ON GEODESIC COORDINATES IN FINSLER SPACES

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§ 1. Introduction.

Let V_n be an n-dimensional Riemann space whose line element ds is given by

$$ds^2 = g_{ij}(x)dx^i dx^j \qquad (1)$$

in local coordinates (x^i). As is well known, the coordinates are called *geodesic* along a geodesic arc $\gamma : x^i = x^i(s)$, $0 \leqq s \leqq l$, if the Christoffel's symbols made by g_{ij} vanish on γ. If (x^i) are geodesic along γ, then

$$x^i(s) = x_0^i + a^i s, \quad g_{ij}(x_0)a^i a^j = 1$$

and any contravariant vector field with constant components defined on γ is parallel displaced along γ.

If we take a coordinate transformation such that

$$x^i = x_0^i + a_\alpha^i \bar{x}^\alpha + a_n^i \bar{x}^n, \quad \alpha = 1, 2, \dots, n-1,$$
$$g_{ij}(x_0)\, a_\lambda^i a_\mu^j = \delta_{\lambda\mu},\ \lambda, \mu = 1, 2, \dots, n,\ a_n^i = a^i,$$

then the coordinates $(\bar{x}^i)$ are also geodesic along γ, γ is written in the coordinates as

$$\bar{x}^\alpha = 0, \qquad \alpha = 1, 2, \dots, n-1,$$
$$\bar{x}^n = s, \qquad 0 \leqq s \leqq l$$

and

$$g_{ij}(\bar{x}(s)) = \delta_{ij}.$$

From this consideration, we can define a unique geodesic coordinate system along a geodesic arc γ which has no self-intersecting points, for a field of orthogonal frames $(x(s), e_1(s), \dots, e_n(s))$, $0 \leqq s \leqq l$ defined on γ, such that each $e_\lambda(s)$ is paralled displaced along γ and $e_n(s)$ is the tangent unit vector to γ at $x(s)$, as follows.

For any point $x(s) \in \gamma$ and any tangent unit vector to V_n at $x(s)$ orthogonal to γ, $\sum_{\alpha=1}^{n-1} e_\alpha(s)\, b^\alpha$, let $\gamma(b^\alpha, s)$ defined by the equation $x = x(t; b^\alpha, s)$ be the geodesic through $x(s)$ and tangent to the vector, where t is arc-length measured on the geodesic from the point $x(s)$. Now, if we put

$$u^\alpha = b^\alpha t, u^n = s,$$

$(u^1, \ldots, u^n)$ become a local coordinate system in a suitable neighborhood of γ. For the coordinates (u^i), we have clearly

$$g_{ij}(0, \ldots, 0, u^n) = \delta_{ij}, \tag{2}$$
$$\Gamma^i_{jn}(0, \ldots, 0, u^n) = 0 \tag{3}$$

since $(x(s), e_1(s), \ldots, e_n(s))$ is a parallel displaced orthogonal frame along γ. Furthermore, we have

$$\Gamma^i_{\alpha\beta}(u)u^\alpha u^\beta = 0 \tag{4}$$

and

$$\Gamma^i_{\alpha\beta}(0, \ldots, 0, u^n) = 0 \tag{5}$$

since $u^\alpha = b^\alpha t$, $u^n = s$ are geodesics.

In the following, we will show that we can also define coordinates (u^i) as above mentioned in Finsler spaces but they are essentially different from the ones in Riemann spaces.

§ 2. Induced coordinates along geodesic arcs

We will use the notations and the equations in E. Cartan's book [1]. Let F_n be an n-dimensional Finsler space whose line element ds is geven by

$$ds = L(x^i, dx^i) \tag{6}$$

in local coordinates (x^i). Let $\gamma : x^i = x^i(s), 0 \leqq s \leqq l$, be a geodesic arc in F_n, then the tangent unit vector $e_n(s)$ of γ is parallel displaced along γ. Let $(x(s), e_1(s), \ldots, e_n(s))$ be a frame defined on γ, such that $e_\lambda(s)$ are parallel displaced along γ and orthogonal each other with respect to the direction element $(x(s), e_n(s))$, $e_n(s) = (dx^i(s)/ds)$. We denote the space of tangent directions of F_n by S. For a direction element (x^i, x'^i), the metric tensor of F_n is defned by

$$g_{ij}(x, x') = \frac{\partial^2 F(x, x')}{\partial x'^i \partial x'^j}, \quad F = \frac{1}{2} L^2. \tag{7}$$

