ON A HOLOMORPHICALLY PROJECTIVE CORRESPONDENCE IN AN ALMOST COMPLEX SPACE

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In a previous paper in collaboration with Prof. Ōtsuki [2], the present author has introduced and investigated holomorphically flat curves in Kählerian spaces and the correspondence between Kählerian spaces, preserving such curves, which is called holomorphically projective (h. p.) correspondence. He has thereby shown that a space h. p. to a Euclidean space is of constant holomorphic curvature.

On the other hand, Prof. K. Yano and I. Mogi [5] has characterized a Kählerian space of constant holomorphic curvature by the axiom of holomorphic planes and also by the holomorphic free mobility. The method of real representation used by them is valid also in a pseudo-Kählerian space.

In the present paper we shall generalize the notions of holomophically flat curves and h. p. correspondence to the case of almost complex spaces with affine connection. Next, a tensor invariant under such a correspondence will be obtained. Finally, in a metric case we shall obtain the tensor of constant holomorphic curvature found by Prof. K. Yano and I. Mogi [5].

As to the notations and conventions, we follow J. A. Schouten [4] and K. Yano [7].

§ 1. Let X_{2n} be a space with an almost complex structure defined by $F_i^{\cdot h}$:

$$F_i^{\cdot j} F_j^{\cdot h} = -A_i^h , \tag{1.1}$$

and let X_{2n} be endowed with a symmetric affine connection Γ_{ji}^h. Denoting by ∇ the covariant differentiation with respect to Γ_{ji}^h, we assume that

$$\nabla_j F_i^{\cdot h} = 0 \tag{1.2}$$

which means geometrically that the fields of proper planes γ_n and $\bar{\gamma}_n$ of the almost complex structure $F_i^{\cdot h}$ are separately parallel with respect to the connection [6].

The geodesics are defined by differential equations of the form

$$\frac{d^2x^h}{dt^2} + \Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt} = \alpha(t)\frac{dx^h}{dt} \tag{1.3}$$

which mean that the tangent deplaced parallelly along the curve remains tangent to the curve.

We now introduce the curves satisfying the differential equations

$$\frac{d^2x^h}{dt^2} + \Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt} = \alpha(t)\frac{dx^h}{dt} + \beta(t)F_i^{\cdot h}\frac{dx^i}{dt}. \tag{1.4}$$

Such a curve is a plane curve and has the property that the tangent holomorphic plane deplaced parallelly along it remains holomorphically tangent to the curve. We call such a curve a holmorphically flat curve.

If, in an almost complex space, there are two connections Γ^h_{ji} and $'\Gamma^h_{ji}$, and if any curve which is holomorphically flat with respect to one of the connections is always holomorphically flat with respect to the other, than they have to be related such as

$$'\Gamma^h_{ji} = \Gamma^h_{ji} + 2P_{(j}A^h_{i)} + 2Q_{(j}F_{i)}^{\cdot h}. \tag{1.5}$$

Under the restriction (1.2) on both of the connections, we have

$$P_kF_i^{\cdot k}A^h_j - P_iF_j^{\cdot h} + Q_kF_i^{\cdot k}F_j^{\cdot h} + Q_iA^h_j = 0,$$

from which, contracting the indices h and j, and taking account of $F_h^{\cdot h}=0$,

$$Q_i = -P_hF_i^{\cdot h}. \tag{1.6}$$

Accordingly the relation (1.5) can be written as

$$'\Gamma^h_{ji} = \Gamma^h_{ji} + 2P_{(j}A^h_{i)} - 2P_kF_{(j}^{\cdot k}F_{i)}^{\cdot h}. \tag{1.7}$$

This correspondence is called a holomorphically projective one (cf. [2]).

If we denote by $R_{kji}{}^h$ the curvature tensor with respect to Γ^h_{ji}:

$$R_{kji}{}^h = 2\partial_{[k}\Gamma^h_{j]i} + 2\Gamma^h_{[k|l|}\Gamma^l_{j]i}, \tag{1.8}$$

then, by a straightforward and rather complicated computation, we obtain

$$'R_{kji}{}^h = R_{kji}{}^h + 2A^h_{[j}P_{k]i} + 2P_{[kj]}A^h_i - 2P_{[k|l|}F_{j]}^{\cdot l}F_i^{\cdot h} - 2P_{[k|l|}F_{j]}^{\cdot h}F_i^{\cdot l}, \tag{1.9}$$

where we have put

(1. 10) $$P_{ji} = \nabla_j P_i - P_j P_i + P_l P_k F_j^{\cdot l} F_i^{\cdot k}.$$

