ON EXTENSIONS OF LIE GROUP, TRANSFOMATION GROUP AND FIBRE BUNDLE

YOSHIHIRO TASHIRO

In a series of his Notes [2, 3] on extensions [*prolongements* in French], C. Ehresmann has given a new view point of differential geometry of higher order. The purpose of the present Note is to show some theorems on extensions of Lie groups, transformation groups and fibre bundles from the view point.

In §1, we shall give preliminaries and a lemma. In §2, we shall show that the extension of a Lie group is also a Lie group, and establish a formula on the first extended group, which is due to Prof. T. Ōtsuki. In §3, we shall consider the extension of a transformation group and prove a theorem on effectiveness. In §4, we clear up the relation between the extensions of a fibre bundle and its associated principal bundle, and at the end of the paragraph we introduce the vertical extension of a bundle, which plays important roles in several places of differential geometry.

§1. Preliminaries. We take C^s-spaces $\mathfrak{X}$ and $\mathfrak{Y}$. If there is a local C^s-map $f: \mathfrak{X}, x \to \mathfrak{Y}, y$, then we denote by $j_x^r(f)$ the *r-jet* $(r \leq s)$ with source x and butt y. If $\mathfrak{R}^k$ is a k-dimensional euclidean space and f is a local map $\mathfrak{R}^k, 0 \to \mathfrak{X}, x$, then the r-jet $X = j_0^r(f)$ is called a *k^r-spread* [vitesse] at x.

Let $\mathfrak{X}$ be a C^r-space of dimension n. The space of k^r-spreads at a point $x \in \mathfrak{X}$ is denoted by $T_k^r(\mathfrak{X}, x)$ and the union $\bigcup_{x \in \mathfrak{X}} T_k^r(\mathfrak{X}, x)$ by $T_k^r(\mathfrak{X})$, which we shall call the *k^r-extension* [prolongement] of the space $\mathfrak{X}$. The k^r-extension $T_k^r(\mathfrak{X})$ has a bundle structure

$$T_k^r(\mathfrak{X}, \ \mathfrak{L}_{n,k}^r, \ \mathfrak{L}_n^r, \ p, \ \Phi^r)$$

with fibre $\mathfrak{L}_{n,k}^r = T_k^r(\mathfrak{R}^n, 0)$, structure group $\mathfrak{L}_n^r = \mathfrak{L}_{n,n}^r$, projection p and family of coordinate functions Φ^r. $T_k^r(\mathfrak{X})$ is a space of class C^{s-r}, and, with respect to a local coordinate system, an element $X \in T_k^r(\mathfrak{X})$ can be represented by

$$X = (x^\lambda,\ X^\lambda_{i_1},\ X^\lambda_{i_1 i_2}, \ldots, X^\lambda_{i_1 \ldots i_r}) \qquad \begin{array}{l}\lambda = 1, \ldots, n,\\ i = 1, \ldots, k.\end{array}$$

In the particular case $k = r = 1$, 1^1-spreads are ordinary tangent vectors and $T(\mathfrak{X}) = T^1_1(\mathfrak{X})$ is equivalent to the bundle of tangent vectors of $\mathfrak{X}$ [1].

If θ is a C^s-mapping $\mathfrak{X} \to \mathfrak{Y}$, then the r-jet $j^r_x(\theta)$ induces a mapping $\theta^r_{*x}: T^r_k(\mathfrak{X}, x) \to T^r_k(\mathfrak{Y}, y)$, $y = \theta(x)$, and $\theta^r_* = j^r(\theta)$ defines a bundle homomorphism of class $C^{s-r}: T^r_k(\mathfrak{X}) \to T^r_k(\mathfrak{Y})$. That is, the commutativity $\theta \circ p = q \circ \theta^r_*$ holds in the diagramm

$$\begin{array}{ccc} T^r_k(\mathfrak{X}) & \xrightarrow{\theta^r_*} & T^r_k(\mathfrak{Y}) \\ p \downarrow & & \downarrow q \\ \mathfrak{X} & \xrightarrow{\theta} & \mathfrak{Y} \end{array}$$

θ^r_* is called the *r-extended map* of θ.