If (a^i_λ) are the components of $e_\lambda(0)$ with respect to the tangent vectors $\partial/\partial x^i$, we have by the above assumption

$$g_{ij}(x^k_0, a^k_n)\, a^i_\lambda a^j_\mu = \delta_{\lambda\mu}. \tag{8}$$

By means of the properties of the Euclidean connection defined by E. Cartan [1] and γ being a geodesic, such constructions of frames are admissible.

According to [1], in local coordinates (x^i), putting

$$\Gamma^k_{ij}(x, x') = g^{kh}(x, x')\ \Gamma_{ihj}(x, x')$$

$$\Gamma_{ihj} = \frac{1}{2}\left(\frac{\partial g_{ih}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h}\right) + C_{ijr}\frac{\partial G^r}{\partial x'^h} - C_{hjr}\frac{\partial G^r}{\partial x'^i} \tag{9}$$

$$C_{ijh} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x'^h} = \frac{1}{2}\frac{\partial^3 F}{\partial x'^i \partial x'^j \partial x'^h}, \tag{10}$$

$$G^r = g^{rh}\ G_h,\ \ 2G_h = \frac{\partial^2 F}{\partial x'^h \partial x^k} x'^k - \frac{\partial F}{\partial x^h}, \tag{11}$$

the Euclidean connection of F_n is given by the Pfaffian forms

$$\begin{aligned} \omega^j_i &= \Gamma^j_{ih}(x, x')dx^h + C_i{}^j{}_h(x, x')\ dx'^h \\ &(= \Gamma^{*j}_{ih}(x, x')dx^h + A_i{}^j{}_h(x, x')\omega^h), \end{aligned} \tag{12}$$

where

$$\Gamma^{*j}_{i\ h} = g^{jk}\Gamma^*_{ikh}, \qquad \Gamma^*_{ikh} = \Gamma_{ikh} - C_{ikr}\frac{\partial G^r}{\partial x'^h} \tag{13}$$

$$A_i{}^j{}_h = LC_i{}^j{}_h,$$

$$\omega^h = Dl^h, \qquad l^h = x'^h/L(x, x'). \tag{14}$$

For the sake of simplicity, we suppose that F_n is analytic. Since $C_i{}^j{}_k(x, x')x'^k = 0$, the equations of deodesics are

$$\frac{d^2x^i}{ds^2} + \Gamma^{*i}_{jk}(x, \frac{dx}{ds})\frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

From this, we get inductively

$$\frac{d^p x^i}{ds^p} + \Gamma^{*i}_{j_1\cdots j_p}(x, \frac{dx}{ds})\frac{dx^{j_1}}{ds}\cdots\cdots\frac{dx^{j_p}}{ds} = 0 \tag{15}$$

$$p = 2, 3. . , \ldots\ldots$$

where $\Gamma^{*i}_{j_1\cdots j_p j_{p+1}}$ are defined by

$$\Gamma^{*i}_{j_1\cdots j_{p+1}} = \frac{1}{(p+1)!}\sum_{\mathfrak{S}_{p+1}\ni\sigma}\sigma\left\{\frac{\partial\Gamma^{*i}_{j_1\cdots j_p}}{\partial x^{j_{p+1}}} - \frac{\partial\Gamma^{*i}_{j_1\cdots j_p}}{\partial x'^h}\frac{\partial G^h}{\partial x'^{j_{p+1}}} - p\Gamma^{*i}_{hj_1\cdots j_{p-1}}\Gamma^{*h}_{j_p j_{p+1}}\right\} \tag{16}$$

and $\mathfrak{S}_{p+1}$ is the permutation group which operates on the indexes $1, 2, \cdots, p+1$ of $j_1, \ldots, j_{p+1}$.

