GALOIS THEORY OF SIMPLE RINGS

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For Galois theory of division rings of infinite degree, recently N. Nobusawa succeeded in constructing his Galois theory under the assumption that the total group is locally finite ([4], [5])[1]. Afterwards, considering the case where the total group is locally compact (or some more general case), T. Nagahara and the present author have generalized the theory in such a way that our theory contains the locally finite case as well as the case of finite degree, further in our investigation, several topological characterizations of the previously treated stages have been presented ([2]).

The main effort of this paper is directed towards extending our previous theory in [2] to simple rings. For a simple ring R which is Galois over a simple subring S, we assume the following conditions, which, in case R is a division ring, take together, are equivalent to that the total group is locally finite-dimensional and locally compact:

(*) For each finite subset F of R, there exists a simple subring N normal, finite over S and containing $S(F)$, and

(*$_f$) $[V_R(S): V_R(R)] < \infty$.

Then, under these assumptions, we can prove that there exists a one-to-one dual correspondence between closed regular subgroups of the total group and intermediate regular subrings, in the usual sense of Galois theory (Theorem 15), further that the so-called extension theorem is still valid (Theorem 16). Our investigation will be restricted at first to the case where the total group is locally finite. The theory in this case is clearly an extension of Nobusawa's and at the same time, it plays a preparatory rôle for our final aim.

§ 1 contains some fundamental definitions and preliminary results, § 2 deals Galois theory for the locally finite case, and it contains also several general results which will be useful in the subsequent sections (Theorems 6, 7). § 3 is devoted to present some structural consequences which will be considered as generalizations of previous results cited in [2], in particular, Theorem 14 may be considered as a topological characterization of our present stage. At last in § 4, our principal results will be stated.

1) Numbers in brackets refer to the references cited at the end of this paper.

We declear here that notations used in this paper are the same as in [2][1].

1. Preliminaries

A ring R is called a *simple ring* if it is a two-sided simple ring with the identity element satisfying minimum condition for one-sided ideals, and we say that S is a *simple subring* of R when S is simple and possesses the same identity element with that of R.

Definition 1. A simple ring R is said to be *locally simple* over a simple subring S, if, for each finite subset F of R, there exists a simple subring containing $S(F)$ (the least subring containing S and F) which is finite over S as a left S-module (accordingly which possesses a finite independent S-basis).

Clearly, the notion of local simplicity coincides with that of local finiteness[2] when R is a division ring. The next fact will be used frequently without notice in our study.

Lemma 1. *If a simple ring R is locally simple over a simple subring S, then so is each intermediate simple subring.*

Proof. Let $R' = \sum_{i,j=1}^{n} D'e_{ij}$ be an intermediate simple subring, where D' is a division ring and e_{ij}'s are matric units. For any finite subset F' of R', there exists a simple subring R^* finite over S and containing $S(\{e_{ij}\}, F')$. Then, as is well-known, $R^* = \sum_{i,j=1}^{n} E^* e_{ij}$ with the simple ring $E^* = V_{R^*}(\{e_{ij}\})$. Clearly, $R^* \cap R' = \sum_{i,j=1}^{n} (E^* \cap D')e_{ij}$ is a simple subring finite over S[3].

Definition 2. For simple rings R, S, we say that R is *Galois* over S (or R/S is *Galois*) if $V_R(S)$ is simple and S is the fixed subring of an

1) See the end of [2, § 1].

2) A division ring K is said to be locally finite over a division subring L if, for each finite subset F of K, $L(F)$ is finite over L.

3) In fact, we can prove that $E \frown D$ is a division ring as follows: If d is a non-zero element in $D \frown E$, there holds that $dd' = d'd = 1$, $d^{m+1}e = d^m$ with some $d' \in D$, $e \in E$ and some positive integer m by minimum condition for E, whence we see that $dd' = 1 = de$. It follows therefore that $d' = e$, which shows that $D \cap E$ is a division ring. This fact will be used often without notice in the sequel.

automorphism group $\mathfrak{H}$ of R: $J(\mathfrak{H}, R) = S$. In the case, the totality $\mathfrak{G}(R/S)$ of S-automorphisms of R is termed the *total group* of R/S, and a subgroup $\mathfrak{H}$ of $\mathfrak{G}(R/S)$ is named a *total subgroup* if $\mathfrak{H}$ is the totality of $J(\mathfrak{H}, R)$-automorphisms. And, if additionally $V_R(V_R(S))$ is simple, we say that R is *s-Galois* over S (or R/S is *s-Galois*).

Definition 3. In case a simple ring R is Galois over a simple subring S, an intermediate simple subring N is said to be *normal* with respect to $\mathfrak{G}(R/S)$ (if there exists no confusion, we say also that N is *normal* over S) if $N^\sigma = N$ for all σ in $\mathfrak{G}(R/S)$. And N is said to be *s-normal* if N is normal over S and $V_R(V_R(N))$ is simple.

Definition 4. Let R be a simple ring which contains a simple subring S. An automorphism group $\mathfrak{H}$ of R is called a *weakly regular group* of R/S, if $J(\mathfrak{H}, R) = S$ and $\mathfrak{H}$ contains all the S-inner automorphisms of R.

Definition 5. Let R be a simple ring. A simple subring R' is called a *regular subring* if $V_R(R')$ is simple. And a group $\mathfrak{H}$ of automorphisms in R is said to be *regular* if $J(\mathfrak{H}, R)$ is a regular subring and $\mathfrak{H}$ is a weakly regular group of $R/J(\mathfrak{H}, R)$[1].

In case R is Galois over a simple subring S, a weakly regular group of R/S is necessarily a regular group.

We set here the following lemma.

Lemma 2. *Let S be a simple subring of a simple ring R. If M is a finitely generated unitary left S-module, then $\mathfrak{M}_R(M)$ is finite over R_r,*[2] *where $\mathfrak{M}_R(M)$ is the totality of S-homomorphisms of M into R, which may be considered as a right R_r-module.*

Proof. Since S is simple, there holds that, for some integers m, n, the direct sum N of n copies of M possesses an independent S-basis of m elements. Thus, to be easily verified, $\mathfrak{M}_R(N)$ possesses an independent R_r-basis of m elements. As N is completely reducible (and so each element of $\mathfrak{M}_R(M)$ can be extended naturally to that of $\mathfrak{M}_R(N)$), $\mathfrak{M}_R(M)$ may be considered as a submodule of $\mathfrak{M}_R(N)$. Hence, noting that $\mathfrak{M}_R(N)$

1) In case R is a division ring, an automorphism group $\mathfrak{H}$ of R is regular if and only if $\mathfrak{H}$ is weakly regular.

2) R_r denotes the totality of right multiplications by elements of R, and x_r denotes the right multiplication in R by $x \in R$. That $\mathfrak{M}_R(M)$ is finite over R_r means that it is finitely generated over R_r as an R_r-module.

is completely reducible, $\mathfrak{M}_R(M)$ is finite over R_r.

