HARMONIC ANALYSIS ON LOCALLY COMPACT GROUPS.

MINORU TOMITA

Introduction.

Let $\mathfrak{G}$ be a separable locally compact group. A continuous function p on $\mathfrak{G}$ such that $\sum_{ij} \bar{\xi}_i \xi_j p(a_i^{-1}a_j) \geqq 0$ for arbitrary $a_1, \cdots\cdots a_n$ in $\mathfrak{G}$ and for arbitrary complex numbers $\xi_1, \cdots\cdots \xi_n$ is called *positive definite*. p is called *normalized* if it takes value 1 at the unit e of $\mathfrak{G}$. p is called *elementary* if we can not devide p to a sum of two positive definte functions q and r except for the case that $q = \alpha p$ and $r = (1-\alpha)p$.

The set $\mathfrak{E}$ of all normalized continuous elementary positive definite functions on $\mathfrak{G}$ is called the *dual space* of $\mathfrak{G}$. In $\mathfrak{E}$ we introduce the *Pontrjagin's topology* as follows. A complete system of neighbourhoods of a $u \in \mathfrak{E}$ is determined by all those sets $\mathfrak{U}(u\,;\ W, \varepsilon) = (v \in \mathfrak{E} : |u(a) - v(a)| < \varepsilon$ for every a in W), where W are compact sub-sets of $\mathfrak{G}$, and ε are positive numbers.

If π is a bounded complex regular measure[1] on $\mathfrak{E}$, a function $\hat{\pi}$ on $\mathfrak{E}$ defined by

$$\hat{\pi}(a) = \int_{\mathfrak{E}} \lambda(a)\, d\, \pi(\lambda)$$

is called the *Fourier transform* of π.

$\hat{\pi}$ is a bounded continuous function on $\mathfrak{G}$, and, if π is non-negative, π is positive definite (Theorem 7). Every continuous positive definite function on $\mathfrak{G}$ is a Fourier transform of a suitable non-negative regular Borel measure on $\mathfrak{E}$ (Theorem 3). If π is a regular measure on $\mathfrak{E}$ and if X is a Borel set in $\mathfrak{E}$, we denote by π_X the relative measure on X: $\pi_X(A) = \pi(X \cap A)$. It is convenient to define a diagonal measure on $\mathfrak{E}$ in connection with a certain property of the Fourier transform $\pi_X \to \hat{\pi}_X$.

A pair of positive definite functions p and q are called *mutually orthogonal* if there is no non-zero positive definite function r such that $p - r$ and $q - r$ are positive definite. A regular non-negative measure π on $\mathfrak{E}$ is called *diagonal* if for every Borel set X in $\mathfrak{E}$, $\hat{\pi}_X$ and $\hat{\pi}_{\mathfrak{E}-X}$

1) In every completely regular topological space we define a *regular measure* as in Definition 2. 1.

are mutually orthogonal.

A sort of reciprocal transform of the above Fourier transform $\pi \to \hat{\pi}$ is considered. Let $L(\mathfrak{G})$ denote the Banach space of all functions on $\mathfrak{G}$ summable by a fixed left invariant Haar measure on $\mathfrak{G}$. The *Fourier transform* of a function f in $L(\mathfrak{G})$ is a function $\hat{f}$ on $\mathfrak{E}$ such that

$$\hat{f}(\lambda) = \int \overline{\lambda(a)} f(a)\, da.$$

$\hat{f}$ is a bounded continuous function on $\mathfrak{E}$ (Theorem 6). Let Δ denote the continuous function on $\mathfrak{G}$ such that $\Delta(a)^{-1} \int f(ba)db = \int f(b)\, db$ for every $f \in L(\mathfrak{G})$. If f, g are two functions in $L(\mathfrak{G})$, we put $f^*(a) = \Delta(a)^{-1} f(a)$ and $(f \circ g)(a) = \int f(b) g(b^{-1}a) db$. Then the Fourier transform $f \to \hat{f}$ satisfies

$$\hat{f^*}(\lambda) = \overline{\hat{f}(\lambda)} \text{ and } (\widehat{f^* \circ f})(\lambda) \geqq 0.$$

The following Plancherel's theorem is asserted.

Theorem 1. *In order that a bounded non-negative regular measure π on $\mathfrak{E}$ be diagonal, it is necessary and sufficient that, for every function φ on $\mathfrak{E}$ measurable and square summable by π, there exist a sequence $\{f_n\}$ in $L(\mathfrak{G})$ such that*

$$\lim_{n\to\infty} \int |\hat{f}_n(\lambda) - \varphi(\lambda)|^2\, d\pi(\lambda) = 0$$

and simultaneously

$$\lim_{n\to\infty} \int (\widehat{f_n^* \circ f_n})(\lambda)\, d\pi(\lambda) = \int |\varphi(\lambda)|^2\, d\pi(\lambda).$$

We call this sequence $\{f_n\}$ in $L(\mathfrak{G})$ an *approximative image of the Fourier transform* of the function φ. The following Parseval's theorem is asserted.

Theorem 2. *Let π be a diagonal measure on $\mathfrak{E}$. If $\{f_n\}$ and $\{g_n\}$ are approximative images of two square summable measurable functions φ and ψ on $\mathfrak{E}$ respectively, then*

$$\lim_{n\to\infty} \int |(\widehat{g_n^* \circ f_n})(\lambda) - \varphi(\lambda)\overline{\psi(\lambda)}|\, d\pi(\lambda) = 0.$$

Other important properties of diagonal measures are related to the theory of the unitary representations for groups.

Given a continuous positive definite function p on $\mathfrak{G}$, the corresponding cyclic unitary representation is determined as follows[1]: There exists a Hilbert stace $L^2(p)$, a strongly continuous homomorphism $a \rightarrow P_a$ of $\mathfrak{G}$ on a suitable unitary groups $\mathfrak{G}(p)$ of operators on $L^2(p)$ and an element $\mathfrak{p}$ in $L^2(p)$ with $p(a) = (P_a\mathfrak{p}, \mathfrak{p})$, for which the closed linear span of the set $(P_a\mathfrak{p} : a \in \mathfrak{G})$ coincides with $L^2(p)$.

The *commutor* $\mathfrak{G}(p)'$ of $\mathfrak{G}(p)$ is the set of all bounded linear operators in $L^2(p)$ which commute to all operators in $\mathfrak{G}(p)$. If q is another continuous function on $\mathfrak{G}$ such that, for a suitable constant γ, $\gamma p - q$ and $\gamma p + q$ are positive definite, there exists a uniquely determined operator K in $\mathfrak{G}(p)'$ such that $q(a) = (P_a K\mathfrak{p}, \mathfrak{p})$.

Let π be a bounded non-negative regular measure on $\mathfrak{X}$, let p denote its Fourier transform $p = \hat{\pi}$ and let $M(\pi)$ denote the Banach space of all bounded functions on $\mathfrak{X}$ summable by π. For every $\varphi \in M(\pi)$ there exists a uniquely determined operator F_φ in $\mathfrak{G}(p)'$ such that $\hat{\pi}_\varphi(a) = (P_a F_\varphi p, p) = \int \lambda(a)\varphi(\lambda)\, d\pi(\lambda)$, where π_φ is the relative measure: $\pi_\varphi(A) = \int_A \varphi(\lambda) d\pi(\lambda)$. This mapping $\varphi \rightarrow F_\varphi$ is called the *Fourier transform* relative to the measure π, and we denote by $\mathfrak{F}(\pi)$ the range $(F_\varphi : \varphi \in M(\pi))$ of this relative Fourier transform.

Theorem 3. *If π is a diagonal measure on $\mathfrak{X}$, then*

(1). *The range $\mathfrak{F}(\pi)$ of the relative Fourier transform is a maximal abelian sub-algebra of $\mathfrak{G}(p)'$.*

(2). *The relative Fourier transform $\varphi \rightarrow F_\varphi$ is an algebraic isomorphism between $M(\pi)$ and $\mathfrak{F}(\pi)$. It is one-to-one, linear, and satisfies the Parseval's equality $F_{\varphi\psi} = F_\varphi F_\psi$ for every $\varphi, \psi \in M(\pi)$.*

Theorem 4. *Given a positive definite function p on $\mathfrak{G}$ and a maximal abelian self-adjoint sub-algebra $\mathfrak{R}$ of $\mathfrak{G}(p)'$, there exists a uniquely determined diagonal measure π on $\mathfrak{X}$ whose Fourier transform $\hat{\pi}$ coincides with p, and for which the range $\mathfrak{F}(\pi)$ of the relative Fourier transform coincides with $\mathfrak{R}$.*

Therefore, $\pi \leftrightarrow (\hat{\pi}, \mathfrak{F}(\pi))$ is a one-to-one correspondence between the system of all diagonal measures on $\mathfrak{X}$ and the system of all couples $(p, \mathfrak{R})$

---

1) Gelfand [2]

of positive definite functions p on $\mathfrak{G}$ and maximal abelian self-adjoint sub-algebras $\mathfrak{K}$ of $\mathfrak{G}(p)'$.

In Chapter 8 we shall establish the general Plancherel's theorem on separable unimodular groups.

Chapter 1. The Pontrjagin's topology and the Raikov's Theorem.

$L(\mathfrak{G})$ is the Banach space of all functions on $\mathfrak{G}$ summable by the fixed left invariant measure. The norm $|f|_1$ of an element f in $L(\mathfrak{G})$ is $|f|_1 = \int |f(a)|\, da$. The dual Banach space of $L(\mathfrak{G})$ is the space $M(\mathfrak{G})$ of all bounded functions on $\mathfrak{G}$ summable in each compact sub-set of $\mathfrak{G}$. The norm $|\varphi|_\infty$ of φ in $M(\mathfrak{G})$ is $|\varphi|_\infty = \text{ess. max. } |\varphi(\lambda)|$. The bilinear form (f, φ) between $L(\mathfrak{G})$ and $M(\mathfrak{G})$ is

$$(f, \varphi) = \int f(a)\varphi(a)da.$$

A function p in $M(\mathfrak{G})$ such that $\iint p(a^{-1}b)\overline{f(a)}f(b)\, dadb \geqq 0$ for every $f \in L(\mathfrak{G})$ is called *positive definite*. If p is continuous, this definition is equivalent to the definition given in the introduction. By the Gelfand's theorem[1] every positive definite function in $M(\mathfrak{G})$ coincides to a continuous positive definite function almost everywhere.

If p is a positive definite function, the representative operator P_f on $L^2(p)$ for an element f in $L(\mathfrak{G})$ is defined by a strong integral $P_f = \int f(a)\, P_a\, da$. Then

(1.1). *The norm* $|P_f|$ *of the operator* P_f *does not exceed* $|f|_1$.

(1.2). $(P_f\mathfrak{p}, \mathfrak{p}) = \int f(a)p(a)\, da = (f, p)$.

