$$\left. + \Gamma^{*j}_{lh} \Gamma^{*l}_{ik} \right\} du^h \wedge du^k + \left\{ -\frac{\partial \Gamma_{ih}^j}{\partial \xi^k} + \frac{\partial C_{ik}^j}{\partial u^h} - \frac{\partial C_{ik}^j}{\partial \xi^t} \Gamma^{*t}_{lh}\, \xi^l + \frac{\partial C_{it}^j}{\partial \xi^k} \Gamma^{*t}_{lh} \xi^l \right.$$

$$\left. + \Gamma^{*j}_{lh}\, C_{ik}^l - C_{lk}^j\, \Gamma^{*l}_{ih} \right\} du^h \wedge \gamma^k + \left\{ \frac{\partial C_{ik}^j}{\partial \xi^h} + C_{lh}^j\, C_{ik}^l \right\} \gamma^h \wedge \gamma^k.$$

Thus, we obtain the formulas

$$R^j_{ihk} = \left(\frac{\partial \Gamma^j_{ik}}{\partial u^h} - \frac{\partial \Gamma^j_{ik}}{\partial \xi^t} \Gamma^{*t}_{lh} \xi^l\right) - \left(\frac{\partial \Gamma^j_{ih}}{\partial u^k} - \frac{\partial \Gamma^j_{ih}}{\partial \xi^t} \Gamma^{*t}_{lk} \xi^l\right)$$
$$+ \Gamma^{*j}_{lh} \Gamma^{*l}_{ik} - \Gamma^{*j}_{lk} \Gamma^{*l}_{ih} \tag{9.4}$$
$$-\left(\frac{\partial C^j_{it}}{\partial u^h} - \frac{\partial C^j_{it}}{\partial \xi^s} \Gamma^{*s}_{mh} \xi^m\right)\Gamma^{*t}_{lk} \xi^l + \left(\frac{\partial C^j_{it}}{\partial u^k} - \frac{\partial C^j_{it}}{\partial \xi^s} \Gamma^{*s}_{mk} \xi^m\right) \Gamma^{*t}_{lh} \xi^l,$$

$$P^j_{ihk} = -\frac{\partial \Gamma^j_{ih}}{\partial \xi^k} + \frac{\partial C^j_{ik}}{\partial u^h} + \left(\frac{\partial C^j_{it}}{\partial \xi^k} - \frac{\partial C^j_{ik}}{\partial \xi^t}\right) \Gamma^{*t}_{lh} \xi^l$$
$$+ \Gamma^{*j}_{lh} C^l_{ik} - C^j_{lk} \Gamma^{*l}_{ih}, \tag{9.5}$$

$$S^j_{ihk} = \frac{\partial C^j_{ik}}{\partial \xi^h} - \frac{\partial C^j_{ih}}{\partial \xi^k} + C^j_{lh} C^l_{ik} - C^j_{lk} C^l_{ih}. \tag{9.6}$$

We call the tensor fields of $\mathfrak{F}$ of the type (1, 3) with the components R^j_{ihk}, P^j_{ihk}, S^j_{ihk} the curvature tensors of the first, second and third kinds of the connection $\widetilde{\Gamma}$ respectively.

Now, we shall define covariant derivatives for vector fields and tensor fields of the vector bundle $\mathfrak{F}$ with respect to $\widetilde{\Gamma}$. Let be given a vector field of $\mathfrak{F}$ whose components are locally V^j. By (9. 1) (9. 2), we get

$$DV^j = dV^j + \omega^j_i V^i$$
$$= \left(\frac{\partial V^j}{\partial u^h} - \frac{\partial V^j}{\partial \xi^t} \Gamma^{*t}_{lh} \xi^l + \Gamma^{*j}_{th} V^t\right) du^h + \left(\frac{\partial V^j}{\partial \xi^h} + C^j_{th} V^t\right)\gamma^h.$$

Hence if we define two covariant differentions of V^j with respect to $\widetilde{\Gamma}$ by

$$V^j{}_{,h} = \frac{\partial V^j}{\partial u^h} - \frac{\partial V^j}{\partial \xi^t} \Gamma^{*t}_{lh} \xi^l + \Gamma^{*j}_{th} V^t, \tag{9.7}$$

$$V^j{}_{;h} = \frac{\partial V^j}{\partial \xi^h} + C^j_{th} V^t, \tag{9.8}$$

the above equations are written as

$$DV^j = V^j{}_{,h}\, du^h + V^j{}_{;h}\, \gamma^h. \tag{9.9}$$

Since we assume that Γ is regular, that $V^j{}_{,h}$ and $V^j{}_{;h}$ are the components of two tensor fields of type (1. 1) of $\mathfrak{F}$ with respect to the frame $\{\partial/\partial u^i\}$ can be easily proved. The covariant derivatives ", h" with respect u^h and " ; h" with respect to ξ^h may be analogously defined for any tensor fields of $\mathfrak{F}$.

We get easily from (4. 12)

$$\frac{\partial \Gamma^j_{ik}}{\partial u^h} = \frac{\partial \Gamma^{*j}_{ik}}{\partial u^h} + \frac{\partial C^j_{il}}{\partial u^h} \Gamma^{*t}_{ik} \xi^l + C^j_{it} \frac{\partial \Gamma^{*t}_{lk}}{\partial u^h} \xi^l, \tag{9.10}$$

$$\frac{\partial \Gamma^j_{ik}}{\partial \xi^h} = \frac{\partial \Gamma^{*j}_{ik}}{\partial \xi^h} + \frac{\partial C^j_{it}}{\partial \xi^h} \Gamma^{*t}_{lk} \xi^l + C^j_{it}\left(\Gamma^{*t}_{hk} + \frac{\partial \Gamma^{*t}_{lk}}{\partial \xi^h} \xi^l\right). \qquad (9.11)$$

Substituting (9.10) into (9.4) and using (9.11), we get

$$\begin{aligned} R^j_{ihk} = &\left(\frac{\partial \Gamma^{*j}_{ik}}{\partial u^h} + C^j_{it} \frac{\partial \Gamma^{*t}_{lk}}{\partial u^h} \xi^l\right) - \left(\frac{\partial \Gamma^{*j}_{ih}}{\partial u^k} + C^j_{it} \frac{\partial \Gamma^{*t}_{lh}}{\partial u^k} \xi^l\right) \\ &- \left(\frac{\partial \Gamma^j_{ik}}{\partial \xi^t} - \frac{\partial C^j_{is}}{\partial \xi^t} \Gamma^{*s}_{mk} \xi^m\right) \Gamma^{*t}_{lh} \xi^l + \left(\frac{\partial \Gamma^j_{ih}}{\partial \xi^t} - \frac{\partial C^j_{is}}{\partial \xi^t} \Gamma^{*s}_{mh} \xi^m\right) \Gamma^{*t}_{lk} \xi^l \\ &+ \Gamma^{*j}_{lh} \Gamma^{*l}_{ik} - \Gamma^j_{lk} \Gamma^{*l}_{ih}, \end{aligned}$$

that is

$$\begin{aligned} R^j_{ihk} = &\left(\frac{\partial \Gamma^{*j}_{ik}}{\partial u^h} - \frac{\partial \Gamma^{*j}_{ik}}{\partial u \xi^t} \Gamma^{*t}_{lh} \xi^l\right) - \left(\frac{\partial \Gamma^{*j}_{ih}}{\partial u^k} - \frac{\partial \Gamma^{*j}_{ih}}{\partial \xi^t} \Gamma^{*t}_{lk} \xi^l\right) \\ &+ \Gamma^{*j}_{lh} \Gamma^{*l}_{ik} - \Gamma^{*j}_{lk} \Gamma^{*l}_{ih} \qquad (9.4') \\ &+ C^j_{it}\left\{\frac{\partial \Gamma^{*t}_{lk}}{\partial u^h} - \frac{\partial \Gamma^{*t}_{lh}}{\partial u^k} + \left(\Gamma^{*t}_{sh} + \frac{\partial \Gamma^{*t}_{mh}}{\partial \xi^s} \xi^m\right)\Gamma^{*s}_{lk}\right. \\ &\left. - \left(\Gamma^{*t}_{sk} + \frac{\partial \Gamma^{*t}_{mk}}{\partial \xi^s} \xi^m\right) \Gamma^{*s}_{lh}\right\}\xi^l. \end{aligned}$$

Then, substituting (9.11) into (9.5), we get

$$\begin{aligned} P^j_{ihk} &= -\frac{\partial \Gamma^{*j}_{ih}}{\partial \xi^k} - \frac{\partial C^j_{it}}{\partial \xi^k} \Gamma^{*t}_{lh} \xi^l - C^j_{it}\left(\Gamma^{*t}_{kh} + \frac{\partial \Gamma^{*t}_{lh}}{\partial \xi^k} \xi^l\right) + \frac{\partial C^j_{ik}}{\partial u^h} \\ &\quad + \left(\frac{\partial C^j_{it}}{\partial \xi^k} - \frac{\partial C^j_{ik}}{\partial \xi^t}\right)\Gamma^{*t}_{lh} \xi^l + \Gamma^{*j}_{lh} C^l_{ik} - C^j_{lk}\Gamma^{*l}_{ih} \\ &= -\frac{\partial \Gamma^{*j}_{ih}}{\partial \xi^k} + \left(\frac{\partial C^j_{ik}}{\partial u^h} - \frac{\partial C^j_{ik}}{\partial \xi^t} \Gamma^{*t}_{lh} \xi^l + C^l_{ik}\Gamma^{*j}_{lh} - C^j_{lk}\Gamma^{*l}_{ih} - C^j_{it}\Gamma^{*t}_{kh}\right) \\ &\quad - C^j_{it} \frac{\partial \Gamma^{*t}_{lh}}{\partial \xi^k} \xi^l, \end{aligned}$$

hence by (9.7) we get

$$P^j_{ihk} = -\frac{\partial \Gamma^{*j}_{ih}}{\partial \xi^k} + C^j_{ik,h} - C^j_{it} \frac{\partial \Gamma^{*t}_{lh}}{\partial \xi^k} \xi^l. \qquad (9.5')$$

Lastly, by means of (9.8) we have

$$\frac{\partial C^j_{ik}}{\partial \xi^h} = C^j_{ik;h} - C^j_{lh} C^l_{ik} + C^l_{ih} C^j_{lk} + C^l_{kh} C^j_{il},$$

hence substituting these into (9.6) we get

$$\begin{aligned} S^j_{ihk} = &C^j_{ik;h} - C^j_{ih;k} - C^j_{lh} C^l_{ik} + C^j_{lk} C^l_{ih} \\ &- C^j_{il} (C^l_{hk} - C^l_{kh}). \qquad (9.6') \end{aligned}$$

Up to now, we did not use the equations (3.7), (3.8) which represent $\widetilde{\Gamma} = \rho^{\diamond} \Gamma$. By means of these equations, (9.4), (9.5) and (9.6) follow immediately

$$\begin{cases} R^j_{ihk}(u,\ t\xi) = R^j_{ihk}(u,\ \xi), \\ P^j_{ihk}(u,\ t\xi) = t^{-1} P^j_{ihk}(u,\ \xi), \\ S^j_{ihk}(u,\ t\xi) = t^{-2} S^j_{ihk}(u,\ \xi), \qquad t > 0. \end{cases} \tag{9.12}$$

Furthermore, by (3.7), (3.8) and Euler's formula, we have

$$\begin{aligned} P^j_{ihk}\xi^k &= -\frac{\partial \Gamma^j_{ih}}{\partial \xi^k}\xi^k + \frac{\partial(C^j_{ik}\xi^k)}{\partial u^h} + \left(\frac{\partial C^j_{it}}{\partial \xi^k}\xi^k - \frac{\partial C^j_{ik}}{\partial \xi^t}\xi^k\right)\Gamma^{*t}_{lh}\xi^l \\ &= -\left(C^j_{it} + \frac{\partial C^j_{ik}}{\partial \xi^t}\xi^k\right)\Gamma^{*t}_{lh}\xi^l = 0, \end{aligned}$$

$$S^j_{ihk}\xi^k = \frac{\partial C^j_{ik}}{\partial \xi^h}\xi^k + C^j_{ih} = 0,$$

that is

$$P^j_{ihk}\xi^k = 0 \tag{9.13}$$

$$S^j_{ihk}\xi^k = S^j_{ihk}\xi^h = 0. \tag{9.14}$$

If we transform the equations by $\tilde{\pi}^\circ$ in $\{\widetilde{\mathfrak{B}}_n - \mathfrak{B}_n,\ \widetilde{\mathfrak{B}}_0\}$, it follows by (3.13) that

$$\widetilde{P}^j_{ihk}\, y^k = 0, \tag{9.15}$$

$$\widetilde{S}^j_{ihk}\, y^k = \widetilde{S}^j_{ihk}\, y^h = 0. \tag{9.16}$$

(9.12) is also written on $\{\widetilde{\mathfrak{B}}_n - \widetilde{\mathfrak{B}}_n,\ \widetilde{\mathfrak{B}}_0\}$ by virtue of (2.13) as

$$\begin{cases} \lambda_t{}^*\, \widetilde{R}^j_{ihk} = \widetilde{R}^j_{ihk}, \\ \lambda_t{}^*\, \widetilde{P}^j_{ihk} = t^{-1}\, \widetilde{P}^j_{ihk}, \\ \lambda_t{}^*\, \widetilde{S}^j_{ihk} = t^{-2}\widetilde{S}^j_{ihk}, \qquad t > 0. \end{cases} \tag{9.17}$$

§ 10. The covariant differentiations on $\tilde{\pi}^{\diamond}\mathfrak{F}$.

The covariant differential of a vector field $\widetilde{\mathfrak{B}} = \mathfrak{z}_i \widetilde{V}^i =$ of $\tilde{\pi}^{\diamond}\mathfrak{F}$ with respect to the connection $\tilde{\pi}^{\diamond}\widetilde{\Gamma}$ is given by

$$D\widetilde{\mathfrak{B}} = \mathfrak{z}_j \otimes D\widetilde{V}^j, \qquad D\widetilde{V}^j = d\widetilde{V}^j + \theta^j_i\, \widetilde{V}^i. \tag{10.1}$$

Especially, if $\mathfrak{B}$ is the image of a vector field $\mathfrak{B}$ of $\mathfrak{F}$ under $\tilde{\pi}^\circ$, that is

$\tilde{\mathfrak{V}} = \tilde{\pi}^\circ \mathfrak{V}$, we have by (1. 17)

$$D\tilde{\mathfrak{V}} = D(\tilde{\pi}^\circ \mathfrak{V}) = (\tilde{\pi}^\circ \otimes \tilde{\pi}^*) D\mathfrak{V}. \tag{10. 2}$$

Let be locally $\mathfrak{V} = V^j \partial/\partial u^j$, then it follows from (9. 9), (2. 4), (2. 5), (2. 11), (3. 10), (6. 12) that

$$\begin{aligned} \mathfrak{z}_j \otimes D\tilde{V}^j &= (\tilde{\pi}^\circ \otimes \tilde{\pi}^*)\left(\frac{\partial}{\partial u^i} \otimes DV^i\right) \\ &= (\tilde{\pi}^\circ \otimes \tilde{\pi}^*)\left(\frac{\partial}{\partial u^i} \otimes (V^i{}_{,h}\, du^h + V^i{}_{;h}\, \gamma^h)\right) \\ &= \mathfrak{z}_j \otimes b^j_i (V^i{}_{,h}\, a^h_k \theta^k + V^i{}_{;h}\, a^h_k \tilde{\gamma}^k), \end{aligned}$$

where we put

$$\tilde{\gamma}^k = \tilde{M}^k_h Dy^h \tag{10. 3}$$

which is locally written by (6. 5), (4. 15) as

$$\tilde{\gamma}^k = b^k_h \gamma^h. \tag{10. 4}$$

The above equation follows immediately the following theorem.

Theorem 10. 1. *For any vector field $\tilde{\mathfrak{V}} = \tilde{\pi}^\circ \mathfrak{V} = \tilde{V}^j \mathfrak{z}_j$, the covariant differential $D\tilde{V}^j$* of $\tilde{V}^j$ *with respect to $\tilde{\pi}^\diamond \tilde{\Gamma}$ is written as*

$$D\tilde{V}^j = \tilde{V}^j{}_{,k}\, \theta^k + \tilde{V}^j{}_{;k}\, \tilde{\gamma}^k, \tag{10. 5}$$

and the coefficients of θ^k, $\tilde{\gamma}^k$ are locally

$$\tilde{V}^j{}_{,k} = b^j_i\, V^i{}_{,h}\, a^h_k, \quad \tilde{V}^j{}_{;k} = b^j_i\, V^i{}_{;h}\, a^h_k \tag{10. 6}$$

and the components of the images of the tensor field $V^j{}_{,k}$, $V^j{}_{;k}$ of $\mathfrak{F}$ under $\tilde{\pi}^\circ$.

This theorem holds clearly for any tensor field of $\mathfrak{F}$.

Making use of this theorem, we shall prove some formulas. Let $\tilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q}$ be the components of a $\tilde{\pi}^\circ$-image tensor field of the type (p, q), then we have

$$\begin{aligned} B_k(\tilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q}) &= \langle B_k, d\tilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q} \rangle = \langle B_k, D\tilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q} \rangle \\ &= \langle B_k, \tilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q, h}\, \theta^h + \tilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q ; h}\, \tilde{\gamma}^h \rangle \\ &= \tilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q, k}, \\ E_k(\tilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q}) &= \langle E_k, d\tilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q} \rangle = \langle E_k, D\tilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q} \rangle \\ &= \tilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q ; h} \langle E_k, \tilde{\gamma}^h \rangle \end{aligned}$$

$$= \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q;h}\, \widetilde{M}^h_k$$

and

$$Q^h_k(\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q}) = \langle Q^h_k,\ d\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q} \rangle$$
$$= \langle Q^h_k,\ D\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q} - \sum_{\alpha=1}^{p} \widetilde{K}^{i_1\cdots l\cdots i_p}_{j_1\cdots j_q}\, \theta^{i_\alpha}_l + \sum_{\beta=1}^{q} \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots l\cdots j_q}\, \theta^l_{j_\beta} \rangle$$
$$= -\sum_{\alpha=1}^{p} \widetilde{K}^{i_1\cdots l\cdots i_p}_{j_1\cdots j_q} \langle Q^h_k,\ \theta^{i_\alpha}_l \rangle + \sum_{\beta=1}^{q} \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots l\cdots j_q} \langle Q^h_k, \theta^l_{j_\beta} \rangle$$
$$= -\sum_{\alpha=1}^{p} \widetilde{K}^{i_1\cdots h\cdots j_p}_{j_1\cdots j_q}\, \delta^{j_\alpha}_k + \sum_{\beta=1}^{q} \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots k\cdots j_q}\, \delta^h_{j_\beta}.$$

Thus, we get the following theorem.

Theorem 10. 2. *For the components* $\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q}$ *of a* $\tilde{\pi}^\circ$ *-image tensor field of type* $(p,\ q)$ *of* $\tilde{\pi}^{\diamond}\ \mathfrak{F}$, *we have the formulas*

$$B_k(\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q}) = \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q,k} \tag{10. 7}$$

$$E_k(\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q}) = \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q;h}\, \widetilde{M}^h_k \tag{10. 8}$$

or

$$Y_k(\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q}) = \widetilde{K}^{i_1\cdots j_p}_{j_1\cdots j_q;k} \tag{10. 8$'$}$$

and

$$Q^h_k(\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q}) = -\sum_{\alpha=1}^{p} \widetilde{K}^{i_1\cdots h\cdots i_p}_{j_1\cdots\cdots j_q}\, \delta^{i_\alpha}_k + \sum_{\beta=1}^{q} \widetilde{K}^{i_1\cdots\cdots i_p}_{j_1\cdots k\cdots j_q}\, \delta^h_{j_\beta}. \tag{10. 9}$$

Since $\mathfrak{z} = \tilde{\pi}^\circ\, \mathfrak{y}$, it follows especially that

$$E_k(y^j) = \langle E_k,\ Dy^j \rangle = \delta^j_k \tag{10. 10}$$

and

$$B_k(y^j) = 0, \quad Q^h_k(y^j) = 0. \tag{10. 11}$$

§ 11. System Σ and Σ_1.

By Theorem 8. 5, the restricted holonomy group $H^0(y)$ is determined by the integral manifolds of the field Π of $(n+1)$-dimensional tangent subspaces of $\widetilde{\mathfrak{B}}$. But the system spanned with $B,\ E_1,\ \cdots,\ E_n$ is not generally involutive. We shall investgate the system.

Lemma 11. 1. *Let* ω^α, $\alpha = 1, 2, \cdots, m$, *be differential forms on an m-dimensional manifold which are linearly independent and* $X_\alpha, \alpha = 1, \cdots, m$ *be the dual tangent vector fields. If we put*

$$d\omega^\alpha = \frac{1}{2} R^\alpha_{\beta\gamma}\, \omega^\beta \wedge \omega^\gamma, \quad R^\alpha_{\beta\gamma} = -R^\alpha_{\gamma\beta} \tag{11.1}$$

then

$$[X_\beta, X_\gamma] = -R^\alpha_{\beta\gamma} X_\alpha \tag{11.2}$$

This lemma is well known, and so we omit the proof.

Now, since (6.10), (6.13), (6.11) are written as

$$\left\{\begin{aligned} d\theta^j &= -\theta^j_i \wedge \theta^i - \frac{1}{2}\widetilde{T}^j_{ik}\,\theta^i \wedge \theta^k - \widetilde{C}^j_{ik}\,\theta^i \wedge \tilde{\gamma}^k, \\ dDy^j &= -\theta^j_i \wedge Dy^i + y^i\,\Theta^j_i, \\ d\theta^j_i &= -\theta^j_k \wedge \theta^k_i + \Theta^j_i, \\ \Theta^j_i &= \frac{1}{2}\widetilde{R}_i{}^j{}_{hk}\,\theta^h \wedge \theta^k + \widetilde{P}_i{}^j{}_{hk}\,\theta^h \wedge \tilde{\gamma}^k + \frac{1}{2}\widetilde{S}^j_{i\,hk}\,\tilde{\gamma}^h \wedge \tilde{\gamma}^k. \end{aligned}\right. \tag{11.3}$$

We obtain by this lemma and (6.11) for Poisson's brackets of the basic tangent vector field B_i, E_i and Q^j_i the following formulas

$$[B_h, B_k] = \widetilde{T}^j_{hk} B_j - y^l \widetilde{R}^j_{l\,hk} E_j - \widetilde{R}^j_{i\,hk} Q^i_j, \tag{11.4}$$

$$[B_h, E_k] = \{\widetilde{C}^j_{hm} B_j - y^l \widetilde{P}_l{}^j{}_{hm} E_j - \widetilde{P}_i{}^j{}_{hm} Q^i_j\}\, \widetilde{M}^m_k, \tag{11.5}$$

$$[E_h, E_k] = \qquad - \{y^l \widetilde{S}_l{}^j{}_{ts} E_j + \widetilde{S}_i{}^j{}_{ts} Q^i_j\}\, \widetilde{M}^t_h \widetilde{M}^s_k, \tag{11.6}$$

$$[B_h, Q^i_j] = -\delta^i_h B_j, \tag{11.7}$$

$$[E_h, Q^i_j] = \qquad -\delta^i_h E_j, \tag{11.8}$$

$$[Q^k_j, Q^i_h] = \delta^k_h Q^i_j - \delta^i_j Q^k_h. \tag{11.9}$$

We denote by $\mathfrak{A}$ the algebra of all differentiable functions on $\widetilde{\mathfrak{B}}_0$ over the real field. Then, in the vector space over $\mathfrak{A}$ of all the tangent vector fields on $\widetilde{\mathfrak{B}}_0$ with the bracket multiplication, we consider the subspace linearly generated by B, $E_1, \cdots, E_n$ which we denote by

$$\Sigma = \{(B = y^h B_h, \quad E_1, \cdots, E_n)\}. \tag{11.10}$$

For any set $\mathfrak{N}$ of tangent vector fields on $\widetilde{\mathfrak{B}}_0$, in the following, we shall denote generally by $\{\mathfrak{N}\} = \{\mathfrak{N}\}_{\mathfrak{A}}$ the subspace which is linearly generated by elements of $\mathfrak{N}$ with coefficients in $\mathfrak{A}$.