By means of this lemma and the method in [4], we can prove the following:

Theorem 1. *Let R be a simple ring which is Galois and finite over a simple subring S. If $\mathfrak{G}$ is a weakly regular group of R/S, then $\mathfrak{G}$ coincides with $\mathfrak{G}(R/S)$ and there holds that $[R:S]=[\mathfrak{G}:\mathfrak{F}]\cdot[V_R(S):Z]$, where $\mathfrak{F}$ is the totality of inner automorphisms contained in $\mathfrak{G}$ and Z is the center of R.*

Proof. By Lemma 2, $\mathfrak{M}_R(R)$ is finite over R_r, whence so is the submodule $\mathfrak{G}R_r$, which is further an R_r-R_r-module. As is well-known, σR_r is R_r-R_r-isomorphic to τR_r if and only if $\sigma\tau^{-1}$ is in $\mathfrak{F}$, where σ, τ are in $\mathfrak{G}$. Since σR_r is an irreducible R_r-R_r-module and $\mathfrak{G}R_r$ is finite over R_r, $[\mathfrak{G}:\mathfrak{F}]$ is finite. To prove that $[V_R(S):Z]$ is finite, it suffices to show that $[D:Z]$ is finite, where D is a division ring containing Z which belongs to $V_R(S)$. The proof of this fact can be given as in that of [4, Theorem 3] (with a slight modification) and it may be left to readers. The rest of the proof is clear from [3, Theorem 1].

Next two theorems are Theorems 5 and 6 of [3], which we shall quote often in the sequel.

Theorem 2. *If a simple ring R is Galois and finite over a simple subring S, then there exists a one-to-one dual correspondence between regular subgroups of the total group and intermediate regular subrings, in the usual sense of Galois theory.*

Theorem 3. *If a simple ring R is Galois and finite over a simple subring S, then for any intermediate regular subring R', each S-isomorphism ρ of R' into R can be extended to an automorphism in the total group of R/S, where $V_R(R'^{\rho})$ is assumed to be simple.*

Definition 6. In case R is a simple ring which is Galois over a simple subring S, $\mathfrak{G}=\mathfrak{G}(R/S)$ is said to be *locally finite* if, for each element $r\in R$, $\{r\}^{\mathfrak{G}}$ is finite. And $\mathfrak{G}$ is said to be *almost outer* [*outer*] if it contains only a finite number of inner automorphisms [no inner automorphisms except the identity].

In the rest of this section we assume that R is a simple ring which is Galois over a simple subring S, Z denotes the center of R and that $\mathfrak{G}$ means the total group of R/S.

Theorem 4. *Let R be locally simple and Galois over S. If the*

total group $\mathfrak{G}$ *is almost outer, then* $\mathfrak{G}$ *is locally finite.*

Proof. Let $R = \sum_{i,j=1}^{n} De_{ij}$, where D is a division ring and e_{ij}'s are matric units. And we take a simple subring T finite over S and containing $S(\{e_{ij}\}, r)$, where r is an arbitrary element in R. By Lemma 2, $\mathfrak{M}_R(T)$ is finite over R_r, accordingly so is $\mathfrak{G}_T R_r$, where $\mathfrak{G}_T$ signifies the restriction of $\mathfrak{G}$ on T. (Similarly, for $\sigma \in \mathfrak{G}$, σ_T signifies the restriction of σ on T.) Hence, there exists the least integer k such that

$$\mathfrak{G}_T R_r = \sum_{i=1}^{k} \sigma_T^{(i)} R_r. \tag{1}$$

Now we shall prove that the sums in (1) are direct sums. We suppose, in contrary, that $\sum_{i=1}^{n} \sigma_T^{(i)} x^{(i)}_r = 0$ with $x^{(1)} \neq 0$. Here we may set $x^{(1)} = (\sum_{i,j=1}^{k} d_{ij} e_{ij})^{\sigma^{(1)}}$ with $d_{\lambda\mu} \neq 0$, where d_{ij}'s in D. As is easily seen, there holds the following:

$$(e_{p\lambda})_{rT}\sigma^{(1)} \left(\sum_{i,j=1}^{n} d_{ij} e_{ij}\right)^{\sigma^{(1)}}_r (d_{\lambda\mu}^{-1} e_{\mu q})^{\sigma^{(1)}}_r = \sigma_T^{(1)} (e_{pq}^{\sigma^{(1)}})_r.$$

From this fact, a brief computation shows that $\sigma_T^{(1)}$ is contained in $\sum_{i=2}^{k} \sigma_T^{(i)} R_r$, being contrary to the minimality of k. We obtain therefore

$$\mathfrak{G}_T R_r = \sum_{i=1}^{k} \oplus \sigma_T^{(i)} R_r. \tag{2}$$

Next, let σ_T be an arbitrary element in $\mathfrak{G}_T$, and $\sigma_T = \sum_{i=1}^{k} \sigma_T^{(i)} u^{(i)}_r$ by (2). As, for each $s \in S$, $s_{rT}\sigma = \sigma_T s_r$, $s_{rT}\sigma^{(i)} = \sigma_T^{(i)} s_r$, we obtain $\sum_{i=1}^{k} \sigma_T^{(i)}(u^{(i)}_r s_r - s_r u^{(i)}_r) = 0$, whence we know that $u^{(i)}$'s are in $V_R(S)$. Let $V_R(S) = \sum_{i',j'=1}^{l} D' e'_{i'j'}$, where D' is a division ring containing Z and $e'_{i'j'}$'s are matric units. Now we distinguish two cases: (I) Z *is finite.* Since $\mathfrak{G}$ is almost outer and Z is finite, D' is finite, accordingly so is $V_R(S)$. The finiteness of $\mathfrak{G}_T$ is a direct consequence of this fact. (II) Z *is infinite.* In this case, $V_R(S)$ has to coincide with Z. For, if $D' \supsetneq Z$, then for any element $d \in D' \backslash Z$, the set $\{1 + dc\,;\, c \in Z\}$ determines an infinite number of S-inner automorphisms of R. This contradiction shows $D' = Z$. Hence $V_R(S) = \sum_{i',j'=1}^{l} Z e'_{i'j'}$. Further if $l > 1$, the set of

regular elements $\{1+ce'_{12};\ c\in Z\}$ determines an infinite number of S-inner automorphisms of R (which send e'_{22} into different images). This contradiction shows that $V_R(S)=Z$. Hence each $\sigma_T\in\mathfrak{S}_T$ is represented in the form uniquely $\sigma_T=\sum_{i=1}^{k}\sigma_T^{(i)}u^{(i)}_r$ with $u^{(i)}$'s in Z. Now let t be an arbitrary element in T, then

$$t_{rT}\sigma=t_{rT}\sum_{i=1}^{k}\sigma^{(i)}u^{(i)}_r=\sum_{i=1}^{k}\sigma_T^{(i)}(t^{\sigma^{(i)}})_r u^{(i)}_r, \tag{3}$$

$$\begin{aligned}\sigma_T(t^\sigma)_r&=(\sum_{i=1}^{k}\sigma_T^{(i)}u^{(i)}_r)(\sum_{j=1}^{k}(t^{\sigma^{(j)}})_r u^{(j)}_r)\\&=\sum_{i=1}^{k}\sigma_T^{(i)}\{\sum_{j=1}^{k}(t^{\sigma^{(j)}})_r u^{(j)}_r\}u^{(i)}_r.\end{aligned} \tag{4}$$

As $t_{rT}\sigma=\sigma_T(t^\sigma)_r$, we obtain from (3) and (4) the next:

$$\sum_{i=1}^{k}\sigma_T^{(i)}\{(t^{\sigma^{(i)}})_r-\sum_{j=1}^{k}(t^{\sigma^{(j)}})_r u^{(j)}_r\}u^{(i)}_r=0,$$

and so $(t^{\sigma^{(i)}}-\sum_{j=1}^{k}t^{\sigma^{(j)}}u^{(j)})u^{(i)}=0$ $(i=1,\ldots,k)$. Here, without loss of generality, we may assume that $u^{(1)}\neq 0$. Therefore, we get

$$t^{\sigma^{(1)}}-t^{\sigma^{(1)}}u^{(1)}-\sum_{j=2}^{k}t^{\sigma^{(j)}}u^{(j)}=0.$$

As t is arbitrary in T, the above equation is equivalent to the next:

$$\sigma_T^{(1)}(1-u^{(1)})_r-\sum_{j=2}^{k}\sigma_T^{(j)}u^{(j)}_r=0. \tag{5}$$

Hence, $u^{(1)}=1$, $u^{(2)}=\ldots.=u^{(k)}=0$, that is, $\sigma_T=\sigma_T^{(1)}$.