(1.3). $P_fP_a = P_{f_a}$ *for every* $a \in \mathfrak{G}$, *where* f_a *denotes the translation* $f_a(b) = f(a^{-1}b)$ *of the function* f.

The space $\mathfrak{N}$ of all normalized continuous positive definite functions on $\mathfrak{G}$ is contained in the dual space $M(\mathfrak{G})$ of $L(\mathfrak{G})$, the weak topology in which is defined in the well-known way. On the other hand we introduce in $\mathfrak{N}$ the Pontrjagin's topology as follows. A complete system of Pontrjagin's neighbourhoods of a $p \in \mathfrak{N}$ is the totality of those sets $\mathfrak{U}(p: W, \varepsilon) = (q \in \mathfrak{N} : |p(a) - q(a)| < \varepsilon$ for every $a \in W)$, where W are com-

1) Gelfand [2]

pact sub-sets of $\mathfrak{G}$, and ε are positive numbers. D. A. Raikov[1] proved the following important theorem.

Theorem 5. *In the space $\mathfrak{N}$, the weak topology and the Pontrjagin's topology coincide with each other. The function $\varphi(p, a) \equiv p(a)$ (defined for $p \in \mathfrak{N}$, $a \in \mathfrak{G}$) is two-sided continuous in the product space $\mathfrak{N} \times \mathfrak{G}$.*

Proof[2]. Let p be an element in $\mathfrak{N}$. For each $f \in L(\mathfrak{G})$ we define a function $p^f(a)$ on $\mathfrak{G}$ by $p^f(a) = (P_a P_f \mathfrak{p}, P_f \mathfrak{p})$. p_f is a continuous positive definite function such that

(1.4). *For every fixed element f in $L(\mathfrak{G})$ and for every positive number ε, there exists a neighbourhood* U *of the unit e of $\mathfrak{G}$ with*

$$|p^f(a) - p^f(e)| < \varepsilon \qquad \textit{for every } p \in \mathfrak{N}.$$

In fact, if $p \in \mathfrak{N}$,

$$\begin{aligned} | p^f(a) - p^f(e) | &= |((P_{f_a} - p_f)\mathfrak{p}, P_f\mathfrak{p})| \\ &\leqq |f_a - f|_1 |f|_1 \, \| \mathfrak{p} \|^2. \end{aligned}$$

Notice that $\| \mathfrak{p} \|^2 = (\mathfrak{p}, \mathfrak{p}) = p(e) = 1$, and that the function $g(a) = |f_a - f|_1 = \int |f(a^{-1}b) - f(b)| \, db$ is continuous and vanishes at e. Then we can choose such a small neighbourhood U of e.

(1.5). *Let p be an element in $\mathfrak{N}$, then for every positive number ε we can choose an $f \in L(\mathfrak{G})$ and a weak neighbourhood $\mathfrak{V}(p)$ of p such that $|q^f(a) - q(a)| < \varepsilon$ for every $a \in \mathfrak{G}$ and $q \in \mathfrak{V}(p)$.*

Proof. We can choose a neighbourhood V of the unit e of $\mathfrak{G}$ so small that $|p(a) - p(e)| < \varepsilon^2/16$ for all $a \in$ V. Let f be a positive function on $\mathfrak{G}$ which vanishes out-side of V, and which satisfies $|f|_1 = \int f(a) \, da = 1$. And consider a weak neighbourhood of $p : \mathfrak{V}(p) = (q \in \mathfrak{N} : |(f, p - q)| < \varepsilon^2/16)$, then f and $\mathfrak{V}(p)$ satisfy the condition in (1.5). In fact,

$$|(f, p) - 1| \leqq \int f(a) |p(a) - p(e)| \, da \leqq \varepsilon^2/16.$$

If $q \in \mathfrak{V}(p)$,

1) D. A. Raikov [9].

2) This proof dues to Yoshizawa [10]

$$|(f,q)-1|\leqq|(f,p-q)|+|(f,p)-1|\leqq\varepsilon^2/8.$$

Notice that $\|\mathfrak{q}\|\leqq 1$, $\|Q_f\mathfrak{q}\|\leqq|f|_1\|\mathfrak{q}\|\leqq 1$ and $(f,q)=\Re e(Q_f\mathfrak{q},\mathfrak{q})$, then

$$\begin{aligned}\|Q_f\mathfrak{q}-\mathfrak{q}\|^2&=\|Q_f\mathfrak{q}\|^2+\|\mathfrak{q}\|^2-2\Re e(Q_f\mathfrak{q},\mathfrak{q})\\&\leqq 2(1-(f,q))\leqq\varepsilon^2/4,\end{aligned}$$

and

$$\begin{aligned}|q^f(a)-q(a)|&\leqq|(Q_aQ_f\mathfrak{q},Q_a\mathfrak{q},\mathfrak{q})-(Q_a\mathfrak{q},\mathfrak{q})|\\&\leqq|(Q_a(Q_f-I)\mathfrak{q},Q_f\mathfrak{q})|+|(Q_a\mathfrak{q},(Q_f-I)\mathfrak{q})|\\&\leqq 2\|Q_f\mathfrak{q}-\mathfrak{q}\|<\varepsilon.\end{aligned}$$

This concludes (1.5).

(1.6). *Let p be an element in $\mathfrak{N}$ and ε be a positive number, then we can choose a neighbourhood* U *of e and a weak neighbourhood $\mathfrak{V}(p)$ of p such that $|q(a)-q(b)|<\varepsilon$ for every $b^{-1}a\in$* U *and every $q\in\mathfrak{V}(p)$.*

In fact, by (1.4) and (1.5) we can choose a neighbourhood U of e and a weak neighbourhood $\mathfrak{V}(p)$ of p such that

$$|q(a)-q(e)|<\varepsilon^2/2 \text{ for every } a\in \mathrm{U} \text{ and every } q\in\mathfrak{V}(p),$$

then by the M. Krein's inequality,

$$|q(a)-q(b)|\leqq(2q(e)(q(e)-\Re e q(b^{-1}a)))^{\frac{1}{2}}<\varepsilon.$$

(1.7). *Every Pontrjagin's neighbourhood $\mathfrak{U}(p: W,\varepsilon)$ of a $p\in\mathfrak{N}$ contains a weak neighbourhood $\mathfrak{W}(p)$ of p.*

Proof. Choose a neighbourhood U of e and a weak neighbourhood $\mathfrak{V}(p)$ of p such that $|q(a)-q(b)|<\varepsilon/8$ for every $b^{-1}a\in$ U and every $q\in\mathfrak{V}(p)$. We now cover the compact space W by finite number of a_1U, ... a_nU, and choose a positive function g on $\mathfrak{G}$ with $|g|_1=\int g(a)\,da=1$, which vanishes out-side of U. Put $\mathfrak{W}(p)=\mathfrak{V}(p)\cap\mathfrak{U}(p)$ and $\mathfrak{U}(p)=(q\in\mathfrak{N}: |(g_{a_i},p-q)|<\varepsilon/4$ for $1\leqq i\leqq$ n). Then $\mathfrak{W}(p)$ is a desired weak neighbourhood of p in the space $\mathfrak{N}$. In fact, if b is a point in W, then b is contained in one of a_iU. Say $b\in a_i$U, then

$$\begin{aligned}|p(a_i) - q(a_i)| &< \int g_{a_i}(x)\,|(p(x) - p(a_i))|\,dx \\ &+ \int g_{a_i}(x)\,|(q(x) - q(a_i))|\,dx \\ &+ \int g_{a_i}(x)\,|(p(x) - q(x))|\,dx \\ &= \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = 3\varepsilon/4,\end{aligned}$$

and

$$\begin{aligned}|p(b) - q(b)| \leqq |p(b) - p(a_i)| + |p(a_i) - q(a_i)| + \\ |q(a_i) - q(b)| < \varepsilon/8 + 3\varepsilon/4 + \varepsilon/8 = \varepsilon.\end{aligned}$$

Thus $\mathfrak{W}(p)$ is contained in $\mathfrak{U}(p: \text{W}, \varepsilon)$.

(1. 8). *Every weak neighbourhood* $\mathfrak{U}(p, f, \varepsilon) = (q \in \mathfrak{R} : |(f, p - q)| < \varepsilon)$ *(where* $f \in L(\mathfrak{G})$ *and* $\varepsilon > 0$) *of a* $p \in \mathfrak{R}$ *contains a suitable Pontrjagin's neighbourhood of p.*

Proof. Choose a bounded measurable function g on $\mathfrak{G}$ with $|g - f|_1 < \varepsilon/3$, which vanishes out-side of a suitable compact set W, and let γ be a constant such that $\int |g(a)|\,da < \gamma$. Then $\mathfrak{U}(p : \text{W}, \varepsilon/3\gamma)$ is a desired Pontrjagin's neighbourhood of p.

In fact, for every $q \in \mathfrak{U}(p : \text{W}, \varepsilon/3\gamma)$ we have

$$\begin{aligned}|(f, p - q)| &\leqq |(g - f, p - q)| + |(g, p - q)| \\ &\leqq 2\varepsilon/3 + \int_W |(p(a) - q(a))|\,|g(a)|\,da < \varepsilon.\end{aligned}$$

(1. 7) and (1. 8) concludes the Raikov's Theorem.

Hereafter, we assume that the Fourier transform of an $f \in L(\mathfrak{G})$ is not only defined on $\mathfrak{C}$ as in the introduction, but is defined on $\mathfrak{R}$ as such a function that

$$\hat{f}(\lambda) = \int \overline{\lambda(a)}\, f(a)\, da \qquad (\lambda \in \mathfrak{R}).$$

$\hat{f}$ is weakly continuous on $\mathfrak{R}$, and by the Raikov's theorem we have

Theorem 6. *The Fourier transform of a function f in* $L(\mathfrak{G})$ *is conitnuous on* $\mathfrak{R}$ *and on* $\mathfrak{C}$ *by the Pontrjagin's topology.*

Chapter 2. Fourier transforms of measures on $\mathfrak{R}$.

Let μ be a regular measure on a compact space Ω. If X is a measurable sub-set of Ω, the relative measure μ_X on X determined by

$\mu_X(A) = \mu(X \cap A)$ is a Borel measure on X such that

(2. 1). *For every measurable sub-set Y of X we can choose a compact sub-set F such that $\mu_X(Y - F) < \varepsilon$.*

Conversely, let X be a topological space which is contained in a comact space Ω. And let μ_X be a Borel measure on X which satisfies (2. 1). Then the measure μ on Ω, defined by $\mu(A) = \mu_X(A \cap X)$ for every Borel set X, is a regular measure on Ω, and X is clearly measurable with respect to μ.

Notice that every completely regular topological space is contained in a suitable compact space, then we define a regular measure on a completely regular space as follows.