Now, we shall calculate the brackets of the elements of Σ. Firstly, we get by (11.5), (4.6) and (10.10) the equations

$$\begin{aligned}[B, E_k] &= [y^h B_h, E_k] = y^h [B_h. E_k] - E_k(y^h) B_h \\ &= \left(y^h \widetilde{C}^j_{hm} B_j - y^l y^h \widetilde{P}_l{}^j{}_{hm} E_j - y^h \widetilde{P}_i{}^j{}_{hm} Q^i_j\right) \widetilde{M}^m_k - B_k,\end{aligned}$$

that is

$$[B,\ E_k] = -\widetilde{M}_k^h \{B_h + y^l\, \widetilde{P}_i{}^j{}_{lh}\, Q_j^i + y^l\, y^m\, \widetilde{P}^j_{lmh}\, E_j\}. \tag{11.11}$$

In order to construct the minimum involutive system containg Σ, we must adjoin the n elements

$$\bar{B}_h = B_h + y^l\, \widetilde{P}_i{}^j{}_{lh}\, Q_j^i \tag{11.12}$$

and $\widetilde{S}_i{}^j{}_{hk}\, Q_j^i$ by (11.6). On the other hand, we have by (9.15)

$$y^h\, \bar{B}_h = y^h\, B_h = B, \tag{11.13}$$

hence if we put

$$\Sigma_1 = \{(\bar{B}_h,\ E_h,\ \widetilde{S}_i{}^j{}_{hk}\, Q_j^i)\}, \tag{11.14}$$

then $\Sigma_1 \supset \Sigma$. By a simple calculation, from (11.8) and (10.9) it follows that

$$[Y_h,\ Q_j^i] = -\delta_h^i\, Y_j. \tag{11.15}$$

§ 12. Ricci formulas

In order to use them in the following, we shall derive the Ricci formulas for $\tilde{\pi}^\circ$-image tensor fields of $\tilde{\pi}^\diamond\mathfrak{F}$ from (11.4)-(11.9) and (10.7)-(10.9). Let $\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q}$ be the components of a $\tilde{\pi}^\circ$-image tensor field of the type $(p,\ q)$. We have firstly

$$\begin{aligned}
\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q,\,k,\,h} - \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q,\,h,\,k} &= (B_h B_k - B_k B_h)\, \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q} \\
&= (\widetilde{T}^t_{hk}\, B_t - y^l\, \widetilde{R}_l{}^t{}_{hk}\, E_t - \widetilde{R}_i{}^j{}_{hk}\, Q_j^i)\, \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q} \\
&= \widetilde{T}^t_{hk}\, \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q,t} - y^l\, \widetilde{R}_l{}^t{}_{hk}\, \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q;\,s}\, \widetilde{M}_t^s \\
&\quad + \widetilde{R}_i{}^j{}_{hk}\, (\sum_\alpha \delta_j^{i_\alpha}\, \widetilde{K}^{i_1\cdots i\cdots i_p}_{j_1\cdots\cdots j_q} - \sum_\beta \delta^i_{j_\beta}\, \widetilde{K}^{i_1\cdots\cdots i_p}_{j_1\cdots j\cdots j_q}),
\end{aligned}$$

hence

$$\begin{aligned}
\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q,k,h} - \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q,h,k} &= \sum_\alpha \widetilde{R}_t{}^{i_\alpha}{}_{hk}\, \widetilde{K}^{i_1\cdots t\cdots i_p}_{j_1\cdots\cdots j_q} - \sum_\beta \widetilde{R}_{j_\beta}{}^t{}_{hk}\, \widetilde{K}^{i_1\cdots\cdots i_p}_{j_1\cdots t\cdots j_q} \\
&\quad + \widetilde{T}^t_{hk}\, \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q,\,t} - y^l\, \widetilde{R}_l{}^t{}_{hk}\, \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q;\,s}\, \widetilde{M}_t^s.
\end{aligned} \tag{12.1}$$

Now, by (5.5) we get from (11.5)

$$\begin{aligned}
[B_h,\ Y_k] = [B_h,\ \widetilde{\Phi}_k^t\, E_t] &= \widetilde{\Phi}_k^t\, [B_h,\ E_t] + \widetilde{\Phi}^t_{k,h}\, E_t \\
&= \widetilde{C}_h{}^j{}_k B_j - y^l\, \widetilde{P}_l{}^j{}_{hk}\, E_j - \widetilde{P}_i{}^j{}_{hk}\, Q_j^i + \widetilde{\Phi}^j_{k,h}\, E_j.
\end{aligned}$$

On the other hand, it is evident that for the natural cross sections dp (3.11) and $d\mathfrak{p}$ (3.12) of $\mathfrak{F}\otimes T^*(\mathfrak{X})$ and $\tilde{\pi}^{\ast}\ \mathfrak{F}\otimes T^*(\widetilde{\mathfrak{B}}_0)$ respectively, their covariant differentials are vanish, that is also represented by

$$D\,\delta^j_i = 0, \tag{12.2}$$

hence we have

$$\widetilde{\Phi}^i_{j,k} = (\delta^i_j + y^l\,\widetilde{C}^i_{lj})_{,k} = y^l\,\widetilde{C}^i_{lj,k}. \tag{12.3}$$

Accordingly, we obtain the formulas

$$[B_h,\ Y_k] = \widetilde{C}^j_{hk}\,B_j + y^l(\widetilde{C}^j_{lk,h} - \widetilde{P}_l{}^j{}_{hk})\,E_j - \widetilde{P}_i{}^j{}_{hk}\,Q^i_j. \tag{12.4}$$

Making use of this formula, (10.7), (10.8′), (10.9), we have

$$\begin{aligned}\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q;k,h} - \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q,h;k} &= (B_hY_k - Y_kB_h)\,\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q}\\ &= \widetilde{C}^j_{hk}\,\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q,j} + y^l(\widetilde{C}^j_{lk.h} - \widetilde{P}_l{}^j{}_{hk})\,\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q;t}\,\widetilde{M}^t_j\\ &+ \widetilde{P}_i{}^j{}_{hk}(\sum_\alpha \delta^{i_\alpha}_j\,\widetilde{K}^{i_1\cdots i\cdots i_p}_{j_1\cdots\cdots j_q} - \sum_\beta \delta^i_{j_\beta}\,\widetilde{K}^{i_1\cdots\cdots i_p}_{j_1\cdots j\cdots j_q}),\end{aligned}$$

that is

$$\begin{aligned}\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q;k,h} - \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q,h;k} &= \sum_\alpha \widetilde{P}_t{}^{i_\alpha}{}_{hk}\,\widetilde{K}^{i_1\cdots t\cdots i_p}_{j_1\cdots\cdots j_q} - \sum_\beta \widetilde{P}_{j_\beta}{}^t{}_{hk}\,\widetilde{K}^{i_1\cdots\cdots i_p}_{j_1\cdots t\cdots j_q}\\ &+ \widetilde{C}^t_{hk}\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q,t} + y^l(\widetilde{C}^t_{lk,h} - \widetilde{P}_l{}^t{}_{hk})\,\widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q;s}\,\widetilde{M}^s_t.\end{aligned} \tag{12.5}$$

Then, we have analogously from (5.5), (11.6) and (10.8′)

$$\begin{aligned}[Y_h,\ Y_k] &= [\widetilde{\Phi}^t_h\,E_t, \widetilde{\Phi}^s_k\,E_s] = \widetilde{\Phi}^t_h\,\widetilde{\Phi}^s_k\,[E_t,\ E_s] + Y_h(\widetilde{\Phi}^s_k)E_s - Y_k(\widetilde{\Phi}^t_h)E_t\\ &= -\,y^l\widetilde{S}_l{}^j{}_{hk}\,E_j - \widetilde{S}_i{}^j{}_{hk}\,Q^i_j + \widetilde{\Phi}^s_{k;h}\,E_s - \widetilde{\Phi}^t_{h;k}\,E_t.\end{aligned}$$

Since we get from (5.5), (10.10) and (12.2) the equations

$$\begin{aligned}\widetilde{\Phi}^i_{h;k} &= (\delta^i_h + y^l\,\widetilde{C}^i_{lh})_{;k} = y^l{}_{;k}\,\widetilde{C}^i_{lh} + y^l\,\widetilde{C}^i_{lh;k}\\ &= \widetilde{\Phi}^l_k\,\widetilde{C}^i_{lh} + y^l\,\widetilde{C}^i_{lh;k} = \widetilde{C}^i_{kh} + y^m\,\widetilde{C}^l_{mk}\,\widetilde{C}^i_{lh} + y^l\,\widetilde{C}^i_{lh;k},\end{aligned}$$

that is

$$\widetilde{\Phi}^i_{h;k} = \widetilde{C}^i_{kh} + y^l(\widetilde{C}^i_{lh;k} + \widetilde{C}_m{}^i{}_h\,\widetilde{C}_l{}^m{}_k), \tag{12.6}$$

we have

$$\begin{aligned}[Y_h,\ Y_k] = \{\,(\widetilde{C}^t_{hk} - \widetilde{C}^t_{kh}) - y^l(\widetilde{S}_l{}^t{}_{hk} + \widetilde{C}^t_{lh;k} - \widetilde{C}_l{}^t{}_{k;h}\\ + \widetilde{C}^t_{mh}\,\widetilde{C}^m_{l\,k} - \widetilde{C}^t_{mk}\,\widetilde{C}^m_{l\,k})\}\,E_t - \widetilde{S}_j{}^t{}_{hk}\,Q^j_t.\end{aligned}$$

The coefficients of E_i of the right hand side of the above equations are rewritten by (9. 6′) as

$$\widetilde{C}^i_{hk} - \widetilde{C}^i_{kh} + y^l\, \widetilde{C}^i_{lm}\, (\widetilde{C}^m_{hk} - \widetilde{C}^m_{kh}) = \widetilde{\Phi}^i_m\, (\widetilde{C}^m_{hk} - \widetilde{C}^m_{kh}),$$

hence we obtain finally the equations

$$[Y_h,\ Y_k] = (\widetilde{C}^i_{hk} - \widetilde{C}^i_{kh})\, Y_i - \widetilde{S}_j{}^i{}_{hk}\, Q^j_i. \tag{12. 7}$$

Making use of (12. 7), we have easily the formulas

$$\begin{aligned} \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q;k;h} - K^{i_1\cdots i_p}_{j_1\cdots j_q;h;k} &= (Y_h Y_k - Y_k Y_h)\, \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q} \\ &= \sum_\alpha \widetilde{S}_t{}^{i_\alpha}{}_{hk}\, \widetilde{K}^{i_1\cdots t\cdots i_p}_{j_1\cdots\cdots j_q} - \sum_\beta \widetilde{S}_{j_\beta}{}^t{}_{hk}\, \widetilde{K}^{i_1\cdots\cdots i_p}_{j_1\cdots t\cdots j_q} \\ &\quad + (\widetilde{C}^t_{hk} - \widetilde{C}^t_{kh})\, \widetilde{K}^{i_1\cdots i_p}_{j_1\cdots j_q;t}. \end{aligned} \tag{12. 8}$$

The formulas (12. 1), (12. 5) and (12. 8) should be called the *Ricci formulas* for the $\bar{\pi}^\circ$ -image tensor fields with respect to $\bar{\pi}^\diamond\, \widetilde{\Gamma}$.

We have obtained the Ricci formulas as an application of the fields B_i, E_i, Q^j_i. But the Ricci formulas for the connection $\widetilde{\Gamma}$ of $\mathfrak{F}$ are similarly true. If we transform the formulas for $\widetilde{\Gamma}$ in $\bar{\pi}^\diamond\, \mathfrak{F}$ by $\bar{\pi}^\circ$, we may obtain (12. 1), (12. 5) and (12. 8). In the following, we shall immediately give the formulas for $\widetilde{\Gamma}$. For the sake of simplicity, we take a tensor field $\mathfrak{K}$ of the type (1. 1). Locally, we put

$$z_{(U)i} = \tau^\circ\, (\partial/u^i), \qquad z_{(U)}{}^j = \tau^\circ\, (du^j). \tag{12. 9}$$

Then we have

$$Dz_{(U)i} = z_{(U)j} \otimes \omega^j_i, \qquad Dz_{(U)}{}^j = -\, z_{(U)}{}^i \otimes \omega^j_i \tag{12. 10}$$

and

$$\nabla^2 z_{(U)i} = z_{(U)j} \otimes \Omega^j_i, \qquad \nabla^2 z_{(U)}{}^j = -\, z_{(U)}^i \otimes \Omega^j_i. \tag{12. 11}$$

Now, let $\mathfrak{K}$ be locally given by

$$\mathfrak{K} = (z_{(U)i} \otimes z_{(U)}^j)\, K^i_j,$$

then from (1. 18), (1. 19) it follows that

$$\begin{cases} \nabla\, \mathfrak{K} = z_{(U)i} \otimes z_{(U)}{}^j \otimes \nabla\, K^i_j \\ \nabla^2 \mathfrak{K} = z_{(U)i} \otimes z_{(U)}{}^j \otimes \nabla^2 K^i_j, \end{cases} \tag{12. 12}$$

where

$$\begin{cases} \nabla K^i_j = DK^i_j, \\ \nabla^2 K^i_j = d(\nabla K^i_j) + \omega^i_t \wedge \nabla K^t_j - \omega^t_j \wedge \nabla K^i_t, \end{cases} \tag{12.13}$$

On the other hand, making use of (1.19) alone, we get

$$\nabla^2 \mathfrak{K} = \left(\nabla^2 z_{(U)i} \otimes z_{(U)}^j + z_{(U)i} \otimes \nabla^2 z_{(U)}^j\right) K^i_j,$$

where the sum of the right hand side should be calculated through a suitable isomorphism on the product $\otimes$. Substituting (12.11) into the equation, it follows that

$$\nabla^2 \mathfrak{K} = z_{(U)i} \otimes z_{(U)}^j \otimes (\Omega^i_t K^t_j - \Omega^t_j K^i_t). \tag{12.14}$$

Thus, we obtain the formulas

$$\nabla^2 K^i_j = \Omega^i_t K^t_j - \Omega^t_j K^i_t$$

and in general

$$\nabla^2 K^{i_1 \cdots i_p}_{j_1 \cdots j_q} = \sum_\alpha \Omega^{i_\alpha}_t K^{i_1 \cdots t \cdots i_p}_{j_1 \cdots \cdots j_q} - \sum_\beta \Omega^t_{j_\beta} K^{i_1 \cdots \cdots i_p}_{j_1 \cdots t \cdots j_q} \tag{12.15}$$

which are equivalent to the Ricci formulas.

On the other hand, we get by (9.9)

$$\begin{aligned} \nabla K^{i_1 \cdots i_p}_{j_1 \cdots j_q} &= K^{i_1 \cdots i_p}_{j_1 \cdots j_q, h} du^h + K^{i_1 \cdots i_p}_{j_1 \cdots j_q; k} \gamma^k \\ \nabla^2 K^{i_1 \cdots i_p}_{j_1 \cdots j_q} &= D(K^{i_1 \cdots i_p}_{j_1 \cdots j_q, h}) \wedge du^h + K^{i_1 \cdots i_p}_{j_1 \cdots j_q, h} \Omega^h \\ &\quad + (D K^{i_1 \cdots i_p}_{j_1 \cdots j_q; k}) \wedge \gamma^k + K^{i_1 \cdots i_p}_{j_1 \cdots j_q; k} \nabla \gamma^k \end{aligned} \tag{12.16}$$

and by (6.8)

$$\begin{aligned} \nabla \gamma^k &= \nabla (M^k_h D\xi^h) = DM^k_h \wedge D\xi^h + M^k_h \nabla^2 \xi^h \\ &= (DM^k_l) \Phi^l_h \wedge \gamma^h + M^k_h \xi^l \Omega^h_l, \end{aligned}$$

that is

$$\nabla \gamma^k = -M^k_l D\Phi^l_h \wedge \gamma^h + M^k_h \xi^l \Omega^h_l. \tag{12.17}$$

Substituting (9.9), (12.3), (12.6), (12.17), (6.6) and (6.7) in the right hand sides of (12.15) and (12.16), we shall obtain the Ricci formulas for the connection $\widetilde{\Gamma}$ of $\mathfrak{F}$.

§. 13. The minimum involutive system Σ_∞ derived from Σ.

In succession we shall calculate the Poisson's brackets of the system Σ_1. By means of (11.4), (11.7), (11.12), (10.7), (10.9), we get

$$[\bar{B}_h, \bar{B}_k] = [B_h + y^{i'} P^{j'}_{i' l' h} Q^{l'}_{j'},\ B_k + y^{i''} P^{j''}_{i'' l'' k} Q^{l''}_{j''}]$$

$$
\begin{aligned}
&= [B_h,\ B_k] + y^l\,\widetilde{P}_i{}^j{}_{lh}\,[Q^i_j,\ B_k] - B_k(y^l\,\widetilde{P}_i{}^j{}_{lh})\,Q^i_j \\
&\qquad\qquad + y^l\,\widetilde{P}_i{}^j{}_{kl}\,[B_h,\ Q^i_j] + B_h(y^l\,\widetilde{P}_i{}^j{}_{lk})\,Q^i_j \\
&+ y^{l'}\,\widetilde{P}_{i'}{}^{j'}{}_{l'h}\,y^{l''}\,\widetilde{P}_{i''}{}^{j''}{}_{l''k}\,[Q^{i'}_{j'},\ Q^{i''}_{j''}] \\
&+ y^{l'}\,\widetilde{P}_{i'}{}^{j'}{}_{l'h}\,Q^{i'}_{j'}(y^{l''}\,\widetilde{P}_{i''}{}^{j''}{}_{l''k})\,Q^{i''}_{j''} - y^{l''}\,\widetilde{P}_{i''}{}^{j''}{}_{l''k}\,Q^{i''}_{j''}(y^{l'}\,P_{i'}{}^{j'}{}_{l'h})\,Q^{i'}_{j'} \\
&= \{\widetilde{T}^j_{hk} - y^l(\widetilde{P}_h{}^j{}_{lk} - \widetilde{P}_k{}^j{}_{lh})\}\,B_j - y^l\,\widetilde{R}_l{}^j{}_{hk}\,E_j \\
&- \{\widetilde{R}_i{}^j{}_{hk} + y^l\,(\widetilde{P}_i{}^j{}_{lh,k} - \widetilde{P}_i{}^j{}_{lk,h}) + y^l\,(\widetilde{P}_h{}^t{}_{lk} - \widetilde{P}_k{}^t{}_{lh})\,y^m\,\widetilde{P}_i{}^j{}_{mt} \\
&\qquad\qquad + y^{l'}y^{l''}(\widetilde{P}_i{}^m{}_{l''k}\,\widetilde{P}_m{}^j{}_{l'h} - \widetilde{P}_i{}^m{}_{l'h}\,\widetilde{P}_m{}^j{}_{l''k})\}\,Q^i_j \\
&= \{\widetilde{T}_h{}^j{}_k - y^l\,(\widetilde{P}_h{}^j{}_{lk} - \widetilde{P}_k{}^j{}_{lh})\}\,\bar{B}_j - y^l\,\widetilde{R}_l{}^j{}_{hk}\,E_j \\
&- \{\widetilde{R}_i{}^j{}_{hk} + y^l\,(\widetilde{P}_i{}^j{}_{lh,k} - \widetilde{P}_i{}^j{}_{lk,h}) + \widetilde{T}^m_{hk}\,y^l\,\widetilde{P}_i{}^j{}_{lm} \\
&\qquad\qquad + y^{l'}y^{l''}(\widetilde{P}_m{}^j{}_{l'h}\,\widetilde{P}_i{}^m{}_{l''k} - \widetilde{P}_m{}^j{}_{l'k}\,\widetilde{P}_i{}^m{}_{l''h})\}\,Q^i_j,
\end{aligned}
$$

hence

$$
\begin{aligned}
[\bar{B}_h,\ \bar{B}_k] \equiv -\{&\widetilde{R}_i{}^j{}_{kh} + y^l(\widetilde{P}_i{}^j{}_{lh,k} - \widetilde{P}_i{}^j{}_{lk,h}) + \widetilde{T}^m_{hk}\,y^l\,\widetilde{P}_i{}^j{}_{lm} \\
&+ y^{l'}\,\widetilde{P}_m{}^j{}_{l'h}\,y^{l''}\,\widetilde{P}_i{}^m{}_{l''k} - y^{l''}\,\widetilde{P}_m{}^j{}_{l''k}\,y^{l'}\,\widetilde{P}_i{}^m{}_{l'h}\}\,Q^i_j \qquad (13.1) \\
&(\mathrm{mod}\ \Sigma_1).
\end{aligned}
$$

We now introduce a convention that for any tensor field of the type (p, q) $(q \geqq 1)$ of $\mathfrak{F}$ with the local components $K^{i_1 \cdots i_p}_{j_1 \cdots j_q}$, $K^{i_1 \cdots\cdots i_p}_{j_1 \cdots 0 \cdots j_q}$ are the local components of the tensor field of the type $(p,\ q-1)$ obtained by the contraction of the tensor product of this tensor field and $\mathfrak{y}$ with respect to its βth covariant sufix and ξ^l. For any tensor field of $\tilde{\pi}^{\diamond}\mathfrak{F}$, we take the same convention by means of the vector field $\mathfrak{z}$, for instance

$$
\widetilde{P}_i{}^j{}_{ok} = y^l\,\widetilde{P}_i{}^j{}_{lk}, \text{ etc. } . \qquad (13.2)
$$

Then, (13.1) is written as

$$
\begin{aligned}
[\bar{B}_h,\ \bar{B}_k] \equiv -\{&\widetilde{R}_i{}^j{}_{hk} + \widetilde{P}_i{}^j{}_{oh,k} - \widetilde{P}_i{}^j{}_{ok,h} + \widetilde{T}^m_{kh}\,\widetilde{P}_i{}^j{}_{om} \\
&+ \widetilde{P}_m{}^j{}_{oh}\,\widetilde{P}_i{}^m{}_{oh} - \widetilde{P}_m{}^j{}_{ok}\,\widetilde{P}_i{}^m{}_{oh}\}\,Q^i_j \\
&(\mathrm{mod}\ \Sigma_1).
\end{aligned}
$$