By contraction over h and k in (1. 9), we have

(1. 11) $$'R_{ji} = R_{ji} - 2(n+1)P_{ji} + 2P_{(ji)} - 2P_{(ba)} F_j^{\cdot b} F_i^{\cdot a}$$

and, multiplying (1. 11) by $F_b^{\cdot j} F_a^{\cdot i}$ and adding the result to (1. 11),

(1. 12) $$'R_{ji} + 'R_{ba} F_j^{\cdot b} F_i^{\cdot a} = R_{ji} + R_{ba} F_j^{\cdot b} F_i^{\cdot a} - 2(n+1)P_{ji} - 2(n+1)P_{ba} F_j^{\cdot b} F_i^{\cdot a},$$

and, eliminating the last terms from (1. 11) and (1. 12), and solving the resulting equations in P_{ji},

(1. 13) $$4(n^2-1)P_{ji} = M_{ji} - 'M_{ji},$$

where we have put

(1. 14) $$M_{ji} = (2n-1)R_{ji} + R_{ij} - 2R_{(ba)} F_j^{\cdot b} F_i^{\cdot a}.$$

Substituting (1. 13) into (1. 9), we see that the tensor

(1. 15) $$P_{kji}{}^h = R_{kji}{}^h + \frac{1}{2(n^2-1)} [M_{[k|i|} A_{j]}^h + M_{[kj]} A_i^h - M_{[k|l|} F_{j]}^{\cdot l} F_i^{\cdot h} - P_{[k|l|} F_{j]}^{\cdot h} F_i^{\cdot l}]$$

is invariant under the h. p. correspondence. We call it the h. p. curvature tensor. It is written down explicitly as follows :

(1. 16) $$\begin{aligned} P_{kji}{}^h = R_{kji}{}^h & \\ + \frac{1}{2(n^2-1)} & [\{(2n-1)R_{[k|i|} + R_{i[k} - 2R_{(ba)} F_{[k}^{\cdot b} F_{|i|}^{\cdot a}\} A_{j]}^h \\ & - \{(2n-1)F_i^{\cdot l} R_{[k|l|} + F_i^{\cdot l} R_{l[k} + 2R_{(bi)} F_{[k}^{\cdot b}\} F_{j]}^{\cdot h}] \\ + \frac{1}{n+1} & [R_{[kj]} A_i^h + F_{[k}^{\cdot l} R_{j]l} F_i^{\cdot h}]. \end{aligned}$$

We can verify that

(1. 17) $$P_{kji}{}^k = 0.$$

§ 2. An almost complex space with an affine connection is said to be h. p. flat if it can be related to a Euclidean space by a h. p. correspondence (1. 7). The necessary condition to be h. p. flat is clearly $P_{kji}{}^h = 0$. Conversely, if $P_{kji}{}^h = 0$, then putting

$$P'_{ji} = \frac{1}{4(n^2-1)}\{(2n-1)R_{ji} + R_{ij} - 2R_{(ba)}F_j^{\cdot b}F_i^{\cdot a}\},$$

the curvature tensor $R_{kji}{}^h$ satisfies the equation

$$R_{kji}{}^h = -2A_{[j}^h P'_{k]i} - 2P'_{[kj]}A_i^h + 2P'_{[k|l|}F_{j]}^{\cdot l}F_i^{\cdot h} + 2P'_{[k|l|}F_{j]}^{\cdot h}F_i^{\cdot l}.$$

On the other hand, if the space is h. p. flat under (1. 7), then P_{ji} satisfies the equation (1. 9) in which the left hand side vanishes. Hence P_{ji} should be equal to the above P'_{ji}. Therefore, in order to prove that the space with $P_{kji}{}^h = 0$ is h. p. flat, it is sufficient that there exists a vector field P_i such that

(2. 1) $$\nabla_j P_i = P_{ji} + P_j P_i - P_b P_a F_j^{\cdot b} F_i^{\cdot a},$$

in the space having the curvature

(2. 2) $$R_{kji}{}^h = -2\{P_{[k|i|}A_{j]}^h + P_{[kj]}A_i^h - F_{[j}^{\cdot a}P_{k]a}F_i^{\cdot h} - F_{[j}^{\cdot h}P_{k]a}F_i^{\cdot a}\}.$$

Taking account of (1. 2), the integrability condition of (2. 1) is

$$-R_{kji}{}^h P_h = 2\{\nabla_{[k}P_{j]i} + \nabla_{[k}P_{j]}P_i + P_{[j}\nabla_{k]}P_i - F_{[j}^{\cdot b}\nabla_{k]}P_b P_a F_i^{\cdot a} - P_b F_{[j}^{\cdot b}\nabla_{k]}P_a F_i^{\cdot a}\}$$

or, substituting (2. 1) and (2. 2),

(2. 3) $$\nabla_{[k}P_{j]i} = 0.$$

Now, if the identity of Bianchi [4, p. 147] is applied to (2. 2), we have

(2. 4) $$0 = \nabla_{[l}P_{k|i|}A_{j]}^h + \nabla_{[l}P_{kj]}A_i^h - \nabla_{[l}P_{k|a|}F_{j]}^{\cdot a}F_i^{\cdot h} - \nabla_{[l}P_{k|a|}F_{j]}^{\cdot h}F_i^{\cdot a}.$$