Definition 2.1. *A bounded non-negative Borel measure μ on a completely regular topological space Ω is called regular if for every Borel set X and for every positive ε, there exists a compact sub-set Y of X such that $\mu(X - Y) < \varepsilon$.*

A complex measure μ on X such that $\mu = \mu_1 - \mu_2 + i\,(\mu_3 - \mu_4)$, where μ_i are non-negative regular measures on Ω, is called a regular measure.

Lemma 2.1. *Let μ be a regular measure on a compact space Ω, and X be a measurable sub-set, then the relative measure μ_X is regular on X. Conversely, if μ is a regular measure of a topological space X which is contained in a compact space Ω, then μ is extended to a regular measure on Ω which vanishes out-side of X.*

Let π be a bounded complex regular measure on the space $\mathfrak{N}$ of all normalized positive definite functions. Then the Fourier transform $\hat{\pi}$ of π is a function on $\mathfrak{G}$ such that

$$\hat{\pi}(a) = \int_{\mathfrak{N}} \lambda(a) d\pi(\lambda).$$

Theorem 7. *The Fourier transform of every bounded complex regular measure on $\mathfrak{N}$ is a bounded continuous function on $\mathfrak{G}$. The Fourier transform of a non-negative measure is positive definite.*

Proof. Let π be a bounded non-negative regular measure on $\mathfrak{N}$. If ε is a positive number, we can choose a compact sub-set $\mathfrak{N}_0$ of $\mathfrak{N}$ such that $\pi(\mathfrak{N} - \mathfrak{N}_0) < \varepsilon$. Now the product space $\mathfrak{N}_0 \times \mathfrak{G}$ is compact, the function $p(a)$ ($p \in \mathfrak{N}$, $a \in \mathfrak{G}$) on which is two-sided continuous; and given each $a \in \mathfrak{G}$, we can choose a suitable open neighbourhood V of a such that $|\lambda(a)-\lambda(b)|<\varepsilon$ for every $b\in V$ and every $\lambda\in\mathfrak{N}_0$. Then for every $b\in V$,

$$|\hat{\pi}(a)-\hat{\pi}(b)|\leqq\int_{\mathfrak{N}_0}|\lambda(a)-\lambda(b)|\,d\pi(\lambda)+\int_{\mathfrak{N}-\mathfrak{N}_0}|\lambda(a)-\lambda(b)|\,d\pi(\lambda)$$
$$\leqq\varepsilon(\pi(\mathfrak{N})+2).$$

Therefore π is continuous at $a\in\mathfrak{G}$.

π is positive definite. In fact, let $a_1,\ldots a_n$ be arbitrary elements in $\mathfrak{G}$, and let $\xi_1,\ldots.\xi_n$ be arbitrary complex numbers. It is sufficient to show that

$$\sum_{i,j}\bar{\xi}_i\xi_j\hat{\pi}(a_i^{-1}a_j)\geqq 0.$$

This follows immediately from the fact that

$$\sum_{i,j}\bar{\xi}_i\xi_j\cdot(a_i^{-1}a_j)\geqq 0$$

for every $\lambda\in\mathfrak{N}$. This concludes the Theorem.

We shall need some preparations to prove that every positive definite function on $\mathfrak{G}$ is a Fourier transform of a measure on $\mathfrak{E}$. But a more weaker result will be easily shown.

Let $\mathfrak{P}$ denote the space of all positive definite functions p on $\mathfrak{G}$ whose norms $p(e)$ do not exceed 1. Then $\mathfrak{P}$ is a bounded regularly convex sub-space of $M(\mathfrak{G})$, and contains $\mathfrak{N}$ and $\mathfrak{E}$.

We denote by $\bar{\mathfrak{E}}$ the closure of $\mathfrak{E}$ by the Pontrjagin's topology, then

Theorem 8. *Every positive definite function on $\mathfrak{G}$ is a Fourier transform of a suitable non-negative bounded regular measure on $\bar{\mathfrak{E}}$.*

This Theorem follows from the next Lemma.

Lemma 2.2. *Let p be a normalized positive definite function, and π be a non-negative regular measure on $\mathfrak{P}$ such that p is a weak integral $p=\int_{\mathfrak{P}}\lambda d\pi(\lambda)$, i. e. $(f,p)=\int_{\mathfrak{P}}(f,\lambda)d\pi(\lambda)$ for every $f\in L(\mathfrak{G})$, then π vanishes out-side of $\mathfrak{N}$, and p is the Fourier transform of π.*

Proof. If f is an element in $L(\mathfrak{G})$, the function (f,λ), where λ varies over $\mathfrak{P}$, is a continuous function on $\mathfrak{P}$. Therefore

$$\lambda(e)=\sup_{|f|_1\leqq 1}|(f,\lambda)|$$

is a lower semi-continuous function on $\mathfrak{P}$, (where λ varies over $\mathfrak{P}$). Now

$$1 = p(e) = \sup_{\|f\|_1 \leqq 1} \left| \int (f, \lambda) \, d\pi(\lambda) \right|$$
$$\leqq \int_{\mathfrak{P}} \lambda(e) \, d\pi(\lambda) \leqq 1.$$

Then $\lambda(e) = 1$ is satisfied almost everywhere. π is a regular measure on $\mathfrak{R}$, and by Theorem 8, the Fourier transform $\hat{\pi}$ is continuous and positive definite on $\mathfrak{G}$. Choose a sub-set $\mathfrak{R}_0$ of $\mathfrak{R}$ with $\pi(\mathfrak{R} - \mathfrak{R}_0) = 0$, which is a countable sum of compact sub-sets of $\mathfrak{R}$. Then $u(a)$ is continuous in the product space $\mathfrak{R}_0 \times \mathfrak{G}$, and measurable by the product measure $d\pi \times da$. Applying the Fubini's theorem, we have

$$(f, p) = \int_{\mathfrak{G}} f(a) p(a) \, da = \int_{\mathfrak{R}_0} d\pi(\lambda) \int_{\mathfrak{G}} \lambda(a) f(a) \, da$$
$$= \int_{\mathfrak{G}} f(a) \, da \int_{\mathfrak{R}_0} \lambda(a) \, d\pi(\lambda) = \int f(a) \hat{\pi}(a) \, da$$
$$= (f, \hat{\pi}).$$

This means the coincidence between two continuous functions p and $\hat{\pi}$.

We now prove Theorem 8. The set $\mathfrak{E}_0$ of all extremal points of $\mathfrak{P}$ consists of the function 0 and the set $\mathfrak{E}$. By the Krein-Milman's theorem, the regularly convex hull of the set $\mathfrak{E}_0$ is $\mathfrak{P}_0$.

We use the next Lemma[1].

Lemma 2.3. *Let $\mathfrak{B}$ be a Banach space, and let $\mathfrak{X}$ be a bounded weakly closed sub-set of the dual space of $\mathfrak{B}$. Then every element x in the regularly convex hull $Co(\mathfrak{X})$ of $\mathfrak{X}$ is a weak integral $x = \int \lambda d\rho(\lambda)$, where ρ is a non-negative regular measure on $\mathfrak{X}$ with total mass* 1.

The weak closure $\overline{\mathfrak{E}}_0$ of $\mathfrak{E}_0$ is weakly compact, and $Co(\overline{\mathfrak{E}}_0) = \mathfrak{P}$, then every element p in $\mathfrak{P}$ is a weak integral

$$p = \int_{\overline{\mathfrak{E}}_0} \lambda d\pi(\lambda), \text{ where } \pi \text{ is a non-negative regular}$$

measure on $\overline{\mathfrak{E}}_0$ with total mass 1. Especially, if p is normalized, by Lemma 2. 2, π vanishes out-side of $\mathfrak{R}$, and p is the Fourier transform of

1) Tomita. Math. Jour. Okayama Univ. Vol. 3 No. 2 (1954)

π. The common part of $\mathfrak{N}$ and $\overline{\mathfrak{F}}_0$ is the closure $\overline{\mathfrak{F}}$ of $\mathfrak{F}$ by the Pontrjagin's topology, then Theorem 8 is concluded.

Chapter 3. Decomposition theorems of operator algebras.

The Main Theorem in the Introduction corresponds to the respective decomposition theorems of states on operator algebras, which is developped by Neumann, Mautner, Godement and Segal. We shall reconstruct some indispensable parts of their results together with some additional results in the following Chapter 3... 6.

We consider a fixed Hilbert space $\mathfrak{H}$ and a separable uniformly closed self-adjoint algebra $\mathfrak{A}$ of bounded linear operators on $\mathfrak{H}$ which contains the identity I. A linear functional s on $\mathfrak{A}$ such that $s(A^*A) \geqq 0$ for every $A \in \mathfrak{A}$ is called a *state*. s is necessarily bounded, and its norm is $s(I)$. s is called *normalized* if $s(I) = 1$. The *canonical representation* of $\mathfrak{A}$ with respect to a state s is determined as follows.

There exists a Hilbert space $L^2(s)$, a uniformly continuous homomorphism $A \rightarrow A_s$ of $\mathfrak{A}$ on a uniformly closed self-adjoint algebra $\mathfrak{A}(s)$ on $L^2(s)$ and an element $\mathfrak{f}$ in $L^2(s)$ with $s(A) = (A_s\mathfrak{f}, \mathfrak{f})$, such that the set $(A_s\mathfrak{f} : A \in \mathfrak{A})$ is uniformly dense in $L^2(s)$. The mapping $A \rightarrow A_s\mathfrak{f}$ is called the canonical representation of $\mathfrak{A}$ in $L^2(s)$.

Let s be a state. If t is a linear functional on $\mathfrak{A}$ such that, for a suitable constant γ, $\gamma s + t$ and $\gamma s - t$ are states on A, then there exists a uniquely determined bounded linear operator K in the commutor $\mathfrak{A}(s)'$ of $\mathfrak{A}(s)$ such that $t(A) = (A_sK\mathfrak{f}, \mathfrak{f})$.

A state s is called *elementary* if we can not devide s to a sum of two states t and u except for the case that $t = \alpha s$ and $u = (1 - \alpha)s$. A state s is elementary if and only if the representative algebra $\mathfrak{A}(s)$ is *irreducible*, i. e. $\mathfrak{A}(s)'$ is the one-dimensional algebra which contains the identity I_s on $L^2(s)$.

We denote by $\mathfrak{S}$ or by $S(\mathfrak{A})$ the set of all normalized states on $\mathfrak{A}$, and by $E(\mathfrak{A})$ the set of all normalized elementary states on $\mathfrak{A}$.

If σ is a regular measure on $\mathfrak{S}$, the linear functional $\hat{\sigma}$ on $\mathfrak{A}$ defined by

$$\hat{\sigma}(A) = \int \lambda(A)\, d\sigma(\lambda)$$

is called the *Fourier transform* of σ. Then

Lemma 3.1. *The Fourier transform of a bounded complex measure on* $\mathfrak{S}$ *is bounded and linear on* $\mathfrak{A}$. *The transform of a bounded non-negative measure is a state on* $\mathfrak{A}$.