Now, a vector field $y^i \partial/\partial x^i$ defined on γ which is parallel displaced along γ with respect to its direction element $(x^i(s), dx^i(s)/ds)$, is given by

$$\frac{dy^i}{ds} + \Gamma^{*i}_{jk}(x, \frac{dx}{ds}) y^j \frac{dx^k}{ds} = 0, \tag{17}$$

From this we get inductively

$$\frac{d^m y^i}{ds^m} + M^i_{jk_1\ldots k_m}(x, \frac{dx}{ds}) y^j \frac{dx^{k_1}}{ds} \cdots\cdots \frac{dx^{k_m}}{ds} = 0,$$
$$m = 1, 2, \ldots.$$

where we put

$$M^i_{jk} = \Gamma^{*i}_{jk},$$

$$M^i_{jk_1\ldots k_{m+1}} = \frac{1}{(m+1)!} \sum_{\mathfrak{S}_{m+1}\ni\sigma} \sigma \left\{ \frac{\partial M^i_{jk_1\ldots k_m}}{\partial x^{k_{m+1}}} - \frac{\partial M^i_{jk_1\ldots k_m}}{\partial x'^h} \frac{\partial G'^h}{\partial x'^{k_{m+1}}} - M^i_{hk_1\ldots k_m} \Gamma^{*h}_{jk_{m+1}} - m\, M^i_{jhk_1\ldots k_{m-1}} \Gamma^{*h}_{k_m k_{m+1}} \right\} \tag{18}$$

We can easily verify the relation

$$\Gamma^{*i}_{k_1\ldots k_m} = \frac{1}{m!} \sum_{\mathfrak{S}_m\ni\sigma} \sigma M^i_{k_1\ldots k_m}$$

From the above assumption, γ is given, near the point (x^i_0), by the equations

$$\xi^i(s) = x^i_0 + s a^i - \sum_{p=2}^{\infty} \frac{s^p}{p!} \Gamma^{*i}_{j_1\ldots j_p}(x_0, a)\, a^{j_1} \ldots a^{j_p} \tag{20}$$
$$a^i = a^i_n.$$

Furthermore, the components of γ^i_α of e_α are

$$\gamma^i_\alpha(s) = a^i_\alpha - \sum_{m=1}^{\infty} \frac{s^m}{m!} M^i_{jk_1\ldots k_m}(x_0, a) a^j_\alpha a^{k_1} \ldots a^{k_m}. \tag{21}$$

Lastly, the geodesic $\gamma(b^\alpha, s)$ through the point $(\xi^i(s))$ at which the tangent vector is

$$\frac{dx^i}{dt} = \gamma^i_\alpha(s) b^\alpha = \gamma^i, \quad (b^\alpha) \neq (0), \tag{22}$$

is given, near the point $(\xi^i(s))$, by the equations

$$x^i = \xi^i(s) + t\eta^i - \sum_{p=2}^{\infty} \frac{t^p}{p!} \Gamma^{*i}_{j_1 \dots j_p}(\xi(s), \eta)\eta^{j_1} \dots \eta^{j_p} \tag{23}$$

If we put

$$u^\alpha = b^\alpha t,$$

the above equations (23) are written as

$$x^i = \xi^i(s) + \eta^i_\alpha(s)u^\alpha - \sum_{p=2}^{\infty} \frac{1}{p!} \Gamma^{*i}_{j_1 \dots j_p} (\xi^h(s), \eta^h_\beta(s)u^\beta)$$
$$\times \eta^{j_1}_{\alpha_1}(s) \dots \eta^{j_p}_{\alpha_p}(s)\, u^{\alpha_1} \dots u^{\alpha_p}. \tag{24}$$

In a sufficiently small neighborhood of γ, $u^1, \dots, u^{n-1}$, $s = u^n$ become a coordinate system of F_n, *which is uniquely determined by the frame* $(x(s), e_1(s), \dots, e_n(s))$. We call it the *induced coordinate system* from the parallel displaced orthogonal frame along the geodesic arc γ. We shall investigate the differentiability class of such coordinate system in the following.

