NOTE ON A PAPER BY J. S. FRAME AND G. DE B. ROBINSON

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1. Introduction. J. S. Frame and G. de B. Robinson have proved [1] the following

1.1 *Let p be a prime. The number of p-regular diagrams with n nodes is equal to the number of p-regular classes of the symmetric group S_n and hence to the number of modular irreducible representations of S_n.*

A diagram is called *p-regular* if no p of its rows are of equal length, otherwise *p-singular.*

Recently this result was refined by G. de B. Robinson [5] as follows:

1.2 *The number of p-regular diagrams in a given block is equal to the number of modular irreducible representations in that block.*

The author has also obtained 1.2 independently by a simple method. We shall also give an alternative proof of 1.1 by our method.

2. Remarks on diagrams. Let $[\alpha] = [\alpha_1, \alpha_2, \ldots, \alpha_h]$ be a diagram with n nodes that contains α_i nodes in its i-th row. We denote the number of nodes in the j-th column of $[\alpha]$ by α'_j. We have evidently

$$\sum_{j=1}^{k} \alpha'_j = n, \qquad \alpha'_1 \geqq \alpha'_2 \geqq \ldots \geqq \alpha'_k > 0 \qquad (k = \alpha_1).$$

We set

$$\rho_j = \alpha'_j - \alpha'_{j+1} \qquad (j = 1, 2, \ldots, k-1),$$
$$\rho_k = \alpha'_k.$$

Then $[\alpha]$ is completely determined by a set of non-negative integers $\{\rho_j\}$ since

$$\alpha'_j = \sum_{i=j}^{k} \rho_i \qquad (j = 1, 2, \ldots, k).$$

It follows from our definition that $[\alpha]$ is p-regular if and only if every ρ_j is less than p. We see also that $[\alpha]$ is p-regular if and only if $[\alpha]$ does not contain a p-hook of leg length $p-1$.

If $[\alpha]$ is p-singular, then there exists at least one ρ_j greater than p. We set

$$\rho_j = \rho_j^{(1)} + \rho_j^{(2)} p \qquad 0 \leqq \rho_j^{(1)} < p. \tag{2.1}$$

Then $[\alpha]$ is completely determined by $\{\rho_j^{(1)}\}$ and $\{\rho_j^{(2)}\}$. Let $[\alpha^{(1)}]$ and $[\alpha^{(2)}]$ be the diagrams determined by $\{\rho_j^{(1)}\}$ and $\{\rho_j^{(2)}\}$ in the above sence respectively. Since $\rho_j^{(1)} < p$, $[\alpha^{(1)}]$ is p-regular and $[\alpha^{(2)}]$ is not vacuous for a p-singular diagram $[\alpha]$. If $[\alpha^{(2)}]$ has a nodes, then $[\alpha^{(1)}]$ has $m = n - ap$ nodes. Moreover we see easily that $[\alpha^{(1)}]$ is obtained by removing a p-hooks of leg length $p-1$ successively from $[\alpha]$. Since the p-regular diagram $[\alpha^{(1)}]$ is determined uniquely by $[\alpha]$, we shall call $[\alpha^{(1)}]$ the p-regular diagram corresponding to $[\alpha]$. We have the

Lemma 1. *$[\alpha]$ and $[\alpha^{(1)}]$ have the same p-core.*

Example. If $[\alpha] = [6, 4, 3^3, 1^4]$ for $p = 3$, then $[\alpha^{(1)}] = [6, 4, 1]$ and $[\alpha^{(2)}] = [3, 1]$. $[\alpha]$ and $[\alpha^{(1)}]$ have the same p-core $[\alpha_0] = [3, 1^2]$.

Let $[\beta]$ be a given p-regular diagram with m nodes and let $[\gamma]$ be an arbitrary diagram with a nodes. Then $[\beta]$ and $[\gamma]$ determine uniquely a diagram $[\alpha]$ with $n = m + ap$ nodes such that

$$[\beta] = [\alpha^{(1)}], \quad [\gamma] = [\alpha^{(2)}]. \tag{2.2}$$

Hence if we denote by $k(n)$ the number of diagrams with n nodes, i.e. the number of classes of S_n, then for a given p-regular diagram $[\beta]$ with m nodes there exist exactly $k(a)$ diagrams $[\alpha]$ with n nodes such that $[\alpha^{(1)}] = [\beta]$. Therefore we obtain the

Lemma 2. *Let $h(n)$ be the number of p-regular diagrams with n nodes. Then*

$$h(n) = k(n) - \sum_{a=1}^{t} h(n-ap)k(a), \tag{2.3}$$

where $n = tp + r$, $0 \leqq r < p$.

3. Proof of 1.1. Let us denote by $k'(n)$ the number of p-regular classes of S_n. We then have [2, Lemma 3]

$$k'(n) = k(n) - \sum_{a=1}^{t} k'(n-ap)k(a). \tag{3.1}$$

Certainly the theorem is true for $n=1$. We shall assume that 1.1 is true for $m < n$. We then have

$$h(n-ap) = k'(n-ap) \qquad (a=1, 2, \ldots, t).$$

It follows immediately from 2.3 and 3.1 that $h(n) = k'(n)$. This proves 1.1.

4. Proof of 1.2. Let B be a block of weight b having a given p-core $[\alpha_0]$. The number $l(b)$ of ordinary irreducible representations in B and the number $l'(b)$ of modular irreducible representations in B are independent of the p-core and we have [2 ; 3 ; 4]

4.1 $$l'(b) = l(b) - \sum_{a=1}^{b} l'(b-a)k(a).$$

If we denote by $g(w)$ the number of p-regular diagrams in a block of weight w having a given p-core $[\alpha_0]$, then we see by Lemma 1 that

4.2 $$g(b) = l(b) - \sum_{a=1}^{b} g(b-a)k(a).$$

Certainly 1.2 is true for $b=1$. We shall assume that 1.2 is true for $w < b$. Then 4.1 and 4.2 yield $g(b) = l'(b)$. Since $l'(b)$ is independent of the p-core, $g(b)$ is also independent of the p-core. This completes the proof of 1.2.

References

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