A pair of states s and t are called mutually *orthogonal* if there is no non-zero state r on $\mathfrak{A}$ such that $s-r$ and $t-r$ are states. A bounded non-negative regular measure σ on $E(\mathfrak{A})$ is called *diagonal* if for every Borel set X in $E(\mathfrak{A})$, two states σ_X and $\sigma_{(E(\mathfrak{A})-X)}$ are mutually orthogonal, where σ_X denotes the relative measure on X.

Let σ be a fixed bounded non-negative regular measure on $\mathfrak{S}$, and s denote its Fourier transform $s=\int\lambda\, d\sigma(\lambda)$. If $\hat{\sigma}_\varphi$ is the Fourier transform of the relative measure σ_φ ($\varphi\in M(\sigma)$), then there exists a uniquely determined operator F_φ in $\mathfrak{A}(s)'$ such that $(A_sF_\varphi s, s)=\hat{\sigma}_\varphi(A)=\int\lambda(A)\varphi(\lambda)\, d\sigma(\lambda)$. The mapping $\varphi\rightarrow F_\varphi$ is called the *Fourier transform relative* to the measure σ. We denote by $\mathfrak{F}(\sigma)$ the range of this mapping $\varphi\rightarrow F_\varphi$. We assert the following theorems which correspond to the respective theorems in the Introduction.

Theorem 9. *In order that a bounded non-negative regular measure* σ *on* $E(\mathfrak{A})$ *be diagonal, it is necessary and sufficient that, for each function* f *on* $E(\mathfrak{A})$ *measurable and square summable by* σ, *there exists a sequence* $\{A_n\}$ *in* $\mathfrak{A}$ *such that*

$$\lim\int|\lambda(A_n)-f(\lambda)|^2\, d\sigma(\lambda)=0$$

and simultaneously

$$\lim\int\lambda(A_n{}^*A_n)\, d\sigma(\lambda)=\int|f(\lambda)|^2\, d\sigma(\lambda).$$

The sequence $\{A_n\}$ in Theorem 9 is called the *approximative image* of the Fourier transform of f.

Theorem 10. *Let* σ *be a diagonal measure on* $E(\mathfrak{A})$, *and let* f *and* g *be two measurable and square summable functions on* $E(\mathfrak{A})$. *If* $\{A_n\}$ *and* $\{B_n\}$ *are approximative images of Fourier transforms of* f *and* g *respectively, then*

$$\lim\int|\lambda(B_n{}^*A_n)-f(\lambda)\overline{g(\lambda)}|\, d\sigma(\lambda)=0.$$

Theorem 11. *Let σ be a diagonal measure on $E(\mathfrak{A})$, then*

(1). *The range $\mathfrak{F}(\sigma)$ of the relalive Fourier transform is a maximal abelian self-adjoint sub-algebra of $\mathfrak{A}(s)'$.*

(2). *The relative Fourier transform $\varphi \rightarrow F_\varphi$ is an algebraic isomorphism between $M(\sigma)$ and $\mathfrak{F}(\sigma)$. It is one-to-one, linear and satisfies the Parseval's equality $F_{\varphi\psi} = F_\varphi F_\psi$.*

Theorem 12. *Let s be a state and $\mathfrak{R}$ be a maximal abelian self-adjoint sub-algebra of $\mathfrak{A}(s)'$, then there exists a uniquely determined diagonal measure σ in $E(\mathfrak{A})$ whose Fourier transform is s, and for which the range $\mathfrak{F}(\sigma)$ of the deformed Fourier transform is $\mathfrak{R}$.*

Therefore, $\sigma \leftrightarrow (\hat{\sigma}, \mathfrak{F}(\sigma))$ is a one-to-noe correspondence between all diagonal measures on $E(\mathfrak{A})$ and all couples $(s, \mathfrak{R})$ of states s on $\mathfrak{A}$ and maximal abelian self-adjoint sub-algebras $\mathfrak{R}$ of $\mathfrak{A}(s)'$.

Chapter 4. The Plansherel's theorem.

Every regular measure on $E(\mathfrak{A})$ is extended to a regular measure on $\mathfrak{S}$ which vanishes out-side of $E(\mathfrak{A})$, Then every diagonal measure on $E(\mathfrak{A})$ is regarded as a regular measure on $\mathfrak{S}$ which satisfies the next two conditions:

(4.1). *For every Borel set X in $\mathfrak{S}$, σ_X and $\sigma_{\mathfrak{S}-X}$ are mutually orthogonal.*

(4.2). *σ vanishes out-side of $E(\mathfrak{A})$.*

We first investigate those measures on $\mathfrak{S}$ which satisfies the condition (4.1).

Theorem 13. *Let σ be a bounded non-negative regular measure on $\mathfrak{S}$. If σ satisfies one of the following three conditions, then the other conditions are necessarily satisfied.*

(4.1). *For every Borel set X in $\mathfrak{S}$, σ_X and $\sigma_{\mathfrak{S}-X}$ are mutually orthogonal.*

(4.3). *The Fourier transform $\varphi \rightarrow F_\varphi$ relative to σ is an algebraic isomorphism.*

(4.4). *Let $L^2(\sigma)$ denote the Hilbert space of all functions on $\mathfrak{S}$ measurable and square summable by σ, then for each $f \in L^2(\sigma)$, we can choose a sequence $\{A_n\}$ in $\mathfrak{A}$ such that*

$$\lim \int |u(A_n) - f(u)|^2 \, d\sigma(u) = 0$$

and simultaneously

$$\lim \int u(A_n{}^*A_n)d\sigma(u) = \int |f(u)|^2 d\sigma(u).$$

Proof. We first notice that the orthogonality relation between states on $\mathfrak{A}$ corresponds to an orthogonality relation between the related canonical representation spaces. That is,

Lemma 4.1. *In order that two states s and t be orthogonal, it is necessary and sufficient that, putting $u = s + t$, the operator K in $\mathfrak{A}(u)'$ determined by $s(A) = (A_u K\mathfrak{u}, \mathfrak{u})$ be a projection operator.*

In fact, let s and t be mutually orthogonal. The operator K in $\mathfrak{A}(u)'$, determined by $s(A) = (A_s K\mathfrak{u}, \mathfrak{u})$, is a definite Hermitian, and $I - K$ is also definite. The state $v(A) = (A_s K(I - K)\mathfrak{u}, \mathfrak{u})$ should be 0, because $s - v$ and $t - v$ are states. Therefore $K(I - K) = 0$, and K is a projection.

Conversely let s and t be two states on $\mathfrak{A}$ such that the operator K in $\mathfrak{A}(u)'$ determined by $s(A) = (A_u K\mathfrak{u}, \mathfrak{u})$ is a projection. If r is a state on $\mathfrak{A}$ such that $s - r$ and $t - r$ are also states, then the corresponding operator R such that $r(A) = (A_u R\mathfrak{u}, \mathfrak{u})$, is a definite Hermitian such that $K - R$ and $I - K - R$ are also definite. This means $R = 0$ and $r = 0$. Hence s and t are mutually orthogonal.

We now show that, if σ satisfies (4. 1), then it satisfies (4. 3). Let s be the Fourier transform of σ: $s = \int \lambda d\sigma(\lambda)$. If φ is a characteristic function on a measurable set X in $\mathfrak{S}$, X coincides with a Borel set X_0 almost everywhere, and the measure σ_{X_0} coincides with the measure σ_φ. Since σ satisties (4. 1), $\hat{\sigma}_\varphi$ and $\sigma_{1-\varphi}$ are mutually orthogonal, and by Lemma 4.1 F_φ is a projection. If φ aud ψ are two characteristic functions on mutually disjoint measurable sub-sets in $\mathfrak{S}$, then $K_\psi \leqq F_{1-\varphi}$, and $F_\varphi F_\psi = 0$. Let φ and ψ be two arbitrary characteristic functions on measurable sets in $\mathfrak{S}$, then $F_\varphi = F_{\varphi\psi} + F_{\varphi(1-\psi)}$, and $F_\psi = F_{\varphi\psi} + F_{\psi(1-\varphi)}$. Therefore $F_\varphi F_\psi = F_{\varphi\psi}$.

Now $F_f F_g = F_{fg}$ is satisfied for every pair (f, g) of measurable step functions f and g. Every bounded measurable function is a uniform limit of a sequence of step functions. Then $F_f F_g = F_{fg}$ is satisfied for every $f, g \in M(\sigma)$. The mapping $f \to F_f$ is one-to-one. In fact, if f in $M(\sigma)$ does not vanish, then $\| F_f \mathfrak{f} \|^2 = (F_{|f|^2} \mathfrak{f}, \mathfrak{f}) = \int |f(\lambda)|^2 d\sigma \neq 0$.

$f \to F_f$ is an algebraic isomorphism between $M(\sigma)$ and $\mathfrak{F}(\sigma)$, hence the measure σ which satisfies (4. 1) satisfies also (4. 3).

We next show that, if σ satisfies (4. 3), then σ satisfies (4. 4). If σ satisfies (4. 3), then the relative Fourier transform $f \to F_f$ is an algebraic isomorphism. Let s be the Fourier transform $s = \int u d\sigma(u)$ of σ. If g is a bounded measurable function on $E(\mathfrak{A})$, and if ε is a positive number, we can choose an element A of $\mathfrak{A}$ such that $\| A_s \mathfrak{f} - F_g \mathfrak{f} \|^2 < \varepsilon$. Using the relation $F_g^* F_g = F_{|g|^2}$, we have

$$\| A_s \mathfrak{f} - F_g \mathfrak{f} \|^2 = \int u(A^*A) - 2\Re e(g(u)u(A)) + |g(u)|^2 d\sigma(u)$$
$$= \int \| A_u \mathfrak{u} - g(u)\mathfrak{u} \|^2 d\sigma(u) < \varepsilon.$$

Then

$$\int | u(A) - g(u) |^2 d\sigma(u) = \int | (A)_u \mathfrak{u} - g(u)\mathfrak{u}, \mathfrak{u}) |^2 d\sigma(\mathfrak{u})$$
$$\leqq \int \| A_u \mathfrak{u} - g(u)\mathfrak{u} \|^2 d\sigma(u) \leqq \varepsilon,$$

and

$$\int | u(A^* A)^{\frac{1}{2}} - |g(u)| |^2 d\sigma(u) = \int \Big| \| A_u \mathfrak{u} \| - \| g(u)\mathfrak{u} \| \Big|^2 d\sigma(u)$$
$$\leqq \int \| A_n \mathfrak{u} - g(u)\mathfrak{u} \|^2 d\sigma(u).$$

If f is an element in $L^2(\sigma)$, we can choose a bounded measuable function g on $\mathfrak{S}$ such that $\int | f(u) - g(u) |^2 d\sigma(u) < \frac{\varepsilon}{4}$. For such g choose an element A in $\mathfrak{A}$ with $\| A_s \mathfrak{f} - K_g \mathfrak{f} \|^2 < \frac{\varepsilon}{4}$, then

$$\int | u(A) - f(u) |^2 d\sigma(u) < \varepsilon,$$

and

$$\int \Big| u(A^* A)^{\frac{1}{2}} - |f(u)| \Big|^2 d\sigma(u) < \varepsilon.$$

Therefore, every regular measure ε on $\mathfrak{S}$ which satisfies the condition (4. 3) satisfies also (4. 4).