The following preliminary lemmas are given in [1].

Lemma 3. *Let $\mathfrak{S}$ be locally finite and $V_R(S)$ be algebraic over Z. If Z is infinite then $V_R(S)=Z$.*

Lemma 4. *If $\mathfrak{S}$ is locally finite and, for each finite subset F of R, there exists a simple subring Galois, finite over S and containing $S(F)$, then $V_R(S)\supsetneqq Z$ implies that $V_R(S)$ is algebraic over Z.*

By making use of these lemmas, we can prove a partial converse of Theorem 4, which has been stated without proof in our previous paper

[1].

Theorem 5. *Let R be a simple ring which is Galois over a simple subring S. If $\mathfrak{G}=\mathfrak{G}(R/S)$ is locally finite and, for each finite subset F of R, there exists a simple subring Galois, finite over S and containing $S(F)$, then $\mathfrak{G}$ is almost outer. Further in the case, either $\mathfrak{G}$ is outer or $V_R(S)$ is finite.*

Proof. If Z is infinite, by Lemmas 3, 4, we shall readily obtain that $V_R(S)=Z$. Hence it suffices to show that, in case Z is finite and properly contained in $V_R(S)$, a division ring belonging to $V_R(S)$ is finite. Let $V_R(S)=\sum_{i,j=1}^{n} D\,e_{ij}$, where D is a division ring containing Z and e_{ij}'s are matric units. By Lemma 4, D is algebraic over Z, and so it is commutative[1], accordingly it is locally finite over Z. Now we select an element $d\in D\setminus Z$, and let r be an element in R such that $dr\neq rd$. Further, let $\{d_i r\, d_i^{-1};\ d_i\in D,\ i=1,\ldots,k\}$ be the set of all images of r by inner automorphisms determined by non-zero elements of D, which is finite by the local finiteness of $\mathfrak{G}(R/S)$, and let $D'=V_D(r)$. Then, for each x in D, there holds that $d_i^{-1}x\in D'$ with some i, from what we see that x belongs to $D'(d_1,\ldots,d_k)=Z(D',\ d_1,\ldots,d_k)$. Since D is locally finite over Z, it suffices to show that D' is finite. To this end, we suppose, in contrary, that D' is infinite. We set here $S'=D'(d,\ r)$. Then d is in $V_{S'}(D'(d))$, but not in C', the center of S'. If $(1+dd')s'(1+dd')^{-1}=(1+dd'')s'(1+dd'')^{-1}$ for all $s'\in S'$ ($d',\ d''\in D'$), then $(1+dd'')^{-1}(1+dd')=c'\in C'\cap D'(d)$ ($\supset D'$). Hence $(c'-1)+d(d''c'-d')=0$. As $D'(d)$ is a field finite over D', $C'\cap D'(d)$ is a field not containing d, accordingly d, 1 are linearly independent over $C'\cap D'(d)$. We get therefore $c'=1$, $d''c'=d'$, that is, $d'=d''$. This fact shows that the set $\{1+dd^*;\ d^*\in D'\}$ determines an infinite number of $D'(d)$-[S-] inner automorphisms of S' [of R]. Hence r possesses an infinite number of images under these automorphisms, being contrary to our assumption.

The next lemma is almost obvious:

Lemma 5. *Let R be Galois and finite over S. If $\mathfrak{G}$ is locally finite, then it is finite.*

Our last lemma in this section is the following, which will be required in the next section.

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1) N. Jacobson: Structure theory for algebraic algebras of bounded degree, Ann. Math., 46 (1945), Theorem 11.

Lemma 6. *If R is Galois and finite over S, then* $[V_R(S) : V_S(S)] < \infty$.

Proof. By Theorem 1, there holds that $[V_R(S) : V_R(R)] < \infty$. As $V_R(R) \subset V_{V_{R'S)}}(V_R(S))$, our assertion will be a direct consequence of $[V_{V_{R(S)}}(V_R(S)) : V_S(S)] < \infty$, which can be proved as follows. To be easily verified, the finite group $\mathfrak{G}^*$, the restriction of $\mathfrak{G}$ on the field $V_{V_R(S)}(V_R(S))$, is the total group of $V_{V_R(S)}(V_R(S))/V_S(S)$, accordingly our required finite dimensionality can be obtained.

2. Galois theory (I. Locally finite case)

In the present and subsequent sections, we assume again R is a simple ring which is Galois over a simple subring S, and we set $\mathfrak{G} = \mathfrak{G}(R/S)$.

In our previous paper [2], we have constructed our Galois theory of division rings under the assumptions that the total group is locally finite-dimensional[1] and that the centralizer of the fixed subring in the division ring considered is finite over the center. Translating the condition that the total group is locally finite-dimensional in the present case, we consider here the following condition:

(*) *For each finite subset F of R, there exists a simple subring N normal, finite over S and containing $S(F)$.*

As was noted previously in the introduction, our conclusive aim is to extend the theory in [2] to simple rings however, in this section, we shall restrict our attention to the case where the total group is locally finite. The results of this section are, of course, generalizations of Nobusawa's in [4] and [5], furthermore which are considered as preliminaries of the final theory.

Our first result is a consequence of the condition (*) which corresponds partially to [2, Theorem 3].

Theorem 6. *If R/S is Galois and the condition* (*) *is satisfied, then either $\mathfrak{G} = \mathfrak{G}(R/S)$ is outer or* $[V_R(S) : V_S(S)] < \infty$.

Proof. To prove our assertion, by Lemma 6, it suffices to show that, in case $\mathfrak{G}$ is non-outer, $V_R(S)$ is contained in some simple subring

1) In case K is a division ring which is Galois over a divisionsubring L, the total group $\mathfrak{G}$ of K/L is said to be locally finite-dimensional if, for each finite subset F of K, $L(F^{\mathfrak{G}})$ is finite over L ([2]).