For $t \neq 0$, the coordinate system is analytic and we have from (24)

$$\frac{\partial x^i}{\partial u^\alpha} = \eta^i_\alpha - \sum_{p=2}^{\infty} \frac{1}{p!} \Big\{ \frac{\partial \Gamma^{*i}_{j_1 \dots j_p}}{\partial x'^h} (\xi, \eta)\eta^h_\alpha \eta^{j_1} \dots \eta^{j_p}$$
$$+ p\Gamma^{*i}_{hj_1 \dots j_{p-1}} (\xi, \eta)\eta^h_\alpha \eta^{j_1} \dots \eta^{j_{p-1}} \Big\} \tag{25}$$

$$\frac{\partial^2 x^i}{\partial u^\beta \partial u^\alpha} = - \sum_{p=2}^{\infty} \frac{1}{p!} \Big\{ \frac{\partial^2 \Gamma^{*i}_{j_1 \dots j_p}}{\partial x'^k \partial x'^h} (\xi, \eta)\eta^h_\alpha \eta^k_\beta \eta^{j_1} \dots \eta^{j_p}$$
$$+ p \frac{\partial \Gamma^{*i}_{hj_1 \dots j_{p-1}}}{\partial x'^k} (\xi, \eta)(\eta^h_\alpha \eta^k_\beta + \eta^h_\beta \eta^k_\alpha)\eta^{j_1} \dots \eta^{j_{p-1}}$$
$$+ p(p-1)\, \Gamma^{*i}_{hkj_1 \dots j_{p-2}}(\xi, \eta)\eta^h_\alpha \eta^k_\beta \eta^{j_1} \dots \eta^{j_{p-2}} \Big\} \tag{26}$$

$$\frac{\partial x^i}{\partial s} = \frac{d\xi^i}{ds} + \frac{\partial \eta^i}{\partial s} - \sum_{p=2}^{\infty} \frac{1}{p!} \Big\{ \Big(\frac{\partial \Gamma^{*i}_{j_1 \dots j_p}}{\partial x^h} \frac{d\xi^h}{ds} + \frac{\partial \Gamma^{*i}_{j_1 \dots j_p}}{\partial x'^h} \frac{\partial \eta^h}{\partial s} \Big) \eta^{j_1} \dots \eta^{j_p}$$
$$+ p\Gamma^{*i}_{hj_1 \dots j_{p-1}} \frac{\partial \eta^h}{\partial s} \eta^{j_1} \dots \eta^{j_{p-1}} \Big\} \tag{27}$$

$$\frac{\partial^2 x^i}{\partial u^\alpha \partial s} = \frac{d\eta^i_\alpha}{ds} - \sum_{p=2}^{\infty} \frac{1}{p!} \Big\{ \Big(\frac{\partial^2 \Gamma^{*i}_{j_1 \dots j_p}}{\partial x'^k \partial x^h} \eta^k_\alpha \frac{d\xi^h}{ds} + \frac{\partial \Gamma^{*i}_{j_1 \dots j_p}}{\partial x'^k \partial x'^h} \eta^k_\alpha \frac{\partial \eta^h}{\partial s}$$
$$+ \frac{\partial \Gamma^{*i}_{j_1 \dots j_p}}{\partial x'^h} \frac{d\eta^h_\alpha}{ds} \Big) \eta^{j_1} \dots \eta^{j_p}$$

$$+ p\left(\frac{\partial \Gamma^{*i}_{kj_1\ldots j_{p-1}}}{\partial x^h}\frac{d\xi^h}{ds} + \frac{\partial \Gamma^{*i}_{kj_1\ldots j_{p-1}}}{\partial x'^h}\frac{\partial \eta^h}{\partial s}\right)\eta^k_\alpha \eta^{j_1}\ldots \eta^{j_{p-1}}$$

$$+ p\frac{\partial \Gamma^{*i}_{hj_1\ldots j_{p-1}}}{\partial x'^k}\eta^k_\alpha\frac{\partial \eta^h}{\partial s}\eta^{j_1}\ldots\eta^{j_{p-1}} + p\Gamma^{*i}_{hj_1\ldots j_{p-1}}\frac{d\eta^h_\alpha}{ds}\eta^{j_1}\ldots\eta^{j_{p-1}}$$

$$+ p(p-1)\Gamma^{*i}_{hkj_1\ldots j_{p-2}}\frac{\partial \eta^h}{\partial s}\eta^k_\alpha\eta^{j_1}\ldots\eta^{j_{p-2}}\Big\} \tag{28}$$