Notice that every diagonal measure satisfies (4. 1), then we have

Lemma 4.2. *Let σ be a diagonal measure on $E(\mathfrak{A})$ whose Fourier transform $\hat{\sigma}$ is s, and let g be a bounded measurable function on $E(\mathfrak{A})$. If $\{A_n\}$ is a sequence in $\mathfrak{A}$ such that its canonical representations $A_{ns}\mathfrak{f}$ converge uniformly to $F_g\mathfrak{f}$, then $\{A_n\}$ is an approximative image of the Fourier trnnsform of g.*

We finally show that, if σ satisfies (4. 4), then σ statisties (4. 1).

Let f be a bounded measurable function and ε be a positive number. If A is an element in $\mathfrak{A}$ such that

$$\int |u(A) - f(u)|^2 \, d\sigma(u) < \varepsilon^2,$$

and simultaneously

$$\int u(A^*A) \, d\sigma(u) \leqq \int |f(u)|^2 \, d\sigma(u) + \varepsilon^2,$$

then

$$\begin{aligned} 0 &\leqq \int \| A_u\mathfrak{u} - f(u)\mathfrak{u} \|^2 \, d\sigma(u) \\ &= \int (u(A^*A) - 2\Re e(f(u)\overline{u(A)}) + |f(u)|^2) \, d\sigma(u) \\ &\leqq (\int |f(u)|^2 \, d\sigma(u) + \varepsilon^2) - 2\int |f(u)|^2 \, d\sigma(u) + 2\varepsilon(\int |f(u)|^2 \, d\sigma(u))^{\frac{1}{2}} \\ &\quad + \int |f(u)|^2 \, d\sigma(u) \\ &\leqq \varepsilon(\varepsilon + 2(\int |f(u)|^2 \, d\sigma(u))^{\frac{1}{2}}). \end{aligned}$$

Therefore we can choose an $A \in \mathfrak{A}$ such that

$$\int \| A_u\mathfrak{u} - f(u)\mathfrak{u} \|^2 \, d\sigma(u) < \varepsilon^2.$$

If g is another bounded measurable function on $\mathfrak{S}$, then

$$\begin{aligned} |(F_g A_s\mathfrak{f} - F_{fg}\mathfrak{f}, B_s\mathfrak{f})| &\leqq |\int g(u)(A_u\mathfrak{u} - f(u)\mathfrak{u}, B_u\mathfrak{u}) \, d\sigma(u)| \\ &\leqq \int |g(u)| \, \| B_u\mathfrak{u} \| \, \| A_u\mathfrak{u} - f(u)\mathfrak{u} \| \, d\sigma(u) \\ &\leqq |g|_\infty (\int \| B_u\mathfrak{u} \|^2 \, d\sigma(u))^{\frac{1}{2}} (\int \| A_u\mathfrak{u} - f(u)\mathfrak{u} \|^2 \, d\sigma(u))^{\frac{1}{2}} \\ &\leqq \varepsilon \, |g|_\infty \, \| B_s\mathfrak{f} \| . \end{aligned}$$

This implies $\| F_g A_\varepsilon \mathfrak{k} - F_{fg}\mathfrak{k} \| \leqq \varepsilon |g|_\infty$. Especially, $\| A_\varepsilon \mathfrak{k} - F_f \mathfrak{k} \| \leqq \varepsilon$. Then $\| F_f F_g \mathfrak{k} - F_{fg}\mathfrak{k} \| \leqq \varepsilon (|f|_\infty + |g|_\infty)$. Now we have $F_f F_g \mathfrak{k} = F_{fg}\mathfrak{k}$ and $F_{fg} = F_f F_g$. If f is a characteristic function on a Borel set X in $\mathfrak{S}$, then $F_f^2 = F_f = F_f^*$. F_f is a projection, and two states $\hat{\sigma}_X$ and $\hat{\sigma}_{\mathfrak{S}-X} = \hat{\sigma} - \hat{\sigma}_X$ are mutually orthogonal. Therefore, if σ satisfies (4. 4), then σ satisfies (4. 1).

Hence, if σ is a non-negative regular measure which satisfies one of the conditions (4. 1), (4. 3) and (4. 4), all other conditions are satisfied.

Theorem 9 and Theorem 10 is an immediate consequence of Theorem 13. Theorem 9 is evident, then we show Theorem 10 only. Let σ be a diagonal measure on $E(\mathfrak{A})$, and extend σ as a regular measure on $\mathfrak{S}$. If $\{A_n\}$ and $\{B_n\}$ are approximative images of Fourier transforms of f and g in $L^2(\sigma)$ respectively, then

$$\lim \int \| A_{nu}\mathfrak{u} - f(u)\mathfrak{u} \|^2 \, d\sigma(u) = 0,$$

and

$$\lim \int \| B_{nu}\mathfrak{u} - g(u)\mathfrak{u} \|^2 \, d\sigma(u) = 0.$$

Hence

$$\begin{aligned}
&\int | u(B_n^* A_n) - f(u)\overline{g(u)} | \, d\sigma(u) \\
&= \int | (A_{nu}\mathfrak{u}, B_{nu}\mathfrak{u}) - (f(u)\mathfrak{u}, g(u)\mathfrak{u}) | \, d\sigma(u) \\
&\leqq \int | (A_{nu}\mathfrak{u}, B_{nu}\mathfrak{u} - g(u)\mathfrak{u}) | \, d\sigma(u) + \\
&\qquad + \int | (A_{nu}\mathfrak{u} - f(u)\mathfrak{u}, g(u)\mathfrak{u}) | \, d\sigma(u) \\
&\leqq \left(\int \| A_{nu}\mathfrak{u} \|^2 \, d\sigma(u) \right)^{\frac{1}{2}} \left(\int \| B_{nu}\mathfrak{u} - g(u)\mathfrak{u} \|^2 \, d\sigma(u) \right)^{\frac{1}{2}} \\
&\quad + \left(\int \| A_{nu}\mathfrak{u} - f(u)\mathfrak{u} \|^2 \, d\sigma(u) \right)^{\frac{1}{2}} \left(\int \| g(u)\mathfrak{u} \|^2 \, d\sigma(u) \right)^{\frac{1}{2}} \to 0
\end{aligned}$$

as $n \to \infty$. This concludes Theorem 10.

Chapter 5. Diagonal measures and maximal abelian sub-algebras.

Let σ be a non-negative regular measure on $\mathfrak{S}$ which satisfies the condition (4.1). Then σ is diagonal if and only if it satisfies the condition (4.2), that is, σ vanishes out-side of $E(\mathfrak{A})$.

Theorem 14. *Let σ be a regular non-negative measure on $\mathfrak{S}$ which satisfies the condition* (4.1). *A necessary and sufficient condition for σ to be diagonal is that the range $\mathfrak{F}(\sigma)$ of the relative Fourier transform $f \to F_f$ should be a maximal abelian self-adjoint sub-algebra of $\mathfrak{A}(s)$.*

Proof of the necessity. Let σ be a diagonal measure on $E(\mathfrak{A})$.

We show that every definite Hermitian K in $\mathfrak{A}(s)'$ which commutes to all elements in $\mathfrak{F}(\sigma)$ belongs to $\mathfrak{F}(\sigma)$. Denote by s the Fourier transform $\hat{\sigma}$ of σ. For each fixed $A \in \mathfrak{A}$ we can determine a bounded measurable function $\gamma_A(\lambda)$ on $\mathfrak{S}$ such that

$$(A_s F_\varphi K \mathfrak{f}, \mathfrak{f}) = \int \gamma_A(\lambda)\varphi(\lambda) d\sigma(\lambda). \quad \text{for every } \varphi \in M(\sigma).$$

In fact, if φ is a non-negative bounded measurable function on $E(\mathfrak{A})$,

$$\begin{aligned} |(A_s F_\varphi K \mathfrak{f}, \mathfrak{f})| &= |(A_s K F_\varphi^{\frac{1}{2}} \mathfrak{f}, F_\varphi^{\frac{1}{2}} \mathfrak{f})| \leqq |A| |K| \| F_\varphi^{\frac{1}{2}} \mathfrak{f} \|^2 \\ &= |A| |K| \int \varphi(\lambda) d\sigma(\lambda). \end{aligned}$$

Then such a function γ_A exists and satisfies $|\gamma_A(\lambda)| \leqq |K| |A|$.

We now put $\lambda'(A) = \gamma_A(\lambda)$, then

$$(A_s F_\varphi K \mathfrak{f}, \mathfrak{f}) = \int u'(A)\varphi(u)\, d\sigma(u) \quad \text{for every} \quad \varphi \in M(\sigma).$$

If γ is the norm of the operator K, then

$$\begin{aligned} 0 \leqq \| A_s F_\varphi^{\frac{1}{2}} K^{\frac{1}{2}} \mathfrak{f} \|^2 &= |(A_s^* A_s F K \mathfrak{f}, \mathfrak{f}) \\ &= \int \lambda'(A^*A)\varphi(\lambda)\, d\sigma(\lambda) \end{aligned}$$

and

$$\begin{aligned} 0 \leqq \| A_s F_\varphi^{\frac{1}{2}} (\gamma I - K)^{\frac{1}{2}} \mathfrak{f} \|^2 &= (A_s^* A_s F_\varphi (\gamma I - K) \mathfrak{f}, \mathfrak{f}) \\ &= \int (\gamma \lambda(A^*A) - \lambda'(A^*A))\varphi(\lambda) d\sigma(\lambda) \end{aligned}$$

for every $A \in \mathfrak{A}$ and for every non-negative φ in $M(\sigma)$.

This implies $\gamma\lambda(A^*A) \geqq \lambda'(A^*A) \geqq 0$ almost everywhere.

By the separability assumption of the algebra $\mathfrak{A}$, we can choose a countable sequence $\{A_n\}$ in $\mathfrak{A}$ which satisfies the next conditions. (1). $\{A_n\}$ is everywhere dense in $\mathfrak{A}$. (2). The sub-system of all definite Hermitians in $\{A_n\}$ is everywhere dense in the space $\mathfrak{A}^H$ of all definite Hermitians in $\mathfrak{A}$. (3). $A_0 = 0$. (4). If A_n is contained in the sequence $\{A_n\}$, then A_n^* is also contained in this sequence.

Except for a set T of $\sigma(\mathrm{T}) = 0$, the following three conditions are satisfied.