which is normal and finite over S. To this end we shall distinguish two cases: (I) $V_R(S)$ *is not a division ring.* In this case, $V_R(S) = \sum_{i,j=1}^{n} D\, e_{ij}$, where D is a division ring, e_{ij}'s are matric units and $n > 1$. Let N be a simple subring normal, finite over S and containing $S(\{e_{ij}\})$. Then, as is well-known, we can write $N = \sum_{i,j=1}^{n} E\, e_{ij}$, where $E = V_N(\{e_{ij}\})$ is simple. For any element $d \in D$, we have $(1 + de_{12})^{-1}e_{21}(1 + de_{12}) = e_{21} - de_{11} + de_{22} - d^2e_{12} \in N$ by assumption. And so we get $de_{11} = -e_{11}(e_{21} - de_{11} + de_{22} - d^2e_{12})e_{11} \in N$. Similarly, there holds that de_{ii} is contained in N for each i. Hence d is in N, whence $V_R(S) \subset N$. (II) $V_R(S)$ *is a division ring.* As $V = V_R(S) \supsetneq V_R(R)$, there exists an element $v \in V$ such that $vr \neq rv$ for some r in R. Now we take a simple subring N normal, finite over S and containing $S(r, v)$. Then there holds that $V = (V \cap N) \cup V_V(N)$. For, if $v_0 \in V$ and 1 are linearly (left) independent over N then $(1 - v_0)n = n^*(1 - v_0)$ $(n \in N)$ implies that $n - n^{**}v_0 = n^* - n^*v_0$, where $v_0 n = n^{**}v_0$, and so we obtain $n = n^* = n^{**}$, that is, v_0 is in $V_V(N)$. On the other hand, if v_0 and 1 are linearly dependent over N: $n_1 v_0 - n_2 = 0$ with non-zero $n_1 \in N$, then, noting that N is simple and normal over S, we can easily obtain that v_0 is in $V \cap N$. If $V \supsetneq V \cap N$, there exists an element $v' \in V \setminus V \cap N$, which is in $V_V(N)$ by the above remark. Since, at the same time, $v' + v$ is in $V \setminus V \cap N$, $v' + v$ is also in $V_V(N)$. From these facts, we have $(v' + v)r = r(v' + v)$ and $v'r = rv'$, from which we get a contradiction $vr = rv$. Hence V has to coincide with $V \cap N$, that is, V is contained in N.

As an easy consequence, it follows the following:

Corollary. *Under the conditions of Theorem 6, for each finite subset F of R, there exists a regular subring N^* normal, Galois and finite over S and containing S(F). In particular, there exists such an N^* that $V_R(N^*)$ is a division ring.*

Proof. The last part is clear from the proof of Theorem 6.

As is done in [4], the condition (*) enables us to introduce a Hausdorff topology in $\mathfrak{G}$, where a fundamental system of neighbourhoods of the identity is defined as the totality of $\mathfrak{G}(N)$, where N runs over all simple subrings normal and finite over S. Then $\mathfrak{G}$ becomes a topological group. In the sequel, whenever $\mathfrak{G}$ is considered as a topological group, the topology should be that noted here.

The next theorem will play an important rôle in our Galois correspondence.

Theorem 7. *Let R be a simple ring which is Galois over a simple subring S. If the condition (*) is satisfied and $[V_R(S): V_R(R)] < \infty$, then any closed regular subgroup of $\mathfrak{G} = \mathfrak{G}(R/S)$ is a regular total subgroup, and conversely.*

Proof. Let $\mathfrak{H}$ be a closed regular subgroup. We set here $T = \sum_{i,j=1}^{n} D\, e_{ij} = J(\mathfrak{H}, R)$, where D is a division ring and e_{ij}'s are matric units. Then, as T is a regular subring, we can set $V_R(T) = \sum_{h,k=1}^{m} D^*\, e^*_{hk}$, where D^* is a division ring and e^*_{hk}'s are matric units. By the condition $[V_R(S): V_R(R)] < \infty$, there exists a simple subring S' of T finite over S, such that $V_R(T) = V_R(S')$. Now, for any given simple subring N normal, finite over S, we take a simple subring N_0 normal, finite over S and containing $S(S', N, \{e_{ij}\}, \{e^*_{hk}\})$. Then, we can write $N_0 = \sum_{i,j=1}^{n} E\, e_{ij} = \sum_{h,k=1}^{m} E^*\, e^*_{hk}$, where $E = V_{N_0}(\{e_{ij}\})$, $E^* = V_{N_0}(\{e^*_{hk}\})$ are simple. We see therefore that $\sum_{i,j=1}^{n} (D \cap E) e_{ij} = T \cap N_0\ (\supset S')$ is simple. Further, $T \supset T \cap N_0 \supset S'$ implies that $V_R(T) \subset V_R(T \cap N_0) \subset V_R(S')$, that is, $V_R(N_0 \cap T) = \sum_{h,k=1}^{m} D^*\, e^*_{hk}$. Thus we obtain that $V_{N_0}(N_0 \cap T) = N_0 \cap V_R(N_0 \cap T) = \sum_{h,k=1}^{m} (D^* \cap E^*) e^*_{hk}$ is simple. By these facts, we have proved that N_0 is Galois and finite over $N_0 \cap T$. While, $V_{N_0}(N_0 \cap T) \subset V_R(N_0 \cap T) = V_R(T)$ shows that $\mathfrak{H}_{N_0}$ is a weakly regular group of $N_0/N_0 \cap T$, whence, by Theorem 1, $\mathfrak{H}_{N_0} = \mathfrak{G}(N_0/N_0 \cap T)$. If $\sigma \in \mathfrak{G}(T)$, then obviously $\sigma_{N_1} \in \mathfrak{G}(N_0/N_0 \cap T)$. Hence there exists some $\tau \in \mathfrak{H}$ such that $\sigma_{N_0} = \tau_{N_0}$, which proves that σ is in $\overline{\mathfrak{H}}$ (the topological closure of $\mathfrak{H}$) $= \mathfrak{H}$, that is, $\mathfrak{G}(T) = \mathfrak{H}$. The converse is almost trivial.

Now we shall introduce the following additional condition:

(**) *$\mathfrak{G}$ is locally finite.*

In virtue of Theorems 4, 5 and Corollary to Theorem 6, one will readily see that, under the condition (*), (**) is equivalent with to say that either $\mathfrak{G}$ is outer or $V_R(S)$ is finite. And on account of Lemma 5 and Corollary to Theorem 6, under the conditions (*) and (**), we may consider the topological group $\mathfrak{G}$ as an inverse limit of finite groups, which shows the following:

Lemma 7. *If the conditions (*) and (**) are satisfied, then $\mathfrak{G}$ is compact.*

Now we shall prove the following theorem, which corresponds to [2,

Theorem 1].

Theorem 8. (*i*) *Under the condition* (*), *the following conditions are equivalent to each other*:

(1) $\mathfrak{G}$ *is compact.*

(2) $\mathfrak{G}$ *is locally finite.*

(3) $\mathfrak{G}$ *is almost outer, or what is the same, either* $\mathfrak{G}$ *is outer or* $V_R(S)$ *is finite.*

(*ii*) *Under the condition* (*), *the following conditions are equivalent to each other*:

(1) $\mathfrak{G}$ *is discrete.*

(2) $[R:S]$ *is finite.*

Proof. (i) (2) $\leftrightarrow$ (3) and (2) $\rightarrow$ (1) are already shown. We shall prove here (1) $\rightarrow$ (2). Let r be an arbitrary element in R, and N be a simple subring normal, finite over S and containing $S(r)$. By definition, $\mathfrak{G}(N)$ is an open invariant subgroup of $\mathfrak{G}$, and so the quotient group $\mathfrak{G}/\mathfrak{G}(N)$ is finite. Hence $\mathfrak{G}_N$ is finite, whence $\{r\}^{\mathfrak{G}}$ is finite.

(ii) Let $\mathfrak{G}$ be discrete, and we shall show that $[R:S]$ is finite. There exists a simple subring N_1 normal, finite over S for which $\mathfrak{G}(N_1)$ consists of only the identity mapping. Here, by Corollary to Theorem 6, we may assume that $V_R(N_1)$ is a division ring, and so that $V_R(N_1) = V_R(R)$. If $R \supsetneqq N_1$, then there exists an element r in $R \setminus N_1$. Let N_2 be a simple subring containing $N_1(r)$ which is normal, Galois and finite over S. Since $\mathfrak{G}_{N_2} = \mathfrak{G}(N_2/S)$ by Theorem 1, and $V_{N_2}(N_1) = N_2 \cap V_R(N_1) = V_{N_2}(N_2)$, Theorem 2 implies a contradiction that $\mathfrak{G}(N_1)$ is not the identity subgroup. The converse part is almost clear.