Hence we get

$$-\lim\frac{\partial^2 x^i}{\partial u^\beta \partial u^\alpha} = \left(\Gamma^{*i}_{hk} + \left(\frac{\partial \Gamma^{*i}_{jh}}{\partial x'^k} + \frac{\partial \Gamma^{*i}_{jk}}{\partial x'^h}\right)x'^j + \right.$$

$$\left. + \frac{1}{2}\frac{\partial^2 \Gamma^{*i}_{j_1j_2}}{\partial x'^h \partial x'^k}x'^{j_1}x'^{j_2}\right)(x^i_0, a^i_\alpha b^\alpha)a^h_\alpha a^k_\beta \tag{29}$$

which depends generally on $(b^\alpha) \neq (0)$. If this quantity is independent of (b^α), the coordinate system is of class C^2 on its domain of definition.

Now, we get from $\Gamma^{*i}_{jk}\, x'^j = \dfrac{\partial G^i}{\partial x'^k}$ and [1, (XV)]

$$\frac{\partial \Gamma^{*i}_{jh}}{\partial x'^k}x'^j = \frac{\partial(\Gamma^{*}_{jh}x'^j)}{\partial x'^k} - \Gamma^{*i}_{kh}$$

$$= \frac{\partial^2 G^i}{\partial x'^k \partial x'^h} - \Gamma^{*i}_{kh} = A^i{}_{kh|j}\, l^j \tag{30}$$

and

$$\frac{\partial^2 \Gamma^{*i}_{j_1j_2}}{\partial x'^h \partial x'^k}x'^{j_1}x'^{j_2} = \left(\frac{\partial}{\partial x'^h}\left(\frac{\partial \Gamma^{*i}_{j_1j_2}}{\partial x'^k}x'^{j_1}\right)\right)x'^{j_2} - \frac{\partial \Gamma^{*i}_{hj_2}}{\partial x'^k}x'^{j_2}$$

$$= \frac{\partial}{\partial x'^h}(A^i{}_{j_2k|j}\, l^j)x'^{j_2} - A^i{}_{hk|j}\, l^j.$$

Since we have $l^i{}_{\|j} = \delta^i_j - l^i l_j$ and $l^i{}_{|j} = 0$, the right hand side is written as

$$= (A^i{}_{j_2k|j_1}\, l^{j_1})_{\|h}\, l^{j_2} - A^i{}_{hk|j}\, l^j$$

$$= A^i{}_{j_2k|j_1\|h}\, l^{j_1}\, l^{j_2} - A^i{}_{hk|j}\, l^j.$$

Now we have

$$A^i{}_{j_2k|j_1\|h}\, l^{j_1}l^{j_2} = g^{il}A_{j_2lk|j_1\|h}\, l^{j_1}l^{j_2}$$

Putting

$$A_{ijhk}=\frac{1}{2}L^2\frac{\partial^4 F}{\partial x'^i\partial x'^j\partial x'^h\partial x'^k}=L^2\frac{\partial C_{ijh}}{\partial x'^k}, \tag{31}$$

we get

$$A_{ijh\|k}=A_{ijhk}+A_{ijh}l_k. \tag{32}$$

Then we have

$$\begin{aligned}A_{ihk|m\|j}l^m &= \left\{\frac{\partial A_{ihk}}{\partial x^m}-\frac{\partial A_{ihk}}{\partial x'^s}\frac{\partial G^s}{\partial x'^m}-A_{shk}\Gamma^{*s}_{im}\right.\\ &\qquad\left.-A_{isk}\Gamma^{*s}_{hm}-A_{ihs}\Gamma^{*s}_{km}\right\}_{\|j}l^m\\ &=\left\{L\frac{\partial^2 A_{ihk}}{\partial x^m\partial x'^j}-L\frac{\partial^2 A_{ihk}}{\partial x'^s\partial x'^j}\frac{\partial G^s}{\partial x'^m}-A_{ihk\|s}G^s_{mi}\right.\\ &\qquad\left.-A_{shk\|j}\Gamma^{*s}_{im}-A_{isk\|j}\Gamma^{*s}_{hm}-A_{ihs\|j}\Gamma^{*s}_{km}\right\}l^m\\ &\qquad-A_{shk}\frac{\partial\Gamma^{*s}_{im}}{\partial x'^j}x'^m-A_{isk}\frac{\partial\Gamma^{*s}_{hm}}{\partial x'^j}x'^m-A_{ihs}\frac{\partial\Gamma^{*s}_{km}}{\partial x'^j}x'^m\\ &=(A_{ihk\|j|m}-A_{shk}A^s_{ij|m}-A_{isk}A^s_{hj|m}-A_{ihs}A^s_{kj|m})\,l^m.\end{aligned}$$