We get similarly from (10.8), (10.10) the equations

$$
\begin{aligned}
[\bar{B}_h,\ E_k] &= [B_h + \widetilde{P}_i{}^j{}_{oh}\,Q^i_j,\ E_k] \\
&= [B_h,\ E_k] + \widetilde{P}_i{}^j{}_{oh}\,[Q^i_j,\ E_k] - E_k(\widetilde{P}_i{}^j{}_{oh})Q^i_j \\
&\equiv \{\widetilde{C}^m_{ht}\,B_m - \widetilde{P}_i{}^j{}_{ht}\,Q^i_j - (\widetilde{P}_i{}^j{}_{oh})_{;t}\,Q^i_j\}\,\widetilde{M}^t_k
\end{aligned}
$$

$$\equiv -\{\widetilde{P}_{i\ ht}^{\ j} + (\widetilde{P}_{i\ oh}^{\ j})_{;t} + \widetilde{C}_{ht}^{m}\, \widetilde{P}_{i\ om}^{\ j}\}\, \widetilde{M}_k^t\, Q_j^i \qquad (\mathrm{mod}\ \Sigma_1),$$

that is

$$[\bar{B}_h,\ E_k] \equiv -\{\widetilde{P}_{i\ ht}^{\ j} + (\widetilde{P}_{i\ oh}^{\ j})_{;t} + \widetilde{C}_{ht}^{m}\, \widetilde{P}_{i\ om}^{\ j}\}\, \widetilde{M}_k^t\, Q_j^i \quad (\mathrm{mod}\ \Sigma_1). \tag{13.3}$$

Nextly, from (11. 2), (11. 7), (11. 9), (10. 9), (10. 11), we get

$$\begin{aligned}
[\bar{B}_h,\ Q_i^j] &= [B_h + \widetilde{P}_{s\ oh}^{\ t}\, Q_t^s,\ Q_i^j] \\
&= [B_h,\ Q_i^j] + \widetilde{P}_{s\ oh}^{\ t}\,[Q_t^s,\ Q_i^j] - Q_i^j(\widetilde{P}_{s\ oh}^{\ t})\, Q_t^s \\
&= -\delta_h^j\, B_i + \widetilde{P}_{i\ oh}^{\ t}\, Q_t^j - \widetilde{P}_{s\ oh}^{\ j}\, Q_i^s \\
&\quad + \widetilde{P}_{s\ oh}^{\ j}\, Q_i^s - \widetilde{P}_{i\ oh}^{\ t}\, Q_t^j - \delta_h^j\, \widetilde{P}_{s\ oi}^{\ t}\, Q_t^s = -\delta_h^j\, \bar{B}_i,
\end{aligned}$$

thus we obtain the equations analogous to (11. 7) as follows

$$[\bar{B}_h,\ Q_i^j] = -\delta_h^j\, \bar{B}_i. \tag{13.4}$$

Furthermore, for any $\bar{\pi}^\circ$ -image tensor field we get from (10. 7), (10. 9)

$$\bar{B}_h(\widetilde{K}_{j_1\cdots j_q}^{i_1\cdots i_p}) = \widetilde{K}_{j_1\cdots j_q, h}^{i_1\cdots i_p} - \sum_{\alpha=1}^{p} \widetilde{P}_{t\ oh}^{\ i_\alpha}\, \widetilde{K}_{j_1\cdots\cdots j_q}^{i_1\cdots t\cdots i_p} + \sum_{\beta=1}^{q} \widetilde{P}_{j_\beta\ oh}^{\ \ t}\, \widetilde{K}_{j_1\cdots t\cdots j_q}^{i_1\cdots\cdots i_p}. \tag{13.5}$$

By means of (13. 4), (13. 5), we get immediately the equations

$$[\bar{B}_h,\ \widetilde{S}_{j\ hl}^{\ i}\, Q_i^j] \equiv (\widetilde{S}_{j\ kl,h}^{\ i} - \widetilde{P}_{t\ oh}^{\ i}\, \widetilde{S}_{j\ kl}^{\ t} + \widetilde{P}_{j\ oh}^{\ t}\, \widetilde{S}_{t\ kl}^{\ i})\, Q_i^j. \quad (\mathrm{mod}\ \Sigma_1). \tag{13.6}$$

Lastly, we have

$$\begin{aligned}
[\widetilde{S}_{i'\ h'k'}^{\ j'}\, Q_{j'}^{i'},\ \widetilde{S}_{i''\ h''k''}^{\ j''}\, Q_{j''}^{i''}] &= \widetilde{S}_{i'\ h'k'}^{\ j'}\, \widetilde{S}_{i''\ h''k''}^{\ j''}\,[Q_{j'}^{i'},\ Q_{j''}^{i''}] \\
&\quad + \widetilde{S}_{i'\ h'k'}^{\ j'}\, Q_{j'}^{i'}\,(\widetilde{S}_{i''\ h''k''}^{\ j''})\, Q_{j''}^{i''} - \widetilde{S}_{i''\ h''k''}^{\ j''}\, Q_{j''}^{i''}\,(\widetilde{S}_{i'\ h'k'}^{\ j'})\, Q_{j'}^{i'} \\
&\equiv \{\widetilde{S}_{m\ h'k'}^{\ j}\, \widetilde{S}_{i\ h''k''}^{\ m} - \widetilde{S}_{i\ h'k'}^{\ m}\, \widetilde{S}_{m\ h''k''}^{\ j} \\
&\quad + \widetilde{S}_{i'\ h'k'}^{\ j'}\,(-\delta_{j'}^j\, \widetilde{S}_{i\ h''k''}^{\ i'} + \delta_i^{i'}\, \widetilde{S}_{j'\ h''k''}^{\ j}) \\
&\quad - \widetilde{S}_{i''\ h''k''}^{\ j''}\,(-\delta_{j''}^j\, \widetilde{S}_{i\ h'k'}^{\ i''} + \delta_i^{i''}\, \widetilde{S}_{j''\ h'k'}^{\ j})\}\, Q_j^i \quad (\mathrm{mod}\ \Sigma_1),
\end{aligned}$$

that is

$$[\widetilde{S}_{i'\ h'k'}^{\ j'}\, Q_{j'}^{i'},\ \widetilde{S}_{i''\ h''k''}^{\ j''}\, Q_{j''}^{i''}] \equiv -(\widetilde{S}_{m\ h'k'}^{\ j}\, \widetilde{S}_{i\ h''k''}^{\ m} - \widetilde{S}_{m\ h''k''}^{\ j}\, \widetilde{S}_{i\ h'k'}^{\ m})\, Q_j^i \quad (\mathrm{mod}\ \Sigma_1). \tag{13.7}$$

From the above calculations, we see that the vertical tangent vector fields of the right hand sides of (13.1), (13.3), (13.6) and (13.7) must be the newly adjoined elements to the system Σ_1 as generators. For the sake of the following discussion, we shall now introduce some concepts and prove lemmas.

We denote by $\mathfrak{M}$ the set of all the differentiable field of square matrixes of degree n defined on $\widetilde{\mathfrak{B}}_0$. Then, $\mathfrak{M}$ is a Lie algebra over $\mathfrak{A}$ with the ordinaly addition and the bracket multiplication

$$[\boldsymbol{K},\ \boldsymbol{H}] = \boldsymbol{KH} - \boldsymbol{HK},\quad \boldsymbol{K},\ \boldsymbol{H} \in \mathfrak{M}.$$

The dimension $\mathfrak{M}$ is n^2 but it is ∞ as a Lie algebra over the real field. Let any tangent vector field X on $\widetilde{\mathfrak{B}}_0$ operate also on any element $\boldsymbol{K} = (K^i_j) \in \mathfrak{M}$ by

$$X(\boldsymbol{K}) = ((X(K^i_j))) \tag{13.9}$$

To any element $\boldsymbol{K} \in \mathfrak{M}$, we correspond a vertical tangent vector field defined by

$$\boldsymbol{K}\, Q = K^i_j Q^j_i. \tag{13.10}$$

Then, we get immediately

Lemma 13.1. *For any $\boldsymbol{K}$, $\boldsymbol{H} \in \mathfrak{M}$ and any tangent vector field X, we have*

$$X([\boldsymbol{K},\ \boldsymbol{H}]) = [X(\boldsymbol{K}),\ \boldsymbol{H}] + [\boldsymbol{K},\ X(\boldsymbol{H})].$$

By (11.7), (13.4), (11.8), (11.5), we get easily

Lemma 13.2. *For any $\boldsymbol{K} = ((K^i_j))$ and the tangent vector field $X_h = B_h,\ \bar{B}_h,\ E_h,\ Y_h$, we have*

$$[X_h,\ \boldsymbol{K}\, Q] = X_h(\boldsymbol{K})\, Q - K^i_h X_i.$$

Furthermore, by (11.9), (10.9) we get also

Lemma 13.3. *Let $\widetilde{K}^{ii_1\cdots i_p}_{jj_1\cdots j_q}$ and $\widetilde{W}^{hh_1\cdots h_v}_{kk_1\cdots k_w}$ be the components of the $\bar{\pi}^\circ$-images of any two tensor fields of the type $(p+1,\ q+1)$ and $(v+1,\ w+1)$ of $\mathfrak{F}$, then we have*

$$[\widetilde{K}^{ii_1\cdots i_p}_{jj_1\cdots j_q} Q^j_i,\ \widetilde{W}^{hh_1\cdots h_v}_{kk_1\cdots k_w} Q^k_h] \equiv -(\widetilde{K}^{ii_1\cdots i_p}_{tj_1\cdots j_q} \widetilde{W}^{th_1\cdots h_v}_{jk_1\cdots k_w} - \widetilde{K}^{ti_1\cdots i_p}_{jj_1\cdots j_q} \widetilde{W}^{ih_1\cdots h_v}_{tk_1\cdots k_w}) Q^j_i$$
$$(\mathrm{mod}\ \{\widetilde{K}^{ii'_1\cdots i'_p}_{jj'_1\cdots j'_q} Q^j_i,\ \widetilde{W}^{hh'_1\cdots h'_v}_{kk'_1\cdots k'_w} Q^k_h\}).$$

From the components $\widetilde{K}^{ii_1\cdots i_p}_{jj_1\cdots j_q}$, we consider an element of $\mathfrak{M}$ which is written as

$$K^{i_1\cdots i_p}_{j_1\cdots j_q} = ((\widetilde{K}^{i i_1\cdots i_p}_{j j_1\cdots j_q}))$$

in the following.

Now, from the right hand sides of (13. 1) and (13. 3), we define two tensor fields of the type (1. 3) of $\tilde{\pi}^{\diamond}\mathfrak{F}$ with the components such that

$$V_j{}^i{}_{hk} = \widetilde{R}_j{}^i{}_{hk} + \widetilde{P}_j{}^i{}_{oh,k} - \widetilde{P}_j{}^i{}_{ok,h} + \widetilde{T}^m_{hk}\,\widetilde{P}_j{}^i{}_{om}$$
$$+ \widetilde{P}_m{}^i{}_{oh}\,\widetilde{P}_j{}^m{}_{ok} - \widetilde{P}_m{}^i{}_{ok}\,\widetilde{P}_j{}^m{}_{oh}, \qquad (13.11)$$

$$\widetilde{W}_j{}^i{}_{hk} = \widetilde{P}_j{}^i{}_{hk} + (\widetilde{P}_j{}^i{}_{oh})_{:k} + \widetilde{C}^m_{hk}\,\widetilde{P}_j{}^i{}_{om}. \qquad (13.12)$$

Clearly the above tensor fields are the images under $\tilde{\pi}^{\circ}$. Since we have from (10. 10), (10. 11)

$$(\widetilde{P}_j{}^i{}_{oh})_{:k} = \widetilde{P}_j{}^i{}_{oh;k} + \widetilde{P}_j{}^i{}_{mh}\,y^h{}_{;k}$$
$$= \widetilde{P}_j{}^i{}_{oh\cdot k} + \widetilde{P}_j{}^i{}_{mh}\,\widetilde{\Phi}^m_k$$
$$= \widetilde{P}_j{}^i{}_{oh;k} + \widetilde{P}_j{}^i{}_{kh} + \widetilde{P}_j{}^i{}_{mh}\,\widetilde{C}^m_{ok},$$

(13. 12) is written also as

$$\widetilde{W}_j{}^i{}_{hk} = \widetilde{P}_j{}^i{}_{hk} + \widetilde{P}_j{}^i{}_{kh} + \widetilde{P}_j{}^i{}_{mh}\,\widetilde{C}^m_{ok} + \widetilde{P}_j{}^i{}_{om}\,\widetilde{C}^m_{hk} + \widetilde{P}_j{}^i{}_{oh;k}. \qquad (13.12')$$

Let us now denote by $\mathfrak{U} = \mathfrak{U}_\Gamma$ all the elements which are obtained by repeatedly but finitely operating $\bar{B}_i$ and Y_i on $\boldsymbol{V}_{hk}$, $\boldsymbol{W}_{hk}$ and $\boldsymbol{S}_{hk}$. Let us denote also by $\mathfrak{M}_\Gamma$ the subalgebra generated by $\mathfrak{U}$, that is the minimum subalgebra containing $\mathfrak{U}$. By this definition, it follows that $\mathfrak{U}$ is invariant under $\bar{B}_i$, Y_i and $\mathfrak{M}_\Gamma$ is also invariant by virtue of Lemma 13. 1.

Lemma 13. 4. $\bar{B}_h(\mathfrak{M}_\Gamma) = Y_h(\mathfrak{M}_\Gamma) = \mathfrak{M}_\Gamma.$

Now, for any point $\tilde{b} \in \widetilde{\mathfrak{B}}_0$ we define a subset of the Lie algebra over the real field of all square matrixes of degree n by

$$\mathfrak{M}_\Gamma(\tilde{b}) = \{\boldsymbol{K}(\tilde{b}) \mid \boldsymbol{K} \in \mathfrak{M}_\Gamma\} \qquad (13.13)$$

which is clearly a subalgebra from the definition of $\mathfrak{M}_\Gamma$.

Theorem 13. 5. *For any* $\tilde{b} \in \widetilde{\mathfrak{B}}_0$, $y = \tilde{\pi}(\tilde{b})$, $H^0(y)(\tilde{b})$ *is the image of the subgroup of* $GL(n)$ *which is generated by* $\mathfrak{M}_\Gamma(b)$ *by the admissible map* $\tilde{b}$: $GL(n) \to \tilde{\pi}^{-1}(y) = G_y$.

Proof. According to Theorem 8. 5, $H^0(y)(\tilde{b})$ is the connected component containing $\tilde{b}$ of the intersection of G_y and $P(\tilde{b})$ which is the integral manifold through $\tilde{b}$ of the field Π. Since Π is the field of the $(n+1)$ -dimensional subspaces spaned by the tangent vectors of the system Σ, the integral manifolds of Π coincide with the integral manifolds of the minimum and involutive system Σ_∞ containing Σ. Owing to § 11, Σ_∞ is also the minimum and involutive system containing Σ_1. By Lemmas 13.2, 13.3, Σ_∞ is a submodule containing $\Sigma_1 + \mathfrak{M}_\Gamma Q$. Furthermore, by Lemmas 13.2, 13.3 and 13.4, $\Sigma_1 + \mathfrak{M}_\Gamma Q$ is involutive. Hence, we have

$$\Sigma_\infty = \Sigma_1 + \mathfrak{M}_\Gamma Q. \tag{13.14}$$

Let us denote by Π_∞ the field of the tangent subspace spaned by the tangent vectors of the system Σ_∞. Then, at $\tilde{b} \in \tilde{\mathfrak{B}}_0$, Π_∞ is the tangent subspace spaned by $\bar{B}_i(\tilde{b})$, $E_i(\tilde{b})$ and the vertical tangent subspace $\mathfrak{M}_\Gamma(\tilde{b})\, Q(\tilde{b})$. Accordingly, $P(\tilde{b}) \cap G_y$ is a manifold whose tangent space at $\tilde{b}$ is $\mathfrak{M}_\Gamma(\tilde{b})\, Q(\tilde{b})$. Since $H^0(y)$ is a Lie group, the image of the subgroup of $GL(n)$ which is generated by $\mathfrak{M}_\Gamma(\tilde{b})$ undr the map $\tilde{b}$ must be contained in $P(\tilde{b}) \cap G_y$, hence it must be the component of $P(\tilde{b}) \cap G_y$ containing $\tilde{b}$ considering on the dimension of this subgroup.

§ 14. Structure of the holonomy group $H^0(y)$.

We shall investigate the structure of the holonomy group $H^0(y)$. We may obtain immediately from the computations of (13.1)

$$[\bar{B}_h, \bar{B}_k] = (\tilde{T}^i_{hk} - \tilde{P}^i_{hok} + \tilde{P}^i_{koh})\bar{B}_i - \tilde{R}^i_{ohk} E_i - \tilde{V}_j{}^i{}_{hk} Q^j_i \tag{14.1}$$

and we had already (12.7)

$$[Y_h, Y_k] = (\tilde{C}^i_{hk} - \tilde{C}^i_{kh})\, Y_i - \tilde{S}_j{}^i{}_{hk}\, Q^j_i. \tag{14.2}$$

Then, by means of (12.4), (11.15), (10.8′) and (13.12) we obtain

$$\begin{aligned}
[\bar{B}_h, Y_k] &= [B_h + \tilde{P}_j{}^i{}_{oh} Q^j_i, Y_k] \\
&= [B_h, Y_k] + \tilde{P}^i_{koh} Y_i - (y^l \tilde{P}^i_{jlh})_{;k}\, Q^j_i \\
&= \tilde{C}_h{}^i{}_k B_i + (\tilde{C}_o{}^j{}_{k,h} - \tilde{P}_o{}^j{}_{hk})\, \tilde{M}^i_j\, Y_i - \tilde{P}_j{}^i{}_{hk}\, Q^j_i \\
&\qquad + \tilde{P}^i_{koh}\, Y_i - (y^l \tilde{P}_j{}^i{}_{lh})_{:k}\, Q^j_i \\
&= \tilde{C}^i_{hk}\, \bar{B}_i + \{\tilde{P}^i_{koh} + (\tilde{C}_o{}^m{}_{k,h} - \tilde{P}_o{}^m{}_{hk})\, \tilde{M}^i_m\}\, Y_i \\
&\qquad - \{\tilde{P}_j{}^i{}_{hk} + (y^l\, \tilde{P}_j{}^i{}_{lh})_{:k} + \tilde{C}^m_{hk}\, \tilde{P}^i_{jom}\}\, Q^j_i,
\end{aligned}$$

that is

$$[\bar{B}_h, Y_k] = \tilde{C}^i_{hk}\, \bar{B}_i + \{\tilde{P}^i_{koh} + (\tilde{C}^m_{ok,h} - \tilde{P}_o{}^m{}_{hk})\, \tilde{M}^i_m\}\, Y_i - \tilde{W}_j{}^i{}_{hk}\, Q^j_i. \tag{14.3}$$

Now, any element of the set $\mathfrak{U}$ defined in § 13 is of the form such that

$$K_{j_1\cdots j_q} = ((\widetilde{K}^t_{j\, j_1\cdots j_q})), \quad q \geqq 2$$

according to the convention for notations. Hence, by the above formulas and (10. 8′), (10. 9) we have the equations

$$\begin{aligned}(\bar{B}_h\bar{B}_k - \bar{B}_k\bar{B}_h)(K_{j_1\cdots j_q}) &= [\bar{B}_h, \bar{B}_k](K_{j_1\cdots j_q}) \\ &= [V_{hk}, K_{j_1\cdots j_q}] - \sum_\beta \widetilde{V}_{j_\beta}{}^t{}_{hk} K_{j_1\cdots t\cdots j_q} \\ &+ (\widetilde{T}^j_{hk} - \widetilde{P}^j_{hok} + \widetilde{P}^j_{koh})\bar{B}_j(K_{j_1\cdots j_q}) - \widetilde{R}^t_{ohk}\widetilde{M}^j_t K_{j_1\cdots j_q;j}, \end{aligned} \quad (14.4)$$

$$\begin{aligned}(\bar{B}_h Y_k - Y_k\bar{B}_h)(K_{j_1\cdots j_q}) &= [\bar{B}_h, Y_k](K_{j_1\cdots j_q}) \\ &= [W_{hk}, K_{j_1\cdots j_q}] - \sum_\beta \widetilde{W}^t_{j_\beta\, hk} K_{j_1\cdots t\cdots j_q} \\ &+ \widetilde{C}^t_{hk}\bar{B}_j(K_{j_1\cdots j_q}) + \{\widetilde{P}^j_{koh} + (\widetilde{C}^m_{ok,h} - \widetilde{P}^m_{o\,hk})\widetilde{M}^j_m\} K_{j_1\cdots j_q;j} \end{aligned} \quad (14.5)$$

and

$$\begin{aligned}(Y_hY_k - Y_kY_h)(K_{j_1\cdots j_q}) &= [Y_h, Y_k](K_{j_1\cdots j_q}) \\ &= [S_{hk}, K_{j_1\cdots j_q}] - \sum_\beta \widetilde{S}^t_{j_\beta hk} K_{j_1\cdots t\cdots j_q} \\ &+ (\widetilde{C}^j_{hk} - \widetilde{C}^j_{kh}) K_{j_1\cdots j_q;j}. \end{aligned} \quad (14.6)$$

We shall divide $\mathfrak{U}$ into three parts. Let us denote by $\mathfrak{B}_{m+2}$, $\mathfrak{W}_{m+2}$ and $\mathfrak{S}_{m+2}$ the subsets of the elements of $\mathfrak{U}$ which are obtained by operating $\bar{B}_1, \cdots, \bar{B}_n, Y_1, \cdots, Y_n$ repeatedly at most m times on the sets (V_{hk}), (W_{hk}) and (S_{hk}) respectively. If we put

$$\mathfrak{B}_\infty = \bigcup_{m=0}^{\infty} \mathfrak{B}_{m+2}, \quad \mathfrak{W}_\infty = \bigcup_{m=0}^{\infty} \mathfrak{W}_{m+2}, \quad \mathfrak{S}_\infty = \bigcup_{m=0}^{\infty} \mathfrak{S}_{m+2},$$

then $\mathfrak{U} = \mathfrak{B}_\infty \cup \mathfrak{W}_\infty \cup \mathfrak{S}_\infty$.

Theorem 14. 1. *The submodules* $\{\mathfrak{B}_\infty\}$, $\{\mathfrak{W}_\infty\}$ *and* $\{\mathfrak{S}_\infty\}$ *are ideals of the Lie algebra* $\mathfrak{M}_\Gamma$ *over* $\mathfrak{A}$, *and* $\{\mathfrak{U}_\Gamma\} = \mathfrak{M}_\Gamma$

Proof. Let $\mathfrak{K}_{m+2}$, $\mathfrak{H}_{m+2}$ be any two of $\{\mathfrak{B}_{m+2}\}$, $\{\mathfrak{W}_{m+2}\}$, $\{\mathfrak{S}_{m+2}\}$, then we see from (14. 4)-(14. 6) that

$$[\mathfrak{K}_{m+2}, \mathfrak{H}_2] \subset \mathfrak{K}_{m+4}, \quad m = 0, 1, 2, \cdots,$$

where the left hand side representes the set of Poisson's brackets of any elements of $\mathfrak{K}_{m+2}$ and $\mathfrak{H}_2$ and we shall use this notation for any two subsets of $\mathfrak{M}$. Furthermore, making use of Lemma 13. 1, we obtain

$$[\mathfrak{K}_{m+2}, \mathfrak{H}_2] \subset [\mathfrak{K}_{m+3}, \mathfrak{H}_2] + \mathfrak{K}_{m+5} = \mathfrak{K}_{m+5}$$

and inductively

$$[\mathfrak{K}_{m+2}, \mathfrak{H}_{p+2}] \subset \mathfrak{K}_{m+p+4}, \qquad m,\ p = 0,\ 1,\ 2, \cdots. \tag{14. 7}$$

Hence, we have the relations

$$\begin{aligned} &[\mathfrak{K}_\infty, \mathfrak{K}_\infty] \subset \mathfrak{K}_\infty \\ &[\mathfrak{K}_\infty, \mathfrak{H}_\infty] \subset \mathfrak{K}_\infty \cap \mathfrak{H}_\infty \\ &\quad \mathfrak{K}_\infty,\ \mathfrak{H}_\infty = \{\mathfrak{V}_\infty\},\ \{\mathfrak{W}_\infty\},\ \{\mathfrak{S}_\infty\}. \end{aligned} \tag{14. 8}$$

These relations will give the proof of the first part of this theorem. On the second part, since $\{\mathfrak{U}\} = \{\mathfrak{V}_\infty\} + \{\mathfrak{W}_\infty\} + \{\mathfrak{S}_\infty\}$, we get from (14. 8) the relation

$$[\{\mathfrak{U}\}, \{\mathfrak{U}\}] \subset \{\mathfrak{U}\},$$

which shows that $\mathfrak{U}$ is a Lie algebra over $\mathfrak{A}$, hence it must be $\mathfrak{M}_\Gamma$.