Remark 1. To prove the implication (2) $\rightarrow$ (3) in Theorem 8, we have used a rather general result Theorem 5, but, in case the condition (*) is satisfied, the method in the proof of Theorem 6 enables us to prove it more simply as follows: We assume that $\mathfrak{G}$ is non-outer and distinguish two cases. (I) $V_R(S)$ *is not a division ring.* Making use of the same notations as in the proof (I) of Theorem 6, we shall prove that D is finite. If D is infinite, the set of regular elements $\{1 + de_{12};\ d \in D\}$ determines an infinite number of S-inner automorphisms of R which send e_{22} into different images, being contrary to the local finiteness of $\mathfrak{G}$. The finiteness of D shows evidently that of $V_R(S)$. (II) $V_R(S)$ *is a division ring.* Making use of the same notations as in the proof (II) of Theorem 6, we shall prove that $V_N(N)$ is finite. Since v is in $V_N(S) \setminus V_N(N)$, in case $V_N(N)$ is infinite, the set $\{1 + cv;\ c \in V_N(N)\}$ determines an infinite

number of S-inner automorphisms of N [of R]. Noting that N is finite over S: $N = \sum_{i=1}^{p} S\, n_i$ ($n_i \in N$), we have that some of n_i's, say n_1, possesses an infinite number of images under these automorphisms, which is a contradiction. Hence $V_N(N)$ is finite, and our assertion is a direct consequence of the fact $[V_R(S) : V_N(N)] = [V_N(S) : V_N(N)] < \infty$.

The next fact will be used frequently also in the sequel.

Lemma 8. *Under the condition* (*), *there holds that* $J(\mathfrak{G}(R'), R) = R'$ *for any intermediate regular subring* R' *finite over* S, *and so the condition* (*) *is satisfied with respect to* R/R'.

Proof. Following to Theorem 6, we distinguish two cases. (I) $\mathfrak{G}$ *is outer.* For an arbitrary element $r \in R \setminus R'$ (if there exists), there exists a simple subring N normal, finite over S and containing $R'(r)$. To be easily seen, there holds that $V_N(R') = V_N(N) = V_N(S)$. Noting that $\mathfrak{G}_N = \mathfrak{G}(N/S)$, Theorem 3 secures the existence of an automorphism $\sigma \in \mathfrak{G}(R')$ such that $r^\sigma \neq r$. (II) $\mathfrak{G}$ *is non-outer.* In this case, for any $r \in R \setminus R'$, there exists a simple subring N normal, finite over S and containing $S(r, V_R(S))$. Clearly we obtain that $\mathfrak{G}_N = \mathfrak{G}(N/S)$ and $V_N(R') = V_R(R')$. Hence, again by Theorem 3, our assertion is clear.

Lemma 9. *Under the conditions* (*) *and* (**), *for any intermediate regular subring* R', *there holds that* $J(\mathfrak{G}(R'), R) = R'$.

Proof. As the condition (*) implies the local simplicity of R over S, by Lemma 1, we can set $R' = \bigcup_{\alpha} R_\alpha$, where R_α is a simple subring finite over S such that $V_R(R_\alpha)$ is simple. In fact, we may, and shall, assume that $V_R(R_\alpha) = V_R(R')$ by the consequence $[V_R(S) : V_R(R)] < \infty$ from the conditions (*) and (**)[1]. Let r be an arbitrary element in $R \setminus R'$, then $\mathfrak{M}_\alpha = \{\sigma \in \mathfrak{G}(R_\alpha) ; r^\sigma \neq r\}$ is a non-empty closed subset of $\mathfrak{G}$ by Lemma 8. If $\bigcap_{\alpha} \mathfrak{M}_\alpha = \phi$ (the empty set) then, by the compactness of $\mathfrak{G}$, there exists a finite number of $\mathfrak{M}_{\alpha_i}$'s such that $\bigcap_{i} \mathfrak{M}_{\alpha_i} = \phi$. Since R' contains a simple subring R^* finite over S and containing R_{α_i}'s, and clearly $V_R(R^*) = V_R(R')$, Lemma 8 gives a contradiction. Hence $J(\mathfrak{G}(R'), R) = R'$.

Combining Theorem 7 and Lemma 9, we obtain the following theorem, which is a generalization of the principal result of [4].

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1) Considering separately the cases where $\mathfrak{G}$ is outer or non-outer, we know that there exists a simple subring R_0 of R' finite over S such that $V_R(R_0) = V_R(R')$. It suffices to consider only such R_α's containing R_0.

Theorem 9. *Let R be a simple ring which is Galois over a simple subring S. If the conditions (*) and (**) are satisfied, then there exists a one-to-one dual correspondence between closed regular subgroups of $\mathfrak{G}(R/S)$ and intermediate regular subrings, in the usual sense of Galois theory.*

The last task of this section is to prove the so-called extension theorem under the conditions (*) and (**). To this end, we set the following:

Lemma 10. *Under the condition (*), each S-isomorphism ρ of a regular subring R' finite over S into R can be extended to an automorphism in $\mathfrak{G}$, where we assume that $V_R(R'^{\rho})$ is simple.*

Proof. If $\mathfrak{G}$ is outer, there exists a simple subring N normal, finite over S and containing $S(R', R'^{\rho})$, for which there hold obviously $V_N(N) = V_N(R') = V_N(R'^{\rho}) = V_N(S)$. Hence Theorem 3 shows the existence of extensions of ρ. While, in case $\mathfrak{G}$ is non-outer, Theorem 6 secures the existence of a simple subring N^* normal, finite over S and containing $S(R', R'^{\rho}, V_R(S))$. Then, we see readily that N^*/S is Galois, $V_R(R') = V_{N^*}(R')$ and $V_R(R'^{\rho}) = V_{N^*}(R'^{\rho})$. Noting the fact $\mathfrak{G}_{N^*} = \mathfrak{G}(N^*/S)$, the rest of the proof is a consequence of Theorem 3.

Theorem 10. *Let R be a simple ring which is Galois over a simple subring S. If the conditions (*) and (**) are satisfied, then any S-isomohphism ρ of R' into R can be extended to an automorphism in $\mathfrak{G} = \mathfrak{G}(R/S)$, where we assume that R' and R'^{ρ} are intermediate regular subrings.*

Proof. Let $R' = \bigcup_{\alpha} R_{\alpha}$, where R_{α} is a simple subring of R' finite over S. Here, without loss of generality, we may assume that $V_R(R_{\alpha}) = V_R(R')$ and $V_R(R_{\alpha}^{\rho}) = V_R(R'^{\rho})$. By Lemma 10, the totality $\mathfrak{N}_{\alpha}$ of extensions of $\rho_{R_{\alpha}}$ to automorphisms in $\mathfrak{G}$ is a non-empty closed subset of $\mathfrak{G}(R/S)$. To prove our theorem, it suffices to show that $\bigcap_{\alpha} \mathfrak{N}_{\alpha} \neq \phi$. If not, the compactness of $\mathfrak{G}(R/S)$ implies that there exists a finite number of $\mathfrak{N}_{\alpha_i}$'s such that $\bigcap_{i} \mathfrak{N}_{\alpha_i} = \phi$. Since there exists a simple subring R^* of R' finite over S and containing R_{α_i}'s, noting that $V_R(R^*) = V_R(R')$, $V_R(R^{*\rho}) = V_R(R'^{\rho})$, Lemma 10 shows that there exists an automorphism σ in $\mathfrak{G}(R/S)$ such that $\sigma_{R^*} = \rho_{R^*}$. This contradiction completes our proof.