In the case $k = r = 1$, the 1-extended map $\theta_* = \theta^1_*$ is the ordinary induced map of the tangent space $T(\mathfrak{X})$ into $T(\mathfrak{Y})$ [1]. If, with respect to local coordinate systems (x^λ) in $\mathfrak{X}$ and (y^μ) in $\mathfrak{Y}$, the mapping θ is represented by functions $y^\mu = \theta^\mu(x^\lambda)$, then the induced map θ_{*x} at x is represented by

$$Y^\mu(y) = \left(\frac{\partial \theta^\mu}{\partial x^\lambda}\right)_x X^\lambda(x), \qquad X \in T(\mathfrak{X}, x).$$

We can easily obtain $T^r_k(\mathfrak{X} \times \mathfrak{Y}) = T^r_k(\mathfrak{X}) \times T^r_k(\mathfrak{Y})$.

Now we consider three C^s-spaces $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ and a C^s-map

$$\theta : \mathfrak{X} \times \mathfrak{Y}, (x, y) \to \mathfrak{Z}, z.$$

For a fixed point $x \in \mathfrak{X}$, we define a map

$$\theta_x : \mathfrak{Y} \to \mathfrak{Z} \qquad \text{by } \theta_x(y) = \theta(x, y),$$

and similarly, for a fixed point $y \in \mathfrak{Y}$,

$$\theta_y : \mathfrak{X} \to \mathfrak{Z} \qquad \text{by } \theta_y(x) = \theta(x, y).$$

Then there give rise to three r-extended maps

$$\begin{aligned} &\theta^r_* : T^r_k(\mathfrak{X}) \times T^r_k(\mathfrak{Y}) \to T^r_k(\mathfrak{Z}), \\ &\theta^r_{x*} : T^r_k(\mathfrak{Y}) \to T^r_k(\mathfrak{Z}), \\ &\theta^r_{y*} : T^r_k(\mathfrak{X}) \to T^r_k(\mathfrak{Z}). \end{aligned}$$

Denoting the natural projections by

$$p : T_k^r(\mathfrak{X}) \to \mathfrak{X}, \qquad q : T_k^r(\mathfrak{Y}) \to \mathfrak{Y}, \qquad r : T_k^r(\mathfrak{Z}) \to \mathfrak{Z},$$

the commutativity

$$r(\theta_*^r(X, Y)) = \theta(p(X), q(Y))$$

holds.

By means of local coordinate systems, we can have

Lemma. In the case $k = r = 1$,

$$\theta_*(X(x), Y(y)) = \theta_{x*}(Y) + \theta_{y*}(X).$$

§2. ***k^r*-extended group.** In place of three manifolds $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ and a map at the end of §1, we now take a Lie group $\mathfrak{G}$ and the composition law of the group $\theta : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$. As usual we write $gg' = \theta(g, g')$. Then the maps θ_x and θ_y turn to the left and right translations. For distinction between them, the left translation will be denoted by θ_g, while the right one by ρ_g, $g \in \mathfrak{G}$.

We consider the k^r-extension $T_k^r(\mathfrak{G})$ of $\mathfrak{G}$ as an analytic space, and denote by $L_k^r(\mathfrak{G})$ the space $T_k^r(\mathfrak{G}, e)$ at the neutral element e of $\mathfrak{G}$. Then $T_k^r(\mathfrak{G})$ is identified with the product $\mathfrak{G} \times L_k^r(\mathfrak{G})$ by the identification

$$G = (g, E),$$

where $G \in T_k^r(\mathfrak{G}, g)$ and we have put

$$E = (\theta_g^{-1})_*^r(G) \in L_k^r(\mathfrak{G}).$$

The composition law θ induces the r-extended map

$$\theta_*^r : T_k^r(\mathfrak{G}) \times T_k^r(\mathfrak{G}) \to T_k^r(\mathfrak{G}).$$

