COMPOSITIONS OF LINEAR TOPOLOGICAL SPACES

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Introduction

A topology of a linear space $\mathscr{L}$ is determined by a system of semi-norms on $\mathscr{L}$, then a conception of a composition of topological representations of $\mathscr{L}$ may be strictly and naturally formed by that of semi-norms of $\mathscr{L}$. In this point of view we shall develop a "composition theory of semi-norms" which lays its foundation on three elementary properties of semi-norms.

(1) A semi-norm is a functional on $\mathscr{L}$, and a space of semi-norms allows some rules of compositions of functionals. For instance, if $s_1, \dots, s_n$ are semi-norms, $q = \max(s_1, \dots, s_n)$ and $r = (s_1^p + s_2^p + \dots + s_n^p)$ $(p \geqq 1)$ are semi-norms. More generally, if a function of n real variables $k(x_1, \dots, x_n)$ satisfies a suitable condition (c. f. §4), the composite functional $k(s_1, \dots, s_n)$ is a semi-norm.

(2) A semi-norm s on $\mathscr{L}$ determines uniquely a representation of $\mathscr{L}$ on another normed space $\mathscr{L}(s)$, then a composition $p = k(s_1, \dots, s_n)$ of semi-norms $s_1, \dots, s_n$ defines a composition of the representative normed spaces $\mathscr{L}(p) = k(\mathscr{L}(s_1), \dots, \mathscr{L}(s_n))$.

(3) A space of semi-norms on $\mathscr{L}$ is a topological space, which has a weak topology as a space of functionals on $\mathscr{L}$. The general Heine-Borel principle (The equivalency between bounded-closedness and compactness) is valid even in the space of semi-norms. (Theorem 1 in §1). It enables us to extend the former algebraical compositions to transcendental compositions $\sup_{\lambda \in \Lambda} \lambda$, $(\int_\Lambda \lambda^p d\rho(\lambda))^{\frac{1}{p}}$: $\sup_{\lambda \in \Lambda} \mathscr{L}(\lambda)$, $(\int_\Lambda \mathscr{L}(\lambda)^p d\rho(\lambda))^{\frac{1}{p}}$ etc. on bounded sets of semi-norms and corresponding sets of normed spaces, respectively.

§1 treats a topological space of semi-norms and the general Heine-Borel principle, §2 and 3 prepare the correspondence theory between semi-norms and representations. In §4 we introduce a concept of a composition of semi-norms in a wide sence. §5 and §6 treats some sets of semi-norms convex by some operations of compositions, and their extremal problems. §7 determines the composition law of representative normed spaces and their dual spaces which correspond to compositions of semi-norms.

§1. Topology and order of semi-norms.

We consider a fixed (real or complex) linear space $\mathscr{L}$. The *totality of complex functionals* on $\mathscr{L}$ is a topological space by its *weak topology*. A *weak neighbourhood* of a functional φ is a set of functionals $U(\varphi: f_1, \ldots, f_n, \varepsilon) = (\psi: |\varphi(f_i) - \psi(f_i)| < \varepsilon,\ 1 \leqq i \leqq n)$. If φ and ψ are real functionals, the *order* $\varphi \geqq \psi$ is defined by $\varphi(f) \geqq \psi(f)$ $(f \in \mathscr{L})$. The *general Heine-Borel Theorem* is valid in a space of functionals.

Lemma 1.1. *A set A of functionals is compact if and only if it is closed and bounded.*

In fact, if A is absolutely bounded by a functional ψ, A is contained in the compact space $U(\psi) = (\varphi: |\varphi| \leqq \psi)$. $U(\psi)$ is the Tychonoff product $\Pi\bigoplus_{f\in\mathscr{L}}(|\varphi(f)| \leqq \psi(f))$ of closed circulars in the complex domain defined for respective f in $\mathscr{L}$. Then a bounded closed set is compact.

Conversely, assume A be compact. Every f in $\mathscr{L}$ determines the (continuous) *coodinate function* $\varphi \to \varphi(f)$ of variable φ defined on the totality of functionals on $\mathscr{L}$. Then the image $(\varphi(f): \varphi \in A)$ of A is bounded closed, and A has an absolute bound $\psi(f) = \sup_{\varphi \in A} |\varphi(f)|$.

A *semi-norm* s of $\mathscr{L}$ is a functional on $\mathscr{L}$ so that (1). $s(f) \geqq 0$, (2). $s(\alpha f) = |\alpha| s(f)$, and (3). $s(f+g) \leqq s(f) + s(g)$. By the continuity of the coodinate functions $s \to s(f)$, the *totality* **P** *of semi-norms on $\mathscr{L}$ is a closed set of functionals*, then the Heine-Borel Theorem is valid in **P** as well.

Theorem 1. *A set of semi-norms is compact if and only if it is bounded and closed.*

§ 2. Relations between semi-norms and representations.

If $f \to f_s$ is a representation (a linear mapping) of $\mathscr{L}$ on another normed linear space $\mathscr{L}(s)$, the functional s on $\mathscr{L}$: $s(f) = |f_s|$ (the norm of f_s) is a semi-norm on $\mathscr{L}$. The *kernel* $\mathscr{N}(s)$ of the representation, $\mathscr{N}(s) = (f \in \mathscr{L}: f_s = 0)$, is identical with the *zero-point-set* $(f \in \mathscr{L}: s(f) = 0)$ of s.

Conversely, if s is a semi-norm on $\mathscr{L}$, its zero-point-set $\mathscr{N}(s)$ is a linear space, and the residue space $\mathscr{L}(s) = \mathscr{L}/\mathscr{N}(s)$ is a normed space. The natural mapping $f \in \mathscr{L} \to f_s = f/\mathscr{N}(s)$ is called the *canonical representation* of $\mathscr{L}$ by s, and the norm $|f_s|$ in $\mathscr{L}(s)$ is determined by $|f_s| = s(f)$.

The totality $\mathscr{L}^*$ of linear functionals on $\mathscr{L}$ is a linear space, $\mathscr{L}^*$ is clearly a closed set of functionals on $\mathscr{L}$, in which the Heine-Borel Theorem is preserved.

Lemma 2.1. *The totality $F(\mathscr{L}^*)$ of weakly closed sub-sets of $\mathscr{L}^*$ is a topological space. A complete system of neighbourhoods of a $W \in F(\mathscr{L}^*)$ consists of those sub-sets of $F(\mathscr{L}^*)$: $\mathfrak{V}(W: f_1, \ldots, f_n, \varepsilon) = (X \in F(\mathscr{L}^*) : X \subseteqq U(W: f_1, \ldots, f_n, \varepsilon)$ and $W \subseteqq U(X: f_1, \ldots, f_n, \varepsilon))$. $U(W: f_1, \ldots, f_n)$, denotes the set theoretical sum of the totality of $U(s: f_1, \ldots, f_n, \varepsilon)$ neighbourhoods of $s \in W$.*

If s is a semi-norm, the *dual Banach space* of $\mathscr{L}(s)$ is denoted by $\mathscr{L}^*(s)$. $\mathscr{L}^*(s)$ is the totality of $\varphi \in \mathscr{L}^*$ so that $|\varphi(f)| \leqq s(f)$ for every $f \in \mathscr{L}$. The *norm* $s^*(\varphi)$ of φ in $\mathscr{L}^*(s)$ is $s^*(\varphi) = \sup_{s(f) \leqq 1} |\varphi(f)|$. The following fundamental relations between semi-norms s and unit spheres $U(s)$ of $\mathscr{L}^*(s)$ is well-known.