3. Some structural consequences

The purpose of this section is to characterize topologically the condition under which our Galois theory will be considered. In case R is a

division ring, Theorem 6 of [2] announced that, under the assumption that $\mathfrak{G}$ is locally finite-dimensional, the condition $[V_R(S): V_R(R)] < \infty$ is equivalent to the local compactness of $\mathfrak{G}$. It is a fact that there holds still Theorem 6 of [2] for simple rings under some natural assumption (Theorem 14). Additionally, the present section contains some structural consequences which may be considered as generalizations of several facts cited in [2].

In §1, we have introduced the notions of s-Galois and s-normal. Clearly these notions are needless in either case where R is a division ring or where R is simple and $[V_R(S): V_R(R)] < \infty$.

We shall introduce here the following condition which coincides evidently with the condition (*) in either of above cases:

($*_1$) *For each finite subset F of R, there exists a simple subring N s-normal, finite over S and containing $S(F)$.*

Lemma 11. *Let R/S be Galois, N be a simple subring s-normal, finite over S and containing $V_R(S)$. Then, for $R' = V_N(N)$, $T = V_R(R')$, there hold the following:*

(1) $V_R(T) = R'$, *and* $T = V_R(V_R(N))$.

(2) $V_T(S) = V_R(S)$, $H = V_T(V_T(S))$ *and* $[V_T(S): V_T(T)] = [T: H] < \infty$, *where* $H = V_R(V_R(S))$.

Proof. (1) From $N \supset V_R(S)$, it follows that $V_N(N) = N \cap V_R(N) = V_R(N)$. Hence we have $V_R(V_R(R')) = V_R(V_R(V_R(N))) = V_R(N)$ and that $V_R(R') = V_R(V_R(N))$ is simple.

(2) As $N \supset V_R(S)$, N/S is Galois and finite over S, which implies that $[V_N(S): R'] < \infty$. Further, $R' = T \cap V_R(T) = V_T(T)$, and clearly $V_T(S) = V_R(S) = V_N(S)$. We have therefore $\infty > [V_N(S): R'] = [V_T(S): V_T(T)]$. And so we have $[T: V_T(V_T(S))] = [V_T(S): V_T(T)] < \infty$. Noting that $V_T(S) = V_R(S)$, we see that $V_T(V_T(S)) = V_T(V_R(S)) \subset H (= V_R(V_R(S))) \subset V_R(V_R(N)) = T$. On the other hand, $H \subset V_T(V_R(H)) = V_T(V_R(S)) = V_T(V_T(S))$. Hence $V_T(V_T(S)) = H$, which proves (2) and that H is simple.

Corollary. *Under the condition ($*_1$), if $\mathfrak{G}$ is non-outer then, for each finite subset F of R, there exists a simple subring T with the following properties:*

(1) *T contains $H(F)$ and is s-normal, Galois over S, where $H = V_R(V_R(S))$.*

(2) $[T: H] = [V_T(S): V_T(T)] < \infty$.

Proof. By Theorem 6, $[V_R(S) : V_S(S)] < \infty$. Accordingly, there exists a simple subring N s-normal, finite over S and containing $S(F, V_R(S))$ by the condition $(*_1)$. Then $T = V_R(V_N(N))$ is a desired one by Lemma 11.

The following theorem corresponds to [2, Theorem 4].

Theorem 11. *Let R/S be Galois. If the condition $(*_1)$ is satisfied then R/S is s-Galois and $(*_1)$ is done with respect to R/H, where $H = V_R(V_R(S))$.*

Proof. In case $\mathfrak{G}$ is non-outer, our assertion is clear from the above corollary. While, if $\mathfrak{G}$ is outer, there is nothing to prove.

By making use of this fact, we can prove the next which corresponds to [2, Theorem 5].

Theorem 12. *If R/S is Galois and the condition $(*_1)$ is satisfied then $\overline{\mathfrak{F}} = \mathfrak{G}(R/H)$, where $\mathfrak{F}$ is the totality of inner automorphisms in $\mathfrak{G}$ and $H = V_R(V_R(S))$.*

Proof. Let σ be in $\mathfrak{G}(R/H)$ and N be a simple subring normal, finite over S. By Theorem 11, there exists a simple subring N' normal, Galois and finite over H. Since $V_R(V_R(H)) = H$, there exists a regular element x in $V_R(H) = V_R(S)$ such that $\tilde{x}_{N'} = \sigma_{N'}$, and of course, that $\tilde{x}_N = \sigma_N$. Hence we have proved our assertion.

For the special case where $[S : V^S(S)]$ is finite, we obtain the following theorem corresponding to [2, Corollary to Theorem 7].

Theorem 13. *Let R/S be Galois and $\mathfrak{G}(R/S)$ be non-outer. If the condition $(*)$ is satisfied and $[S : V_S(S)]$ is finite, then $H = V_R(V_R(S))$ is finite over S.*

Proof. There exists a simple subring M finite over S and containing $S(V_R(S))$ such that $V_R(M)$ is a division ring. (Cf. Corollary to Theorem 6.) For any r in $R \setminus M$ (if there exists), there is a simple subring N normal, finite over S and containing $S(r, M)$. Then there holds that $V_N(N) \subset V_R(S) \subset M$. Since N is finite over S and $[S : V_S(S)] \subset \infty$, it follows that $[N : V_N(N)] < \infty$ by Lemma of [6][1]. Clearly $V_N(M)$ is a division ring, and so N is Galois and finite over M by Lemma 8. Combining this fact with $[N : V_N(N)] < \infty$ and $V_N(N) \subset M \subsetneq N$, we know (that $\mathfrak{G}(N/M)$ is inner, and so) that there exists an element $v \in V_N(M)$ $(\subset V_R(S))$ such that $rv \neq vr$, which shows that r is not in H. We have

1) In the proof of Lemma in [6], $[R' : Z'] = g^2$ is an error and it should be read as $[R' : Z'] = ng$, where n is the capacity of R'.

proved therefore that $H \subset M$.

Remark 2. Combining Theorem 12 with Theorem 13, we shall readily see that, in case R/S is Galois and the condition $(*_1)$ is satisfied, if $[S: V_S(S)]$ is finite, then either $\mathfrak{G}$ is outer or $[\mathfrak{G}: \overline{\mathfrak{F}}]$ is finite. Hence, we may say roughly that, in the case, either $\mathfrak{G}$ is outer or essentially inner. In particular, if $\mathfrak{G}$ is locally finite, then either $\mathfrak{G}$ is outer or R is finite.

Now we are going to prove the principal result of this section.

Theorem 14. *Let R/S be Galois. Under the condition $(*_1)$, the following conditions are equivalent to each other:*

(1) *$\mathfrak{G}$ is locally compact.*

(2) $[V_R(S): V_R(R)] < \infty$.