(1). $\lambda'(A_n^*) = \overline{\lambda'(A_n)}$

(2). $|\lambda'(A_i) + \lambda'(A_j) + \lambda'(A_k)| \leqq \gamma |A_i + A_j - A_k|$.

(3). If A_n is a definite Hermitian, then $\gamma u(A_n) \geqq u'(A_n) \geqq 0$.

If $\{B_n\}$ is a sub-sequence of $\{A_n\}$ which converges to an element B in $\mathfrak{A}$, and if λ is a point in $E(\mathfrak{A}) - \mathrm{T}$, then $\overline{\lim\limits_{m,n\to\infty}} |\lambda'(B_n) - \lambda'(B_m)| \leqq \lim |B_n - B_m| = 0$, and $\{\lambda(B_n)\}$ converges to a complex number $\lambda''(B)$. The value $\lambda''(B)$ is uniquely determined for each $B \in \mathfrak{A}$, and does not depend how to choose such a sub-sequence $\{B_n\}$ of $\{A_n\}$ which converges to B. In fact, if $\{B_n\}$ and $\{C_n\}$ are two sub-sequences of $\{A_n\}$ which converges to B, then

$$\overline{\lim} |\lambda'(B_n) - \lambda'(C_n)| \leqq \gamma \lim |B_n - C_n| = 0.$$

This $\lambda''(A)$ satisfies also

$$(AF_\varphi K\mathfrak{f}, \mathfrak{f}) = \int \lambda''(A)\varphi(\lambda)d\sigma(\lambda) \text{ for every } \varphi \in M(\sigma).$$

We show that all those λ'' are states on $\mathfrak{A}$. If B and C are two elements in $\mathfrak{A}$, there exists three sub-sequences $\{B_n\}$, $\{C_n\}$ and $\{D_n\}$ of $\{A_n\}$ such that $\lim B_n = B$, $\lim C_n = C$ and $\lim D_n = B + C$. Then

$$\begin{aligned} &|\lambda''(B) + \lambda''(C) - \lambda''(B + C)| = \lim |\lambda'(B_n) + \lambda'(C_n) - \lambda'(D_n)| \\ &\leqq \gamma \overline{\lim} |B_n + C_n - D_n| = 0, \end{aligned}$$

and $\lambda''(B) + \lambda''(C) = \lambda''(B + C)$.

If $\{B_n\}$ is a sub-sequence of $\{A_n\}$ which converges to B, then B_n^* is a sub-sequence of $\{A_n\}$, and

$$\lambda''(B^*) = \lim \lambda'(B_n^*) = \lim \overline{\lambda'(B_n)} = \overline{\lambda''(B)}.$$

If B is a definite Hermitian, there exists a sequence $\{B_n\}$ of definite Hermitians contained in $\{A_n\}$ which converges to B. Then $\gamma\lambda(B_n) \geqq \lambda'(B_n) \geqq 0$ and $\gamma\lambda(B) \gtrsim \lambda''(B) \gtrsim 0$.

Each $\lambda''(\lambda \in E(\mathfrak{A}) - \mathrm{T})$ is a bounded additive functional on the space $\mathfrak{A}$. By the Banach's theorem, λ'' is linear on the real Banach space $\mathfrak{A}^H$ of all definite Hermitians in $\mathfrak{A}$. Then λ'' is linear on $\mathfrak{A}$, and is a state such that $\gamma\lambda - \lambda''$ is also a state. Every elements in $E(\mathfrak{A}) - \mathrm{T}$ are elementary, then those λ'' satisfy $\lambda''(A) = \lambda''(I)\lambda(A)$, where $\varphi_0(\lambda) = \lambda''(I)$ is a bounded measurable function on $E(\mathfrak{A})$, hence

$$(A_s K\mathfrak{f}, \mathfrak{f}) = \int \lambda''(A)\, d\sigma(\lambda) = \int \lambda(A)\varphi_0(\lambda)\, d\sigma(\lambda)$$
$$= (A_s F_{\varphi_0}\mathfrak{f}, \mathfrak{f}).$$

Therefore K coincides with F_{φ_0} and belongs to $\mathfrak{F}(\sigma)$.

Let $\mathfrak{A}(s)' \cap \mathfrak{F}(\sigma)'$ denote the uniformly closed self-adjoint algebra of all operators in $\mathfrak{A}(s)'$ which commute to every elements in $\mathfrak{F}(\sigma)$. Then $\mathfrak{F}(\sigma)$ and $\mathfrak{A}(s)' \cap \mathfrak{F}(\sigma)'$ contains every definite Hermitian elements comments commonly. It is easiely shown that such two uniformly closed self-adjoint sub-algebras coincide with each other. Then $\mathfrak{F}(\sigma)$ coincides with $\mathfrak{A}(s)' \cap \mathfrak{F}(\sigma)'$, and is a maximal abelian self-adjoint sub-algebra of $\mathfrak{A}(s)'$. The necessary part in the Theorem is thus proved.

To prove the sufficiency part in the Theorem, it may be convenient to use an elementary results for Radon extentions of the Lebesgue's integrals. Let Ω be an abstract space on which a bounded non-negative measure μ is given. Let $B(\Omega)$ denote the Banach space of all real bounded functions on Ω. A positive linear functional ρ on $B(\Omega)$ is called a Radon integral on Ω, and denoted symbolicly by $\rho(f) = \int f\, d\rho$. If $f = g + ih$ is a complex bounded function on Ω, then the integral of f is defined by $\int f\, d\rho = \int g\, d\rho + i \int h\, d\rho$.

The Lebesgue's outer-integral of an element f in $B(\Omega)$ is defined in the usual way. That is,

$$\mu^*(f) = \overline{\int} f\, d\mu = \inf_{f \leqq g \in M(\mu)} \int g d\mu.$$

Then

(5. 1). $\mu^*(\alpha f)=\alpha\mu^*(f)$ *if* $\alpha\geqq 0$.

(5. 2). $\mu^*(f+g)\leqq\mu^*(f)+\mu^*(g)$.

(5. 3). $\mu^*(f)\leqq 0$ *if* $f\leqq 0$.

(5. 4). *An element* f *in* $B(\Omega)$ *is measurable by* μ *if and only if* $\mu^*(f)=-\mu^*(-f)$.

Lemma 5.1. *Gi en* $f_0\in B(\Omega)$ *and given a number* t *such that* $\mu^*(f_0)\geqq t\geqq -\mu^*(-f_0)$. *The integral* $\int f\,d\mu$ *is extended to a suitable Radon integral* ρ *such that* $\int f_0\,d\rho=t$.

By the Hahn-Banach's theorem there exists a linear functional ρ on $B(\Omega)$ such that $\rho(f_0)=t$ and $\mu^*(f)\geqq\rho(f)\geqq-\mu^*(-f)$ for every $f\in B(\Omega)$. ρ is a desired Radon extention of the measure μ. In fact, if f is non-negative, then $\rho(f)\geqq-\mu^*(-f)\geqq 0$, being a positive linear functional. If f is a bounded measurable function, then $\mu^*(f)=-\mu^*(-f)=\int f\,d\mu$, and $\int f\,d\mu=\int f\,d\rho$. Then ρ is an extended Radon integral of μ. q. e. d.

Assume that σ is a non-negative regular measure on $\mathfrak{S}$ which satisfies the condition (4. 1), and for which the set $\mathfrak{F}(\sigma)$ is a maximal abelian self-adjoint sub-algebra of $\mathfrak{A}(s)'$.

We shall show that σ vanishes out-side of $E(\mathfrak{A})$.

For each $\lambda\in\mathfrak{S}$ we define a new state λ' as follows.

(5. 5). *If* $\lambda\in E(\mathfrak{A})$, *then* $\lambda=\lambda'$.

(5. 6). *If* $\lambda\overline{\in} E(\mathfrak{A})$, *then* λ *is a sum of two states* u *and* v *such that* $u\neq\alpha\lambda$ *and* $v\neq(1-\alpha)\lambda$.

We can assume here that $u(I)\geqq\frac{1}{2}$. Then we put $\lambda'=u/u(I)$.

This correspondence $\lambda\to\lambda'$ satisfies the next two conditions:

(5. 7) $\lambda=\lambda'$ *if and only if* $\lambda\in E(\mathfrak{A})$.

(5. 8) *Every* λ' *is normalized, and* $2\lambda-\lambda'$ *is a state.*

Let ρ be an arbitrary positive Radon integral defined for every bounded complex functions on $\mathfrak{S}$, which is extended from the Lebesgue's integral $\int f\,d\sigma$. Then obviously

$$t(A)=\int\lambda'(A)\,d\rho(\lambda)$$

is a state on $\mathfrak{A}$. More generally, if φ is a bounded non-negative measurable function on $\mathfrak{S}$, then

$$t_\varphi(A) = \int \lambda'(A)\varphi(\lambda)d\rho(\lambda)$$

is a state on $\mathfrak{A}$. Further

$$2\hat{\sigma}_\varphi(A) - t_\varphi(A) = \int (2\lambda(A) - \lambda'(A))\varphi(\lambda)\, d\rho(\lambda)$$

is non-negative for every definite Hermitian A in $\mathfrak{A}$. Then $2\sigma_\varphi - t_\varphi$ is also a state, and there exists a definite Hermitian T_φ in $\mathfrak{A}(s)'$ such that $t_\varphi(A) = (A, T_\varphi\mathfrak{f}, \mathfrak{f})$, where $2F_\varphi - T_\varphi$ is a definite Hermitian. Especially, if φ is a characteristic function on a measurable set in $\mathfrak{S}$, then F_φ is a projection, and we have $F_\varphi T_\varphi = T_\varphi F_\varphi = T_\varphi$. Analogously $F_{(1-\varphi)}T_{(1-\varphi)} = T_{(1-\varphi)}F_{(1-\varphi)} = T_{(1-\varphi)}$. Because $T_1 = T_\varphi + T_{(1-\varphi)}$, we have

(5. 9). $$F_\varphi T_1 = T_1 F_\varphi = T_\varphi.$$

Now the operator T_φ in $\mathfrak{A}(s)'$ such that $(A, T_\varphi\mathfrak{f}, \mathfrak{f}) = \int \lambda'(A)\varphi(\lambda)\, d\rho(\lambda)$ is defined not only positive φ, but evey φ in $M(\sigma)$, where $\varphi \to T_\varphi$ is a linear mapping. Then (5. 9) is satisfied by every measurable step function $\varphi = \sum \alpha_i\varphi_i$, where α_i are complex numbers and φ_i are characteristic functions on measurable sub-sets. Every bounded measurable function on $\mathfrak{S}$ is a uniform limit of a sequence of measurable step functions, then every $\varphi \in M(\sigma)$ satisfies (5. 9), and T_1 commutes to every elements in $\mathfrak{F}(\sigma)$. By the assumption that $\mathfrak{F}(\sigma)$ is a maximal abelian sub-algebra of $\mathfrak{A}(s)'$, T_1 belongs to $\mathfrak{F}(\sigma)$, and there exists a $\psi \in M(\sigma)$ such that $T_1 = F_\varphi$. Then

$$t_\varphi(A) = \int \lambda'(A)\varphi(\lambda)\, d\rho(\lambda) = \int \lambda(A)\varphi(\lambda)\, \psi(\lambda)\, d\rho(\lambda),$$

and for every $\varphi \in M(\sigma)$

$$\int \varphi(\lambda)\, d\dot{\sigma}(\lambda) = \int \varphi(\lambda)\, d\rho(\lambda) = t_\varphi(I) = \int \varphi(\lambda)\,.\,(\lambda)\, d\sigma(\lambda).$$