Now, we denote by $\mathfrak{S}'_{m+2}$ the set of the elements of $\mathfrak{S}_{m+2}$ which are obtained by operating $Y_1, \cdots, Y_n$ alone and put $\mathfrak{S}'_\infty = \bigcup_{m=0}^{\infty} \mathfrak{S}'_{m+2}$. Then, we obtain analogously from (14. 6)

$$[\{\mathfrak{S}'_{m+2}\}, \{\mathfrak{S}'_{p+2}\}] \subset \{\mathfrak{S}'_{m+p+4}\},$$

hence

$$[\{\mathfrak{S}'_\infty\}, \{\mathfrak{S}'_\infty\}] \subset \{\mathfrak{S}'_\infty\} \tag{14. 9}$$

Thus we obtain the following

Theorem 14. 2. *$\{\mathfrak{S}'_\infty\}$ is a Lie subalgebra of $\mathfrak{M}_\Gamma$ over $\mathfrak{A}$.*

For any point $y \in T_0(\mathfrak{X})$, $x = \tau(y)$, the set of $h_{\bar{c}}$ corresponding to all the closed α-curves at y in $T_x(\mathfrak{X}) - x$ must be a Lie subgroup of $H^0(y)$ since the fundamental group of the sphere of dimension $n-1$ $(n \geqslant 3)$ is vanish. We shall denote this group by $H^*(y)$. Then, we have a theorem analogous to Theorem 13. 5.

Theorem 14. 3. *For any $\tilde{b} \in \tilde{\mathfrak{B}}_0$, $y = \tilde{\pi}(\tilde{b})$, $H^*(y)(\tilde{b})$ is the image of the subgroup of $GL(n)$ which is generated by the Lie algebra $\{\mathfrak{S}'_\infty\}(\tilde{b})$ over the real field under the admissible map $\tilde{b}$.*

§ 15. Derived connections.

According to § § 11 — 14, the homogeneous holonomy group of the connection Γ was determined by the tangent vector fields $\bar{B}_i$, E_i over

$\widetilde{\mathfrak{B}}_0$ and owing to the right hand side of (11. 11) we define a connection of $\tilde{\pi}^{\diamond}\ \mathfrak{F}$ determined by the differential forms on $\widetilde{\mathfrak{B}}_0$ such that

$$'\theta_i^j = \theta_i^j - \widetilde{P}_i{}^j{}_{ok}\,\theta^k \tag{15. 1}$$

in place of θ_i^j for Γ. On the other hand, if we take for each coordinate neighborhood $(U,\ u)$ of $\mathfrak{X}$ the differential forms

$$'\omega_i^j = \omega_i^j - P_i{}^j{}_{ok}\,du^k \tag{15. 2}$$

then $'\omega_i^j$ satisfy (3. 3), (3. 7), (3. 8) by means of (3. 3), (9. 12) with respect to ω_i^j and so the system of $'\omega_i^j$ determine a connection of $\mathfrak{F}$ which is induced from a connection $'\Gamma$ of the vector bundle $\{\mathfrak{Z},\ \mathfrak{S}(\mathfrak{X})\}$ by ρ. We shall call this connection $'\Gamma$ the *derived conection* of Γ. Then, we can easily see that $'\theta_i^j$ are the differential forms for $'\Gamma$.

If we put $'\omega_i^j = {}'\Gamma_{ik}^j\, du^k + {}'C_{ik}^j\, d\xi^k$, we have

$$'\Gamma_{ik}^j = \Gamma_{ik}^j - P_i{}^j{}_{ok}, \quad 'C_{ik}^j = C_{ik}^j, \tag{15. 3}$$

hence

$$'\Phi_i^j = \Phi_i^j, \quad 'M_i^j = M_i^j, \tag{15. 4}$$

that is $\Phi_{'\Gamma} = \Phi_\Gamma$. If we denote by $'D$ the covariant differential with respect to $'\Gamma$, we get from (15. 1)

$$'Dy^j = Dy^j - P_o{}^j{}_{ok}\,\theta^k. \tag{15. 5}$$

Accordingly, the tangent vector fields $'B_i$, $'E_i$, $'Q_i^j$ dual to $'\theta^j$, $'Dy^j$, $'\theta_i^j$ are given by

$$'B_i = B_i + \widetilde{P}_o{}^h{}_{oi}\,E_h + \widetilde{P}_k{}^h{}_{oi}Q_h^k, \quad 'E_i = E_i, \quad 'Q_i^j = Q_i^j. \tag{15. 6}$$

Hence, by (15. 4) and (11. 2) we have immediately

$$'Y_i = Y_i, \tag{15. 7}$$

$$'B_i = \bar{B}_i + \widetilde{P}_o{}^h{}_{oi}\,E_h. \tag{15. 8}$$

These equations show that

$$\{('B_1, \cdots, 'B_n, 'E_1, \cdots, 'E_n)\} = \{(\bar{B}_1, \cdots, \bar{B}_n, E_1, \cdots, E_n)\}. \tag{15. 9}$$

In the following, we shall denote by $H_\Gamma(y)$, etc. with the symbol Γ the homogeneous holonomy group at a point $y \in T_0(\mathfrak{X})$ for the connection $\rho^{\diamond}\ \Gamma$ of $\mathfrak{F}$, etc.

Theorem 15. 1. *For any connection $\widetilde{\Gamma} = \rho^{\diamond}\ \Gamma$, we have*

$$H_\Gamma(y) = \widetilde{H}_{'\Gamma}(y), \qquad H_\Gamma^0(y) = \widetilde{H}_{'\Gamma}^0(y).$$

Proof. On the system Σ and Σ_1 of Γ defined by (11. 10) and (11. 14), we have by (15. 9) the relation

$$\Sigma \subset \{('B_1, \cdots, 'B_n, \ 'E_1, \cdots, 'E_n)\} \subset \Sigma_1,$$

hence the minimul involutive system containing the system $\{('B_1, \cdots, 'B_n, 'E_1, \cdots, 'E_n)\}$ coincides with Σ_∞. $'B_1, \cdots, 'B_n, 'E_1, \cdots, 'E_n$ are the basic horizontal tangent vector fields with respect to $'\Gamma$, the integral manifold of the system $\{('B_1, \cdots, 'B_n, 'E_1, \cdots, E_n)\}$ containing a point $\tilde{b} \in \tilde{\mathfrak{B}}_0$ coincides with the integral manifold $P_\Gamma(\tilde{b}) = P(\tilde{b})$ (§8) of the system Σ. In general, for any connection Γ, we shall denote by $\tilde{P}_\Gamma(b)$ the integral manifold containing a point $\tilde{b} \in \tilde{\Gamma}_0$ of the system $\{(B_1, \cdots, B_n, E_1, \cdots, E_n)\}$ of Γ. Clearly, $\tilde{P}_\Gamma(\tilde{b})$ is the locus of the end points of all horizontal curves with respect to Γ through the point $\tilde{b}$, hence we have the relation analogous to (8. 7)

$$\tilde{H}_\Gamma(y)(\tilde{b}) = \tilde{P}_\Gamma(\tilde{b}) \cap \tilde{\pi}^{-1}(y). \tag{15. 10}$$

As stated above, we have

$$\tilde{P}_{'\Gamma}(\tilde{b}) = P_\Gamma(\tilde{b}).$$

Thus, we obtain from these relations and (8. 7)

$$H_\Gamma(y) = \tilde{H}_{'\Gamma}(y).$$

In the next place, we shall prove the second part of the theorem. The group $H_\Gamma^0(y)$ as the image of $\Omega^0(y)$ under the homormorphism $h_{\bar{c}}$ is the connected component of the identity of $H_\Gamma(y)$ according to Proposition 8. 3. On the other hand, $\tilde{H}_\Gamma^0(y)$ is the image of $\tilde{\Omega}^0(y)$ under $h_{\bar{c}}$ by definition but we have a proposition similar to Proposition 8. 3 on this group considering the homomorphisms

$$\tilde{\Omega}(y)/\tilde{\Omega}^0(y) \to \tilde{H}(y)/\tilde{H}^0(y) \to \text{the fundamental group of } T_0(\mathfrak{X})$$

under the condition that $\mathfrak{X}$ is separable. Hence $\tilde{H}^0(y)$ is the connected component of $\tilde{H}(y)$ containing the identity element. Thus, we obtain the relation

$$H_\Gamma^0(y) = \tilde{H}_{'\Gamma}(y).$$

By virtue of this theorem, with respect to the homogeneous holonomy groups $H_\Gamma(y)$ of a connection Γ of the vector bundle $\{\mathfrak{Z}, \mathfrak{S}(\mathfrak{X})\}$, it is sufficient that we may consider the homogeneous holonomy groups $\widetilde{H}_{'\Gamma}(y)$ in a wide sense of the derived connection $'\Gamma$ of Γ which are obtained regarding $\mathfrak{S}(\mathfrak{X})$ merely as a differentiable manifold.

Theorem 15. 2. *In order that for a connection Γ of $\{\mathfrak{Z}, \mathfrak{S}(\mathfrak{X})\}$ we have $H_\Gamma(y) = \widetilde{H}_\Gamma(y)$, for any $y \in T_0(\mathfrak{X})$, it is necessary and sufficient that $\boldsymbol{P}_{oh} = (\widetilde{P}_j{}^i{}_{oh})$ defined by the curvature tensor of the second kind of Γ belong to $\{\mathfrak{U}_\Gamma\}$.*

Proof. By means of (8. 7) and (15. 10), for any $y \in T_0(\mathfrak{X})$, in order that $H_\Gamma(y) = \widetilde{H}_\Gamma(y)$, it is necessary and sufficient that the integral manifolds of the system $\sum = \{(B, E_1, \cdots, E_n)\}$ and the system $\{B_1, \cdots, B_m, E_1, \cdots, E_n)\}$ have always the same intersections with the fibres of $\{\widetilde{\mathfrak{B}}_0, T_0(\mathfrak{X})\}$. Furthermore, the two systems of integral manifolds contains horizontal curves over a-curves of $T_0(\mathfrak{X})$. Hence, it is equivalent to that the integral manifolds of the system Σ and $\{(B_1, \cdots, B_m, E_1, \cdots, E_n)\}$ coincide with each others. Hence it is necesary and sufficient that Σ and the system $\{(B_1, \cdots, B_m, E_1, \cdots, E_n)\}$ have the same minimum involutive system containing them. On the other hand, from (11. 4) — (11. 9), non vertical elements of the minimum involutive systems of Σ and $\{(B_1, \cdots, B_L, E_1, \cdots, E_n)\}$ are linear combinations of $\bar{B}_1, \cdots, \bar{B}_m, E_1, \cdots, E_n$ and $B_1, \cdots, B_m, E_1, \cdots, E_n$ respectively with coefficients in $\mathfrak{A}$. Hence the condition that the two minimum involutive systems are the same system is

$$y^l \widetilde{P}_i{}^j{}_{lh} Q^i_j \in \mathfrak{M}_\Gamma Q \quad \text{or } \boldsymbol{P}_{oh} \in \mathfrak{M}_\Gamma$$

by virtue of (11. 12). According to Theorem 14. 1, we have $\{\mathfrak{U}_\Gamma\} = \mathfrak{M}_\Gamma$, hence we get the condition $\boldsymbol{P}_{oh} \in \{\mathfrak{U}_\Gamma\}$.

Corollary 15. 3. *If, for the curvature tensor $P_i{}^j{}_{hk}$ of the second kind of Γ, $y^l P_i{}^j{}_{lk}$ vanishes everywhere, then we have*

$$H_\Gamma(y) = \widetilde{H}_\Gamma(y).$$

From the definition of $'\Gamma$ we have immediately

Theorem 15. 4. *For the derived connection $'\Gamma$ of Γ, in order that $'\Gamma = \Gamma$ it is necessary and sufficient* that $y^l P_i{}^j{}_{lk}$ of Γ *vanishes everywhere.*

We shall say that a connection Γ of $\{\mathfrak{Z}, \mathfrak{X}(\mathfrak{X})\}$ is *h-proper* if $\Gamma = {'\Gamma}$.

Theorem 15. 5. *For any connection Γ of $\{\mathfrak{Z}, \mathfrak{S}(\mathfrak{X})\}$, we have*

$$''\Gamma = '('\Gamma) = '\Gamma.$$

Proof. By means of (15.1) — (15.7), if we consider the formulas (11.4) — (11.9) made for the derived connection $'\Gamma$ of Γ, we have

$$\begin{aligned}[][{}'B_h,\ {}'E_k] &= \{\widetilde{C}^j_{hm}\,{}'B_j - {}'\widetilde{P}_o{}^j{}_{hm}\,E_j - {}'\widetilde{P}_i{}^j{}_{hm}\,Q^i_j\}\ \widetilde{M}^m_k \\ &\equiv -\,{}'\widetilde{P}_i{}^j{}_{hm}\,\widetilde{M}^m_k\,Q^i_j \quad (\mathrm{mod}\ \{(\bar{B}_1, \cdots, \bar{B}_n,\ E_1, \cdots, E_n)\}).\end{aligned}$$

Furthermore, from (15.8), (13.3) we get

$$\begin{aligned}[][{}'B_h,\ {}'E_k] &= [\bar{B}_h + \widetilde{P}^l_{ooh}\,E_l,\ E_k] \\ &= [\bar{B}_h,\ E_k] + \widetilde{P}^l_{ooh}\,[E_l,\ E_k] - E_k(\widetilde{P}_o{}^i{}_{oh})E_i \\ &\equiv -(\widetilde{W}_i{}^j{}_{hm}\,\widetilde{M}^m_k + \widetilde{P}^l_{ooh}\,\widetilde{S}_i{}^j{}_{ts}\,\widetilde{M}^t_l\,\widetilde{M}^s_k)\,Q^i_j \\ &\qquad\qquad (\mathrm{mod}\ \{(\bar{B}_1, \cdots, \bar{B}_n,\ E_1, \cdots, E_n)\}.\end{aligned}$$

Accordingly, we have

$$'\widetilde{P}_i{}^j{}_{hk} = \widetilde{W}_i{}^j{}_{hk} + \widetilde{P}_o{}^l{}_{oh}\,\widetilde{M}^t_l\,\widetilde{S}_i{}^j{}_{tk}. \tag{15.11}$$

By (9.16) we have

$$'\widetilde{P}_i{}^j{}_{ok} = y^h\,{}'\widetilde{P}_i{}^j{}_{hk} = y^h\,\widetilde{W}_i{}^j{}_{hk}. \tag{15.12}$$

On the other hand, we get from (13.12)

$$\begin{aligned}\widetilde{W}_j{}^i{}_{ok} &= \widetilde{P}_i{}^j{}_{ok} + y^h\,(\widetilde{P}_i{}^j{}_{oh})_{;k} + \widetilde{C}^m_{ok}\,\widetilde{P}_i{}^j{}_{om} \\ &= \widetilde{P}_i{}^j{}_{ok} + (y^h\,\widetilde{P}_i{}^j{}_{oh})_{;k} - \widetilde{P}_i{}^j{}_{oh}\,y^h{}_{;k} + \widetilde{C}^m_{ok}\,\widetilde{P}_i{}^j{}_{om} \\ &= \widetilde{P}_i{}^j{}_{ok} - \widetilde{P}_i{}^j{}_{oh}\,\widetilde{\Phi}^h_k + \widetilde{C}^m_{ok}\,\widetilde{P}_i{}^j{}_{om} = 0,\end{aligned}$$

that is

$$'P_i{}^j{}_{ok} = 0. \tag{15.13}$$

According to Theorem 15.4, it must be $''\Gamma = {}'\Gamma$.

Part II.

§ 16. A projective vector bundle $\mathfrak{F}_0$ over $T_0(\mathfrak{X})$.

To make clear the following discussions, we shall now prepare some concepts. Let $\{\hat{\mathfrak{B}}, T(\mathfrak{X})\boxtimes T_0(\mathfrak{X}), \hat{\pi}, GL(n), GL(n)\}$ be the induced principal fibre bundle of the principal fibre bundle $\{\widetilde{\mathfrak{B}}_0, T_0(\mathfrak{X}), \bar{\pi}_0, GL(n), GL(n)\}$ which is the associated principal fibre bundle of the vector bundle $\mathfrak{F} = \{T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X}), T_0(\mathfrak{X}), \bar{\tau}_0, R^n, GL(n)\}$ by the map $\bar{\tau}_0 : T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X}) \to T_0(\mathfrak{X})$ and denote by $\bar{\tau}_p : \hat{\mathfrak{B}} \to \widetilde{\mathfrak{B}}_0$ the induced bundle map. Since $\bar{\tau}_0 \cdot \hat{\pi} = \bar{\pi}_0 \cdot \bar{\tau}_p$, we denote this map by $\bar{\mu} : \hat{\mathfrak{B}} \to T_0(\mathfrak{X})$. Now, for any point

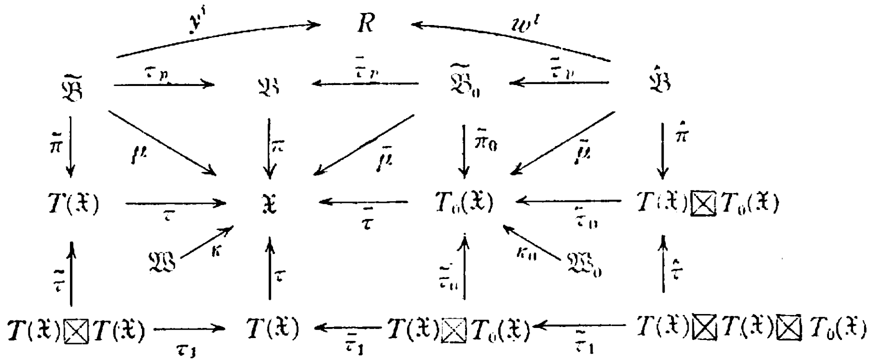

Diagram 3

$\hat{b} \in \hat{\mathfrak{B}}$, if we put $b = \tau_p(\bar{\tau}_p(\hat{b}))$, $y = \bar{\mu}(\hat{b})$, $w = \tau_1(\hat{\pi}(\hat{b}))$, then y and w are written uniquely as

$$y = y^i \mathrm{e}_i(b), \qquad w = w^i \mathrm{e}_i(b).$$

Hence w^i may be regarded as n maps

$$w^i : \hat{\mathfrak{B}} \to R. \tag{16.1}$$

Clearly, it follows that $\hat{\mathfrak{B}} = \tilde{\mathfrak{B}}_0 \times R^n$ as a manifold. For a coordinate neighborhood (U, u) of $\mathfrak{X}$, let (u^i, ξ^i) be its canonical coordinates of $T(\mathfrak{X})$ by (3.1), then w is written as

$$w = \eta^i \frac{\partial}{\partial u^i}(x). \tag{16.2}$$

Hence we may regard $(u^i ; \xi^i ; \eta^i)$ as local coordinates of $T(\mathfrak{X}) \boxtimes T(\mathfrak{X})$ for (U, u), and so we shall call them its canonical coordinates in $T(\mathfrak{X}) \boxtimes T(\mathfrak{X})$. By means of (2.4) we may also consider $(u^j ; \xi^j ; \eta^j ; a_i^j)$ for $b \in (\bar{\tau} \cdot \bar{\mu})^{-1}(U)$ as its canonical coordinates in $\hat{\mathfrak{B}}$. By virtue of (3.14) we have of course

$$y^j = b_i^j \xi^i, \quad w^j = b_i^j \eta^i, \quad \mathrm{e}_i(b) = a_i^j \partial / \partial u^j. \tag{16.3}$$

Now, let $(\bar{U}, \bar{u})$ be another coordinate neighborhood of $\mathfrak{X}$ such that $U \cap \bar{U} \neq \emptyset$ and $(\bar{u}^j ; \bar{\xi}^j ; \bar{\eta}^j ; \bar{a}_i^j)$ be the canonical coordinates for $(\bar{U}, \bar{u})$, then we have on $(\bar{\tau} \cdot \bar{\mu})^{-1}(U \cap \bar{U})$ the equations

$$\bar{u}^j = \bar{u}^j(u), \quad \bar{\xi}^j = \frac{\partial \bar{u}^j}{\partial u^i} \xi^i, \quad \bar{\eta}^j = \frac{\partial \bar{u}^j}{\partial u^i} \eta^i, \quad \bar{a}_i^j = \frac{\partial \bar{u}^j}{\partial u^k} a_i^k. \tag{16.4}$$

If we put

$$a_0^j = \eta^j, \tag{16.5}$$

then (16.4) is written as

$$\bar{u}^j = \bar{u}^j(u), \quad \bar{\xi}^j = \frac{\partial \bar{u}^j}{\partial u^i} \xi^i, \quad \bar{a}_\alpha^j = \frac{\partial \bar{u}^j}{\partial u^k} a_\alpha^k, \tag{16.4'}$$

$$\alpha = 0, 1, \cdots, n.$$

In the following, we assume that $\alpha, \beta, \cdots$ run over $0, 1, \cdots, n$.