Proof. (2) $\rightarrow$ (1). (2) implies that $[V_R(S): V_R(R)] = [R: H]$, where $H = V_R(V_R(S))$, hence there exists a finite linearly independent H-basis $\{r_1, \ldots, r_n\}$ of R. By the condition $(*_1)$, there exists a simple subring N normal, finite over S and containing $S(r_1, \ldots, r_n)$. Then, to be easily verified, $V_R(N) = V_R(R)$. Noting that the condition $(*_1)$ is satisfied with respect to R/N and that the topology of $\mathfrak{G}(R/N)$ $(= \mathfrak{G}(N))$ is equivalent to the relative topology as a snbspace of $\mathfrak{G}$, by Theorem 8, we see that $\mathfrak{G}(N)$ is a required neighbourhood of the identity.

(1) $\rightarrow$ (2). As $\mathfrak{G}$ is locally compact, there exists a simple subring N normal, Galois and finite over S such that $\mathfrak{G}(N)$ is compact. Here, withont loss of generality, we may assume that $\mathfrak{G}(N) = \mathfrak{G}(R/N)$ is outer. (See the first part of this proof.) By Theorem 11, there exists a simple subring R_1 normal, finite over H and containing $H(N)$ such that $V_R(R_1)$ is a division ring. If $R \supsetneq R_1$, again by Theorem 11, there exists a simple subring R_2 normal, Galois and finite over H such that $R_2 \supsetneq R_1$. Then, evidently $V_{R_2}(R_1)$ is a division ring. Since $V_R(V_R(H)) = H$, $\mathfrak{G}(R_2/H)$ is given by the restriction on R_2 of the totality of inner automorphisms determined by regular elements in $V_R(H)$. Hence, by Theorem 2, $\mathfrak{G}(R_1)$ contains an inner automorphism different from the identity mapping, being contrary to our assumption that $\mathfrak{G}(N)$ is outer. Hence R has to coincide with R_1, whence R is Galois and finite over H. We obtain therefore $[V_R(S): V_R(R)] < \infty$.

3. Galois theory (II. General case)

It is the purpose of this section to present a generalization of our

Galois theory for division rings constructed in [2]. Throughout the study of this section, we assume the condition (*) and the next:

$$(*_f) \qquad [V_R(S) : V_R(R)] < \infty.$$

Under the condition $(*_f)$, there exists a finite independent H-basis $\{r_1, \ldots, r_n\}$ of R, which will be fixed in the sequel. And H will mean $V_R(V_R(S))$ throughout this section. Our principal theorems are Theorem 15 and Theorem 16, the former announces the existence of Galois correspondence, and the latter is concerned with extensions of isomorphisms.

To prove Theorem 15, we shall require a chain of lemmas, the first of which is the next:

Lemma 12. *Under the conditions* (*) *and* $(*_f)$, *the condition* (*) *is satisfied with respect to* H/S. *And* $\mathfrak{G}(H/S)$ *is outer.*

Proof. We set here $H = \sum_{i,j=1}^{l} D e_{ij}$, where D is a division ring and e_{ij}'s are matric units. By (*), for any finite subset F of H, there exists a simple subring N_0 normal, finite over S and containing $S(\{e_{ij}\}, \{r_k\}, F)$. As is familiar, $N_0 = \sum_{i,j=1}^{l} S_0 e_{ij}$, where $S_0 = V_{N_0}(\{e_{ij}\})$ is simple. Clearly, $H_0 = N_0 \cap H = \sum_{i,j=1}^{l} (D \cap S_0) e_{ij}$ is a simple subring of H normal, finite over S and containing $S(F)$. Further, H_0 is normal with respect to $\mathfrak{G}(H/S)$. For, if not, there exists an automorphism $\sigma \in \mathfrak{G}(H/S)$ such that $H_0^\sigma \not\subset H_0$. Then, there exists a simple subring H^* of H normal, finite over S and containing H_0 as well as H_0^σ. Hence, by Theorem 3, there is an automorphism $\tau \in \mathfrak{G}(H^*/S) = \mathfrak{G}_{H^*}$ such that $\sigma_{H_0} = \tau_{H_0}$. But this is a contradiction.

The next lemma will play such an essential rôle in our present theory as [2, Lemma 9] did.

Lemma 13. *Under the conditions* (*) *and* $(*_f)$, *there holds that* $\mathfrak{G}(H/S) = \mathfrak{G}_H$.

Proof. Let $H = \sum_{i,j=1}^{l} D e_{ij}$ as in Lemma 12, and let N_1 be an arbitrary simple subring normal, finite over S and containing $S(\{e_{ij}\}, \{r_k\})$. To be easily verified, $V_R(N_1) = V_R(R)$, and hence $\mathfrak{G}^{(1)} = \mathfrak{G}(R/N_1)$ is compact by Lemmas 8, 12 and Theorem 8. And, as is readily seen from the proof of Lemma 12, $H_1 = H \cap N_1 = J(\mathfrak{G}_H^{(1)}, H)$ is a simple subring finite over S and normal with respect to $\mathfrak{G}(H/S)$. Further, we obtain that $\mathfrak{H}^{(1)} = \mathfrak{G}(H/H_1)$ is compact by Lemma 12. For any simple subring

H_2 of H finite over S and normal with respect to $\mathfrak{H}^{(1)}$, there exists a simple subring N_2 normal, finite over S and containing $N_1(H_2)$. Then evidently there holds that $\mathfrak{G}^{(1)}(N_2)_H \subset \mathfrak{H}^{(1)}(H_2)$, which shows that the mapping $\varphi : \sigma \to \sigma_H$ of $\mathfrak{G}^{(1)}$ into $\mathfrak{H}^{(1)}$ is continuous. Recalling that $\mathfrak{G}^{(1)}$ and $\mathfrak{H}^{(1)}$ are compact Hausdorff spaces, we obtain that $\mathfrak{G}_H^{(1)}$ is closed in $\mathfrak{H}^{(1)}$. On the other hand, $H^*_2 = H \cap N_2$ is a simple subring finite over S and containing H_2 which is normal with respect to $\mathfrak{H}^{(1)}$, and $\mathfrak{G}^{(1)}{}_{H^*_2} = \mathfrak{G}(H^*_2/H_2)$. Thus, for any $\rho \in \mathfrak{H}^{(1)}$, there exists an automorphism $\tau \in \mathfrak{G}^{(1)}$ such that $\rho_{H^*_2} = \tau_{H^*_2}$, which shows that $\mathfrak{G}_H^{(1)}$ is dense in $\mathfrak{H}^{(1)}$. Combining this fact with that $\mathfrak{G}_H^{(1)}$ is closed in $\mathfrak{H}^{(1)}$, we arrive to the equality $\mathfrak{H}^{(1)} = \mathfrak{G}_H^{(1)}$. Clearly, the outer group $\mathfrak{G}(H_1/S) = \mathfrak{G}_{H_1}$ is finite by Lemma 5. Now we shall assume that $\mathfrak{G}(H_1/S)$ is induced by $\{\sigma^{(1)}, \cdots, \sigma^{(t)}\} \subset \mathfrak{G}$. Then, for any $\sigma \in \mathfrak{G}(H/S)$, there holds that $\sigma_{H_1}^{(i)} = \sigma_{H_1}$ with some i, and so $(\sigma^{(i)})_H^{-1}\sigma = \tau \in \mathfrak{H}^{(1)} = \mathfrak{G}_H^{(1)}$. Hence there exists an automorphism $\tau' \in \mathfrak{G}^{(1)}$ such that $\tau = \tau'_H$, and evidently $\sigma = (\sigma^{(i)}\tau')_H$. We have proved therefore that $\mathfrak{G}(H/S) = \mathfrak{G}_H$.