From definitions, for

$$G = j_0^r(\gamma) = (g, E), \qquad \gamma : \mathfrak{R}^k, 0 \to \mathfrak{G}, g,$$
$$G' = j_0^r(\gamma') = (g', E'), \qquad \gamma' : \mathfrak{R}^k, 0 \to \mathfrak{G}, g',$$

the image $\theta_*^r(G', G)$, denoted sometimes by $G'G$, is given by

$$G'G = j_0^r(\gamma'') = (g'g,\ E''),$$

where γ'', defined by $\gamma''(u) = \gamma'(u)\gamma(u),\ u \in \mathfrak{R}^k$, is a map $\mathfrak{R}^k, 0 \to \mathfrak{G}, g'g$ and $E'' = (\theta_{g'g}{}^{-1})_*^r j_0^r(\gamma'')$.

The k^r-spread of the constant map $\mathfrak{R}^k \to e$, which will be indicated by e too, gives a left and right neutral element, *i. e.*,

$$eG = Ge = G \qquad\qquad \text{for any} \quad G \in T_k^r(\mathfrak{G}).$$

Moreover, for an element $G \in T_k^r(\mathfrak{G})$, the inverse element G^{-1} is given by

$$G^{-1} = j_0^r(\bar{\gamma}) = (g^{-1},\ \bar{E}),$$

where $\bar{\gamma}(u) = (\gamma(u))^{-1} : \mathfrak{R}^k, 0 \to \mathfrak{G}, g^{-1}$ and $\bar{E} = j_0^r(\theta_g \bar{\gamma})$. Thus we have

Theorem 1. *The k^r-extension $T_k^r(\mathfrak{G})$ of a Lie group $\mathfrak{G}$ is a Lie group with the r-extended map θ_*^r as composition law.* We call it the *k^r-extended group* of $\mathfrak{G}$.

In the case $k = r = 1$, $L(\mathfrak{G}) = L_1^1(\mathfrak{G})$ is homeomorphic to the Lie algebra of $\mathfrak{G}$. Taking account of Lemma in §1, we have formula

$$G'G = \theta_*(G', G) = \theta_{g'*}(G) + \rho_{g*}(G') \qquad \begin{array}{l} G \in T(\mathfrak{G}, g), \\ G' \in T(\mathfrak{G}, g'), \end{array}$$

the right hand side of which is equal to

$$\begin{aligned} \theta_{g'*}(G) + \rho_{g*}(G') &= \theta_{g'*} \circ \theta_{g*}(E) + \rho_{g*} \circ \theta_{g'*}(E') \\ &= \theta_{g'*} \circ \theta_{g*}(E + \theta_{g*}{}^{-1} \circ \rho_{g*}(E')) \\ &= \theta_{g'g*}(E + \mathrm{ad}(g^{-1})E'). \end{aligned}$$

Consequently,

Theorem 2. *The multiplication law of the first extended group $T(\mathfrak{G}) = \mathfrak{G} \times L(\mathfrak{G})$ is explicitly given by*

$$(g',\ E')\ (g,\ E) = (g'g,\ E + \mathrm{ad}(g^{-1})E').$$

The neutral element is $(e,\ 0)$ and the inverse of an element $(g,\ E)$ is $(g^{-1},\ -\mathrm{ad}(g)E)$.

§3. Extension of transformation group. Now let $\mathfrak{G}$ be a Lie group, $\mathfrak{Y}$ a C^s-space and τ a map

$$\tau : \mathfrak{G} \times \mathfrak{Y} \to \mathfrak{Y}.$$

We write sometimes $g(y)$ for $\tau(g, y)$. It is said that $\mathfrak{G}$ C^s-operates effectively on $\mathfrak{Y}$ and τ is the operation law, if the following properties are satisfied:

i) τ is of class C^s in g and y,

ii) for any $g \in \mathfrak{G}$, τ_g is an automorphism of $\mathfrak{Y}$,

iii) (transitivity) for any $g, g' \in \mathfrak{G}$ and any $y \in \mathfrak{Y}$,

$$\tau(g', \tau(g, y)) = \tau(\gamma(g', g), y),$$

or

$$g'(g(y)) = (g'g)(y).$$

iv) (effectiveness) $\tau(g, y) = y$ holds for every $y \in \mathfrak{Y}$ if and only if $g = e$.