Nextly we define a vector bundle $\mathfrak{F} = \{\mathfrak{W}, \mathfrak{X}, \kappa, R^{n+1}, G\}$ as follows: The group of bundle G is the subgroup of $GL(n+1)$ such that

$$a_0^0 = 1, \qquad a_i^0 = 0 \tag{16.6}$$

regarding $x^0, x^1, \cdots, x^n$ are the coordinates of R^{n+1}. For any two coodinate neighborhoods (U, u), $(\bar{U}, \bar{u})$ of $\mathfrak{X}$ such that $U \cap \bar{U} \neq \emptyset$ the coordinate transformation $g_{\bar{U}U} : U \cap \bar{U} \to G$ of $\mathfrak{F}$ is given by

$$g_{\bar{U}U} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{\partial \bar{u}^j}{\partial u^i} \end{pmatrix}. \tag{16.7}$$

Clearly (16. 7) satisfy (1. 2), hence $\mathfrak{F}$ may be defind. We denote by φ_U the coordinate function of $\mathfrak{F}$ corresponding to U. Let $\delta_\alpha = (\delta_\alpha^\beta)$ be the αth unit vectors of the $(n+1)$ axises of R^{n+1}. Then we have

$$\varphi_U(x,\ g\delta_\alpha) = \varphi_U(x,\ \delta_\beta)a_\alpha^\beta(g), \quad g \in G,$$

in general, for a vector bundle. If we put for $\bar{g} = g_{\bar{U}U}(x)g$

$$\varphi_{\bar{U}}(x,\ \delta_\beta)a_\alpha^\beta(\bar{g}) = \varphi_U(x,\ \delta_\beta)a_\alpha^\beta(g),$$

then we get immediately

$$a_\alpha^\beta(\bar{g}) = a_\gamma^\beta(g_{\bar{U}U}(x))a_\alpha^\gamma(g) \tag{16. 8}$$

which is the last equations of (16. 4′). Thus, we shall easily obtain

Proposition 16. 1. *$\{\widetilde{\mathfrak{B}},\ \mathfrak{X},\ \mu\}$ and $\{\hat{\mathfrak{B}},\ T_0(\mathfrak{X}),\ \tilde{\mu}\}$ is the associated principal fibre bundles of $\mathfrak{F}$ and $\bar{\tau}^\diamond \mathfrak{F} = \mathfrak{F}_0 = \{\mathfrak{W}_0,\ T_0(\mathfrak{X}),\ \kappa_0,\ R^{n+1},\ G\}$ respectively and $\{\hat{\mathfrak{B}},\ T_0(\mathfrak{X}),\ \tilde{\mu}\} = \bar{\tau}^\diamond \{\widetilde{\mathfrak{B}},\ \mathfrak{X},\ \mu\}$.*

We shall call $\mathfrak{F}$ and $\mathfrak{F}_0$ the *induced projective vector bundles* of $T(\mathfrak{X})$ and $\mathfrak{F}$. Let $\lambda_k(k>0)$ operate naturally on $\mathfrak{W}_0$ and $\hat{\mathfrak{B}}$ so as to commute with κ_0 and $\hat{\pi}$, then $\{\mathfrak{W}_0/R^+,\ \mathfrak{S}(\mathfrak{X})\}$ is similarly considered as the induced projective vector bundle of $\{\mathfrak{Z},\ \mathfrak{S}(\mathfrak{X})\}$. Since the fibre of $\bar{\tau}^\diamond\mathfrak{F}$ at $(w,\ y) \in T_x(\mathfrak{X}) \times (T_x(\mathfrak{X}) - x)$ is clearly $T_x(\mathfrak{X}) \times (w,\ y)$, we can define a natural cross section $\mathfrak{w}$: $T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X}) \to T(\mathfrak{X}) \boxtimes T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X})$ of $\bar{\tau}^\diamond \mathfrak{F}$ by corresponding $(w,\ y)$ to $(w,\ w,\ y)$. Let $\bar{\tau}_1$: $T(\mathfrak{X}) \boxtimes T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X}) \to T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X})$ be the induced bundle map of the induced bundle $\bar{\tau}^\diamond\mathfrak{F}$, and $\bar{\tau}_p : \hat{\mathfrak{B}} \to \widetilde{\mathfrak{B}}_0$ be the induced bundle map of the induced bundle $\bar{\tau}^\diamond\{\widetilde{\mathfrak{B}}_0,\ T_0(\mathfrak{X})\}$. In the following, we shall denote $\bar{\tau}^\circ\mathfrak{y}$ and $y^i \cdot \bar{\tau}_p$ by the same symbols $\mathfrak{y}$ and y^i respectively.

§ 17. The induced projective connection Γ_0 of Γ.

From a connection $\widetilde{\Gamma}$ of $\mathfrak{F}$, we shall define a connection $\widetilde{\Gamma}_0$ of $\mathfrak{F}_0$ by the system of differential forms on each coordinate neighborhood $\bar{\tau}^{-1}(U)$ of $\mathfrak{F}_0$ such that

$$\omega_0^0 = \omega_i^0 = 0, \quad \omega_0^j = du^j, \quad \omega_i^j \tag{17. 1}$$

where ω_i^j are the differential forms on $\bar{\tau}^{-1}(U)$ of the connection $\widetilde{\Gamma}$. We see easily that (17. 1) satisfies (1. 9) and (1. 10) by (3. 3) and (16. 7). Hence, the connection $\widetilde{\Gamma}_0$ can be defined. We shall call $\widetilde{\Gamma}_0$ the *induced projective connection* of $\widetilde{\Gamma}$. The differential forms $\hat{\theta}_\alpha^\beta$ of $\widetilde{\Gamma}_0$ which are defined on $\hat{\mathfrak{B}}$

by (1.13) are written locally as

$$\hat{\theta}^{\beta}_{\alpha} = b^{\beta}_{\gamma}(da^{\gamma}_{\alpha} + \omega^{\gamma}_{\rho}a^{\rho}_{\alpha})$$

regarding u^j, ξ^j, η^j, a^j_i as local coordinates of $\hat{\mathfrak{B}}$. Making use of (16.6), (17.1), we have

$$\hat{\theta}^0_{\alpha} = 0, \quad \hat{\theta}^j_i = \theta^j_i,$$

$$\hat{\theta}^j_0 = \theta^j + b^j_k(da^k_0 + \omega^k_h a^h_0) \tag{17.2}$$

where θ^j_i, θ^j are differential forms on $\widetilde{\mathfrak{B}}_0$ but we used the same symbols for their $\tilde{\tau}^*_p$ images.

If $\widetilde{\Gamma} = \rho^{\diamond}\Gamma$, there exists a connection Γ_0 of $\{\mathfrak{W}_0/R^+, \mathfrak{S}(\mathfrak{X})\}$ such that $\widetilde{\Gamma}_0 = \rho^{\diamond}\Gamma_0$. We call Γ_0 also the induced projective connection of Γ.

On the other hand, the natural vector field $\mathfrak{w}$ of $\tilde{\tau}^{\diamond}\mathfrak{F}$ has locally η^j as its components with respect to the canonical coordinates $(u^j\,;\ \xi^j\,;\ \eta^j)$, and so its covariant differential $D\mathfrak{w}$ with respect to the induced connection $\hat{\Gamma} = \tilde{\tau}^{\diamond}\widetilde{\Gamma}$ is written as

$$D\mathfrak{w} = \frac{\partial}{\partial u_j} \otimes D\eta^j, \quad D\eta^j = d\eta^j + \eta^i\omega^j_i. \tag{17.3}$$

The vector field $\hat{\pi}^c\mathfrak{w}$ of the vector bundle $\hat{\pi}^{\diamond}(\tilde{\tau}^{\diamond}\mathfrak{F})$ has w^j as its components with respect to its natural base, hence we get from (16.3) the equations

$$Dw^j = dw^j + w^i\theta^j_i = b^j_i D\eta^i. \tag{17.4}$$

Accordingly, we get by means of (16.5) the equation

$$\hat{\theta}^j_0 = \theta^j + Dw^j. \tag{17.5}$$

Thus, from the above considerations, regarding $\hat{\mathfrak{B}}$ as the bundle spaces of the principal fibre bundles $\{\hat{\mathfrak{B}},\ T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X}),\ \hat{\pi},\ GL(n),\ GL(n)\}$ and $\{\hat{\mathfrak{B}},\ T_0(\mathfrak{X}),\ \bar{\mu},\ G,\ G\}$, we obtain two set of $n^2 + 3n$ differential forms

$$(\theta^j,\ Dy^j,\ Dw^j,\ \theta^j_i) \tag{17.6}$$

and

$$(\theta^j,\ Dy^j,\ \hat{\theta}^j_0,\ \hat{\theta}^j_i) \tag{17.7}$$

which have geometrical significances corresponding to their structures respectively, when Γ is regular.

Theorem 17. 1. *The affine holonomy group of a connection* $\widetilde{\Gamma}$ *of the vector bundle* $\mathfrak{F} = \{T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X}),\ T_0(\mathfrak{X})\}$ *is isomorphic with the holonomy group of the induced projective connection* $\widetilde{\Gamma}_0$ *of* $\widetilde{\Gamma}$.

Proof. To prove this theorem, it is sufficient to say the following. Let a curve $\bar{C}$ of $T_0(\mathfrak{X})$ be given by $\bar{f}:\ I \rightarrow T_0(\mathfrak{X})$. The developments of $\bar{C}$ and affine frames along $\bar{C}$ are obtained from the solutions of the Pfaffian equations on $R^n \times GL(n) \times \bar{\pi}^{-1}(\bar{C})$

$$dp^i = \mathrm{e}^i_j\theta^j, \qquad d\mathrm{e}^i_i = \mathrm{e}^i_j\theta^j_i \tag{17.8}$$

as stated in § 6. On the other hand, the developments of frames along $\bar{C}$ with respect to $\widetilde{\Gamma}_0$ may be likewise obtained from the solutions of the Pfaffian equations on $GL(n+1) \times \bar{\mu}^{-1}(\bar{C})$

$$dA^i_\alpha = A^i_\beta\,\hat{\theta}^\beta_\alpha. \tag{17.9}$$

Clearly, as a manifold, the affine transformation group of dimension n

$$\bar{x}^j = a^j_i x^i + a^j, \qquad |\ a^j_i\ | \neq 0$$

is $R^n \times GL(n)$. Putting $a^j_0 = a^j,\ a^0_0 = 1,\ a^0_i = 0,$ the group is isomorphic with $G \subset GL\ (n+1)$. In this sense, if we regard any solution $p^i(\bar{b}),\ \mathrm{e}^i_i(\bar{b})$ of (17. 8) as functions on $\bar{\mu}^{-1}(\bar{C})$ by setting $p^i(\bar{\tau}_p(\hat{b})),\ \mathrm{e}^i_i(\bar{\tau}_p(\hat{b}))$, we can compare them with solutions of (17. 9). By means of (17. 2) and (17. 5), in fact, (17. 9) is written as

$$dA^i_0 = A^i_j(\theta^j + Dw^j),\ \ dA^i_i = A^i_j\theta^j_i,$$

hence if we put

$$A^i = A^i_0 - A^i_j w^j, \tag{17.10}$$

we get

$$dA^i = dA^i_0 - A^i_j Dw^j = A^i_j\theta^j.$$

In place of (17. 9), we may consider

$$dA^i = A^i_j\theta^j,\ dA^i_i = A^i_j\theta^j_i. \tag{17.9$'$}$$

Accordingly, if we put for any solution $p^i,\ \mathrm{e}^i_i$ of (17. 8)

$$A^i = p^i + \delta_0, \qquad A^i_i = \mathrm{e}^i_i,$$

then $A^i,\ A^i_i$ are a solution of (17. 9′). This shall immediately lead the verification of the theorem by virtue of the definitions of holonomy groups.

§ 18. The basic tangent vector fields of Γ_0 on $\hat{\mathfrak{B}}$.

Let $(\hat{B}_i,\ \hat{E}_i,\ \hat{Q}^0_i,\ \hat{Q}^j_i)$ and $(B_i,\ E_i,\ W_i,\ Q^j_i)$ be the suits of tangent vector fields on $\hat{\mathfrak{B}}$ dual to $(\theta^j,\ Dy^j,\ \hat{\theta}^j_0,\ \hat{\theta}^j_i)$ and $(\theta^j,\ Dy^j,\ dw^j,\ \theta^j_i)$ respectively. Since $\hat{\mathfrak{B}} = \widetilde{\mathfrak{B}}_0 \times R^n$, B_i, E_i and Q^j_i are given locally by the same equations as (5. 1), (5. 2), (5. 3) with respect to local coordinates u^j, ξ^j, a^j_α of $\hat{\mathfrak{B}}$ and we have

$$W_i = \partial/\partial w^i = a^k_i \partial/\partial a^k_0. \tag{18. 1}$$

Hence, by means of (17. 4), (17. 5) it must be

$$\hat{B}_i = B_i - W_i,\quad \hat{E}_i = E_i,\quad \hat{Q}^0_i = W_i,\quad \hat{Q}^j_i = Q^j_i - w^j W_i. \tag{18. 2}$$

On the other hand, the equations of structure on $\hat{\mathfrak{B}}$ for the connection $\hat{\Gamma}$ as an affine connection are

$$d\theta^j = -\theta^j_i \wedge \theta^i + \Theta^j,\qquad d\theta^j_i = -\theta^j_k \wedge \theta^k_i + \Theta^j_i \tag{18. 3}$$

by (6. 10), (6. 11). Furthermore, from the covariant differentials of the fields $\bar{\tau}^{\circ}_{p}\mathfrak{z}$, $\hat{\tau}^{\circ}\mathfrak{w}$ of $\bar{\mu}^{\diamond}\mathfrak{T}$, we can obtain similarly to (6. 13)

$$dDy^j = -\theta^j_i \wedge Dy^i + y^i\Theta^j_i,\qquad dDw^j = -\theta^j_i \wedge Dw^i + w^i\Theta^j_i. \tag{18. 4}$$

From the above definitions and Lemma 13. 1 we have

$$[W_i,\ B_j] = [W_i,\ E_j] = [W_i,\ Q^k_h] = [W_i,\ W_j] = 0. \tag{18. 5}$$

Needless to say, (11. 4) — (11. 9) hold good for the Poisson's brackets of B_i, E_i, Q^j_i. Furthermore, the equations of structure on $\hat{\mathfrak{B}}$ for the induced projective connection $\widetilde{\Gamma}_0$ are clearly

$$\begin{cases} d\hat{\theta}^j_0 = -\hat{\theta}^j_i \wedge \hat{\theta}^i_0 + \Theta^j + w^i\Theta^j_i, \\ d\hat{\theta}^j_i = -\hat{\theta}^j_k \wedge \hat{\theta}^k_i + \Theta^j_i \end{cases} \tag{18. 6}$$

by (17. 2), (17. 5). The tangent vector fields on $\hat{\mathfrak{B}}$ dual to $(\theta^j,\ Dy^j,\ Dw^j,\ \theta^j_i)$ are $(B_i,\ E_i,\ W_i,\ \hat{Q}^j_i)$.

Nextly, we shall define covariant differentiations for tensor fields of $\bar{\tau}^{\diamond}\mathfrak{T}$ analogous to (9. 7) and (9. 8). For instance, let a vector field $\mathfrak{v}$ of the type (1, 0) have V^i as its components with respect to $\partial/\partial u_i$. Then, by (4. 8), (6. 5), (17. 3) we have

$$\begin{aligned} DV^j &= dV^j + \omega^j_i V^i \\ &= \frac{\partial V^j}{\partial u^h}du^h + \frac{\partial V^j}{\partial \xi^h}d\xi^h + \frac{\partial V^j}{\partial \eta^h}d\eta^h + \Gamma^{*j}_{ih}V^i du^h + C^j_{ih}V^i\gamma^h \\ &= \frac{\partial V^j}{\partial u^h}du^h + \frac{\partial V^j}{\partial \xi^h}(\gamma^h - \Gamma^{*h}_{lk}\xi^l du^k) \end{aligned}$$

$$+\frac{\partial V^j}{\partial \eta^h}(D\eta^h - \Gamma^{*h}_{ik}\eta^l du^k - C^h_{lk}\eta^l \gamma^k) + \Gamma^{*j}_{ih} V^i du^h + C^j_{ih} V^i \gamma^h.$$

We define the covariant derivatives $V^j_{,h}$, $V^j_{;h}$, $V^j_{:h}$ by

$$V^j_{,h} = \frac{\partial V^j}{\partial u^h} - \frac{\partial V^j}{\partial \xi^t}\Gamma^{*t}_{lh}\xi^l - \frac{\partial V^j}{\partial \eta^t}\Gamma^{*t}_{lh}\eta^l + \Gamma^{*j}_{ih} V^i, \qquad (18.7)$$

$$V^j_{;h} = \frac{\partial V^j}{\partial \xi^h} - \frac{\partial V^j}{\partial \eta^t} C^t_{lh}\eta^l + C^j_{ih} V^i, \qquad (18.8)$$

$$V^j_{:h} = \frac{\partial V^j}{\partial \eta^h}. \qquad (18.9)$$

Then DV^j are written as

$$DV^j = V^j_{,h}\, du^h + V^j_{;h}\gamma^h + V^j_{:h} D\eta^h. \qquad (18.10)$$

We can easily verify that $V^j_{,h}$, $V^j_{;h}$ and $V^j_{:h}$ are the components of tensor fields of $\tilde{\tau}^{\diamond}\mathfrak{F}$ of the type (1.1). Let $\hat{V}^j$ be the components of $\hat{\pi}^{\circ}\mathfrak{v}$ with respect to the natural bases $(\tilde{\tau}_p)^{\circ}\mathfrak{z}_i$, then $\hat{V}^j$ are represented locally as

$$\hat{V}^j = b^j_i V^i \qquad (18.11)$$

and by means of (1.17), (17.4) we have

$$D\hat{V}^j = \hat{V}^j_{,h}\theta^h + \hat{V}^j_{;h}\tilde{\gamma}^h + \hat{V}^j_{:h} Dw^h \qquad (18.12)$$

$\hat{V}^j_{,h}$, $\hat{V}^j_{;h}$ and $\hat{V}^j_{:h}$ are the components of $\hat{\pi}^{\circ}$-images of the above mentioned tensor fields of $\tilde{\pi}^{\diamond}\mathfrak{F}$.

Lastly, making use of the relations between $(\theta^j, Dy^j, Dw^j, \theta^j_i)$ and $(B_i, E_i, W_i, \hat{Q}^i_j)$ dual each other to, we shall obtain the formulas analogous to (10.7)—(10.9). Let $\hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q}$ be the components of a $\hat{\pi}^{\circ}$-image tensor field of the type (p, q), then we have easily

$$B_k(\hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q}) = \hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q,k}, \qquad (18.13)$$

$$E_k(\hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q}) = \hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q;h}\widetilde{M}^h_k, \qquad (18.14)$$

$$Y_k(\hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q}) = \hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q;k} \qquad (18.14')$$

where

$$Y_k = \widetilde{\Phi}^h_k E_h. \qquad (18.15)$$

Furthermore, we have

$$W_k(\hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q}) = \hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q:k} = \partial\hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q}/\partial w^k \qquad (18.16)$$

and

$$\hat{Q}^h_k\,(\hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q}) = -\sum_\alpha \hat{K}^{i_1\cdots h\cdots i_p}_{j_1\cdots\cdots j_p}\,\delta^{i_\alpha}_k + \sum_\beta \hat{K}^{i_1\cdots\cdots\cdots i_p}_{j_1\cdots k\cdots j_q}\,\delta^h_{j_\beta}. \tag{18.17}$$

§ 19. The holonomy group of Γ_0 and the system $\hat{\Sigma}$.

Let us define the holonomy groups of the projective connection $\widetilde{\Gamma}_0$ by means of α-curves only, that is, the curves in $\hat{\mathfrak{B}}$ which are horizontal with respect to $\widetilde{\Gamma}_0$ and whose images under the projection $\tilde{\mu}$ of $\{\hat{\mathfrak{B}},\ T_0(\mathfrak{X})\}$ are α-curves.

Following the manner in § 7, we denote by $\hat{\Xi}$ the field of n-dimensional tangent subspaces of $\hat{\mathfrak{B}}$ spanned by $\hat{E}_1, \cdots, \hat{E}_n$. Setting $\hat{B} = y^j\hat{B}_j$, we denote by $\hat{\Pi}$ the fields of $(n+1)$-dimensional tangent spaces of $\hat{\mathfrak{B}}$ spanned by $\hat{B}$ and $\hat{\Xi}$. The same calculations as in § 7 will give the following lemmas.

Lemma 19.1. *A necessary and sufficient condition that a curve $\bar{C}$ in $T_0(\mathfrak{X})$ is the lift of a curve in $\mathfrak{X}$ is that the tangent vectors of a horizontal lift of $\bar{C}$ in $\hat{\mathfrak{B}}$ with respect to $\widetilde{\Gamma}_0$ belong always to the field $\hat{B}+\hat{\Xi}$.*

Lemma 19.2. *A necessary and sufficient condition that a curve $\bar{C}$ in $T_0(\mathfrak{X})$ has the property such that $\rho(\bar{C})$ or $\rho(\varepsilon\bar{C})$ is the lift of the curve $\tau(\bar{C})$ in $\mathfrak{S}(\mathfrak{X})$ is that the tangent vector of a horizontal lift of $\bar{C}$ in $\hat{\mathfrak{B}}$ with respect to $\widetilde{\Gamma}_0$ belong always to the field $\hat{\Pi}$.*

Lemma 19.3. *For any α-curve $\bar{C}$, the tangent vectors of a horizontal lift $\hat{C}$ of $\bar{C}$ in $\hat{\mathfrak{B}}$ with respect to $\widetilde{\Gamma}_0$ belong always to the field $\hat{\Pi}$ and the converse is also true.*

Now, we shall denote by $\hat{\mathfrak{A}}$ the algebra over the real field of all scalar fields on $\hat{\mathfrak{B}}$ and we may consider $\mathfrak{A}$ defined in § 11 naturally as a subalgebra of $\hat{\mathfrak{A}}$. From these lemmas, we see that the locus of the points of horizontal lifts with respect to $\widetilde{\Gamma}_0$ of α-curves through a point $\hat{b} \in \hat{\mathfrak{B}}$ is the integral manifold $\hat{P}(\hat{b})$ containing $\hat{b}$ of the system

$$\hat{\Sigma} = \{(\hat{B},\ \hat{E}_1,\ \cdots,\ \hat{E}_n)\}_{\hat{\mathfrak{A}}} \tag{19.1}$$

with coefficients in $\hat{\mathfrak{B}}$.

We shall calculate the Poisson's brackets of $\hat{\Sigma}$. We get from (18.2) (18.5), (11.11) and (10.10)

$$[\hat{B}, \hat{E}_k] = [B - y^h W_h, E_k] = [B, E_k] + E_k(y^h) W_k$$
$$= -\widetilde{M}^h_k(\bar{B}_h + \widetilde{P}_o{}^t{}_{oh} E_t) + W_k$$
$$= -\widetilde{M}^h_k(\bar{B}_h - \widetilde{\Phi}^t_h W_t + \widetilde{P}_o{}^t{}_{oh} E_t).$$

Hence the elements to be newly adjoined to $\hat{\Sigma}$ are

$$\bar{B}_h - \widetilde{\Phi}^t_h W_t = B_h + \widetilde{P}_j{}^t{}_{oh} Q^j_t - \widetilde{\Phi}^t_h W_t. \tag{19.2}$$

Since we have by (4.9), (11.13) the equation

$$y^h(\bar{B}_h - \widetilde{\Phi}^t_h W_t) = B - y^t W_t = y^t(B_t - W_t)$$
$$= y^t \hat{B}_t = \hat{B}, \tag{19.3}$$

the system

$$\hat{\Sigma}_1 = \{\bar{B}_1 - \widetilde{\Phi}^t_1 W_t, \cdots, \bar{B}_n - \widetilde{\Phi}^t_n W_t, E_1, \cdots, E_n\}_{\hat{\mathfrak{A}}} \tag{19.4}$$

includes the system $\hat{\Sigma}$ and have the same minimum involutive system containing $\hat{\Sigma}_1$ with the one of $\hat{\Sigma}$. We may use $Y_1, \cdots, Y_n$ in place of $E_1, \cdots, E_n$ of $\hat{\Sigma}$.