The next is only an easy consequence of the above lemma, but the proof of Lemma 14 can be reduced to it.

Corollary. *Under the conditions* (*) *and* ($*_f$), *there holds that* $J(\mathfrak{G}(H'), R) = H'$ *for any simple subring* H' *with* $H \supset H' \supset S$.

Proof. As, by Lemma 12, the condition (*) is satisfied with respect to H/S, Theorem 9 proves that $H' = J(\mathfrak{G}_H(H'), H) = H \cap J(\mathfrak{G}(H'), R)$. On the other hand, $H = V_R(V_R(H))$ shows that $J(\mathfrak{G}(H'), R) \subset H$, and so we have $H' = J(\mathfrak{G}(H'), R)$.

Lemma 14. *Under the conditions* (*) *and* ($*_f$), *there holds that* $J(\mathfrak{G}(R'), R) = R'$ *for any intermediate regular subring* R'.

Proof. Since $[V_R(S) : V_R(R)] < \infty$ and R' is locally simple over S by Lemma 1, the same argument as in the last part of the proof of [2, Lemma 9] enables us to find a simple subring R'' of R' finite over S such that $V_R(R') = V_R(R'')$, which shows that $R'' \subset R' \subset V_R(V_R(R''))$. By Lemma 8, the condition (*) is satisfied with respect to R/R''. We can apply therefore Corollary to Lemma 13 for R'', R', $V_R(V_R(R''))$ instead of S, H', H respectively, and obtain our conclusion.

Combining Lemma 14 with Theorem 7, we obtain our first principal result:

Theorem 15. *Let* R *be a simple ring which is Galois over a simple subring* S. *Under the conditions* (*) *and* ($*_f$), *there exists a one-to-one dual correspondence between closed regular subgroups of*

the total group and intermediate regular subrings, in the usual sense of Galois theory.

In front of our last theorem, we shall set the following:

Lemma 15. *Under the conditions* (*) *and* ($*_f$), *if* H' *is a simple subring with* $S \subset H' \subset H$, *for any* S*-isomorphism* ρ *of* H' *into* R, *there holds that* $H'^{\rho} \subset H$, *where we assume that* $V_R(H'^{\rho})$ *is simple.*

Proof. We have necessarily $V_R(S) = V_R(H')$. If $H'^{\rho} \not\subset H$, by the condition ($*_f$) and the local simplicity of H' and H'^{ρ} over S, H' contains a simple subring F finite over S such that $F^{\rho} \not\subset H$ and $V_R(F^{\rho})$ is simple. By Lemma 10, $\rho_F = \tau_F$ with some $\tau \in \mathfrak{G}$, but this is contrary to that H is normal over S.

Now we are going to prove our last theorem.

Theorem 16. *Let* R *be a simple ring which is Galois over a simple subring* S. *Under the conditions* (*) *and* ($*_f$), *for any regular subring* R', *each* S*-isomorphism* ρ *of* R' *into* R *can be extended to an automorphism in* $\mathfrak{G} = \mathfrak{G}(R/S)$, *where we assume that* $V_R(R'^{\rho})$ *is simple.*

Proof. By the condition ($*_f$) and the fact that R' and R'^{ρ} are locally simple over S, R' contains a simple subring R'' finite over S such that $V_R(R') = V_R(R'')$ and $V_R(R'^{\rho}) = V_R(R''^{\rho})$. Since the condition (*) is satisfied with respect to R/R'' by Lemma 8, in virtue of Lemma 10, we see that $\rho_{R''} = \sigma_{R''}$ with some $\sigma \in \mathfrak{G}$. Clearly $\rho\sigma^{-1}$ is an R''-isomorphism of R' into R and $V_R(R'^{\rho\sigma^{-1}}) = (V_R(R'^{\rho}))^{\sigma^{-1}}$ is simple. And as $R'' \subset R' \subset H'' = V_R(V_R(R''))$, by Lemma 15, there holds that $R'^{\rho\sigma^{-1}} \subset H''$.[1] Hence, applying Theorem 10 for R'', H'' instead of S, R respectively, we obtain that $\rho\sigma^{-1} = \tau_{R'}$ with some $\tau \in \mathfrak{G}(H''/R'') = \mathfrak{G}(R/R'')_{H''}$, accordingly $\rho\sigma^{-1} = \tau'_{R'}$ with some $\tau' \in \mathfrak{G}(R/R'')$. Clearly $\tau'\sigma$ is a required extension of ρ.

Remark 3. Under the conditions (*) and ($*_f$), we can generalize Theorem 13 of [2] as follows: For an arbitrary intermediate regular subring T such that $V_T(S)$ is simple, we set $\mathfrak{T} = \{\sigma \in \mathfrak{G};\ T^{\sigma} = T\}$ and denote by $\widetilde{\mathfrak{T}}$ the composite of $\mathfrak{T}$ and the totality of $J(\mathfrak{T}, R)$-inner automorphisms of R. Under these notatiodns, T/S *is Galois if and only if* $\widetilde{\mathfrak{T}}$ *is dense in* $\mathfrak{G}$. The proof is the same with that of [2, Theorem 13] by the validity of Theorems 7, 15 and 16, but this fact will be not so of importance in the present stage.

1) Recall here that, by Lemmas 8, 12, the condition (*) is satisfied with respect to H''/R''.

REFERENCES

[1] T. NAGAHARA and H. TOMINAGA: A note on Galois theory of division rings of infinite degree, Proc. Japan Acad., 31 (1955), 655—658.
[2] T. NAGAHARA and H. TOMINAGA: On Galois theory of division rings, Math. J. Okayama Univ., 6 (1956), 1—21.
[3] T. NAKAYAMA: Galois theory of simple rings, Trans. Amer. Math. Soc., 73 (1952), 276—292.
[4] N. NOBUSAWA: An extension of Krull's Galois theory to division rings, Osaka Math. J., 7 (1955), 1—6.
[5] N. NOBUSAWA: On compact Galois groups of division rings, Osaka Math. J., 8 (1956), 43—50.
[6] H. TOMINAGA: On a theorem of N. Jacobson, Poc. Japan Acad., 31 (1955), 653—654.

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Added in proof.

Recently the present author has found the following:

(1) If a simple ring R is Galois over a simple subring S and (*) is satisfied then $(*_1)$ is satisfied.

(2) Local simplicity and (*) are equivalent to local finiteness and local finite-dimensionality similarly defined as in [2] respectively.

From these facts we can restate Theorem 14 as follows:

Theorem 14*. *Let R be a simple ring which is Galois over a simple subring S and $\mathfrak{G}(R/S)$ be l.f.d. Then $\mathfrak{G}(R/S)$ is locally compact if and only if $V_R(S)$ is finite over the center of R.*

(3) Let a simple ring R be Galois over a simple subring S. If $\mathfrak{G}(R/S)$ is l. f. d. and locally compact then so is $\mathfrak{G}(R/T)$ for any intermediate regular subring T.

(4) More generally, Lemma 14, Theorem 16 and (3) are still true under the conditions similar to those considered in [2, §3].

Furthermore, by a remark from Mr. F. Kasch (F. Kasch: Eine Bemerkung über innere Automorphismen, to appear in this journal.), Theorem 5 is true without our complicated assumption:

Theorem 5*. *Let a simple ring R be Galois over a simple subring S. If $\mathfrak{G}(R/S)$ is locally finite then it is almost outer.*