Lemma 2.2. *The unit sphere $U(s)$ of $\mathscr{L}^*(s)$ is a weakly compact convex and symmetric (i. e. $\varphi \in U(s)$ and $|\alpha| = 1$ imply $\alpha s \in U(s)$) subset of $\mathscr{L}^*$. $s \leftrightarrow U(s)$ is a one-to-one correspondence between the totality $\mathbf{P}$ of semi-norms and the totality $\mathbf{U}$ of weakly compact convex and symmetric sub-sets of $\mathscr{L}^*$.*

$\mathbf{U}$ is a topological space, as a sub-space of $F(\mathscr{L}^*)$. Our interest is how the order and the topology of $\mathbf{P}$ are transposed to those of $\mathbf{U}$.

Theorem 2. *The correspondence $s \leftrightarrow U(s)$ between $\mathbf{P}$ and $\mathbf{U}$ is an order-preserving ($r \geqq s$ if and only if $U(r) \supseteqq U(s)$) homeomorphism. A neighbourhood $U(s: f_1, \ldots, f_n, \varepsilon)$ of an $s \in \mathbf{P}$ corresponds to the neighbourhood $\mathfrak{V}(U(s): f_1, \ldots, f_n, \varepsilon)$ of $U(s)$.*

Proof. We only need to prove that $|r(f) - s(f)| < \varepsilon$ is equivalent to $U(U(r): f, \varepsilon) \supseteqq U(s)$ and $U(U(s): f, \varepsilon) \supseteqq U(r)$. If $|r(f) - s(f)| < \varepsilon$, every $\varphi \in U(s)$ satisfies $|\varphi(f)| \leqq s(f) \leqq r(s) + \varepsilon$, while we can choose a $\psi \in U(r)$ with $r(f) = \psi(f)$. There is a suitable number $\alpha (|\alpha| \leqq 1)$ so that $|\varphi(f) - \alpha\psi(f)| < \varepsilon$ and $\varphi \in U(\alpha\psi: f, \varepsilon) \subseteqq U(U(r): f, \varepsilon)$. This means $U(s) \subseteqq U(U(r): f, \varepsilon)$ and $U(r) \subseteqq U(U(s): f, \varepsilon)$. Conversely if $U(s) \subseteqq U(U(r): f, \varepsilon)$ and $U(r) \subseteqq U(U(s): f, \varepsilon)$, $s(f) = \max_{\varphi \in U(s)} |\varphi(f)|$ and $r(f) = \max_{\psi \in U(r)} |\psi(f)|$ satisfies $|r(f) - s(f)| < \varepsilon$. Hence $s \rightarrow U(s)$ is an homeomorphism which mapps the set $U(s: f_1, \ldots, f_n, \varepsilon)$ to the set $\mathfrak{V}(U(s): f_1, \ldots, f_n, \varepsilon)$.

§ 3. Orderly semi-norms on vector lattices.

We apply the result of the former § to semi-norms on vector lattices. Let Ω denote a compact space and $C(\Omega)$ denote the totality of continuous functions on it. A semi-norm s on $C(\Omega)$ is said to be *orderly* if $s(f) \geqq s(g)$ whenever $|f(\lambda)| \geqq |g(\lambda)|$ on Ω. Generally, a normed lattice with a unit is represented as a uniformly dense sub-lattice of a suitable $C(\Omega)$ with an orderly norm, then we shall treat a normed lattice under such a concrete representation.

Among many orderly semi-norms of $C(\Omega)$, the following semi-norms are well-known.

(1) Let μ be a regular measure on Ω. Then $|\mu|_p(f) = (\int |f(\lambda)|^p d\mu(\lambda))^{\frac{1}{p}}$ $(p \geqq 1)$ is an orderly semi-norm. We call it an *L^p-semi-norm* on $C(\Omega)$.

(2) If X is a closed sub-set of Ω, $|X|(f) = \sup_{\lambda \in X} |f(\lambda)|$ is an orderly semi-norm. We call it an *L^∞-semi-norm.*

$C(\Omega)$ is always treated as a Banach space with the norm $\|f\| = |\Omega|(f) = \sup |f(\lambda)|$. The dual Banach space $C^*(\Omega)$ of $C(\Omega)$ consists of the totality of completely additive set functions on the Borel field in Ω. Every $\varphi \in C^*(\Omega)$ has the Lebesgue decomposition $\varphi = \varphi^+ + \varphi^-$. The *total variation* of $\varphi : |\varphi| = \varphi^+ - \varphi^-$ is a regular measure on Ω.

A regular measure is called *normalized* if its total mass is 1. The totality $\mathscr{M}(\Omega)$ of normalized regular measures on Ω is a bounded regularly convex sub-set of $C^*(\Omega)$.

We recall some elementary properties of orderly semi-norms.

(3.1) *An orderly semi-norm s is smaller than the norm* $s(1)|\Omega|$. In fact, $|f(\lambda)| \leqq \|f\|$ implies $s(f) \leqq s(\|f\| 1) = \|f\| s(1)$.

(3.2) *An orderly semi-norm s is said to be normalized if* $s(1) = 1$. *The totality* $\mathbf{P}_1$ *of normalized orderly semi-norms on* $C(\Omega)$ *is bounded, closed and compact; the totaliiy* $\mathbf{P}$ *of orderly semi-norms s with* $s(1) \leqq 1$ *is as well.*

Lemma 3.1. *The totality* $\mathbf{L}^p$ *of normalized L^p-semi-norms on* $C(\Omega)$ *is compact (for each* $1 \leqq p \leqq \infty$).

Proof. In case $p < \infty$, the Lemma follows from that $\mathbf{L}^p$ is the range of the weakly continuous mapping $\mu \to |\mu|_p$ on the weakly compact space $\mathscr{M}(\Omega)$ of normalized regular measures.

In case $p = \infty$, the Lemma follows from the I. Gelfand's theorem on normed algebras that a necessary and sufficient condition for an (orderly) semi-norm s to be an L^∞-semi-norm is $s(fg) \leqq s(f)s(g)$ and $s(f^2) = s(f)^2$ for f, g in $C(\Omega)$.

(3.3) *We denote by $C(s)$ the normed space canonically represented by an orderly semi-norm s. The dual space $C^*(s)$ of $C(s)$ is contained in $C^*(\Omega)$. The total variation $|\varphi| = \varphi^+ - \varphi^-$ of every element φ in $C^*(s)$ belongs to $C^*(s)$ and has the same norm $s^*(|\varphi|) = s^*(\varphi)$ with φ.*

In fact, $|\varphi|(|f|)$ is determined by $|\varphi|(|f|) = \sup_{|f| \geqq |g|} |\varphi(g)|$. Thus $|\varphi|(f) \leqq |\varphi|(|f|) = \sup_{|f| \geqq |g|} |\varphi(g)| \leqq \sup_{s(f) \geqq s(g)} |\varphi(g)| = s^*(\varphi)s(f)$. Then $s^*(|\varphi|) \leqq s^*(\varphi)$. On the other had, $|\varphi(f)| \leqq |\varphi|(|f|)$ implies $s^*(\varphi) \leqq s^*(|\varphi|)$ and $s^*(\varphi) = s^*(|\varphi|)$.