From the transitivity iii), it is observed that the mappings $\tau_g : \mathfrak{Y} \to \mathfrak{Y}$, $\tau_y : \mathfrak{G} \to \mathfrak{Y}$ satisfy relations

$$\tau_{g'} \circ \tau_g = \tau_{g'g},$$

$$\tau_{g'} \circ \tau_y = \tau_y \circ \theta_{g'},$$

$$\tau_{\tau_y(g)} = \tau_y \circ \rho_g.$$

We now prove the following

Theorem 3. *If a Lie group $\mathfrak{G}$ C^s-operates effectively on a space $\mathfrak{Y}$, then the k^r-extended group $T_k^r(\mathfrak{G})$ C^{s-r}-operates effectively on the k^r-extension $T_k^r(\mathfrak{Y})$.*

The fact that $T_k^r(\mathfrak{G})$ C^{s-r}-operates on $T_k^r(\mathfrak{Y})$ follows from the composition of r-jets. The effectiveness is proved as follows: If $\tau_*^r(G, Y) = Y$ for any point $Y \in T_k^r(\mathfrak{Y})$, then, denoting the natural projections $p : T_k^r(\mathfrak{G}) \to \mathfrak{G}$ and $q : T_k^r(\mathfrak{Y}) \to \mathfrak{Y}$,

$$\tau(p(G), q(Y)) = q(\tau_*^r(G, Y)) = q(Y)$$

holds for any point $y = q(Y) \in \mathfrak{Y}$. Hence, by effectiveness of $\mathfrak{G}$ on $\mathfrak{Y}$,

$p(G)$ should be identical with the neutral element e of $\mathfrak{G}$. We may therefore confine ourselves to a neighborhood of e in $\mathfrak{G}$ and that of any point y in $\mathfrak{Y}$ with local coordinate systems (g^α) and (y^μ) respectively. Taking a coorinate system (u^i) in $\mathfrak{R}^k$, $G \in L_k^r(\mathfrak{G})$ has local coordinates $(e^\alpha, G_i^\alpha, \cdots, G_{i_1 \cdots i_r}^\alpha)$ and Y has $(y^\mu, Y_i^\mu, \cdots, Y_{i_1 \cdots i_r}^\mu)$. The first set $(r=1)$ of local coordinates of $Y' = \tau_*(G, Y)$ is related to those of G and Y by the equations

$$Y_i'^\mu = \left(\frac{\partial \tau^\mu}{\partial g^\alpha}\right)_{e,y} G_i^\alpha + \left(\frac{\partial \tau^\mu}{\partial y^\lambda}\right)_{e,y} Y_i^\lambda.$$

If these equations replaced for $Y_i'^\mu$ by Y_i^μ hold for any Y_i^μ, we have

$$\left(\frac{\partial \tau^\mu}{\partial g^\alpha}\right)_{e,y} G_i^\alpha = 0.$$

The effectiveness of group $\mathfrak{G}$ on $\mathfrak{Y}$ implies the essentiality of the parameter g^α in transformation functions $\tau^\mu(g^\alpha, y^\lambda)$, and therefore $\xi_\alpha^\mu(y) = \left(\frac{\partial \tau^\mu}{\partial g^\alpha}\right)_{e,y}$ are linearly independent with constant coefficients. Since G_i^α are independent of y, we have $G_i^\alpha = 0$. By induction on r, we can see that $\xi_\alpha^\mu(y) G^\alpha_{i_1 \cdots i_r} = 0$ should hold and therefore all $G^\alpha_{i_1 \cdots i_r}$ vanish. It is finally concluded that G is $(e, 0) = e$.

§4. **k^r-extended bundle.** We now consider a fibre bundle $\mathfrak{B}(\mathfrak{X}, \mathfrak{Y}, \mathfrak{G}, \pi, \Phi)$ of class C^s, that is, we assume that the base space $\mathfrak{X}$, the fibre space $\mathfrak{Y}$, the bundle space $\mathfrak{B}$, the projection π, and the coordinate functions $\varphi_i \in \Phi$ are all of class C^s, and the structure group $\mathfrak{G}$ is a Lie group.