Making use of (32), this equation is written as

$$\begin{aligned}A_{ihk|m\|j}l^m=(A_{ihkj|m}&+A_{ihk|m}l_j-A_{shk}A^s_{ij|m}\\ &-A_{isk}A^s_{hj|m}-A_{ihs}A^s_{kj|m})l^m.\end{aligned} \tag{33}$$

From the equations above we get easily

$$A_{j_2ik|j_1\|h}l^{j_1}l^{j_2}=A_{ikhj_2|j_1}l^{j_1}l^{j_2}. \tag{34}$$

On the other hand, we have from (32)

$$A_{ijhk}l^k=-A_{ijh}, \tag{35}$$

hence

$$A^i{}_{khj_2|j_1}l^{j_1}l^{j_2}=-A^i{}_{kh|j_1}l^{j_1}.$$

By virtue of these equations, we obtain

$$\frac{\partial^2\Gamma^{*i}_{j_1j_2}}{\partial x'^h\partial x'^k}x'^{j_1}x'^{j_2}=A^i{}_{khj_2|j_1}l^{j_1}l^{j_2}-A^i{}_{kh|j}l^j=-2A^i{}_{hk|j}l^j, \tag{36}$$

hence

$$\Gamma^{*i}_{hk} + \left(\frac{\partial \Gamma^{*i}_{jh}}{\partial x'^k} + \frac{\partial \Gamma^{*i}_{jk}}{\partial x'^h}\right)x'^j + \frac{1}{2}\frac{\partial^2 \Gamma^{*i}_{j_1 j_2}}{\partial x'^h \partial x'^k} x'^{j_1} x'^{j_2}$$
$$= \Gamma^{*i}_{hk} + A^i{}_{hk|j}\, l^j . \tag{37}$$

By means of this relation, we get lastly

$$-\lim_{t\to 0} \frac{\partial^2 x^i}{\partial u^\beta \partial u^\alpha} = (\Gamma^{*i}_{hk} + A^i{}_{hk|j}\, l^j)(x^j_0, a^j_\gamma b^\gamma) a^h_\alpha a^k_\beta . \tag{38}$$

In order that the above quantity is independent of $(b^\alpha) \neq (0)$, it is sufficient that

$$\left(\frac{\partial}{\partial x'^m}(\Gamma^{*i}_{hk} + A^i{}_{hk|j}\, l^j)\right)(x^j_0, a^j_\gamma b^\gamma) a^h_\alpha a^k_\beta a^m_\gamma = 0. \tag{39}$$

If the condition hold good for any geodesic arc of F_n and $n > 4$, it follows that

$$L \frac{\partial}{\partial x'^j} \Gamma^{*i}_{hk} + (A^i_{hk|m}\, l^m)_{\|j}$$
$$= L \frac{\partial}{\partial x'^j} \Gamma^{*i}_{hk} + A^i{}_{hk|m\|j} l^m + A^i{}_{hk|j} - A^i{}_{hk|m} l^m l_j = 0.$$

By (33) and [1, (44)], we get

$$A^i{}_{hk|m\|j}\, l^m = -2A^{it}{}_j A_{thk|m} l^m + g^{it} A_{thk|m\|j} l^m$$

hence

$$L \frac{\partial}{\partial x'^j} \Gamma^{*i}_{hk} + (A^i{}_{hk|m} l^m)_{\|j} =$$
$$= A^i_{hj|k} + A^i_{kj|h} - g^{im} A_{hkj|m} - A_h{}^i{}_s A_k{}^s{}_{j|m}\, l^m - A_k{}^i{}_s A_h{}^s{}_{j|m}\, l^m$$
$$+ A_h{}^s{}_k A_s{}^i{}_{j|m}\, l^m - 2A^{it}{}_j A_{thk|m} l^m \tag{40}$$
$$+ A^i{}_{hkj|m}\, l^m - A_{shk} A^{si}{}_{j|m}\, l^m - A^i{}_{sk} A^s{}_{hj|m}\, l^m - A^i{}_{hs} A^s{}_{kj\, m}\, l^m + A^i{}_{hk\, j}$$