Nextly, making use of (14.1) — (14.3) and the formulas of the last section, we have

$$[\bar{B}_h - \widetilde{\Phi}^t_h W_t, \bar{B}_k - \widetilde{\Phi}^j_k W_j] = [\bar{B}_h, \bar{B}_k] - (\bar{B}_h(\widetilde{\Phi}^j_k) - \bar{B}_k(\widetilde{\Phi}^j_h)) W_j$$
$$= (\widetilde{T}^j_{hk} - \widetilde{P}_h{}^j{}_{ok} + \widetilde{P}_k{}^j{}_{oh}) \bar{B}_j - \widetilde{R}_o{}^j{}_{hk} E_j - \widetilde{V}_t{}^j{}_{hk} Q^t_j$$
$$- (\bar{B}_h(\widetilde{\Phi}^j_k) - \bar{B}_k(\widetilde{\Phi}^j_h)) W_j$$
$$= (\widetilde{T}^j_{hk} - \widetilde{P}_h{}^j{}_{ok} + \widetilde{P}_k{}^j{}_{oh}) (\bar{B}_j - \widetilde{\Phi}^t_j W_t) - \widetilde{R}_o{}^j{}_{hk} E_j - \widetilde{V}_t{}^j{}_{hk} \hat{Q}^t_j$$
$$- \{w^t \widetilde{V}_t{}^j{}_{hk} - \widetilde{\Phi}^j_t(\widetilde{T}^t_{hk} - \widetilde{P}_h{}^t{}_{ok} + \widetilde{P}_k{}^t{}_{oh}) + \bar{B}_h(\widetilde{\Phi}^j_k) - \bar{B}_k(\widetilde{\Phi}^j_h)\} \hat{Q}^o_j.$$

On the other hand, $\bar{B}_h$ is written on $\widetilde{\mathfrak{B}}$ as

$$\bar{B}_h = B_h + \widetilde{P}_j{}^t{}_{ok} Q^j_t = B_h + \widetilde{P}_j{}^t{}_{oh} \hat{Q}^j_t + w^j \widetilde{P}_j{}^t{}_{oh} W_t, \tag{19.5}$$

hence for the components $\hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q}$ of a $\hat{\pi}^o$-image tensor field of the type (p, q), making use of (18.13), (18.17), we get

$$\bar{B}_h(\hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q}) = \hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q}{}_{,h} - \sum_\alpha \widetilde{P}_t{}^{i_\alpha}{}_{oh} \hat{K}^{i_1\cdots t\cdots i_p}_{j_1\cdots\cdots j_q} + \sum_\beta \widetilde{P}_{j_\beta}{}^t{}_{oh} \hat{K}^{i_1\cdots\cdots\cdots i_p}_{j_1\cdots t\cdots j_q}$$
$$+ w^t \widetilde{P}_t{}^k{}_{oh} \hat{K}^{i_1\cdots i_p}_{j_1\cdots j_q}{}_{:k} \tag{19.6}$$

analogous to (13.5). By (19.6), (12.3) we get

$$\bar{B}_h(\widetilde{\Phi}_k^j) = \widetilde{\Phi}_{k,h}^j - \widetilde{P}_t{}^j{}_{oh}\,\widetilde{\Phi}_k^t + \widetilde{P}_k{}^t{}_{oh}\,\widetilde{\Phi}_t^j$$
$$= \widetilde{C}_{ok,h}^j - \widetilde{P}_t{}^j{}_{oh}\,\widetilde{\Phi}_k^t + \widetilde{P}_{koh}^t\,\widetilde{\Phi}_t^j.$$

Hence we obtain

$$[\bar{B}_h - \widetilde{\Phi}_h^i W_i, \bar{B}_k - \widetilde{\Phi}_k^j W_j] = (\widetilde{T}_{hk}^j - \widetilde{P}_{ohk}^j + \widetilde{P}_k{}^j{}_{oh})(\bar{B}_j - \widetilde{\Phi}_j^i W_i) - \widetilde{R}_o{}^j{}_{hk} E_j$$
$$- \widetilde{V}_i{}^j{}_{hk}\hat{Q}_j^i - \{w^i \widetilde{V}_i{}^j{}_{hk} - \widetilde{\Phi}_t^j \widetilde{T}_{hk}^t + \widetilde{\Phi}_h^t \widetilde{P}_t{}^j{}_{ok} - \widetilde{\Phi}_k^t P_{toh}^j \qquad (19.7)$$
$$- \widetilde{C}_o{}^j{}_{h,k} + \widetilde{C}_{ok,h}^j\}\,\hat{Q}_j^o$$

Nextly, we have

$$[\bar{B}_h - \widetilde{\Phi}_h^j W_j,\ Y_k] = [\bar{B}_h, Y_k] + Y_k(\widetilde{\Phi}_h^j)\, W_j$$
$$= \widetilde{C}_{hk}^j \bar{B}_j + \{\widetilde{P}_k{}^j{}_{oh} + (\widetilde{C}_{ok,h}^t - \widetilde{P}_o{}^t{}_{hk})\,\widetilde{M}_t^j\}\, Y_j - \widetilde{W}_i{}^j{}_{hk} Q_j^i + \widetilde{\Phi}_{h;k}^j W_j$$
$$= \widetilde{C}_{hk}^j(\bar{B}_j - \widetilde{\Phi}_j^i W_i) + \{\widetilde{P}_{koh}^j + (\widetilde{C}_{ok,h}^t - \widetilde{P}_{ohk}^t)\,\widetilde{M}_t^j\}\, Y_j - \widetilde{W}_{ihk}^j \hat{Q}_j^i$$
$$- \{w^i \widetilde{W}_i{}^j{}_{hk} - \widetilde{\Phi}_t^j \widetilde{C}_{hk}^t - \widetilde{\Phi}_{h;k}^j\}\hat{Q}_j^o,$$

that is

$$[\widetilde{B}_h - \widetilde{\Phi}_h^j W_j,\ Y_k] = \widetilde{C}_{hk}^j(\bar{B}_j - \widetilde{\Phi}_j^i W_i) + \{\widetilde{\Phi}_t^j \widetilde{P}_{koh}^t + \widetilde{C}_{ok,h}^j - \widetilde{P}_{ohk}^j\} E_j$$
$$- \widetilde{W}_i{}^j{}_{hk}\hat{Q}_j^i - \{w^i \widetilde{W}_{ihk}^j - \widetilde{C}_{hk}^j - \widetilde{C}_{ot}^j \widetilde{C}_{hk}^t - (\widetilde{C}_{ok}^j)_{;k}\}\,\hat{Q}_j^o. \qquad (19.8)$$

Lastly, by means fo (12. 7) we get

$$[Y_h, Y_k] = (\widetilde{C}_{hk}^j - \widetilde{C}_{kh}^j)\, Y_j - S_i{}^j{}_{hk} Q_j^i$$
$$= (\widetilde{C}_{hk}^j - \widetilde{C}_{kh}^j)\, Y_j - \widetilde{S}_{ihk}^j \hat{Q}_j^i - w^i \widetilde{S}_{ihk}^j \hat{Q}_j^o. \qquad (19.9)$$

Thus, the elements to be newly adjoined to $\hat{\Sigma}_1$ are the vertical vector fields $\widetilde{V}_i{}^j{}_{hk}\hat{Q}_j^i + \hat{V}^j{}_{hk}\hat{Q}_j^o$, $\widetilde{W}_i{}^j{}_{hk}\hat{Q}_j^i + \hat{W}^j{}_{hk}\hat{Q}_j^o$, $\widetilde{S}_i{}^j{}_{hk}\hat{Q}_j^i + w^t\,\widetilde{S}_t{}^j{}_{hk}\hat{Q}_j^o$ in the right hand sides of (19. 7), (19. 8), (19. 9) where we set

$$\hat{V}^j{}_{hk} = w^t \widetilde{V}_t{}^j{}_{hk} - \widetilde{\Phi}_t^j \widetilde{T}^t{}_{hk} + \widetilde{\Phi}_h^t \widetilde{P}_t{}^j{}_{ok} - \widetilde{\Phi}_k^t \widetilde{P}_t{}^j{}_{oh}$$
$$- \widetilde{C}_{oh,k}^j + \widetilde{C}_{ok,h}^j \qquad (19.10)$$

and

$$\hat{W}^j{}_{hk} = w^i \widetilde{W}_i{}^j{}_{hk} - \widetilde{C}_{hk}^j - \widetilde{C}_o{}^j{}_t \widetilde{C}_{hk}^t - (\widetilde{C}_{oh}^j)_{;k}. \qquad (19.11)$$

§ 20. The minimum involutive system derived from $\hat{\Sigma}$.

For the connection $\widetilde{\Gamma}_o$, successively we shall treat the analogy to the argument in § 13. For brevity, we set

$$\bar{B}^*_h = \bar{B}_h - \widetilde{\Phi}^j_h W_j. \tag{20.1}$$

Firstly, we get by (19.6).

$$\bar{B}_h(w^i) = w^i{}_{,h} - \widetilde{P}_t{}^i{}_{oh} w^t + w^t \widetilde{P}^k_t{}_{oh} w^i{}_{:k}.$$

Since we have easily

$$w^i{}_{,h} = 0, \quad w^i{}_{;h} = 0, \quad w^i{}_{:h} = \delta^i_h, \tag{20.2}$$

and so

$$\bar{B}_h(w^i) = 0. \tag{20.3}$$

We have immediately

$$[\bar{B}^*_h, \hat{Q}^o_i] = [\bar{B}_h - \widetilde{\Phi}^j_h W_j, W_i] = 0. \tag{20.4}$$

Furthermore, by means of (13.4), (18.5), (20.3), (18.17), we get

$$\begin{aligned}[\bar{B}^*_h, \hat{Q}^j_i] &= [\bar{B}_h - \widetilde{\Phi}^k_h W_k, \hat{Q}^j_i] = [\bar{B}_h, \hat{Q}^j_i] - \widetilde{\Phi}^k_h [W_k, \hat{Q}^j_i] + \hat{Q}^j_i(\widetilde{\Phi}^k_h) W_k \\ &= [\bar{B}_h, Q^j_i] + \widetilde{\Phi}^k_h [W_k, w^j W_i] + (\widetilde{\Phi}^k_i \delta^j_h - \widetilde{\Phi}^j_h \delta^k_i) W_k \\ &= - \delta^j_h \bar{B}_i + \widetilde{\Phi}^j_h W_i + \delta^j_h \widetilde{\Phi}^k_i W_k - \widetilde{\Phi}^j_h W_i \\ &= - \delta^j_i (\bar{B}_i - \widetilde{\Phi}^k_i W_k),\end{aligned}$$

that is

$$[\bar{B}^*_h, \hat{Q}^j_i] = -\delta^j_h \bar{B}^*_i. \tag{20.5}$$

We have easily

$$[Y_h, \hat{Q}^j_i] = [Y_h, Q^j_i - w^j W_i] = - \delta^j_h Y_i, \tag{20.6}$$

$$[Y_h, \hat{Q}^o_i] = 0. \tag{20.7}$$

Lastly, by (18.3), (18.4), (18.5), (20.2) and Lemma 11.1 we get

$$[\hat{Q}^j_i, \hat{Q}^k_h] = \delta^j_h \hat{Q}^k_i - \delta^k_i \hat{Q}^j_h \tag{20.8}$$

and

$$[\hat{Q}^j_i, \hat{Q}^o_h] = [Q^j_i - w^j W_i, W_h] = \delta^j_h \hat{Q}^o_i. \tag{20.9}$$

For the components $\hat{K}^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ of a $\hat{\pi}^o$-image tensor field, we get from (19.6), (18.16) the formulas

$$\begin{aligned}\bar{B}^*_h (\hat{K}^{i_1 \cdots i_p}_{j_1 \cdots j_q}) = \hat{K}^{i_1 \cdots i_p}_{j_1 \cdots j_q , h} - \sum_\alpha \widetilde{P}_t{}^{i_\alpha}{}_{oh} \hat{K}^{i_1 \cdots t \cdots i_p}_{j_1 \cdots \cdots j_q} + \sum_\beta \widetilde{P}^t_{j_\beta oh} \hat{K}^{i_1 \cdots \cdots \cdots i_p}_{j_1 \cdots t \cdots j_q} \\ + (w^t \widetilde{P}^k_t{}_{oh} - \widetilde{\Phi}^k_h) \hat{K}^{i_1 \cdots i_p}_{j_1 \cdots j_q : k}.\end{aligned} \tag{20.10}$$

Now, let $\hat{K}^{t}{}_{k_1\cdots k_p}$, $\hat{K}^{t}_{jk_1\cdots k_p}$, $\hat{H}^{t}{}_{h_1\cdots h_q}$, $\hat{H}^{t}_{jh_1\cdots h_q}$ be the components of $\hat{\pi}^{\circ}$-image tensor fields of the type (1, p), (1, $p+1$), (1, q), (1, $q+1$) respectively. Making use of the above formulas. we calculate the following Poisson's brackets.

$$
\begin{aligned}
&[\hat{K}^{t}{}_{k_1\cdots k_p}\hat{Q}^{0}_{t} + \hat{K}^{t}_{jk_1\cdots k_p}\hat{Q}^{j}_{t},\ \hat{H}^{l}{}_{h_1\cdots h_q}\hat{Q}^{0}_{l} + \hat{H}^{l}_{mh_1\cdots h_q}\hat{Q}^{m}_{l}]\\
&= (\hat{K}^{t}{}_{k_1\cdots k_p}\hat{H}^{i}{}_{h_1\cdots h_q:t} - \hat{K}^{i}{}_{k_1\cdots k_p:t}\hat{H}^{t}{}_{h_1\cdots h_q}\\
&\quad + \hat{K}^{i}_{tk_1\cdots k_p}\hat{H}^{t}{}_{h_1\cdots h_q} - \hat{K}^{t}{}_{k_1\cdots k_p}\hat{H}^{i}_{th_1\cdots h_q})\hat{Q}^{0}_{i}\\
&\quad + \hat{K}^{t}_{sk_1\cdots k_p}\hat{Q}^{s}_{t}(\hat{H}^{i}{}_{h_1\cdots h_q})\hat{Q}^{0}_{i} - \hat{K}^{i}_{jk_1\cdots k_p:t}\hat{H}^{t}{}_{h_1\cdots h_q}\hat{Q}^{j}_{i}\\
&\quad + \hat{K}^{t}{}_{k_1\cdots k_p}\hat{H}^{i}_{jh_1\cdots h_q:t}\hat{Q}^{j}_{i} - \hat{Q}^{s}_{t}(\hat{K}^{i}{}_{k_1\cdots k_p})\hat{H}^{t}_{sh_1\cdots h_q}\hat{Q}^{0}_{i}\\
&\quad + (\hat{K}^{i}_{tk_1\cdots k_p}\hat{H}^{t}_{jh_1\cdots h_q} - \hat{K}^{t}_{jk_1\cdots k_p}\hat{H}^{i}_{th_1\cdots h_q})\hat{Q}^{j}_{i}\\
&\quad + (-\hat{K}^{i}_{tk_1\cdots k_p}\hat{H}^{t}_{jh_1\cdots h_q} + \hat{K}^{t}_{jk_1\cdots k_p}\hat{H}^{i}_{th_1\cdots h_q} + \sum_{\beta}\hat{K}^{t}_{h_\beta k_1\cdots k_p}\hat{H}^{i}_{jh_1\cdots t\cdots h_q}\\
&\qquad + \hat{K}^{t}_{jk_1\cdots k_p}\hat{H}^{i}_{th_1\cdots h_q} - \hat{K}^{i}_{tk_1\cdots k_p}\hat{H}^{t}_{jh_1\cdots h_q} - \sum_{\alpha}\hat{K}^{i}_{jk_1\cdots t\cdots k_p}\hat{H}^{t}_{k_\alpha h_1\cdots h_q})\hat{Q}^{j}_{i}\\
&= -\sum_{\alpha}\hat{H}^{t}_{k_\alpha h_1\cdots h_q}(\hat{K}^{i}{}_{k_1\cdots t\cdots k_p}\hat{Q}^{0}_{i} + \hat{K}^{i}_{jk_1\cdots t\cdots k_p}\hat{Q}^{j}_{i})\\
&\quad + \sum_{\beta}\hat{K}^{t}_{k_\beta k_1\cdots k_p}(\hat{H}^{i}{}_{h_1\cdots t\cdots h_q}\hat{Q}^{0}_{i} + \hat{H}^{i}_{jh_1\cdots t\cdots h_q}\hat{Q}^{j}_{i})\\
&\quad - \{(\hat{K}^{i}{}_{k_1\cdots k_p:t}\hat{H}^{t}{}_{h_1\cdots h_q} - \hat{H}^{i}{}_{h_1\cdots h_q:t}\hat{K}^{t}{}_{k_1\cdots k_p})\hat{Q}^{0}_{i}\\
&\qquad + (\hat{K}^{i}_{tk_1\cdots k_p}\hat{H}^{t}_{jh_1\cdots h_q} - \hat{K}^{t}_{jk_1\cdots k_p}\hat{H}^{i}_{th_1\cdots h_q} + \hat{K}^{i}_{jk_1\cdots k_p:t}\hat{H}^{t}{}_{h_1\cdots h_q}\\
&\qquad\qquad - \hat{K}^{t}{}_{k_1\cdots k_p}\hat{H}^{i}_{jh_1\cdots h_q:t})\hat{Q}^{j}_{i}\}. \qquad (20.11)
\end{aligned}
$$

We denote by $\hat{\mathfrak{M}}$ the set of all the differentiable fields of (n, $n+1$)-matrixes defined on $\hat{\mathfrak{B}}$. $\hat{\mathfrak{M}}$ is a module over $\hat{\mathfrak{A}}$ of dimension $n(n+1)$. We define a *bracket multiplication* for any $\boldsymbol{K} = ((K^i, K^i_j))$, $\boldsymbol{H} = ((H^i, H^i_j)) \in \hat{\mathfrak{M}}$ by

$$
\begin{aligned}
[\boldsymbol{K}, \boldsymbol{H}] &= \boldsymbol{F} = ((F^i, F^i_j)),\\
F^i &= W_t(K^i)H^t - W_t(H^i)K^t, \qquad (20.12)\\
F^i_j &= K^i_t H^t_j - H^i_t K^t_j + W_t(K^i_j)H^t - W_t(H^i_j)K^t.
\end{aligned}
$$

This multiplication is not dependent on connections since the differential operators W_t are tangent vector fields defined only according to the structure $\hat{\mathfrak{B}} = \widetilde{\mathfrak{B}}_0 \times R^n$.

Lemma 20.1. *The bracket multiplication of* $\hat{\mathfrak{M}}$ *defined by* (20.12) *has the following properties. For any* $\boldsymbol{K}$, $\boldsymbol{H}$, $\boldsymbol{L} \in \hat{\mathfrak{M}}$, $f \in \hat{\mathfrak{A}}$

$$[\boldsymbol{K},\ \boldsymbol{H}] = -[\boldsymbol{H},\ \boldsymbol{K}]. \tag{20.13}$$

$$[\boldsymbol{K}+\boldsymbol{H},\ \boldsymbol{L}] = [\boldsymbol{K},\ \boldsymbol{L}] + [\boldsymbol{H},\ \boldsymbol{L}], \tag{20.14}$$

$$[f\boldsymbol{K},\ \boldsymbol{H}] = f[\boldsymbol{K},\ \boldsymbol{H}] + W_t(f)H^t\boldsymbol{K}, \tag{20.15}$$

$$[[\boldsymbol{K},\ \boldsymbol{H}],\ \boldsymbol{L}] + [[\boldsymbol{H},\ \boldsymbol{L}],\ \boldsymbol{K}] + [[\boldsymbol{L},\ \boldsymbol{K}],\ \boldsymbol{H}] = 0. \tag{20.16}$$

Proof. The first three of these formulas are evident from the definition. We shall give the verification of (20.16). Setting $[\boldsymbol{K},\ \boldsymbol{H}] = \boldsymbol{F}$, we have

$$\begin{aligned} & W_s(F^i)L^s - W_s(L^i)F^s \\ &= \{W_s(W_t(K^i))H^tL^s - W_s(W_t(H^i))L^sK^t\} \\ &+ \{W_t(K^i)W_s(H^t)L^s - W_s(L^i)W_t(K^s)H^t\} \\ &+ \{W_s(L^i)W_t(H^s)K^t - W_t(H^i)W_s(K^t)L^s\} \end{aligned}$$

and

$$\begin{aligned} & F^i_sL^s_j - L^i_sF^s_j + W_s(F^i_j)L^s - W_s(L^i_j)F^s \\ &= \{K^i_tH^t_sL^s_j - L^i_sK^s_tH^t_j - H^i_tK^t_sL^s_j + L^i_sH^s_tK^t_j\} \\ &+ \{W_t(K^i_s)H^tL^s_j - W_s(H^i_t)L^sK^t_j - W_t(H^i_s)K^tL^s_j + W^s(K^i_t)L^sH^t_j\} \\ &- \{L^i_sW_t(K^s_j)H^t - K^i_tW_s(H^t_j)L^s - L^i_sW_t(H^s_j)K^t + H^i_tW_s(K^t_j)L^s\} \\ &+ \{W_s(W_t(K^i_j))H^tL^s - W_s(W_t(H^i_j))L^sK^t\} \\ &+ \{W_t(K^i_j)W_s(H^t)L^s - W_s(L^i_j)W_t(K^s)H^t - W_t(H^i_j)W_s(K^t)L^s \\ &\qquad + W_s(L^i_j)W_t(H^s)K^t\}. \end{aligned}$$

Cyclicly changing on $\boldsymbol{K}$, $\boldsymbol{H}$, $\boldsymbol{L}$ of the right hand sides of the above equations and adding up them, the sums vanish since $W_1, \cdots, W_n$ are commuative each others.

Lemma 20.1 shows that *the module $\hat{\mathfrak{M}}$ over $\hat{\mathfrak{A}}$ with the bracket multiplication is a Lie algebra over $\mathfrak{A}$ since $\mathfrak{A}$ vanisches under W_h.*

From (20.15), it follows that

$$[f\boldsymbol{K},\ \boldsymbol{H}] \equiv f[\boldsymbol{K},\ \boldsymbol{H}] \quad \text{mod } \boldsymbol{K}. \tag{20.15}$$

Now, to any $\boldsymbol{K} = ((K^i,\ K^i_j)) \in \hat{\mathfrak{M}}$, we correspond a vertical tangent vector field on $\hat{\mathfrak{B}}$ with respect to $\tilde{\Gamma}_0$ by

$$\boldsymbol{K}\hat{Q} = K^i\hat{Q}^0_i + K^i_j\hat{Q}^j_i. \tag{20.17}$$

Let any tangent vector field X on $\hat{\mathfrak{B}}$ operate also on any element $\boldsymbol{K} = ((K^i,\ K^i_j))$ by

$$X(\boldsymbol{K}) = ((X(K^i),\ X(K^i_j))). \tag{20.18}$$

Making use of (20. 12), we get immediately

Lemma 20. 2. *For any* $\boldsymbol{K},\ \boldsymbol{H} \in \hat{\mathfrak{M}}$ *and any tangent vector field* X *on* $\hat{\mathfrak{B}}$ *which is commutative with* $W_1, \cdots, W_\nu$, *we have*

$$X([\boldsymbol{K},\ \boldsymbol{H}]) = [X(\boldsymbol{K}),\ \boldsymbol{H}] + [\boldsymbol{K},\ X(\boldsymbol{H})].$$

Furthermore, making use of (20. 5) — (20. 7), the following lemma can be obtained.