(3.4) *Let $U^+(s)$ denote the totality of regular measures contained in the unit sphere of $C^*(s)$ of an orderly semi-norm s. For each $f \in C(\Omega)$ we can choose $\mu \in U^+(s)$ so that $s(f) = \mu(|f|) = |\mu|_1(f)$.*

In fact, let φ be an element of $C^*(s)$ so that $s^*(\varphi) = 1$ and $s(f) = \varphi(f)$, then its total variation $|\varphi|$ belongs to $U^+(s)$ and satisfies $|\varphi|(|f|) \geqq |\varphi(f)| = s(f)$ and $|\varphi|(|f|) = s(f)$.

When μ, ν are regular measures, we say $\mu \leqq \nu$ if $\nu - \mu$ is a measure. A set V of regular measures is said to be a *star* if $\mu \in V$ and $\mu \geqq \nu$ implies $\nu \in V$. The $U^+(s)$ of an orderly semi-norm s is clearly a star.

Theorem 3. *$s \leftrightarrow U^+(s)$ is a one-to-one correspondence between $\mathbf{P}^+$ and the totality $\mathbf{U}^+$ of regularly convex stars of regular measures on Ω with masses $\leqq 1$.*

Proof. By 3.7, every $s \in \mathbf{P}^+$ satisfies $s(f) = \sup_{\mu \in U^+(s)} |\mu|_1(f)$ and $s = \sup_{\mu \in U^+(s)} |\mu|_1$, then $s \leftrightarrow U^+(s)$ is one-to-one. It is sufficient to say that every $V \in \mathbf{U}^+$ is a $U^+(s)$ of an $s \in \mathbf{P}^+$. The set $U = (\varphi \in C^*(\Omega) : |\varphi| \in V)$ is regularly convex and symmetric in $C^*(\Omega)$, then it determines a semi-norm s on $C(\Omega)$ whose unit sphere $U(s)$ of the space $C^*(s)$ is U. s is orderly. In fact, $\varphi \in U$ implies $|\varphi| \in U$, where $|\varphi|(|f|) = \sup_{|\varphi| \geqq |\psi|} \psi(f)$ and $s(f) = \sup_{\varphi \in U} |\varphi(f)| = \sup_{\varphi \in U} |\varphi|(|f|) = \sup_{\varphi \in U} \int |f(\lambda)| \, d|\varphi|(\lambda)$. Then V is identical with $U^+(s)$, the totality of measures in V.

Corollary. *$s \leftrightarrow U^+(s)$ is an order-preserving homeomorphism between $\mathbf{P}^+$ and $\mathbf{U}^+$.*

The topology of $\mathbf{U}^+$ is in the sense of Lemma 2.1.

§ 4. Concept of a composition.

We turn once more to the linear space $\mathscr{L}$ in §1. Let k be an orderly semi-norm on the n-dimensional Euclidean space $\mathscr{R}^n$. k is a function of n variables $k(x_1, \ldots, x_n)$ so that (1) $k(x_1, \ldots, x_n) \leqq k(y_1, \ldots, y_n)$ whenever $|x_i| \leqq |y_i|$ $(1 \leqq i \leqq n)$. (2) $k(\alpha x_1, \ldots, \alpha x_n) = |\alpha| k(x_1, \ldots, x_n)$ and (3) $k(x_1 + y_1, \ldots x_n + y_n) \leqq k(x_1, \ldots, x_n) + k(y_1, \ldots, y_n)$. If $s_1 \ldots s_n$ are n semi-norms on $\mathscr{L}$, the composite functional $q = k(s_1, \ldots, s_n)$ (that is, $q(f) = k(s_1(f), \ldots, s_n(f))$) is a semi-norm. We call it a *composite semi-norm* or *composition* of $s_1, \ldots, s_n$. A composition is called *normalized* if $k(1, \ldots, 1) = 1$. In partiqular, $q = (\sum \alpha_i s_i^p)^{\frac{1}{p}}$ and $r = \max(s_1, \ldots, s_n)$ are called an *L^p-sum* and an *L^∞-sum* of those s_i, respectively.

More generally a composition is defined in a bounded set of semi-norms. Let Λ be a bounded set of semi-norms. A function on Λ is uniformly continuous[1] if and only if it is extended to a continuous function on the closure $\bar{\Lambda}$ of Λ. Then the space $C(\bar{\Lambda})$ is identical with the totality of uniformly continuous functions on Λ. Let k be an orderly semi-norm on $C(\bar{\Lambda})$. If $f \in \mathscr{L}$ is fixed, the function $|f_\lambda| = \lambda(f)$ of variable λ in Λ is uniformly continuous on Λ. A *composition* on Λ is a semi-norm on $\mathscr{L}$ determined by $k^\Lambda(f) = k(|f_\lambda|)$. A composition k is called *normalized* if the semi-norm k is normalized. A composition by an L^p-semi-norm on $C(\Omega)$ is called an *L^p-sum* of Λ. If Λ is compact, every L^p-sums are expressed as follows.

Lemma 4.1. *If Λ is a compact set of semi-norms, then*

(1) *An L^p-sum $(p < \infty)$ $|\mu|_p^\Lambda$ on Λ is a weak integral by a regular measure μ on Λ;* $|\mu|_p^\Lambda = (\int \lambda^p \, d\mu(\lambda))^{\frac{1}{p}}$.

(2) *An L^∞-sum $|X|^\Lambda$ is the least upper bound* $|X|^\Lambda = \sup_{\lambda \in X} \lambda$ (i.e. $|X|^\Lambda(f) = \sup_{\lambda \in \Lambda} \lambda(f)$ *on a closed set X in Λ.*

Usually every L^p-sum on a bounded set Λ of semi-norms is treated as an L^p-sum on the closure $\bar{\Lambda}$. But, if necessary, we represent an L^p-sum $(p < \infty)$ by a *weak Radon integral* on Λ.

A finitely additive set-function defined on the totality of sub-sets of a topological space Ω is called a *Radon measure*.

The Radon integral exists for every bounded function f on Ω.

Lemma 4.2. *If s is an L^p-sum $(p < \infty)$ on a bounded set Λ of semi-norms, s is a weak Radon integral* $s = (\int_\Lambda \lambda^p d\rho(\lambda))^{\frac{1}{p}}$ *by a suitable Radon measure ρ on Λ.*

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1) Uniformly continuous by the uniform structure of the weak topology.

Proof. The space $C(\bar{\Lambda})$, regarded as the totality of uniformly continuous functions on Λ, is contained in the totality $B(\Lambda)$ of bounded functions on Λ. If ρ is a Radon measure, the Radon integral $\rho(f) = \int f\,d\rho$ is a positive linear functional on $B(\Lambda)$. Its restriction on $C(\bar{\Lambda})$ is a positive linear functional on $C(\bar{\Lambda})$, and represents a regular measure μ on $\bar{\Lambda}$. Then the weak Radon integral $(\int \lambda(f)^p d\rho(\lambda))^{\frac{1}{p}}$ is equal to an L^p-sum $(\int \lambda(f)^p\, d\mu(\lambda))^{\frac{1}{p}} = |\mu|_p^\wedge (f)$.

The "Extention theorem of the measure" mentions that (See the next Remark) every positive linear functional on $C(\bar{\Lambda})$ is extended to a positive linear functional (a Radon integral) on $B(\Lambda)$. Then every L^p-sum is a weak Radon integral on Λ.