We define a tensor of F_n

$$M^i{}_{jhk} = A^i{}_{hk|j} + A^i{}_{kj|h} + A^i{}_{jh|k} - 2(A^i{}_{sj} A^s{}_{hk|m} + A^i{}_{sh} A^s{}_{kj|m} \tag{41}$$
$$+ A^i{}_{sk} A^s{}_{jh|m}) l^m - g^{im} A_{hkj|m} + A^i{}_{hkj|m} l^m$$

which is symmetric with respect to j, h, k. Thus we obtain the following theorem.

Theorem 1. *In an n-diemensional Finsler space, in order that* *any induced coordinate system along each geodesic arc is of class* C^2, *it is necessary and sufficient that the tensor* $M^i{}_{jhk}$ *vanishes everywhere* $(n > 4)$.

§ 3. **$V_n(\xi)$ and geodesic coordinates.**

For a given field ξ of tangent directions defined on a domain of F_n, we obtain there a Riemann space $V_n(\xi)$ whose metric tensor $\bar{g}_{ij}(x)$ is

$$\bar{g}_{ij}(x) = g_{ij}(x^k, \varphi^k(x)) \tag{42}$$

in local coordinates (x^i) where $\varphi^k(x)$ represents ξ at the point (x^i). Since we have

$$\begin{aligned}\frac{\partial \bar{g}_{ih}}{\partial x^j} &= \bar{\Gamma}_{ihj} + \bar{\Gamma}_{hij} \\ &= \frac{\partial g_{ih}}{\partial x^j} + \frac{\partial g_{ih}}{\partial x'^k}\frac{\partial \varphi^k}{\partial x^j} = \Gamma_{ihj} + \Gamma_{hij} + 2C_{ihk}\frac{\partial \varphi^k}{\partial x^j},\end{aligned}$$

the parameters $\bar{\Gamma}^i_{jk}$ of the connection of Levi-Civita of $V_n(\xi)$ are by means of (9), (10), (13)

$$\begin{aligned}\bar{\Gamma}_{ihj} = \Gamma^*_{ihj} &+ C_{ihk}\left(\frac{\partial G^k}{\partial x'^j} + \frac{\partial \varphi^k}{\partial x^j}\right) + C_{jhk}\left(\frac{\partial G^k}{\partial x'^i} + \frac{\partial \varphi^k}{\partial x^i}\right) \\ &- C_{ijk}\left(\frac{\partial G^k}{\partial x'^h} + \frac{\partial \varphi^k}{\partial x^h}\right)\end{aligned} \tag{43}$$

where x'^i in the right hand side are $\varphi^i(x)$.

Now, we assume that for a given geodesic arc $\gamma : x^i = x^i(s)$ of F_n in the domain of definition of $V_n(\xi)$, the tangent directions of γ are elements of ξ, that is

$$\varphi^k(x(s)) = \frac{dx^k}{ds}, \qquad 0 \leqq s \leqq l.$$

Then we get from (43) on γ

$$\bar{\Gamma}_{ihj}(x)\frac{dx^i}{ds}\frac{dx^j}{ds} = \Gamma^*_{ihj}\left(x, \frac{dx}{ds}\right)\frac{dx^i}{ds}\frac{dx^j}{ds}.$$

Hence we have

Theorem 2. *For a field ξ of tangent directions of F_n which contains the tangent directions of a given curve C, C is simultaneously a geodesic arc in F_n and $V_n(\xi)$.*

Now, if C is a geodesic arc, then we have on $C=\gamma$

$$\left(\frac{\partial G^i}{\partial x'^j}+\frac{\partial\varphi^i}{\partial x^j}\right)\frac{dx^j}{ds}=\frac{d^2x^i}{ds^2}+\Gamma^{*i}_{jk}\left(x,\frac{dx}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds}=0$$

and

$$\bar{\Gamma}^h_{ij}\frac{dx^j}{ds}=\Gamma^{*h}_{ij}\frac{dx^j}{ds}=\Gamma^h_{ij}\frac{dx^j}{ds}=\frac{\partial G^h}{\partial x'^i}. \tag{44}$$

Furthermore, we assume that the field ξ is transversal to a family of hyper surfaces $f(x)=\text{constant}$.