Lemma 20. 3. *For any* $\boldsymbol{K} = ((K^i,\ K^i_j)) \in \hat{\mathfrak{M}}$ *and* $X_h = \bar{B}^*_h,\ Y_h$, *we have*

$$[X_h,\ \boldsymbol{K}\,\hat{Q}] = X_h(\boldsymbol{K})\,\hat{Q} - K^i_h X_i.$$

For a set of functions $K^i{}_{j_1 \cdots j_p},\ K^i_{j j_1 \cdots j_p}$ on $\hat{\mathfrak{B}}$, we shall put

$$\boldsymbol{K}_{j_1 \cdots j_p} = ((K^i{}_{j_1 \cdots j_p},\ K^i_{j j_1 \cdots j_p})) \in \hat{\mathfrak{M}}.$$

Then, formula (20. 11) for $\boldsymbol{K}_{k_1 \cdots k_p},\ \boldsymbol{H}_{h_1 \cdots h_q}$ corresponding to the components of $\hat{\pi}^\circ$-image tensor fields is written as

$$\begin{aligned}[\boldsymbol{K}_{k_1 \cdots k_p}\,\hat{Q},\ \boldsymbol{H}_{h_1 \cdots h_q}\,\hat{Q}] &= -\sum_\alpha \hat{H}^t_{k_\alpha h_1 \cdots h_q}\,\boldsymbol{K}_{k_1 \cdots t \cdots k_p}\,\hat{Q} \\ &+ \sum_\beta \hat{K}^t_{h_\beta k_1 \cdots k_p}\,\boldsymbol{H}_{h_1 \cdots t \cdots h_q}\,\hat{Q} \\ &- [\boldsymbol{K}_{k_1 \cdots k_p},\ \boldsymbol{H}_{h_1 \cdots h_q}]\,\hat{Q}.\end{aligned} \tag{20.11'}$$

Now, by means of $\hat{V}^i{}_{hk},\ \hat{W}^i{}_{hk}$ defined by (19. 10), (19. 11) and $\widetilde{V}_j{}^i{}_{hk}$, $\widetilde{W}^i_j{}_{hk}$ defined by (13. 11), (13. 12), we define the elements of $\hat{\mathfrak{M}}$ such that

$$\boldsymbol{V}_{hk} = ((\hat{V}^i{}_{hk},\ \widetilde{V}^i_j{}_{hk})),\ \boldsymbol{W}_{hk} = ((\hat{W}^i{}_{hk},\ \widetilde{W}^i_j{}_{hk}))$$
$$\boldsymbol{S}_{hk} = ((w^t\ \widetilde{S}_t{}^i{}_{hk},\ \widetilde{S}^i_j{}_{hk})).$$

Let us denote by $\hat{\mathfrak{U}} = \hat{\mathfrak{U}}_\Gamma$ the set of all the elements which are obtained by repeatedly but finitly operating $\bar{B}^*_h,\ Y_h$ on $\boldsymbol{V}_{hk},\ \boldsymbol{W}_{hk}$ and $\boldsymbol{S}_{hk}$. Let us denote also by $\hat{\mathfrak{M}}'_\Gamma$ the submodule generated by $\hat{\mathfrak{U}}$ with respect to the summation and the bracket multiplication with coefficients in $\hat{\mathfrak{A}}$. Since $\hat{\mathfrak{U}}$ is invariant under $\bar{B}^*_h$ and Y_h, $\hat{\mathfrak{M}}'_\Gamma$ is also invariant by Lemma 20. 1 and Lemma 20. 2.

Lemma 20. 4. $\bar{B}^*_h(\hat{\mathfrak{M}}'_\Gamma) = Y_h(\hat{\mathfrak{M}}'_\Gamma) = \hat{\mathfrak{M}}'_\Gamma$.

Then, in order to obain an analogy of Theorem 13.5, we denote by $\hat{\mathfrak{M}}_\Gamma$ the subalgebra generated by $\hat{\mathfrak{U}}$ with coefficients in $\mathfrak{A}$. We get immediately the following

Lemma 20.5. $\bar{B}_h^*(\hat{\mathfrak{M}}_\Gamma) = Y_h(\hat{\mathfrak{M}}_\Gamma) = \hat{\mathfrak{M}}_\Gamma$.

Theorem 20.6. *For any* $\boldsymbol{K} = ((K^i, K^i_j)) \in \hat{\mathfrak{M}}_\Gamma$, *we have*

$$W_j(K^i) = K^i_j, \quad W_h(K^i_j) = 0. \tag{20.19}$$

Proof. Since $\hat{Q}^0_i = W_i$ by (18.2) and W_i are commutative with $\bar{B}^*_j$, Y_j by (20.4), (20.7), the above equations are satisfied by $\boldsymbol{V}_{hk}$, $\boldsymbol{W}_{hk}$, $\boldsymbol{S}_{hk}$ and also by any element of $\hat{\mathfrak{U}}$. If any $\boldsymbol{K}$, $\boldsymbol{H} \in \hat{\mathfrak{M}}$ satisfy (20.19), then $f\boldsymbol{K} + g\boldsymbol{H}$, f, $g \in \mathfrak{A}$ is so. For $[\boldsymbol{K}, \boldsymbol{H}] = \boldsymbol{F}$, we get from (20.12)

$$\begin{aligned} W_j(F^i) &= W_j(K^i_t H^t - H^i_t K^t) \\ &= K^i_t W_j(H^t) - H^i_t W_j(K^t) + W_j(K^i_t) H^t - W_j(H^i_t) K^t \\ &= K^i_t H^t_j - H^i_t K^t_j + W_j(K^i_t) H^t - W_j(H^i_t) K^t = F^i_j \end{aligned}$$

and

$$W_h(F^i_j) = W_h(K^i_t H^t_j - H^i_t K^t_j) = 0.$$

Since $\hat{\mathfrak{M}}_\Gamma$ is generated by $\hat{\mathfrak{U}}$ with coefficients in $\mathfrak{A}$, any element of $\hat{\mathfrak{M}}_\Gamma$ must satisfy (20.19).

Now, for any point $\hat{b} \in \hat{\mathfrak{B}}$, we define a set of $(n, n+1)$-matrixes by

$$\hat{\mathfrak{M}}_\Gamma(\hat{b}) = \{\boldsymbol{K}(\hat{b}) = ((K^i(\hat{b}), K^i_j(\hat{b}))) \mid \boldsymbol{K} \in \hat{\mathfrak{M}}_\Gamma\}. \tag{20.20}$$

Then, we have

Theorem 20.7. $\hat{\mathfrak{M}}_\Gamma(\hat{b})$ *is the Lie algebra of a connected Lie subgroup* $A(\hat{b})$ *of the affine transformation group* A_n *of dimension* n.

Proof. By means of Theorem 20.6, for any $\boldsymbol{K}$, $\boldsymbol{H} \in \hat{\mathfrak{M}}_\Gamma$, (20.12) is written as

$$F^i = K^i_t H^t - H^i_t K^t, \quad F^i_j = K^i_t H^t_j - H^i_t K^t_j \tag{20.21}$$

which coincide with the formulas of the multiplacation of the Lie algebra LA_n of A_n, for if we write an affine transformation as

$$x'^i = a^i_j(g) x^j + a^i(g), \quad g \in A_n,$$

then the multiplication of A_n is given by

$$a^i_j(g_1 g_2) = a^i_k(g_1) a^k_j(g_2), \qquad a^i(g_1 g_2) = a^i_j(g_1) a^j(g_2) + a^i(g_1)$$

and hence the equations imply the same formulas of the multiplication of the Lie algebra LA_n. As stated above, $\hat{\mathfrak{M}}_\Gamma$ is closed under the bracket multiplication $[\boldsymbol{K}, \boldsymbol{H}]$, hence $\hat{\mathfrak{M}}_\Gamma(\hat{b})$ is a Lie subalgebra of LA_n. We shall denote by $A(\hat{b})$ the Lie subgroup generated by $\hat{\mathfrak{M}}_\Gamma(\hat{b})$.

Now, we shall denote by $\widetilde{AH}(y)$ the *affine holonomy group at* $y \in T_0(\mathfrak{X})$ of the connection $\tilde{\Gamma} = \rho^{\diamond} \Gamma$, regarding it as an affine connection and $T_0(\mathfrak{X})$ as a differentiable manifold merely and by $\widetilde{AH^0}(y)$ the *restricted affine holonomy group at* y which correspond to $\tilde{\Omega}^0(y)$. Furthermore, we denote by $AH(y)$ and $AH^0(y)$ the subgroups of $\widetilde{AH}(y)$ and $\widetilde{AH^0}(y)$ which correspond to the groups $\Omega(y)$ and $\Omega^0(y)$ of α-curves defined in § 8 respectivly. Owing to Theorem 17.1, these groups $\widetilde{AH}(y)$, $\widetilde{AH^0}(y)$ and $AH(y)$, $AH^0(y)$ are isomorphic with the homogeneous holonomy group, the restricted homogeneous holonomy group *in a wide sense* and the homogeneous holonomy group, the restricted homogeneous group *in a narrow sense* at y, of the connection $\tilde{\Gamma}_0$ of the vector bundle $\mathfrak{F}_0 = \{\mathfrak{B}_0, T_0(\mathfrak{X}), \kappa_0, R^{n+1}, G\}$ respectively and we shall identify them respectively.

Let a curve $\bar{C}$ of class C^1 in $T_0(\mathfrak{X})$ be given by $\bar{f}: I \to T_0(\mathfrak{X})$ and set $\bar{f}(0) = y_0$, $\tau(y_0) = x_0$, $\bar{f}(1) = y_1$, $\tau(y_1) = x_1$. Then, by the development of affine frames along $\bar{C}$, we may obtain the affine transformation

$$\varphi_{\bar{C}} : \quad T_{x_1}(\mathfrak{X}) \times y_1 \to T_{x_0}(\mathfrak{X}) \times y_0$$

of the fibres of $\mathfrak{F}$. The induced linear transformation on vectors from $\varphi_{\bar{C}}$ is $h_{\bar{C}}$ defined in § 8. For any $\hat{b}_1 \in \tilde{\mu}^{-1}(y_1)$, setting $\bar{b}_1 = \bar{\tau}_p(\hat{b}_1)$, $w_1 = \tau_1 \cdot \hat{\pi}(\hat{b})$, we define $\bar{b}_0$, w_0 by

$$\bar{b}_0 = h_{\bar{C}}(\bar{b}_1) \in \bar{\pi}^{-1}(y_0), \qquad w_0 = \varphi_{\bar{C}}(w_1)$$

then the point $\hat{b}_0 = (\bar{b}_0, w_0) \in \tilde{\mu}^{-1}(y_0)$. Thus setting $\hat{b}_0 = \varphi_{\bar{C}}(\hat{b}_1)$, $\varphi_{\bar{C}}$ may be considered as an isomorphism of $\tilde{\mu}^{-1}(y_1) = \hat{G}_{y_1}$ onto $\tilde{\mu}^{-1}(y_0) = \hat{G}_{y_0}$.

Theorem 20.8. *For any point* $\hat{b} \in \hat{\mathfrak{B}}$, $\tilde{\mu}(\hat{b}) = y$, $AH^0(y)(\hat{b})$ *is the image of the subgroup* $A(\hat{b})$ *of* A_n *under the admissible map* $\hat{b}: A_n \to \tilde{\mu}^{-1}(y)$.

Proof. By the similar manner to Theorem 13. 5, we can prove that $AH^0(y)(\hat{b})$ is the connected component containing $\hat{b}$ of the intersection of $\hat{\mu}^{-1}(y) = \hat{G}_y$ and the integral manifold $\hat{P}(\hat{b})$ containing b of the field $\hat{\Pi}$. Since $\hat{\Pi}$ is the field of the tangent subspaces of $\hat{\mathfrak{B}}$ spanned by the elements of $\hat{\Sigma}$, the integral manifolds of $\hat{\Pi}$ coincide with the integral manifolds of the system $\hat{\Sigma}'_\infty$ which is the minimum involutive system containg $\hat{\Sigma}$ over $\hat{\mathfrak{A}}$. $\hat{\Sigma}'_\infty$ is also the minimum involutive system derived from $\hat{\Sigma}_1$. By means of Lemma 20. 1 — 20. 3, we have $\hat{\Sigma}'_\infty \supset \hat{\Sigma}_1 + \hat{\mathfrak{M}}'_\Gamma\hat{Q}$. Considering Lemma 20. 4, the system $\hat{\Sigma}_1 + \hat{\mathfrak{M}}'_\Gamma\hat{Q}$ must be involutive. Hence we have

$$\hat{\Sigma}'_\infty = \hat{\Sigma}_1 + \hat{\mathfrak{M}}'_\Gamma\hat{Q}. \tag{20. 22}$$

At each point $\hat{b}$, the tangent subspace $\hat{\Pi}'_\infty(\hat{b})$ spanned by the elements of $\hat{\Sigma}'_\infty$ is spanned by $\bar{B}^*_h$, Y_h and the vertical tangent subspace spanned by the elements of $\hat{\mathfrak{M}}'_\Gamma\ \hat{Q}$. On the other hand, we denote by $\hat{\Pi}_\infty$ the field of tangent subspace spanned by $\bar{B}^*_h$, Y_h and $\hat{\mathfrak{M}}_\Gamma\ \hat{Q}$. Then it is evident that $\hat{\Pi}_\infty(\hat{b}) \subset \hat{\Pi}'_\infty(\hat{b})$. But, $\hat{\mathfrak{M}}_\Gamma$ and $\hat{\mathfrak{M}}'_\Gamma$ are spaned linearly by the elements of $\hat{\mathfrak{U}}_\Gamma$ with coefficients in $\mathfrak{A}$ and $\hat{\mathfrak{A}}$ respectively. Accordingly it must be $\hat{\Pi}_\infty = \hat{\Pi}'_\infty$. Thus, we see that $\hat{P}(\hat{b}) \cap \hat{G}_y$ is a differentiable manifold of dimension dim $\hat{\mathfrak{M}}_\Gamma(\hat{b})$ which is tangent to the vertical tangent subspace $T_{\hat{b}}(\hat{G}_y) \cap \hat{\Pi}_\infty$. Since $AH^0(y)$ is a Lie group, $AH^0(y)(\hat{b})$ must include the image of the group $A(\hat{b})$ generated by the Lie algebra $\hat{\mathfrak{M}}_\Gamma(\hat{b})$ under the admissible map $\hat{b}$. By virtue of the coincidence of the dimensions of $AH^0(y)$ and $A(\hat{b})$, this image must be the connected component containing $\hat{b}$ of $\hat{P}(\hat{b}) \cap \hat{G}_y = AH(y)(\hat{b})$. Since we assume that $\mathfrak{X}$ is separable, the component of $AH(y)$, containing the identity, is $AH^0(y)$. Hence we have

$$AH^0(y)(\hat{b}) = \hat{b}(A(\hat{b})). \tag{20. 23}$$

§ 21. Structure of $\hat{\mathfrak{M}}_\Gamma$.

We shall treat the same investigations of $\hat{\mathfrak{M}}_\Gamma$ as the ones of $\mathfrak{M}_\Gamma$ doned in § 14. We denote by $\hat{\mathfrak{B}}_{m+2}$, $\hat{\mathfrak{W}}_{m+2}$, $\hat{\mathfrak{S}}_{m+2}$ the sets of elements of $\hat{\mathfrak{U}}$ which are operated at most m times by $\bar{B}^*_i$, Y_i upon V_{hk}, W_{hk}, S_{hk} respectively and set

$$\hat{\mathfrak{V}}_\infty = \bigcup_{m=0}^{\infty}\hat{\mathfrak{V}}_{m+2},\quad \hat{\mathfrak{W}}_\infty = \bigcup_{m=0}^{\infty}\hat{\mathfrak{W}}_{m+2},\quad \hat{\mathfrak{S}}_\infty = \bigcup_{m=0}^{\infty}\hat{\mathfrak{S}}_{m+2}.$$

Then we have $\hat{\mathfrak{U}} = \hat{\mathfrak{V}}_\infty \cup \hat{\mathfrak{W}}_\infty \cup \hat{\mathfrak{S}}_\infty$. For any subset $\mathfrak{N}$ of $\hat{\mathfrak{M}}_\Gamma$, as previously, we denote by $\{\mathfrak{N}\} = \{\mathfrak{N}\}_{\mathfrak{A}}$ the submodule spanned by $\mathfrak{N}$ over $\mathfrak{A}$.

Theorem 21. 1. *$\{\hat{\mathfrak{V}}_\infty\}$, $\{\hat{\mathfrak{W}}_\infty\}$ and $\{\hat{\mathfrak{S}}_\infty\}$ are ideals of the Lie algebra $\hat{\mathfrak{M}}_\Gamma$ and $\{\hat{\mathfrak{U}}_\Gamma\} = \hat{\mathfrak{M}}_\Gamma$*

Proof. According to Theorem 20. 6, any element of $\hat{\mathfrak{U}}$ is of the form $\boldsymbol{K}_{k_1\cdots k_p} = (\hat{K}^t{}_{k_1\cdots k_p},\ \widetilde{K}^t_{jk_1\cdots k_p})$. By (18. 17), we have

$$\hat{Q}^s_t(\hat{K}^t{}_{k_1\cdots k_p}) = -\delta^t_t\hat{K}^s{}_{k_1\cdots k_p} + \sum_\alpha \hat{K}^t{}_{k_1\cdots t\cdots k_p}\delta^s_{k_\alpha},$$

$$\hat{Q}^s_t(\widetilde{K}^t_{jk_1\cdots k_p}) = -\delta^t_t\widetilde{K}^s_{jk_1\cdots p_k} + \widetilde{K}^t_{t\,k_1\cdots k_p}\delta^s_j + \sum_\alpha \widetilde{K}^t_{jk_1\cdots t\cdots k_p}\delta^s_{k_\alpha}$$

which are written as

$$\hat{Q}^s_t(\boldsymbol{K}_{k_1\cdots k_p}) = \sum_\alpha \boldsymbol{K}_{k_1\cdots t\cdots k_p}\delta^s_{k_\alpha} - ((\delta^t_t\hat{K}^s{}_{k_1\cdots k_p},\ \delta^t_t\widetilde{K}^s{}_{jk_1\cdots k_p} - \widetilde{K}^t_{tk_1\cdots k_p}\delta^s_j)). \tag{21. 1}$$

By (18. 14′), (18. 16) and (19. 7), we have

$$\begin{aligned}(\bar{B}^*_h\bar{B}^*_k - \bar{B}^*_k\bar{B}^*_h)(\boldsymbol{K}_{j_1\cdots j_p}) &= (\widetilde{T}^t_{hk} - \widetilde{P}_h{}^t{}_{ok} + \widetilde{P}_k{}^t{}_{oh})\bar{B}^*_t(\boldsymbol{K}_{j_1\cdots j_p})\\ &- \widetilde{R}_o{}^s{}_{hk}\widetilde{M}^t_s\boldsymbol{K}_{j_1\cdots j_p;t} - \hat{V}^t_{hk}\boldsymbol{K}_{j_1\cdots j_p;t} - \widetilde{V}_s{}^t{}_{hk}\hat{Q}^s_t(\boldsymbol{K}_{j_1\cdots j_p})\\ &= (\widetilde{T}^t_{hk} - \widetilde{P}_h{}^t{}_{ok} + \widetilde{P}_k{}^t{}_{oh})\bar{B}^*_t(\widetilde{\boldsymbol{K}}_{j_1\cdots j_p}) - \widetilde{R}_o{}^s{}_{hk}\widetilde{M}^t_s\boldsymbol{K}_{j_1\cdots j_p;t}\\ &- \hat{V}^t_{hk}\boldsymbol{K}_{j_1\cdots j_p;t} - \sum_\alpha \hat{V}_{j_\alpha}{}^t{}_{hk}\boldsymbol{K}_{j_1\cdots t\cdots j_p}\\ &+ ((\widetilde{V}^t_{thk}\hat{K}^t{}_{j_1\cdots j_p},\ \widetilde{V}^t_{t\,hk}\hat{K}^t_{j\,j_1\cdots j_p} - \widetilde{K}^t_{tj_1\cdots j_p}\widetilde{V}^t_{j\,hk})).\end{aligned}$$

By virtue of Theorem 20. 6, it must be

$$\hat{V}^t{}_{hk}\boldsymbol{K}_{j_1\cdots j_p;t} = ((\hat{V}^t{}_{hk}\widetilde{K}^t_{tj_1\cdots j_p},\ 0)).$$

Hence, by (20. 21) we obtain the formulas

$$\begin{aligned}(\bar{B}^*_h\bar{B}^*_k - \bar{B}^*_k\bar{B}^*_h)(\boldsymbol{K}_{j_1\cdots j_p}) &= [\boldsymbol{V}_{hk},\ \boldsymbol{K}_{j_1\cdots j_p}] - \sum_\alpha \widetilde{V}_{j_\alpha}{}^t{}_{hk}\boldsymbol{K}_{j_1\cdots t\cdots j_p}\\ &+ (\widetilde{T}^t_{hk} - \widetilde{P}_h{}^t{}_{ok} + \widetilde{P}_k{}^t{}_{oh})\bar{B}^*_t(\boldsymbol{K}_{j_1\cdots j_p}) - \widetilde{R}_o{}^s{}_{hk}\widetilde{M}^t_s\boldsymbol{K}_{j_1\cdots j_p;t}\end{aligned} \tag{21. 2}$$

which are identical with (14. 4) for $\mathfrak{M}_\Gamma$. Nextly, by (19. 8) and (21. 1) we have

$$\begin{aligned}(\bar{B}^*_hY_k - Y_k\bar{B}^*_h)(\boldsymbol{K}_{j_1\cdots j_p}) &= \widetilde{C}^t_{hk}\bar{B}^*_t(\boldsymbol{K}_{j_1\cdots j_p}) +\\ &+ \{\widetilde{P}_h{}^t{}_{ok} + (\widetilde{C}_o{}^s{}_{k,h} - \widetilde{P}_o{}^s{}_{hk})\widetilde{M}^t_s\}\boldsymbol{K}_{j_1\cdots j_p;t} - \hat{W}^t{}_{hk}\boldsymbol{K}_{j_1\cdots j_p;t}\end{aligned}$$

$$- \sum_{\alpha} \widetilde{W}_{j_\alpha}{}^{t}{}_{hk} \boldsymbol{K}_{j_1 \cdots t \cdots j_p}$$
$$+ ((\widetilde{W}_t{}^t{}_{hk} \hat{K}^t{}_{j_1 \cdots j_p},\ \widetilde{W}_t{}^i{}_{hk} \widetilde{K}^t_{jj_1 \cdots j_p} - \widetilde{K}^t_{tj_1 \cdots j_p} \widetilde{W}_j{}^t{}_{hk})),$$

hence

$$(\bar{B}_h^* Y_k - Y_k \bar{B}_h^*)(\boldsymbol{K}_{j_1 \cdots j_p}) = [\boldsymbol{W}_{hk},\ \boldsymbol{K}_{j_1 \cdots j_p}] - \sum_{\alpha} \widetilde{W}_{j_\alpha}{}^{t}{}_{hk} \boldsymbol{K}_{j_1 \cdots t \cdots j_p}$$
$$+ \widetilde{C}_h{}^t{}_k \bar{B}_t^* (\boldsymbol{K}_{j_1 \cdots j_p}) + \{\widetilde{P}^t_{koh} + (\widetilde{C}_o{}^s{}_{k,h} - \widetilde{P}_o{}^s{}_{hk}) \widetilde{M}^t_s\} \boldsymbol{K}_{j_1 \cdots j_p; t}, \tag{21.3}$$

which are identical with (14. 5). Lastly, by (19. 9) we have

$$(Y_h Y_k - Y_k Y_h)(\boldsymbol{K}_{j_1 \cdots j_p}) = (\widetilde{C}_h{}^t{}_k - \widetilde{C}_k{}^t{}_h) \boldsymbol{K}_{j_1 \cdots h_p; t} - w^t \widetilde{S}_t{}^t{}_{hk} \boldsymbol{K}_{j_1 \cdots j_p; t}$$
$$- \sum_{\alpha} \widetilde{S}_{j_\alpha}{}^{t}{}_{hk} \boldsymbol{K}_{j_1 \cdots t \cdots j_p}$$
$$+ ((\widetilde{S}_t{}^t{}_{hk} \hat{K}^t{}_{j_1 \cdots j_p},\ \widetilde{S}_t{}^i{}_{hk} \widetilde{K}^t{}_{j j_1 \cdots j_p} - \widetilde{K}^t_{tj_1 \cdots j_p} \widetilde{S}_j{}^t{}_{hk})),$$

that is

$$(Y_h Y_k - Y_k Y_h)(\boldsymbol{K}_{j_1 \cdots j_p}) = [\boldsymbol{S}_{hk},\ \boldsymbol{K}_{j_1 \cdots j_p}] - \sum_{\alpha} \widetilde{S}_{j_\alpha}{}^{t}{}_{hk} \boldsymbol{K}_{j_1 \cdots t \cdots j_p}$$
$$+ (\widetilde{C}_h{}^t{}_k - \widetilde{C}_k{}^t{}_h) \boldsymbol{K}_{j_1 \cdots j_p; t}, \tag{21.4}$$

which are identical with (14. 6).