Then we have the k^r-extension $T_k^r(\mathfrak{B})$ of the bundle space $\mathfrak{B}$ as a C^s-space, and there are induced the r-extended maps

$$\pi_*^r : T_k^r(\mathfrak{B}) \to T_k^r(\mathfrak{X})$$

of the projection π and

$$\varphi_{i*}^r : T_k^r(V_i) \times T_k^r(\mathfrak{Y}) \to T_k^r(\mathfrak{B})$$

of the coordinate functions φ_i.

If $X = j_0^r(\alpha)$, $\alpha : \mathfrak{R}^k, 0 \to \mathfrak{X}, x$ and $Y = j_0^r(\beta)$, $\beta : \mathfrak{R}^k, 0 \to \mathfrak{Y}, y$ and if $x \in V_i \cap V_j$, then we have

$$\varphi^r_{i*X}(Y) = \varphi^r_{i*}(X, Y)$$

and

$$\begin{aligned}(\varphi^r_{j*X})^{-1} \circ \varphi^r_{i*X}(Y) &= (\varphi^r_{j*})^{-1}(\varphi^r_{i*}(X, Y)) \\ &= j^r(\varphi_j^{-1}) j^r(\varphi_i)(j^r_0(\alpha),\ j^r_0(\beta)) \\ &= j^r(g_{ji})(j^r_0(\alpha),\ j^r_0(\beta)) \\ &= g^r_{ji*}(X, Y) = g^r_{ji*X}(Y).\end{aligned}$$

Hence $(\varphi^r_{j*})^{-1} \circ \varphi^r_{i*}$ is identical with the r-extended map $g^r_{ji*} : T^r_k(V_i) \cap T^r_k(V_j) \to T^r_k(\mathfrak{G})$. Since $T^r_k(\mathfrak{G})$ operates effectively on $T^r_k(\mathfrak{Y})$, k^r-extension $T^r_k(\mathfrak{B})$ has a bundle structure

$$T^r_k(\mathfrak{B})\ (T^r_k(\mathfrak{X}),\ T^r_k(\mathfrak{Y}),\ T^r_k(\mathfrak{G}),\ \pi^r_*,\ \Phi^r_*),$$

where Φ^r_* is the family of the r-extended coordinate functions φ^r_{i*}. The bundle $T^r_k(\mathfrak{B})$ is called the k^r-extension of $\mathfrak{B}$.

Let us denote the associated principal bundle of a bundle $\mathfrak{B}(\mathfrak{X},\ \mathfrak{Y},\ \mathfrak{G},\ \pi,\ \Phi)$ by $\mathfrak{B}'(\mathfrak{X},\ \mathfrak{G},\ \mathfrak{G}, \pi',\ \Phi')$, which is also of class C^s. The coordinate functions $\varphi'_i : V_i \times \mathfrak{G} \to \mathfrak{B}'$ is given by

$$\varphi'_i(x,\ g) = \varphi_{i,x} \circ g.$$

The right hand side of this expression means an admissible map

$$\varphi_{i,x} \circ g : \mathfrak{Y} \to \mathfrak{Y}_x \qquad (\mathfrak{Y}_x \text{ being the fibre over } x)$$

of $\mathfrak{B}$, and the associated principal bundle $\mathfrak{B}'$ may be identified with the set of admissible maps of the bundle $\mathfrak{B}$.