By contraction over h and i, we have

(2. 5) $$\nabla_{[l}P_{kj]} = 0$$

and, by contraction over h and j,

(2. 6) $$(2n-1)\nabla_{[l}P_{k]i} + 2\nabla_{[b}P_{a][l}F_{k]}^{\cdot b}F_i^{\cdot a} = 0.$$

Alternating indices i, k, l in (2. 6) and considering (2. 5), we have

(2. 7) $$2\nabla_{[b}P_{a][l}F_{k]}^{\cdot b}F_i^{\cdot a} + \nabla_{[b}P_{a]i}F_l^{\cdot b}F_k^{\cdot a} = 0,$$

substituting (2. 7) into (2. 6),

(2. 8) $$(2n-1)\nabla_{[l}P_{k]i} - \nabla_{[b}P_{a]i}F_l^{\cdot b}F_k^{\cdot a} = 0,$$

and finally, solving this equation with respect to $\nabla_{[l} P_{k]i}$,

$$(2.9) \qquad (2n-3)(2n-1)\nabla_{[l} P_{k]j} = 0.$$

Hence, the integrability condition (2. 3) of (2. 1) is a consequence of (2. 2). This proves

Theorem 1. *An almost complex space with an affine connection is holomorphically projectively flat if and only if the h. p. curvature* $P_{kji}{}^h$ *vanishes.*

§ **3.** In Hermitian metric case, our restriction (1. 2) implies that the space is pseudo-Kählerian [7]. Consequently, Ricci tensor is symmetric and satisfies [3]

$$(3.1) \qquad R_{ji} = R_{ba} F_j{}^{\cdot b} F_i{}^{\cdot a},$$

and the h. p. curvature tensor may be reduced to

$$(3.2) \qquad \begin{aligned} P_{kjih} &\equiv P_{kji}{}^l g_{lh} \\ &= R_{kjih} + \frac{1}{2(n^2-1)} [2(n-1) R_{[k|i|} g_{j]h} - (2n-1) R_{a[k} F_{j]h} F_i^{\cdot a}] \\ &\quad - \frac{1}{n-1} R_{a[k} F_{j]}^{\cdot a} F_{ih}. \end{aligned}$$

If the space is h. p. flat, i. e., $P_{kjih} = 0$, then contracting by g^{ji}, we have

$$(3.3) \qquad 2nR_{kh} = Rg_{kh}.$$

Hence R is a constant, and we put

$$(3.4) \qquad R = n(n+1)k$$

or

$$(3.5) \qquad R_{ji} = \frac{n+1}{2} k g_{ji}.$$

Then we obtain

$$(3.6) \qquad R_{kjih} = \frac{k}{4}(g_{ji}g_{kh} - g_{ki}g_{jh} + F_{ji}F_{kh} - F_{ki}F_{jh} - 2F_{kj}F_{ih}),$$

which is the expression of constant holomorphic curvature, found by Prof.

K. Yano and I. Mogi [5], and corresponding to one introduced by S. Bochner [1], see also [2]. Thus we have

Theorem 2. *If a Kählerian space is h. p. flat, then it is of constant holomorphic curvature.*

REFERENCES

[1] S. BOCHNER, Curvature and Betti numbers, II, Annals of Math., vol. 50 (1949), pp. 77—93.

[2] T. ŌTSUKI and Y. TASHIRO, On curves in Kählerian spaces, Math. J. Okayama Univ., vol. 4 (1954), pp. 57—78.

[3] S. SASAKI and K. YANO, Pseudo-analytic vectors on pseudo-Kählerian manifolds, Pacific J. Math., vol. 5 (1955), pp. 987—993.

[4] J. A. SCHOUTEN, Ricci Calculus, 2nd ed., Springer-Verlag, Berlin, 1954.

[5] K. YANO and I. MOGI, On real representations of Kählerian manifolds, Annals of Math., vol. 61 (1955), pp. 170—189.

[6] K. YANO, Quelques remarques sur les variétés à structure presque complexe, Bull. Soc. Math. France, tome 83 (1955), pp. 57—80.

[7] K. YANO, On three remarkable affine connections in almost Hermitian spaces Proc. Kon. Ned. Akad. Amsterdam, A 58 (Indag. Math. 17) (1955), pp. 24—32.

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