Remark. The following extention theorem is often useful.

Theorem 4. *Let Λ be a topological space, Ω be a bounded set in the dual space of a Banach space and $Co(\Omega)$ be the smallest regularly convex set which contains Ω. If $\lambda \to T\lambda$ is a mapping of Λ on Ω, every element φ of $Co(\Omega)$ is a weak Radon integral $\varphi = \int T\lambda\, d\rho(\lambda)$ by a suitable Radon measure with the total mass* 1.

If Ω is compact and if $\lambda \to T_\lambda$ is continuous, the measure ρ can be chosen as a normalized regulur measure on Ω.

Proof. The totality $\mathbf{R}$ of Radon measures on Ω with the total mass 1 is the smallest regularly convex sub-set of the dual space $B^*(\Omega)$ of $B(\Omega)$, which contains the totality of point measures δ_λ: $\int f\,d\delta_\lambda = f(\lambda)$ $(\lambda \in \Omega)$. Since $\rho \in \mathbf{R} \to \int T\lambda d\rho(\lambda)$ is weakly continuous on $\mathbf{R}$, its Range is weakly compact, convex and contains every $T\lambda = \int T\nu d\delta_\lambda(\nu)$ $(\lambda \in \Omega)$. Then it is identical with the set $Co(\Lambda)$.

If Ω is compact and if the mapping is continuous, a weak Radon integral $\int T\lambda d\rho(\lambda)$ is equal to a weak integral $\int T\lambda d\mu(\lambda)$ by a regular measure μ. μ is the restriction of the positive linear functional ρ on $B(\Omega)$ within the space $C(\Omega)$.

Corollary. (Extention theorem of the measure). *Let Ω be a topological space, and Λ be a compact space. If $\lambda \to T\lambda$ is a mapping of Ω in a dense sub-space of Λ, every regular measure μ on Λ, as a positive linear functional on $C(\Lambda)$, is extended to a Radon integral ρ*

on $B(\Omega)$ *so that*

$$\int_{\Omega} f(T\lambda)d\rho(\lambda) = \int_{\Lambda} f(\nu)d\mu(\nu).$$

If Ω *is compact, and if* T *is continuous,* ρ *is chosen as a regular measure on* Ω.

Proof. We apply Theorem 4 to the mapping $\lambda \to \delta_{T\lambda}$. The point mass $\delta_{T\lambda}$ at $T\lambda$ in Λ is a positive linear functional on $C(\Omega)$. Then every positive linear functional μ on $C(\Lambda)$ in the set $X = Co(\delta_{T\lambda} : \lambda \in \Omega)$ is a weak Radon integral $\int \delta_{T\lambda}\, d\rho(\lambda)$. X is the totality of normalized regular measures on Λ, and every normalized regular measure μ is a weak integral $\mu = \int \delta_{T\lambda} d\rho(\lambda)$. Then for every $f \in C(\Lambda)$,

$$\int f d\mu = \int \delta_{T\lambda}(f)d\rho(\lambda) = \int f(T\lambda)d\rho(\lambda).$$

§ 5. **Convex sets of semi-norms.**

A set Λ of semi-norms is called *L^p-sum convex* $(1 \leqq p \leqq \infty)$ if every normalized L^p-sum of every pair of elements in Λ belongs to Λ. Λ is called *universally convex* if it is L^1-sum convex, and $s,\ t \in \Lambda$ imply $\max(s, \alpha t) \in \Lambda$ for evey $0 \leqq \alpha \leqq 1$.

Theorem 5. *The totality* $Co^p(\Lambda)$ *of normalized* L^p*-sums on a bounded set* Λ *of semi-norms on* $\mathscr{L}$ *is the smallest closed* L^p*-sum convex set which contains* Λ.

Theorem 6. *The totality* $Co^u(\Lambda)$ *of normalized compositions on a bounded set* Λ *of semi-norms on* $\mathscr{L}$ *is the smallest closed universally convex set which contains* Λ.

Proof of Theorem 5. In case $p < \infty$, $Co^p(\Lambda)$ is the range of the weakly continuous mapping $\mu \to |\mu|^{\Lambda}_{p} = (\int \lambda^p\, d\mu(\lambda))^{\frac{1}{p}}$ defined on the totality $\mathscr{M}(\Lambda)$ of normalized regular measures on Λ. This mapping transposes the addition $\alpha\mu + \beta\nu (\alpha + \beta = 1)$ in $\mathscr{M}(\Lambda)$ to the L^p-sum operation $\{\alpha(|\mu|^{\Lambda}_{p})^p + \beta(|\nu|^{\Lambda}_{p})^p\}^{\frac{1}{p}}$ in $Co^p(\Lambda)$. $\mathscr{M}(\Lambda)$ is the smallset weakly compact convex set which contains those point measures δ_λ at λ in Λ, then $Co^p(\Lambda)$ is the smallest compact L^p-sum convex set which contains all elements $\nu = (\int \lambda^p d\delta_\nu(\lambda))^{\frac{1}{p}}$ in Λ.

In case $p=\infty$, $Co^\infty(A)$ is compact as the image of the continuous mapping $k \rightarrow |k|^\Lambda$ of the compact set of the totality of L^∞-semi-norms on $C(\bar{A})$. (c. f. Lemma 3.2), then it is sufficient to see that $Co^\infty(A)$ contains everywhere densely those L^∞-sums $\max(\lambda_1, \dots, \lambda_n)$ of finite elements in A, that is, every neighbourhood $U(|X|^\Lambda: f_1, \dots, f_n, \varepsilon)$ of an L^∞-sum $|X|^\Lambda = \sup_{\lambda \in X} \lambda$ (the least upper bound on a closed set X in A) contains such an L^∞-sum $\max(\lambda_1, \dots, \lambda_n)$. Let ν_i be a point in X so that $\nu_i(f_i) = \sup_{\lambda \in X} \lambda(f_i) = |X|^\Lambda(f_i)$, then we can choose $\lambda_1, \dots, \lambda_n$ in A so that $|\nu_i(f_j) - \lambda_i(f_j)| < \varepsilon$ $(i, j = 1, 2, \dots, n)$. The $\max(\lambda_1, \dots, \lambda_n)$ is a requiered one. Thus the Theorem is valid in every case.

Proof of Theorem 6. By (3.4) in §3 an orderly seminorm k satisfies $k(f) = \sup_{\mu \in U^+(k)} \int |f| \, d\mu = \sup_{\mu \in U^+(k)} |\mu|_1(f)$. Then the composition k^Λ is a L^∞-sum of L^1-sums on A:

$$k^\Lambda(f) = \sup_{\mu \in U^+(k)} |\mu|_1 (|f_\lambda|) = \sup_{\mu \in U^+(k)} |\mu|_1^\Lambda (f).$$

Lemma 5.2. *Every composition k^Λ on A is a L^∞-sum of a system of L_1-sums on A.*

Let A be a bounded set and let W denote the smallest universally convex set which contains A. Then $W = Co^p(W)$ for $p=1$ and $p=\infty$. Let k^Λ be an arbitrary normalized composition. Then k is a L^∞-sum of a suitable set Ω of L^1-sums $|\mu|_1^\Lambda$ on A by measures μ with masses $\mu(A) \leqq 1$. On the other hand k^Λ is normalized, and there is at least one measure ν in $U^+(k)$ so that $k(1) = |\nu|_1(1) = 1$. Then k is a L^∞-sum $k = \sup_{\mu \in \Omega} (\max(|\nu|_1^\Lambda, |\mu|_1^\Lambda))$. Every $|\mu|_1^\Lambda$ is a multiplication $|\mu|_1^\Lambda = \alpha \mu_0$ of $0 \leqq \alpha \leqq 1$ and a normalized L^1-sum μ_0 in A. Since $|\nu|_1^\Lambda$ and μ_0 belong to $Co^1(A) \subseteqq W$, $\max(|\nu|_1^\Lambda, |\mu|_1^\Lambda) = \max(|\nu|_1^\Lambda, \alpha\mu_0)$ belongs to W (by the definition of the univresal convexity), then $k = \sup_{|\mu|_1 \in \Omega} \max(|\nu|_1^\Lambda, |\mu|_1^\Lambda))$ belongs to $Co^\infty(W) = W$.