Since we take coefficients in $\mathfrak{A}$, it follows from (21. 2) — (21. 4) that the verification of Theorem 14. 1 may hold good in this case. q. e. d.

Similarly to Theorem 14. 2, if we denote by $\hat{\mathfrak{S}}'_\infty$ the set of elements which are obtained by operating finitely Y_t upon $\boldsymbol{S}_{hk}$, we shall get

Theorem 21. 2. *$\hat{\mathfrak{S}}'_\infty$ is a Lie subalgebra over $\mathfrak{A}$ of $\hat{\mathfrak{M}}_\Gamma$.*

§ 22. Relation between $\widetilde{AH}$ and AH.

We take an element of $\hat{\mathfrak{M}}$ defined by

$$\boldsymbol{P}_{hk} = ((\hat{P}^t{}_{hk},\ \widetilde{P}_j{}^t{}_{hk}))$$
$$\hat{P}^t{}_{hk} = w^t \hat{P}_t{}^t{}_{hk} - \widetilde{C}_h{}^t{}_k. \tag{22.1}$$

Theorem 22. 1. *For any $y \in T_0(\mathfrak{X})$, a necessary and sufficient condition in order that $A\,H(y) = \widetilde{A\,H}(y)$ is that $y^h \boldsymbol{P}_{hk} \in \{\hat{\mathfrak{U}}_\Gamma\}_{\mathfrak{A}}$.*

Proof. By virtue of the above arguments, a necessary and sufficient condition in order that $A\,H(y) = \widetilde{A\,H}(y)$ is that the integral manifolds of the minimum involutive system over $\hat{\mathfrak{A}}$ containing the system

$$\{(\hat{B}_1, \cdots, \hat{B}_m, E_1, \cdots, E_n)\}_{\hat{\mathfrak{A}}}$$

and the integral manifolds of the system $\hat{\Sigma}'_\infty$ defined in § 20 have the same intesections with fibres $\tilde{\mu}^{-1}(y)$ of $\{\hat{\mathfrak{B}}, T_0(\mathfrak{X})\}$, $y \in T_0(\mathfrak{X})$. Since we can join any two points $y_0, y_1 \in T_0(\mathfrak{X})$ by an α-curve $\bar{C}$ and the any horizontal lift $\hat{C}$ of $\bar{C}$ with respect to the projective connection $\widetilde{\Gamma}_0$ must be contained in two integral manifolds of the both systems. Hence, the above condition is equivalent to that the integral manifolds of the both involutive systems coincide with each others, that is the both involutive systems are so. Accordingly the condition above is equivalent to that

$$\hat{B}_h \in \hat{\Sigma}_1 + \hat{\mathfrak{M}}'_\Gamma \hat{Q}.$$

From (20. 1), (19. 5), (18. 2), we have

$$\begin{aligned}\bar{B}^*_h &= \bar{B}_h - \widetilde{\Phi}^i_h W_j = \hat{B}_h + (w^t \widetilde{P}_t{}^i{}_{oh} - \widetilde{C}^i_{oh}) \hat{Q}^o_i + \widetilde{P}_j{}^i{}_{oh} \hat{Q}^j_i \\ &= \hat{B}_h + \boldsymbol{P}_{oh}\hat{Q},\end{aligned} \tag{22. 2}$$

hence it must be $\boldsymbol{P}_{oh} \in \hat{\mathfrak{M}}'_\Gamma$. On the other hand, it is clear $\hat{\mathfrak{M}}'_\Gamma = \{\hat{\mathfrak{U}}_\Gamma\}_{\hat{\mathfrak{A}}}$ and $W_j(\hat{P}^i{}_{hk}) = \widetilde{P}_j{}^i{}_{hk}$, $W_h(\widetilde{P}_j{}^i{}_{hk}) = 0$. Hence in order that $AH(y) = \widetilde{AH}(y)$, it is necessary and sufficient that $\hat{\boldsymbol{P}}_{oh} \in \{\hat{\mathfrak{U}}_\Gamma\}_{\mathfrak{A}}$. q. e. d.

We call a connection Γ being *a-proper*, if $\boldsymbol{P}_{oh} = 0$, that is

$$y^h \widetilde{C}_h{}^i{}_k = 0, \qquad y^h \widetilde{P}_j{}^i{}_{hk} = 0. \tag{22. 3}$$

Now, if a connection Γ is h-proper, that is $\Gamma = {}'\Gamma$, then we have

$$y^h \widetilde{P}_j{}^i{}_{hk} = 0.$$

Accordingly, in order that an h-proper connection Γ be a-proper, it is necessary and sufficient that $\Phi_\Gamma = 1$. By means of (15. 3), we obtain immediately the following

Theorem 22. 2. *In order that the derived connection $'\Gamma$ of a regular connection Γ is a-proper, it is necessary and sufficient $\Phi_\Gamma = 1$ that is*

$$\widetilde{C}_o{}^i{}_j = y^h \widetilde{C}_h{}^i{}_j = 0. \tag{22. 4}$$

Theorem 22. 3. *If $\Phi_\Gamma = 1$, we have*

$$AH_\Gamma(y) = \widetilde{AH}_{'\Gamma}(y), \quad AH^0_\Gamma(y) = \widetilde{AH}^0_{'\Gamma}(y), \quad y \in T_0(\mathfrak{X}).$$

Proof. We shall denote by the same symbols with primes the quantites of the derived connection $'\Gamma$ of Γ corresponding to those of Γ. By means of (15. 8), (18. 2), (20. 1) and $\Phi_i^j = \delta_i^j$, we have

$$'\hat{B}_i = 'B_i - W_i = \bar{B}_i + \widetilde{P}_o{}^h{}_{oi} E_h - W_i = \bar{B}^*_i + \widetilde{P}_o{}^h{}_{oi} E_h,$$
$$'\hat{E}_i = 'E_i = E_i = \hat{E}_i.$$

Hence, the system $\hat{\Sigma}_1$ given by (19. 4) is equivalent to the system

$$\{'\hat{B}_1, \cdots, '\hat{B}_n, 'E_1, \cdots, 'E_n\}_{\mathfrak{A}}.$$

This will prove the theorem.

Lastly we state a remark on an h-proper connection Γ. In this case, since $\widetilde{P}_i{}^j{}_{oh} = 0$, we have from (13. 12), (19. 11), (22. 1)

$$\widetilde{W}_j{}^t{}_{hk} = \widetilde{P}_j{}^t{}_{hk},$$
$$\hat{W}^t{}_{hk} = w^j \widetilde{P}_j{}^t{}_{hk} - \widetilde{C}_h{}^t{}_k - \widetilde{C}_o{}^t{}_t \widetilde{C}_h{}^t{}_k - (\widetilde{C}_o{}^t{}_h)_{;k}$$
$$= \hat{P}^t{}_{hk} - \widetilde{C}_o{}^t{}_t \widetilde{C}^t_{hk} - (\widetilde{C}_o{}^t{}_h)_{;k}.$$

Thus, we obtain the relation

$$\boldsymbol{W}_{hk} = \boldsymbol{P}_{hk} - ((\widetilde{C}_o{}^t{}_t \widetilde{C}^t_{hk} + (\widetilde{C}^t_{oh})_{;k}, \ 0)). \tag{22. 5}$$

Accordingly, if Γ is a-proper, it must hold

$$\boldsymbol{W}_{hk} = \boldsymbol{P}_{hk}. \tag{22. 6}$$

Conversely, if we have (22. 6), it must be

$$\widetilde{C}^t_{ot} \widetilde{C}^t_{hk} + (\widetilde{C}^t_{oh})_{;k} = 0.$$

Multiplying the equation by y^h, we obtain

$$\widetilde{C}^t_{ot} \widetilde{C}^t_{ok} - \widetilde{C}_o{}^t{}_t \widetilde{\Phi}^t_k = - \widetilde{C}^t_{ok} = 0.$$

Thus, the following theorem has been proved.

Theorem 22. 4. *When a connection Γ is h-proper, a necessary and sufficient condition that Γ is also a-proper is that $\boldsymbol{W}_{hk} = \boldsymbol{P}_{hk}$ holds good.*

§ 23. Modified connections.

In the last section, we generalized Theorem 15. 1 on the homogeneous holonomy group of a regular connection Γ in the case of affine holonomy group under the condition $\Phi_\Gamma = 1$. In this section, we shall attempt to remove this restriction in Theorem 22. 3.

The homomorphism $\Phi = \Phi_\Gamma$ of $\mathfrak{F} = \{T(\mathfrak{X}) \boxtimes T_0(\mathfrak{X}),\ T_0(\mathfrak{X})\}$ defined in § 4 for a regular connection Γ is an isomorphism such that $\tilde{\tau} \cdot \Phi = 1 \cdot \tilde{\tau}$ and $\lambda_k \cdot \Phi = \Phi \cdot \lambda_k$ for any $k > 0$. Accordingly, we may regard Φ is induced by the bundle map ρ_1 from an isomorphism of the vector bundle $\{\mathfrak{Z},\ \mathfrak{S}(\mathfrak{X})\}$. We shall denote this isomorphism of $\{\mathfrak{Z},\ \mathfrak{S}(\mathfrak{X})\}$ by the same symbole Φ. Then we have

$$\rho_1 \cdot \Phi = \Phi \cdot \rho_1 \tag{23. 1}$$

Now, since Φ is a bundle map, we can define a connection $\Phi \# \tilde{\Gamma}$ of $\mathfrak{F}$, $\tilde{\Gamma} = \rho^{\diamond}\Gamma = \rho_1 \# \Gamma$, by means of (1. 16). Using (23. 1), we have immediately

$$(\Phi) \# \tilde{\Gamma} = (\rho_1 \cdot \Phi) \# \Gamma = \rho_1 \# (\Phi \# \Gamma) = \rho^{\diamond}(\Phi \# \Gamma) \tag{23. 2}$$

We call $\Phi \# \Gamma$, $\Phi \# \tilde{\Gamma}$ the modified connections of Γ, $\tilde{\Gamma}$ and denote them by ${}^{\vartriangle}\Gamma$, ${}^{\vartriangle}\tilde{\Gamma}$ respectively.

Owing to the relation (1. 17), a vector field along a curve in $T_0(\mathfrak{X})$ which is parallel with respect to $\tilde{\Gamma}$ is transformed another vector field along the curve under $\tilde{\Phi}^{\ominus}$ which is parallel with respect to ${}^{\vartriangle}\Gamma$, and the converse is also true. Hence we have

Theorem 23. 1. *For a regular connection Γ and its modified connection ${}^{\vartriangle}\Gamma$, the following isomorphisms hold good through Φ_Γ*

$$H_\Gamma(y) \approx H_{{}^{\vartriangle}\Gamma}(y), \qquad \tilde{H}_\Gamma(y) \approx \tilde{H}_{{}^{\vartriangle}\Gamma}(y).$$

Now, we shall write the Pfaffian forms of ${}^{\vartriangle}\tilde{\Gamma}$ with respect to a local canonical coordinate system $(u^j,\ \xi^j)$. Making use of $\tilde{\tau}^{\circ}\partial/\partial u^i (= \partial/\partial u^i$ according to our convention), we have at each point of $\tilde{\tau}^{-1}(U)$

$$\Phi\,(\partial/\partial u^i) = \Phi_i^j \partial/\partial u^j, \tag{23. 3}$$

hence by (1. 15) we have

$$\left(\Phi^{\ominus}\frac{\partial}{\partial u^i}\right)(y) = (\Phi \mid T_x(\mathfrak{X}) \times y)^{-1}\left(\frac{\partial}{\partial u^i}(y)\right)$$
$$= \Phi^{-1}\left(\frac{\partial}{\partial u^i}(y)\right) = \left(M_i^j \frac{\partial}{\partial u^j}\right)(y),$$

that is

$$\Phi^{\ominus}\frac{\partial}{\partial u^i} = M_i^j \frac{\partial}{\partial u^j}. \tag{23. 4}$$

Accordingly, the covariant differentiation of $^{\triangle}\widetilde{\Gamma}$ is given by the equations

$$\bar{D}\left(M_i^j \frac{\partial}{\partial u^j}\right) = M_j^k \frac{\partial}{\partial u^k} \otimes \omega_i^j$$

making use of (1. 16). This equations is written as

$$\bar{D}\frac{\partial}{\partial u^i} = \frac{\partial}{\partial u^j} \otimes \bar{\omega}_i^j,$$

$$\bar{\omega}_i^j = M_k^j (d\Phi_i^k + \omega_h^k \Phi_i^h) = M_k^j D\Phi_i^k + \omega_i^j. \tag{23. 5}$$

Let $\bar{\theta}_i^j$ be the differential forms on $\widetilde{\mathfrak{B}}_0$ representing $\tilde{\pi}^{\diamond}(^{\triangle}\widetilde{\Gamma})$, then these are written locally as

$$\bar{\theta}_i^j = b_k^j (da_i^k + \bar{\omega}_h^k a_i^h),$$

in which we substitute (23. 5), we get the relations

$$\bar{\theta}_i^j = \widetilde{M}_k^j (d\widetilde{\Phi}_i^k + \theta_h^k \widetilde{\Phi}_i^h). \tag{23. 6}$$

For any curve $\bar{C}$ of class C^1 in $T_0(\mathfrak{X})$, its developments are given by the solutions of the Pfaffian equations (17. 8) on $R^n \times GL(n) \times \tilde{\pi}^{-1}(\bar{C})$

$$dp' = e_j'\theta^j, \quad de_i' = e_j'\theta_i^j. \tag{23. 7}$$

For any solutions (p', e_i'), if we put $\bar{e}_i' = e_j' \widetilde{\Phi}_i^j$, then $(p', \bar{e}'_i)$ is, as is easily verified, a solution of the following equations

$$dp' = \bar{e}_j' \widetilde{M}_i^j \theta^i, \quad d\bar{e}_i' = \bar{e}_j' \bar{\theta}_i^j. \tag{23. 8}$$

Furthermore, if $\bar{C}$ is an α-curve, we have locally the relations

$$\frac{du^j}{dt} = \psi\xi^j,$$

hence it must be $\theta^j = \psi y^j\, dt$ on $\tilde{\pi}^{-1}(\bar{C})$. Accordingly, by virtue of (4. 10) we have

$$dp' = \bar{e}_j'\theta^j, \quad d\bar{e}_i' = \bar{e}_j' \bar{\theta}_i^j \tag{23. 8'}$$

for a solution (p', e_i') of (23. 7) when $\bar{C}$ is an α-curve. From this, it follows immediately

Theorem 23. 2. *For any regular conection Γ and its modified connection $^{\triangle}\Gamma$, the following relation holds through Φ_Γ regarding it as an affine transformation on each fibre $T_x(\mathfrak{X}) \times y$, $x = \tau(y)$:*

$$A\, H_\Gamma(y) \approx A\, H_{^{\triangle}\Gamma}(y)$$

Lemma 23. 3. *For any regular connection Γ, the isomophism $\Phi_{\triangle\Gamma}$ of its derived connection $^{\triangle}\Gamma$ is the identity transformation.*

Proof. Making use of canonical local coordinates (u^j, ξ^j), if we put

$$\bar{\omega}_i^j = \bar{\Gamma}_i{}^j{}_k du^k + \bar{C}_i{}^j{}_k d\xi^k,$$

then by (23. 5) we have

$$\bar{C}_i{}^j{}_k = M_h^j\left(\frac{\partial\Phi_i^h}{\partial\xi^k} + C_l{}^h{}_k\,\Phi_i^l\right),$$

hence by (4. 10)

$$\bar{C}_0{}^j{}_k = \xi^i\bar{C}_i{}^j{}_k = M_h^j\left(\xi^i\frac{\partial\Phi_i^h}{\partial\xi^k} + C_l{}^h{}_k\,\xi^l\right)$$
$$= M_h^j\,(\delta_k^h - \Phi_k^h + C_{ok}^h) = 0.$$

Thus it must be $\Phi_{\triangle\Gamma} = 1$ by the definition of this homormorphism.

Corollary 23. 4. *For any regular connection Γ, we have*

$$^{\triangle}(^{\triangle}\Gamma) = {}^{\triangle}\Gamma.$$

Theorem 23. 2, Lemma 23. 3 and Theorem 22. 3 will immediately imply the following main theorem.

Theorem 23. 5. *For any regular connection Γ of* $(\mathfrak{Z}, \mathfrak{S}(\mathfrak{X}))$ *and the derived connetion $^*\Gamma = {}'(^{\triangle}\Gamma)$ of its modified connection $^{\triangle}\Gamma$, the following relations hold*

$$AH_\Gamma(y) \approx \widetilde{AH}_{*\Gamma}(y),\ \ AH_\Gamma^0(y) \approx \widetilde{AH}_{*\Gamma}^0(y),\ \ y \in T_0(\mathfrak{X}).$$

In the following, we shall make some remarks on $^{\triangle}\Gamma$. By means of (23. 6) we have

$$\bar{D}y^j = dy^j + y^i\bar{\theta}_i^j = dy^j + y^i\widetilde{M}_k^j\,(d\widetilde{\Phi}_i^k + \theta_h^k\,\widetilde{\Phi}_i^h)$$
$$= dy^j - y^i\widetilde{\Phi}_i^k\,d\widetilde{M}_k^j + \widetilde{M}_k^j y^h\theta_h^k$$
$$= \widetilde{M}_k^j(dy^k + y^h\theta_h^k),$$

that is

$$\bar{D}y^j = \widetilde{M}_k^j Dy^k = \bar{\gamma}^j,\quad \bar{D}\xi^j = M_k^j D\xi^k = \gamma^j. \tag{23. 9}$$

On the other hand, from (23. 5), (4. 8), we have

$$\bar{\omega}_i^j = \omega_i^j + M_h^j\,(C_o{}^h{}_i)_{,k}du^k + M_h^j(C_o{}^h{}_i)_{;k}\gamma^k$$
$$= (\Gamma^*{}_i{}^j{}_k + M_h^j(C_o{}^h{}_i)_{,k})du^k + (C_i{}^j{}_k + M_h^j(C_{oi}^h)_{;k})\,\bar{D}\xi^k.$$

Making use of Lemma 23. 3, we have

$$\bar{\Gamma}^{*j}_{ik} = \Gamma^{*j}_{ik} + M^j_h (C^h_{oi})_{,k}, \quad \bar{C}_i{}^j{}_k = C_i{}^j{}_k + M^j_h (C_o{}^h{}_i)_{;k}. \tag{23.10}$$

Since the curvature forms $\bar{\Omega}^j_i$ of ${}^{\triangle}\Gamma$ are written as

$$\bar{\Omega}^j_i = d\bar{\omega}^j_i + \bar{\omega}^j_k \wedge \bar{\omega}^k_i = M^j_k \Omega^k_h \Phi^h_i \tag{23.11}$$

by (23.5), if we put

$$\bar{\Omega}^j_i = \frac{1}{2} \bar{R}_i{}^j{}_{hk}\, du^h \wedge du^k + \bar{P}_i{}^j{}_{hk}\, du^h \wedge \bar{D}\xi^k + \frac{1}{2} \bar{S}_i{}^j{}_{hk}\, \bar{D}\xi^h \wedge \bar{D}\xi^h, \tag{23.12}$$

then we get from the above equations and (23.9)

$$\begin{cases} \bar{R}_i{}^j{}_{hk} = M^j_s \Phi^t_i R_t{}^s{}_{hk}, & \bar{P}_i{}^j{}_{hk} = M^j_s \Phi^t_i P_t{}^s{}_{hk} \\ \bar{S}_i{}^j{}_{hk} = M^j_s \Phi^t_i S_t{}^s{}_{hk}. \end{cases} \tag{23.13}$$

Furthermore, the differential forms on $\tilde{\mathfrak{B}}_0$ for the connection ${}^*\Gamma = {}'({}^{\triangle}\Gamma)$ are given by

$$\begin{aligned} &\bar{\theta}^j_i - \tilde{M}^j_k \tilde{\Phi}^h_i \tilde{P}_h{}^k{}_{om}\theta^m \\ &= \tilde{M}^j_k\, d\tilde{\Phi}^k_i + \tilde{M}^j_k(\theta^k_h - \tilde{P}_h{}^k{}_{om}\theta^m)\tilde{\Phi}^h_i \\ &= \tilde{M}^j_k(d\tilde{\Phi}^k_i + {}'\theta^k_h \tilde{\Phi}^h_i), \end{aligned} \tag{23.14}$$

where ${}'\theta^k_h = \theta^k_h - \tilde{P}_h{}^k{}_{om}\theta^m$, making use of (23.13) and (15.1). ${}'\theta^k_h$ being the differential forms on $\tilde{\mathfrak{B}}_0$ for the derived connection ${}'\Gamma$ of Γ and $\Phi_\Gamma = \Phi_{'\Gamma}$ by means of (15.4), the right hand side of (23.14) must be the differential forms on $\tilde{\mathfrak{B}}_0$ for the modified connection ${}^{\triangle}({}'\Gamma)$ of ${}'\Gamma$. Thus, we have proved the following

Theorem 23.6. *For any regular connection Γ of $\{\mathfrak{Z}, \mathfrak{S}(\mathfrak{X})\}$, the following relation holds good*

$${}'({}^{\triangle}\Gamma) = {}^*\Gamma = {}^{\triangle}({}'\Gamma).$$

Corollary 23.7. *For any regular connection Γ, ${}^*\Gamma$ is a-proper.*

Proof. This corollary follows immediately from this theorem, Theorem 15.5, Lemma 23.3 and Theorem 22.2.

DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

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