This proves $\psi = 1$, then the integral value

$$\int \lambda'(A)\varphi(\lambda)d\rho(\lambda) = \int \lambda(A)\varphi(\lambda)d\sigma(\lambda) = \hat{\sigma}_\varphi(A)$$

is invariant even if the extended Radon-integral ρ varies. Let A be a fixed Hermitian operator in $\mathfrak{A}$. By Lemma 5. 1 we can choose two extended Radon integrals whose integral values for the integrand $\lambda'(A)$

are respectively the outer-integral and the inner-integral of $\lambda'(A)$ by σ. Then those values are coincident, and $\lambda'(A)$ should be measurable by σ. Even if A is not Hermitian, $\lambda'(A) = \lambda'(A + A^*)/2 - \lambda'((A - A^*)/i)/2i$ is measurable, and satisfies

$$\int \lambda'(A)\varphi(\lambda)d\sigma(\lambda) = \int \lambda(A)\varphi(\lambda)d\sigma(\lambda).$$

Then $\lambda'(A)$ and $\lambda(A)$ coincide with each other almost everywhere.

Let $\{A_n\}$ be a countable sequence in $\mathfrak{A}$ everywhere dense in $\mathfrak{A}$. Except for a null-set $\mathfrak{T}$, we have $\lambda'(A_n) = \lambda(A_n)$ $n = 1, 2, \ldots.$ Then except for $\mathfrak{T}$, we have $\lambda'(A) = \lambda(A)$ for every $A \in \mathfrak{A}$, hence $\lambda' = \lambda$. Applying (5. 7), $\mathfrak{S} - \mathfrak{T}$ is contained in $E(\mathfrak{A})$, then σ vanishes out-side of $E(\mathfrak{A})$.

Hence the sufficiency part of the Theorem is proved.

Chapter 6. Existence and uniqueness of diagonal measures.

We consider a fixed state s on $\mathfrak{A}$ and a fixed maximal abelian self-adjoint sub-algebra $\mathfrak{K}$ of $\mathfrak{A}(s)'$. By the theorem of Gelfand-Raikov, $\mathfrak{K}$ is isomorphic to the Banach algebra $C(\Omega)$ of all continuous functions on a suitable compact space Ω which is called the spectrum of $\mathfrak{K}$. We express every element in $\mathfrak{K}$ and the representative continuous function in Ω by the same letters. The linear functional μ on $C(\Omega)$ defined by $\mu(K) = (K\mathfrak{f}, \mathfrak{f})$ is positive, and it determines a non-negative regular measure μ on Ω such that

$$(K\mathfrak{f}, \mathfrak{f}) = \int K(\lambda)\, d\mu(\lambda) \quad (K \in \mathfrak{K}).$$

Let $\mathfrak{R}$ denote the smallest uniformly closed self-adjoint algebra which contains $\mathfrak{K}$ and $\mathfrak{A}(s)$, then $\mathfrak{K}$ is the center of $\mathfrak{R}$. If X is an element in $\mathfrak{R}$, for every positive function K in $C(\Omega)$,

$$(KX\mathfrak{f}, \mathfrak{f}) = (XK^{\frac{1}{2}}\mathfrak{f}, K^{\frac{1}{2}}\mathfrak{f}) \leqq |X|(K\mathfrak{f}, \mathfrak{f}) = |X| \int K(\lambda)\, d\mu(\lambda).$$

By the theorem of Radon-Nikodym there exists a bounded measurable function $\xi_X(\lambda)$ on Ω such that $(KX\mathfrak{f}, \mathfrak{f}) = \int \xi_X(\lambda)K(\lambda)\, d\mu(\lambda)$. Putting $\xi_X(\lambda) = \lambda^{\gamma}(X)$, we have

$$(KX\mathfrak{f},\ \mathfrak{f}) = \int \lambda^{\gamma}(X)K(\lambda)d\mu(\lambda).$$

If X is a definite Hermitian in $\mathfrak{R}$, we have

$$\| K^{\frac{1}{2}}X^{\frac{1}{2}}\mathfrak{f}\|^2 = (KX\mathfrak{f},\ \mathfrak{f}) = \int \lambda^{\gamma}(X)K(\lambda)\ d\mu(\lambda) \geqq 0$$

for every positive function K in $C(\Omega)$. Then $\lambda^{\gamma}(X)$ is non-negative on Ω almost everywhere.

Let r denote the state on $\mathfrak{R}$ such that $r(X) = (X\mathfrak{f},\ \mathfrak{f})$.

If φ is a bounded non-negative measurable function on K, the linear functional r_{φ} on $\mathfrak{R}$ defined by

$$r_{\varphi}(X) = \int \lambda^{\gamma}(X)\varphi(\lambda)d\mu(\lambda)$$

is a state. Choose a constant γ such that $\psi = \gamma - \varphi$ is non-negative, then $r_{\psi} = \gamma r - r_{\varphi}$ is also a state. Notice that $L^2(r) = L^2(s)$, $\mathfrak{r} = \mathfrak{f}$, and $\mathfrak{R}(r) = \mathfrak{R}$ satisfies the conditions of the canonical representation of $\mathfrak{R}$ with respect to the state r, then we can choose a definite Hermitian K_{φ} in $\mathfrak{R}' = \mathfrak{K} = C(\Omega)$ such that

$$r_{\varphi}(X) = (XK_{\varphi}\mathfrak{f},\ \mathfrak{f}) = \int \lambda^{\gamma}(X)K_{\varphi}(\lambda)d\mu(\lambda).$$

Every element K in $\mathfrak{K}$ satisfies $K(\lambda) = \lambda^{\gamma}(K)$, and

$$r_{\varphi}(K) = \int K(\lambda)K_{\varphi}(\lambda)d\mu(\lambda) = \int K(\lambda)\varphi(\lambda)d\mu(\lambda).$$

Then φ coincides with the continuous function K_{φ} almost everywhere. We can assume that every $\lambda^{\gamma}(X)$ is continuous on Ω. For each fixed $X \in \mathfrak{R}$, the value $\lambda^{\gamma}(X)$ is uniquely determined, because the carrier of the measure μ is Ω itself. In fact, if K is a continuous function on Ω which vanishes almost everywhere, then

$$\| K\mathfrak{f}\|^2 = \int |K(\lambda)|^2 d\mu(\lambda) = 0,\quad K\mathfrak{f} = 0 \text{ and } K = 0.$$

Every λ^{γ} $(\lambda \in \Omega)$ is a state on $\mathfrak{R}$, and $\lambda \to \lambda^{\gamma}$ is a one-to-one continuous mapping of Ω in a sub-space Ω^{γ} of the space $S(\mathfrak{R})$ of all normalized states on $\mathfrak{R}$. In fact, if λ and μ are two different elements in Ω, then

$\lambda^{\gamma}(K) = K(\lambda) \neq K(\mu) = \mu^{\gamma}(K)$ for a suitable $K \in \mathfrak{K}$.

If K, L belong to $\mathfrak{K}$ and X belongs to $\mathfrak{N}$, then

$$(KLX\mathfrak{f},\ \mathfrak{f}) = \int K(\lambda)\lambda^{\gamma}(LX)d\mu(\lambda) = \int K(\lambda)L(\lambda)\lambda^{\gamma}(X)d\mu(\lambda).$$

This means $L(\lambda)\lambda^{\gamma}(X) = \lambda^{\gamma}(LX)$ for $L \in \mathfrak{K}$ and for $X \in \mathfrak{N}$.

Lemma 6. 1. *Let s be a state on $\mathfrak{A}$, and $\mathfrak{K}$ be a maximal abelian self-adjoint sub-algebra of $\mathfrak{A}(s)'$. Let Ω denote the spectrum of $\mathfrak{K}$, and embedd the algebra $\mathfrak{K}$ on $C(\Omega)$. Let μ be a regular measure on $\mathfrak{K}$ determined by $(K\mathfrak{f},\ \mathfrak{f}) = \int K(\lambda)d\mu(\lambda)$. Then every bounded measurable function on Ω coincides to a continuous function on Ω almost everywhere. Let $\mathfrak{N}$ denote the smallest uniformly closed self-adjoint algebra which contains $\mathfrak{A}(s)$ and $\mathfrak{K}$, then there exists a homeomorphism $\lambda \rightarrow \lambda^{\gamma}$ of Ω on a compact sub-space Ω^{γ} of the set $S(\mathfrak{N})$ such that*

(6. 1). $\lambda^{\gamma}(KX) = K(\lambda)\lambda^{\gamma}(X)$ *for every* $K \in \mathfrak{K}$ *and* $X \in \mathfrak{N}$.

(6. 2). $(KX\mathfrak{f},\ \mathfrak{f}) = \int \lambda^{\gamma}(X)d\mu(\lambda)$.

Such a homeomorphism $\lambda \rightarrow \lambda^{\gamma}$ is uniquely determined.

The uniqueness of the homeomorphism is shown as follows:

Let $\lambda \rightarrow \lambda^{\tau}$ be another homeomorphism which satisfies two condition (6. 1) and (6. 2). Then

$$(KX\mathfrak{f},\ \mathfrak{f}) = \int K(\lambda)\lambda^{\gamma}(X)d\mu(\lambda) = \int K(\lambda)\lambda^{\tau}(X)d\mu(\lambda).$$

Therefore $\lambda^{\tau}(X)$ and $\lambda^{\gamma}(X)$ coincide with each other.

Let $\lambda \rightarrow \lambda^{\gamma}$ be the homeomorphism between Ω and Ω^{γ} determined by the conditions (6. 1) and (6. 2). At each $\lambda \in \Omega$ a state λ^{σ} on $\mathfrak{A}$ is determined by $\lambda^{\sigma}(A) = \lambda^{\gamma}(A_s)$. $\lambda \rightarrow \lambda^{\sigma}$ is a weakly continuous mapping of the compact space Ω in a suitable compact sub-space Ω^{σ} of $\mathfrak{S} = S(\mathfrak{A})$.