§ 6. An extremal theorem of semi-norms.

An element of an L^p-sum convex compact set A of semi-norms is called *L^p-sum indecomposable* if, whenever s be a normalized L^p-sum of two elements q, r in A, we have either $s=q$ or $s=r$. The next Theorem will be applied to decomposing operator algebras in §8.

Theorem 7. *If A is a compact L^p-sum convex set of semi-norms,*

Λ contains sufficiently many indecomposable elements in it. Then every element of Λ is an L^p-sum on the totality E^p of L^p-sum indecomposable elements in Λ (i. e. $\Lambda = Co^p(E^p)$).

Proof. Let q denote an upper bound of Λ. In case $p < \infty$, the Theorem follows the Krein-Milman's theorem. Replace each λ in Λ by a bounded functional $(\lambda(f)/q(f))^p$ on $\mathscr{L}$ provided $(\lambda(f)/q(f))^p = 0$ whenever $q(f) = 0$, then the set Λ and its L^p-sum operation are transposed to a set Λ^p of bounded functionals and the usual addition of functionals, respectively. $\Lambda^p = ((\lambda/q)^p : \lambda \in \Lambda)$ is a bounded regularly convex sub-set of the Banach space B[1] of the totality of bounded functionals on $\mathscr{L}$, and $\lambda \to (\lambda/q)^p$ is a weak homeomorphism between Λ and Λ^p. Λ^p contains sufficiently many extremal elements in it, while $(\lambda/q)^p$ in Λ^p is extremal if and only if λ is L^p-sum indecomposable in Λ. Λ contains therefore sufficiently many indecomposable elements in it.

We next consider in the case of $p = \infty$. Given each s in Λ and each f in $\mathscr{L}$, we can choose an L^∞-indecomposable element t_f in Λ so that $t_f = s$ and $t_f(f) = s(f)$. In fact the totality of elements t in Λ so that $t \leqq s$ and $t(f) = s(f)$ is inductive by the inverse order of semi-norms, and contains a minimal element t_f by the Zorn's lemma. t_f has no semi-norm r so that $t_f > r$ and $t_f(f) = r(f) = s(f)$. This t_f is indecomposable since $t_f = \max(q, r)$ $(q, r \in \Lambda)$ implies either $t_f(f) = q(f) = s(f)$ or $t_f(f) = r(f) = s(f)$, and either $t_f = q$ or $t_f = r$. The given semi-norm s is an L^∞-sum $s = \sup_{f \in \mathscr{L}} t_f$ of L^∞-indecomposable elements t_f in Λ, hence Λ contains sufficiently many L^∞-indecomposable elements.

Example. The totality $\mathbf{P}^+$ of normalized orderly semi-norms on the space $C(\Omega)$ of a compact space Ω is L^∞-sum convex. It is shown that

Theorem 8. *A necessary and sufficient condition for an orderly semi-norm s to be indecomposable to any non-trivial L^∞-sum of orderly semi-norms is that s be an L^1-semi-norm.*

Proof. If an L^1-semi-norm $|\mu|_1$ is an L^∞-sum $|\mu|_1 = \max(q, r)$ of two orderly semi-norms q and r, then either $|\mu|_1(1) = q(1)$ or $|\mu|_1(1) = r(1)$. We can assume $|\mu|_1(1) = q(1)$ without loss of generality, then every f in $C(\Omega)$ with $0 \leqq f \leqq 1$ satisfies $|\mu|_1(1) = |\mu|_1(f) + |\mu|_1(1-f)$,

1) B is considered as a dual Banach space of a suitable normed space $l(\mathscr{L})$, the totality of functionals $\varphi(f)$ which vanishes except for finite elements in $\mathscr{L}$, and whose norm is $\sum_{f \in L} \varphi(f)|$.

$q(1) \leqq q(f) + q(1-f)$, $q(f) \leqq |\mu|_1(f)$ and $q(1-f) \leqq |\mu|_1(1-f)$. This means $q(f) = |\mu|_1(f)$ and $q = |\mu|_1$, the indecomposability of $|\mu|_1$ to non-trivial L^∞-sums.

Conversely let s be an L^∞-indecomposable orderly semi-norm in $\mathbf{P}^+$. By (3.4) in §3 s is an L^∞-sum $s = \sup_{\mu \in U^+(s)} |\mu|_1 = \sup_{\lambda \in \Lambda} \lambda$ on a suitable set Λ of L^1-semi-norms. Eevery neighbourhood $U = U(s: f_1, \ldots, f_n, \varepsilon)$ contains an element of Λ. In fact, let s_0 denote the least upper bound of semi-norms in the common-part of U and Λ, and let s_i denote the least upper bound of semi-norms r in Λ so that $r(f_i) \leqq s(f_i) - \varepsilon$. Then s is an L^∞-sum $s = \max(s_0, s_1, \ldots, s_n)$. By the indecomposability of s, s is coincident with s_0, and $U \cap \Lambda$ is non-empty. Then s is an L^1-semi-norm as a limit element of L^1-semi-norms (Lemma 3.1).

§ 7. Decompositions of normed spaces and its dual spaces.

Let k be a composite semi-norm on a set Λ of semi-norms on $\mathscr{L}$.

Then the representation $\mathscr{L} \rightarrow \mathscr{L}(k^\Lambda)$ is said to *be composed* of the system of those representations $(\mathscr{L} \rightarrow \mathscr{L}(\lambda) : \lambda \in \Lambda)$. The consistency of such a definition is asserted by the one-to-oneness of the correspondence between semi-norms and representations. To study the relation between $\mathscr{L}(k^\Lambda)$ and $\mathscr{L}(\lambda)$ $(\lambda \in \Lambda)$, we define the carrier of a composition.

The *carrier* of an orderly semi-norm k on the space $C(\Omega)$ of a compact space Ω is the smallest closed sub-set X of Ω so that every f in $C(\Omega)$ which vanishes on X belongs to $\mathscr{N}(k)$ (i. e. $k(f) = 0$). The existence of the carrier is asserted by the next Lemma.

Lemma 7.1. *The carrier $D(k)$ of an orderly semi-norm k on $C(\Omega)$ is the closure of the set-theoretical sum of carriers of all regular measures in the space $U^+(k)$ ($U^+(k)$ is defined in* 3.4, §3).

In fact, if a continuous function f vanishes on the carrier of every regular measure μ in $U^+(k)$, then by 3.4 $|\mu|_1(f) = \int |f| d\mu = 0$ and $k(f) = \sup_{\mu \in U^+(k)} \int |f| d\mu = 0$. Conversely, if a continuous function f does not vanish on the carrier of a measure μ in $U^+(k)$, then $k(f) \geqq \int |f| d\mu > 0$. Q. E. D.