On the one hand, a mapping φ^r_{i*X} is an admissible one of the fibre $T^r_k(\mathfrak{Y})$ onto the fibre over $X \in T^r_k(\mathfrak{X})$. On the other hand, we have the k^r-extension of the associated principal bundle $\mathfrak{B}'$:

$$T^r_k(\mathfrak{B}')(T^r_k(\mathfrak{X}),\ T^r_k(\mathfrak{Y}),\ T^r_k(\mathfrak{G}),\ \pi'^r_*,\ \Phi'^r_*).$$

By use of α, β in this paragraph and γ in §2 defining X, Y and G respectively, we have

$$\begin{aligned}\varphi'^r_{i*}(X, G)\ (Y) &= j^r(\varphi'_i)(j^r_0(\alpha),\ j^r_0(\gamma)) j^r_0(\beta) \\ &= j^r_0(\varphi'_i(\alpha,\ \gamma))\ j^r_0(\beta) \\ &= j^r_0(\varphi'_i(\alpha,\ \gamma)\ (\beta))\end{aligned}$$

$$
\begin{aligned}
&= j_0^r(\varphi_i(\alpha, \gamma(\beta))) \\
&= j^r(\varphi_i)(j_0^r(\alpha), j_0^r(\gamma)(j_0^r(\beta))) \\
&= \varphi_{i*}^r(X, G(Y)).
\end{aligned}
$$

Since $T_k^r(\mathfrak{G})$ operates effectively on $T_k^r(\mathfrak{Y})$, we have

$$\varphi'^r_{i*}(X, G) = \varphi^r_{i*X} \circ G.$$

Thus the following theorem is obtained.

Theorem 4. *The k^r-extension of the associated principal bundle $\mathfrak{B}'$ of a bundle $\mathfrak{B}$ is equivalent to the associated principal bundle of the k^r-extension $T_k^r(\mathfrak{B})$ of the bundle $\mathfrak{B}$.*

In the k^r-extension $T_k^r(\mathfrak{X})$ of a space $\mathfrak{X}$, the neutral k^r-spreads form a trivial cross-section under its projection p, which may be identifies with the space $\mathfrak{X}$. Then we consider the portion over $\mathfrak{X}$ of the k^r-extension $T_k^r(\mathfrak{B})$ of a bundle $\mathfrak{B}$, and denote it by $\widetilde{\mathfrak{B}}(\mathfrak{X})$. Its coordinate transformations $\tilde{g}_{ji}$ are given by

$$\tilde{g}_{ji} = g^r_{ji*} \mid V_i \cap V_j$$

and they are maps

$$\tilde{g}_{ji} : V_i \cap V_j \to \mathfrak{G},$$

$\mathfrak{G}$ operating on $T_k^r(\mathfrak{Y})$ as a subgroup of $T_k^r(\mathfrak{G})$, i. e.

$$g(Y) = \tau^r_*(g, Y).$$

Its coordinate functions $\tilde{\varphi}_i$ are given by

$$\tilde{\varphi}_i = \varphi^r_{i*} \mid V_i \times T_k^r(\mathfrak{Y}),$$

and the projection $\tilde{\pi}$ by

$$\tilde{\pi}(\tilde{\varphi}_i(x, Y)) = x.$$

Consequently, the portion $\mathfrak{B}(\mathfrak{X})$ has a bundle structure

$$\widetilde{\mathfrak{B}}(\mathfrak{X}, T_k^r(\mathfrak{Y}), \mathfrak{G}, \tilde{\pi}, \widetilde{\Phi})$$

and we call it the *verticall k^r-extension* of the bundle $\mathfrak{B}$.

Bibliography

[1] S. S. Chern, Lecture note on differential geometry, Chicago, 1955.

[2] C. Ehresmann, Les prolongements d'une variété différentiable,

I) Calcul des jets, prolongement principal, C. R. Acad. Sci., Paris, tome 233, 1951, pp. 598—600.

II) L'espace des jets d'order r de V_n dans V_m, ibid., tome 233, 1951, pp. 777—779.

III) Transitivité des prolongements, ibid., tome 233, 1951, pp. 1081—1083.

IV) Eléments de contact et éléments d'envelope, ibid., tome 234, 1952, pp. 1028—1030.

V) Covariants différentiables et prolongements d'une structure infinitésimale, ibid., tome 234, 1952, pp. 1424—1425.

[3] C. Ehresmann, Structures locales et structures infinitésimales, ibid., tome 234, 1952, pp. 587—589.

Department of Mathematics,
Okayama University

(Received September 27, 1956)