By the assumption, we get

$$\bar{g}_{ij}(x)\varphi^j(x)=\rho(x)\frac{\partial f(x)}{\partial x^i}, \tag{45}$$

from which

$$\left(\frac{\partial g_{ij}}{\partial x^k}+2C_{ijh}\frac{\partial\varphi^h}{\partial x^k}\right)\varphi^j+g_{ij}\frac{\partial\varphi^j}{\partial x^k}=\rho\frac{\partial^2 f}{\partial x^i\partial x^k}+\frac{\partial\rho}{\partial x^k}\frac{\partial f}{\partial x^i},$$

hence

$$(\Gamma_{ijk}+\Gamma_{jik})\varphi^j+g_{ij}\frac{\partial\varphi^j}{\partial x^k}=\rho\frac{\partial^2 f}{\partial x^i\partial x^k}+\frac{\partial\rho}{\partial x^k}\frac{\partial f}{\partial x^i}$$

Along γ, we get by (44)

$$g^{im}\Gamma_{mjk}\frac{dx^j}{ds}+\left(\frac{\partial\varphi^i}{\partial x^k}+\frac{\partial G^i}{\partial x'^k}\right)=\rho\frac{\partial^2 f}{\partial x^m\partial x^k}g^{mi}+\frac{1}{\rho}\frac{\partial\rho}{\partial x^k}\frac{dx^i}{ds}$$

Putting the relations into (43), we have the relation

$$\begin{aligned}\bar{\Gamma}_{ihj}=\Gamma^*_{ihj}&+C_{ih}{}^k\left(\rho\frac{\partial^2 f}{\partial x^k\partial x^j}-\Gamma_{kmj}\frac{dx^m}{ds}\right)\\&+C_{jh}{}^k\left(\rho\frac{\partial^2 f}{\partial x^k\partial x^i}-\Gamma_{kmi}\frac{dx^m}{ds}\right)-C_{ij}{}^k\left(\rho\frac{\partial^2 f}{\partial x^k\partial x^h}-\Gamma_{kmh}\frac{dx^m}{ds}\right)\end{aligned} \tag{46}$$

on γ. If f is a linear function of $x^1,\dots,x^n$ or more generally a function such that $\partial^2 f/\partial x^i\partial x^j$ vanish along γ, we have

$$\begin{aligned}\bar{\Gamma}_{ihj}=\Gamma_{ihj}-C_{ihm}\frac{\partial G^m}{\partial x'^j}-C_{ih}{}^k\Gamma_{kmj}\frac{dx^m}{ds}-C_{jh}{}^k\Gamma_{kmi}\frac{dx^m}{ds}\\+C_{ij}{}^k\Gamma_{kmh}\frac{dx^m}{ds}\end{aligned} \tag{47}$$

and

$$\Gamma_{ihj} = \bar{\Gamma}_{ihj} + C_{ihm}\bar{\Gamma}^{m}_{kj}\frac{dx^k}{ds} + C_{ih}{}^{k}\bar{\Gamma}_{kmj}\frac{dx^m}{ds} + C_{jh}{}^{k}\bar{\Gamma}_{kmi}\frac{dx^m}{ds} - C_{ij}{}^{k}\bar{\Gamma}_{kmh}\frac{dx^m}{ds} \tag{48}$$

Thus we obtain the following theorem.

Theorem 3. *For a given geodesic arc γ, a family of hypersurfaces $f = constant$ transversal to γ such that $\partial^2 f/\partial x^i \partial x^j = 0$ along γ and a field ξ of tangent directions of F_n transversal to the hypersurfaces defined on a neighborhood of γ, if the coordinates (x^i) are geodesic along γ in the Riemann space $V_n(\xi)$, then $\Gamma^{i}_{jk}(x, \varphi(x))$ and $\Gamma^{*i}_{jk}(x, \varphi(x))$ vanish along γ, the converse is also true.*

Remark. We have assume that F_n is analytic, but the theorems in the present paper will hold good if F_n have a suitable differentiability.

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