If two functions f and g in $C(\Omega)$ has the same values on the carrier $D(k)$ of an orderly semi-norm k, then $k(f) = k(g)$. *k is thus determined as an orderly norm on $C(D(k))$.*

The *carrier* of a composition k^Λ on a compact space Λ of semi-norms is defined as the carrier of the orderly semi-norm k on $C(\Lambda)$.

Lemma 7.2. *If k^{Λ} is a composition on a compact set Λ of semi-norms on $\mathscr{L}$, the zero-point-set $\mathscr{N}(k^{\Lambda})$ of k^{Λ} is contained in each zero-point set $\mathscr{N}(\lambda)$ of semi-norms in the carrier of k^{Λ}.*

In fact, if $f \in \mathscr{N}(k^{\Lambda})$ and $k(|f_{\lambda}|)=0$, then $|f_{\lambda}|=\lambda(f)=0$ and f belongs to $\mathscr{N}(\lambda)$ for each λ in $D(k)$.

A composition of normed spaces is now defined.

Definition. 1. Let $\mathfrak{R}(\lambda)$ be a system of normed spaces defined at each point λ in a compact space Λ. A normed space $\mathfrak{R}$ is said to *be composed of* $(\mathfrak{R}(\lambda) : \lambda \in \Lambda)$ by an orderly semi-norm k on $C(\Lambda)$ if

(1). For each $\mathfrak{R}(\lambda)$ a representation $f \to f_{\lambda}$ of $\mathfrak{R}$ in $\mathfrak{R}(\lambda)$ exists.

(2). If an f in $\mathfrak{R}$ is fixed, a numerical function $|f_{\lambda}|$ of the variable λ is continuous on Λ.

(3). Every two elements λ, μ in Λ are distinguished $(|f_{\lambda}| \neq |f_{\mu}|)$ by the norm-function $|f_{\lambda}|$ of a suitable f in $\mathfrak{R}$.

(4). The norm $|f|$ of f in $\mathfrak{R}$ is determined by $|f|=k(|f_{\lambda}|)$.

Definition 2. If the space $\mathfrak{R}$ in Definition 1 is composed by an L^p-semi-norm ($p < \infty$) by a regular measure μ on Λ, then $\mathfrak{R}$ is said to be an *L^p-sum* and denoted by $\mathfrak{R} = (\int \mathfrak{R}(\lambda)^p d\mu(\lambda))^{\frac{1}{p}}$.

If the space $\mathfrak{R}$ is composed by the L^{∞}-semi-norm $|\Lambda|$, then $\mathfrak{R}$ is said to be the *L^{∞}-sum* (or *the least upper bound*) of those spaces $(\mathfrak{R}(\lambda) : \lambda \in \Lambda)$, and denoted by $\mathfrak{R} = \sup_{\lambda \in \Lambda} \mathfrak{R}(\lambda)$.

Theorem 9. *Let k^{Λ} be a composite semi-norm on a compact set Λ of semi-norms on a linear space $\mathscr{L}$. Then the representative space $\mathscr{L}(k^{\Lambda})$ is a composition of those normed spaces $(\mathscr{L}(\lambda) : \lambda \in D(k))$.*

Proof. If $\lambda \in D(k)$, $\mathscr{N}(\lambda) \supseteqq \mathscr{N}(k^{\Lambda})$ implies that $f_k \to f_{\lambda} = (f_k)_{\lambda}$ defines a representation of $\mathscr{L}(k^{\Lambda}) = \mathscr{L}/\mathscr{N}(k^{\Lambda})$ on $\mathscr{L}(\lambda) = \mathscr{L}/\mathscr{N}(\lambda)$. k is a semi-norm on $C(D(k))$, and the representation $f_k \to (f_k)_{\lambda}$ satisfies clearly (1), (2), (3) and

(4); $|f_k| = k^{\Lambda}(f) = k(|f_k|_{\lambda})$.

We now treat our final problem, the composition rules of the dual space of a composite normed space.

Let Λ be a compact set of semi-norms on $\mathscr{L}$. A *spherical system* $\mathrm{U}(\Lambda)$ on Λ is the totality of those pairs (λ, φ) so that $\lambda \in \Lambda$ and $\varphi \in$

$U(\lambda)$ ($U(\lambda)$ is the unit sphere of $\mathscr{L}^*(\lambda)$.). $\mathrm{U}(\Lambda)$ is a topological space, as a sub-space of the product space $\Lambda \times \mathscr{L}^*$.

Theorem 10. *The spherical system* $\mathrm{U}(\Lambda)$ *of a compact set* Λ *of semi-norms on* $\mathscr{L}$ *is weakly compact.*

In fact, let q denote the least upper bound of Λ. The $U(\lambda)$ of every λ in Λ is contained in $U(q)$, then $\mathrm{U}(\Lambda)$ is contained in the compact space $\Lambda \times U(q)$. If f is a fixed element in $\mathscr{L}$, the functions $\lambda(f) - |\varphi(f)|$ is continuous on the product space $\Lambda \times U(q)$ with respect to the variables $\lambda \in \Lambda$ and $\varphi \in U(q)$. $\mathrm{U}(\Lambda)$ is the common part of all those closed sets $((\lambda, \varphi) \in \Lambda \times U(q) : \lambda(f) - |\varphi(f)| \geqq 0)$, then it is closed and compact in $U(q) \times \Lambda$ as well. Q. E. D.

If μ is a regular measure on $\mathrm{U}(\Lambda)$, μ determines a regular measure μ_Λ on Λ so that $\int f(\lambda)\, d\mu_\Lambda(\lambda) = \int f(\lambda) d\mu(\lambda, \varphi)$. μ_Λ is the ***restriction*** of the measure μ on Λ by the mapping $(\lambda, \varphi) \to \lambda$.

Theorem 11. *Let* k^Λ *be a composite semi-norm on a compact set* Λ *of semi-norms on* $\mathscr{L}$, *and let* $\mathrm{U}(\Lambda)$ *denote the spherical system on* Λ. *Then every* ψ *in* $U(k_\Lambda)$ *is a weak integral* $\psi = \int \varphi d\mu(\lambda, \varphi)$ *by a suitable regular measure* μ *on* $\mathrm{U}(\Lambda)$ *whose restrictiou* μ_Λ *on* Λ *belongs to* $U^+(k)$.

Proof. If μ varies on a set of regular measures on $\mathrm{U}(\Lambda)$, the mapping $\mu \to \mu_\Lambda$ is weakly continuous. Then the totality W of those regular measures μ on $\mathrm{U}(\Lambda)$ whose restriction μ_Λ belongs to $U^+(k)$ is regularly convex and bounded (since $\int 1\, d\mu = \int 1 d\mu_\Lambda \leqq 1$.). If $\mu \in W$, then the weak integral $\int \varphi d\mu(\lambda, \varphi)$ belongs to $U(k^\Lambda)$. In fact by the relation $\lambda(f) \geqq |\varphi(f)|$ for $(\lambda, \varphi) \in \mathrm{U}(\Lambda)$, we have

$$|\varphi(f)| = \left|\int \varphi(f) d\mu(\lambda, \varphi)\right| \leqq \int \lambda(f) d\mu(\lambda, \varphi)$$
$$= \int |f_\lambda|\, d\mu_\Lambda(\lambda) \leqq k(|f_\lambda|) = k^\Lambda(f).$$

The weakly continuous mapping $\varphi \to \int \varphi d\mu(\lambda, \varphi)$ maps therefore the set W in a regularly convex symmetric sub-set U of $U(k^\Lambda)$.

Now suffice it to say $U = U(k^\Lambda)$. This follows the fact that for each f in $\mathscr{L}$ we can choose a weak integral $\psi = \int \varphi d\mu(\lambda, \varphi)$ in U with

$k^{\Lambda}(f)=\psi(f)$, or $k(|f_{\lambda}|)=\int\varphi(f)d\mu(\lambda,\varphi)$. The existence of such a measure μ is shown in the more stronger condition that

(1). μ vanishes out-side of a compact set $V=((\lambda,\varphi)\in \mathrm{U}(\Lambda):\lambda(f)=\varphi(f))$.

(2). The restriction μ_{Λ} of μ within the space Λ belongs to $U^{+}(k)$ and satisfies $k(|f_{\lambda}|)=\int|f_{\lambda}|\,d\mu_{\Lambda}(\lambda)\ (=\int\varphi(f)\,d\mu(\lambda,\varphi))$.

In fact, by (3.4) we can choose a measure ν in $U^{+}(k)$ so that $k(|f_{\lambda}|)=\int|f_{\lambda}|\,d\nu(\lambda)$. It is sufficient to see the extensibility of the measure ν to a measure μ in the space V. This is done by the extension theorem of measures (Theorem 4 Remark in §4), and by the fact that the mapping $(\lambda,\varphi)\to\lambda$ maps the set $V=((\lambda,\varphi)\in \mathrm{U}(\Lambda):\lambda(f)=\varphi(f))$ onto the set Λ, that is, for each λ in Λ, we can choose a φ in $U(\lambda)$ so that $\varphi(f)=\lambda(f)$. Thus such a measure μ exists, and U coincides with $U(k^{\Lambda})$.

Theorem 12.[1] *Let k^{Λ} be a composition on a separable compact set Λ of semi-norms, then every element ψ in the dual space $L^{*}(k^{\Lambda})$ is a weak integral $\psi=\int\psi_{\lambda}\,d\nu(\lambda)$ by a suitable regular measure ν in Λ, where ψ_{λ} is a weakly measurable function on Λ so that each ψ_{λ} is an element in $L^{*}(\lambda)$ of norm $\lambda^{*}(\psi_{\lambda})\leqq 1$, and ν is a measure in $U^{+}(k)$ of norm $k^{*}(\nu)=k^{\Lambda *}(\psi)$.*

Proof. J. Dieudonné [2] extended the Doob's theorem to the following result. "*If $\lambda\to T\lambda$ is a continuous mapping of a compact space Ω to another separable compact space Λ, every regular measure μ on Ω is a weak integral $\mu=\int\mu_{\lambda}\,d\nu(\lambda)$, where each μ_{λ} is a normalized regular measure on the space $\Omega_{\lambda}=(x:Tx=\lambda)$, and the measure ν is the restriction of the measure μ on the space Λ*". We now assume the space Λ in Theorem 11 be separable.

If ψ is an element of $\mathscr{L}^{*}(k^{\Lambda})$ with the norm $k^{\Lambda *}(\psi)=1$, ψ is a weak integral $\psi=\int\varphi\,d\mu(\lambda,\varphi)$ so that the restriction ν of μ within the space Λ belongs to $U^{+}(k)$. The Dieudonné's theorem is applicable to this measure μ with respect to the mapping $(\lambda,\varphi)\to\lambda$, and μ is a weak integral $\mu=\int_{\Lambda}\mu_{\lambda}\,d\nu(\lambda)$ of a weakly measurable function μ_{λ}. Each μ_{λ} is a normalized regular measure on the sphere $U(\lambda)$ (= the complete

1) Analogous cares to those of [2] must be exercised with the measurablility of the norm functions $\lambda^{*}(\psi_{\lambda})$, as well as the difference between weak equivalency and usual equivalency, of those functions ψ_{λ} in Theorem 12 and its Corollary.

inverse image of the point λ). Then

$$\psi(f) = \int_{U(\Lambda)} \varphi(f) d(\lambda, \varphi) = \int_{\Lambda} \left(\int_{U(\lambda)} \varphi(f)\, d\mu_\lambda(\varphi) \right) d\mu(\lambda)$$
$$= \int \psi_\lambda(f) d\nu(\lambda).$$

Each ψ_λ is a weak integral $\psi_\lambda = \int \varphi d\mu_\lambda(\varphi)$ on the unit sphere $U(\lambda)$ by a normalized regular measure μ_λ and belongs to $U(\lambda)$ as well, then ψ is a weak integral $\int_\Lambda \psi_\lambda\, d\nu(\lambda)$. The norm $k^*(\nu)$ of ν is 1 because it is not smaller than the norm $k^{\Lambda*}(\psi) = 1$ of ψ. Q. E. D.

Corollary. *Let s be a L^p-sum ($p < \infty$) on a compact set Λ of semi-norms; $s = (\int_\Lambda \lambda^p\, d\nu(\lambda))^{\frac{1}{p}}$. Then every ψ in $\mathscr{L}^*(s)$ is a weak integral $\psi = \int \psi_\lambda\, d\nu(\lambda)$ so that*

(1). *In case* $p = 1$, $s^*(\psi) = \text{ess. max } \lambda^* (\psi_\lambda)$.

(2). *In case* $p > 1$, $s^*(\psi) = (\int \lambda^*(\psi_\lambda)^q\, d\nu(\lambda))^{\frac{1}{q}}$. $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.

If s is the least upper bound on Λ, every ψ in $\mathscr{L}^*(s)$ is a weak integral $\psi = \int \psi_\lambda\, d\mu(\lambda)$ by a suitable regular measure μ on Λ so that

$$s^*(\psi) = \int \lambda^*(\psi_\lambda)\, d\mu(\lambda).$$

§ 8. Applications to decomposing operator algebras.

Let $\mathfrak{R}$ be a normed space, and $\mathscr{A}$ be a linear algebra of bounded operators on $\mathfrak{R}$ which contains the identity I. A semi-norm r on $\mathscr{A}$ so that $r(I) = 1$ and $r(AB) \leqq r(A)r(B)$ is said to be *algebraic*. The norm $n(A) = |A|$ of A as an operator is clearly algebraic. *The totality of algebraic semi-norms on $\mathscr{A}$ is L^∞-sum convex.*

If x is an element of $\mathfrak{R}$, the semi-norm $p(A) = |Ax|$ on $\mathscr{A}$ satisfies $p(AB) \leqq n(A)p(B)$. In general, a semi-norm p on $\mathscr{A}$ so that $\sup p(B) \leqq 1$ $p(AB) < \infty$ ($A \in \mathscr{A}$), is said to be *operative*. And if r is an algebraic semi-norm so that $p(AB) \leqq r(A)p(B)$, r is said to *operate on p*, and p is said to *be operated* by r.

The canonical representation of $\mathscr{A}$ by an algebraic semi-norm r

is an algebraic representation of $\mathscr{A}$ to another normed algebra $\mathscr{A}(r)$. If p is an operative semi-norm, then $\mathscr{A}$ operates bounded linearly on the normed space $\mathscr{A}(p)$. That is, there is an algebraic representation $A\in\mathscr{A} \to A(p)$ on a linear algebra of bounded operators on $\mathscr{A}(p)$ so that $p(A) = |A(p)x|$ for a suitable cyclic element x in $\mathscr{A}(p)$. If p is operated by an algebraic semi-norm r moreover, then $A_r \to A(p)$ determines a topologico-algebraic representation of the normed algebra $\mathscr{A}(r)$ on an operator algebra on $\mathscr{A}(p)$. The totality of semi-norms operated by a fixed algebraic semi-norm r is universally convex.

An operative semi-norm p on $\mathscr{A}$ is said to be *normalized* if $p(I) = 1$. If p is normalized and operated by an algebraic semi-norm r, then $p \leqq r$. (In fact, $p(A) = p(AI) \leqq r(A)p(I) = r(A)$). Thus

Lemma 8.1. *The totality of semi-norms on a linear algebra $\mathscr{A}$ operated by a fixed algebraic semi-norm r on A is compact, universally convex and bounded by r.*

The present problem is to decompose a given (algebraic or operative) semi-norm p into simpler pieces.

The past works for operator-algebras were almost restricted to that of C^*-algebras, but for the G. E. Šilov's several early works. Then we shall reconstruct once more the decomposition theory of C^*-algebras in a new point of view. It may be perhaps useful to extend the result to general Banach algebras in a feature.

Let $\mathscr{A}$ be a *C^*-algebra* (i. e. a uniformly closed self-adjoint algebra of bounded linear operators on a Hilbert space.) with the identity I. An algebraic semi-norm p on $\mathscr{A}$ so that $p(A^*A) = p(A)^2$ and $p(A^*) = p(A)$ is said to be a *B^*-semi-norm.* If $\mathscr{N}$ is a two-sided closed ideal of $\mathscr{A}$, by the theorem of Gelfand-Kaplansky ([4] and [6]) the residue algebra $\mathscr{A}/\mathscr{N}$ with the norm $|A/\mathscr{N}| = \inf_{X\in\mathscr{N}} |A - X|$ is a C^*-algebra. Those natural representations $A \to A/\mathscr{N}$ by closed ideals $\mathscr{N}$ exhaust the totality of representations of $\mathscr{A}$ to C^*-algebras. Then the zero-point-set $\mathscr{N}(p)$ of a B^*-semi-norm p is a two-sided ideal, and $A \to A/\mathscr{N}(p)$ is the canonical representation of $\mathscr{A}$ on the C^*-algebra $\mathscr{A}/\mathscr{N}(p)$ so as to be $p(A) = \inf_{X\in\mathscr{N}(p)} |A - X| = |A/\mathscr{N}(p)|$.

In such a way, B^*-semi-norms and two-sided closed ideals of $\mathscr{A}$ correspond one-to-one with each other, however, the correspondence reverses their respective orders. *The totality* $\mathbf{B}^*$ *of B^*-semi-norms on $\mathscr{A}$ is L^∞-sum convex, closed and bounded by the norm $n(A)$ of $\mathscr{A}$.* If p is a least upper bound $p = \sup_{\lambda\in\Lambda}\lambda$ on a set Λ of B^*-semi-norms, then the ideal $\mathscr{N}(p)$ should be the greatest lower bound (the common ideal) of the system of corresponding ideals $(\mathscr{N}(\lambda): \lambda \in \Lambda)$. In particular, *$p \in \mathbf{B}^*$ is L^∞-sum indecomposable if and only if the ideal $\mathscr{N}(p)$ (and the algebra) $\mathscr{A}/\mathscr{N}(p)$ is irreducible* (c. f. [7]), that is, $\mathscr{N}(p)$ has no expression as the common ideal of any pair of ideals containing $\mathscr{N}(p)$ properly. Thus

Lemma 8.2. *Every C^*-algebra with the identity I is a L^∞-sum of (ideal theoretically) irreducible algebras.*

If $\mathscr{A}$ is a projective C^*-algebra (A C^*-algebra $\mathscr{A}$ is said to be *projective* if it is spaned uniformly by its projection elements.), then the result is more strengthened.

Theorem 13. *Every projective C^*-algebra $\mathscr{A}$ with the identity I is a L^∞-sum of several sub-direct sum irreducible C^*-algebras.*

Proof. $\mathscr{A}$ is said to be *sub-direct sum irreducible* (c. f. [1], [7]) if it contains the smallest closed two-sided ideal. For each projection E, we consider a minimal B^*-semi-norm p_E so that $p_E = E| = 1$. If q is a B^*-semi-norm properly smaller than p_E, then $q(E) < p(E)$ and $q(E) = q(E^2) = q(E)^2$ imply $q(E) = 0$. The least upper bound r of all those B^*-semi-norms properly smaller than p_E vanishes at E as well. r is then the largest B^*-semi-norm properly smaller than p_E, and $\mathscr{N}(p_E)$ is the smallest closed ideal which contains $\mathscr{N}(p_E)$ properly. Thus $\mathscr{N}(p_E)$ and $\mathscr{A}/\mathscr{N}(p_E)$ are totally irreducible. Since $\mathscr{A}$ is projective, the least upper bound s of those B^*-semi-norms p with the totally irreducible zero-point-sets (ideals) $\mathscr{N}(p)$ does not vanish at any projections in $\mathscr{A}$. This means $\mathscr{N}(s) = 0 = \mathscr{N}(n)$, the equality of the semi-norm s and the norm n of $\mathscr{A}$. Q. E. D.

If $\mathscr{A}$ is a C^*-algebra with the identity on a Hilbert space $\mathfrak{H}$, every element x in $\mathfrak{H}$ determines an semi-norm p on $\mathscr{A}$ operated by the norm $n(A)$ of $\mathscr{A}$. An *H-semi-norm* on $\mathscr{A}$ operated by the norm n is defined in the following condition. The space $\mathscr{A}(p)$ of the canonical representation $A \rightarrow A_p$ is a Hilbert space, $\mathscr{A}$ operates on which as a self-adjoint algebra, that is, $((AB)_p, C_p) = (B_p, (A^*C)_p)$ for every A, B, C in $\mathscr{A}$.

Notice that $p(A) = (A_p, A_p)^{\frac{1}{2}}$, then an H-semi-norm p determines the inner-product $(A_p, B_p) = \frac{1}{4}\{p(A+B)^2 - p(A-B)^2 + ip(A+iB)^2 - ip(A-iB)^2\}$ of $\mathscr{A}_p$, and the totality of normalized H-semi-norms on $\mathscr{A}$ operated by the norm of $\mathscr{A}$ is compact and L^2-sum convex. An normalized H-semi-norm p is L^2-sum indecomposable if and only if the algebra $\mathscr{A}$ operates irreduciblly on the Hilbert space $\mathscr{A}(p)$.

The above-stated is of cause merely a reproduction of the I. Gelfand's theory. His theory of the canonical representation of a positive definite function consists of two relation-theories. One treats the relation between positive definite functions and H-semi-norms. Another treats the relation between H-semi-norms and the canonical representations. We can eliminate its somewhat superfluous former half.

Coincidence between indecomposability of (operative and algebraic) semi-norms and algebraic irreduciblity of canonical representations observed in B^*- and H-semi-norms may not be expectable for general operator-algebras. However, to investigate the propereies of general algebraic and operative semi-norms may be an important remained problem.

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