Let μ^{σ} denote the measure on Ω^{σ} induced from μ by the mapping $\lambda \rightarrow \lambda^{\sigma}$, which is defined as follows: For every continuous function f on Ω^{σ} we put $\mu^{\sigma}(f) = \int f(\lambda^{\sigma})d\mu(\lambda)$. μ^{σ} is a positive linear functional on $C(\Omega^{\sigma})$, then it determines a non-negative regular measure μ^{σ} such that

(6. 3). $$\mu^{\sigma}(f) = \int f(\lambda)d\mu^{\sigma}(\lambda) = \int f(\lambda^{\sigma})d\mu(\lambda).$$

The measure μ^σ is extended as a regular measure on $\mathfrak{S}$ which vanishes out-side of Ω^σ.

Lemma 6.2. *The measure μ^σ induced by the mapping $\lambda \to \lambda^\sigma$ is a diagonal measure, whose Fourier transform is s, and whose set $\mathfrak{F}(\mu^\sigma)$ coincides with the algebra $\mathfrak{K}$.*

Proof. Every continuous function f on Ω^σ satisfies (6. 3). Then every bounded Baire's function on Ω^σ satisfies also (6. 3). Ω^σ is a separable compact space, as a sub-space of the unit-sphere of the dual space of a separable Banach space $\mathfrak{A}$. Every measurable function on Ω coincides with a bounded Baire's function almost everywhere. If f is a bounded Baire's function on Ω^σ, then $f(\lambda^\sigma)$ is a Baire's function on Ω. By Lemma (6. 1), $f(\lambda^\gamma)$ coincides to a continuous function $K_f(\lambda)$ on Ω almost everywhere.

Embedding the algebra $\mathfrak{K}$ on $C(\Omega)$, K_f expresses an element in $\mathfrak{K}$. $f \to K_f$ is an algebraic isomorphism. It is one one-to-one, linear and satisfies $K_{fg} = K_f K_g$. Further, for every $A \in \mathfrak{A}$, we have

$$(K_f A_s \mathfrak{f},\ \mathfrak{f}) = \int K_f(\lambda)\lambda^\gamma(A_s)d\mu(\lambda) = \int f(\lambda^\sigma)\lambda^\sigma(A)d\mu(\lambda)$$
$$= \int f(\nu)\nu(A)d\mu^\sigma(\nu) = (F_f A_s \mathfrak{f},\ \mathfrak{f}),$$

where $f \to F_f$ is the Fourier transform relative to the measure μ^σ. K_f and F_f belong to $\mathfrak{A}(s)'$, then they are coincident with each other. So, $f \to F_f$ is an algebraic isomorphism, and the set $\mathfrak{F}(\mu^\sigma)$ is contained in $\mathfrak{K}$.

We shall show that the set $\mathfrak{F}(\mu^\sigma)$ coincides with $\mathfrak{K}$. Let K be an arbitrary definite Hermitian in $\mathfrak{K}$. Then the linear functional τ_K on $C(\Omega^\sigma)$ such that

$$\tau_K(f) = \int f(\lambda^\sigma)K(\lambda)d\mu(\lambda)$$

is a positive linear functional on $C(\Omega^\sigma)$. Denote by γ the norm of the operator K, then $\gamma\mu^\sigma - \tau_K$ is also a positive linear functional. By the Radon'-Nykodym's theorem there exists a bounded Baire's function ψ on Ω^σ such that

$$\tau_K(f) = \int f(\lambda)\psi(\lambda)d\mu^\sigma(\lambda) \text{ for every } f \in C(\Omega^\sigma).$$

If A is an element in $\mathfrak{A}$, then $\lambda(A)$ is continuous on Ω^σ, and

$$(A_s K\mathfrak{f},\ \mathfrak{f}) = \int \lambda^{\gamma}(A_s)K(\lambda)d\mu(\lambda) = \int \lambda^{\sigma}(A)K(\lambda)d\mu(\lambda)$$
$$= \int \nu(A)\psi(\nu)d\mu^{\sigma}(\nu) = (A_s F_{\psi}\mathfrak{f},\ \mathfrak{f}).$$

K coincides with F_{ψ} and belongs to $\mathfrak{F}(\mu^{\sigma})$. $\mathfrak{F}(\mu^{\sigma})$ is a uniformly closed self-adjoint sub-algebra of $\mathfrak{K}$ which contains all definite Hermitians in $\mathfrak{K}$. Hence $\mathfrak{F}(\mu^{\sigma})$ coincides with $\mathfrak{K}$.

The regular measure μ^{σ} satisfies (4. 3) and (4. 2), then by Theorem 13 and Theorem 14 it is a diagonal measure, which vanishes out-side of $E(\mathfrak{A})$.

Lemma 6. 2 includes the former half of Theorem 11; that is, given a state s and a maximal abelian self-adjoint sub-algebra $\mathfrak{K}$ of $\mathfrak{A}(s)'$, there exists a diagonal measure μ^{σ} on $E(\mathfrak{A})$ whose Fourier transform is s and whose set $\mathfrak{F}(\mu^{\sigma})$ is the algebra $\mathfrak{K}$. To complete Theorem 11, we must show that such a diagonal measure μ^{σ} is uniquely determined.

Lemma 6.3. *Let σ be a diagonal measure on $E(\mathfrak{A})$. Let s denote the Fourier transform of σ, and $\mathfrak{K}$ denote the set $\mathfrak{F}(\sigma)$, which is a maximal abelian-self-adjoint sub-algebra of $\mathfrak{A}(s)'$. Extend the measure σ as a regular measure on $\mathfrak{S}$ which vanishes out-side of $E(\mathfrak{A})$. Then σ coincides with the measure μ^{σ} which is determined in Lemma* 6. 2.

Proof. Let the spectrum Ω of $\mathfrak{K}$, the measure μ on Ω, the algebra $\mathfrak{R}$ on $L^2(s)$, the homeomorphism $\lambda \rightarrow \lambda^{\gamma}$ between Ω and a sub-set Ω^{γ} in $S(\mathfrak{R})$, and the continuous mapping $\lambda \rightarrow \lambda^{\sigma}$ of Ω on a sub-set Ω^{σ} in $\mathfrak{S}$ be as in Lemma 6. 1 and 6. 2.

The carrier $\mathfrak{F}$ of the regular measure σ on $\mathfrak{S}$ is the smallest closed sub-set of $\mathfrak{S}$ whose mass coincides with the total mass of σ. Let $f \rightarrow F_f$ denote the Fourier transform relative to the measure σ. If λ is an element in Ω, the linear functional λ^{θ} on $C(\mathfrak{F})$ defined by $\lambda^{\theta}(f) = F_f(\lambda)$ satisfies $\lambda^{\theta}(fg) = \lambda^{\theta}(f)\lambda^{\theta}(g)$. Then we can choose a point λ^{θ} in $\mathfrak{F}$ such that $F_f(\lambda) = f(\lambda^{\theta})$. $\lambda \rightarrow \lambda^{\theta}$ is a weakly continuous mapping whose range is $\mathfrak{F}$. In fact, if $\mathfrak{U}$ is an open set in $\mathfrak{F}$, and if $f \neq 0$ is a non-negative continuous function on $\mathfrak{F}$ which vanishes out-side of $\mathfrak{U}$, then $\int f(\lambda)d\sigma(\lambda) = (F_f\mathfrak{f}, \mathfrak{f}) \neq 0$ and $F_f \neq 0$. Choose an element u in Ω such that $F_f(u) \neq 0$, then $f(u^{\theta}) = F_f(u) \neq 0$, and u^{θ} is contained in $\mathfrak{U}$. $\lambda \rightarrow \lambda^{\theta}$ maps the compact set Ω everywhere densely in $\mathfrak{F}$, then its range coincides with $\mathfrak{F}$. $\lambda \rightarrow \lambda^{\theta}$ induces the measure μ to the measure σ on $\mathfrak{F}$, because

$$\int f(\lambda)d\sigma(\lambda) = (F_f\mathfrak{f}, \mathfrak{f}) = \int F_f(\lambda)d\mu(\lambda) = \int f(\lambda^\theta)d\mu(\lambda).$$

we shall show that this mapping $\lambda \to \lambda^\theta$ coincides with the mapping $\lambda \to \lambda^\sigma$ in Lemma 6. 2. The mapping $\lambda \to \lambda^\theta$ possesses the following properties. If K is a continuous function on Ω such that $\int K(\lambda)f(\lambda^\theta)d\mu(\lambda) = 0$ for every continuous function f on $\mathfrak{F}$, then $K=0$. In fact, if K is such a continuous function on Ω, K expresses an element in $\mathfrak{K} = \mathfrak{F}(\sigma)$, and there exists a bounded measurable function φ on F such that $K = F_\varphi$. Then

$$(KF_f\mathfrak{f}, \mathfrak{f}) = \int \varphi(\lambda)f(\lambda)d\sigma(\lambda) = \int K(\nu)f(\nu^\theta)d\mu(\nu) = 0$$

for every $f \in C(\mathfrak{F})$. This means $\varphi = 0$ and $K = F_\varphi = 0$.

Let A be an element in $\mathfrak{A}$, and f be a continuous function on $\mathfrak{F}$. Then

$$\int \lambda^\theta(A) f(\lambda^\theta)d\mu(\lambda) = \int \nu(A)f(\nu)d\sigma(\nu) = (A_s F_f\mathfrak{f}, \mathfrak{f})$$
$$= \int \lambda(F_f A_s)d\mu(\lambda) = \int \lambda^\gamma(A_s)F_f(\lambda)d\mu(\lambda) = \int \lambda^\sigma(A)f(\lambda^\theta)d\mu(\lambda).$$

$\lambda^\sigma(A)$ and $\lambda^\theta(A)$ are continuous functions on Ω, then they are coincident with each other. The mapping $\lambda \to \lambda^\sigma$ coincides with the mapping $\lambda \to \lambda^\theta$, which induces the measure μ to the measnre σ. Then σ coincides with the measure μ^σ constructed in Lemma 6. 2.

By Lemma 6. 2, given each couple $(s, \mathfrak{K})$ of a state s and a maximal abelian self-adjoint sub-algebra $\mathfrak{K}$ of $\mathfrak{A}(s)'$, the diagonal measure σ on $E(\mathfrak{A})$ such that $\hat{\sigma} = s$ and $\mathfrak{F}(\sigma) = \mathfrak{K}$ is uniquely determined. Then Theorem 12 is completed.

Chapter 7. Positive definite functions and related operator algebras.

We return to the separable locally compact group $\mathfrak{G}$ considered in the Introduction. If p is a positive definite function on $\mathfrak{G}$, we denote by $\mathfrak{A}(p)$ the smallest uniformly closed self-adjoint algebra which contains the identity I and all the operators $P_f = \int f(a)P_a da$ on $L^2(p)$. As is well-known, (7.1). *The set* $(P_f\mathfrak{p} : f \in L(G))$ *is uniformly dense in* $L^2(p)$.

(7.2). $\mathfrak{G}(p)' = \mathfrak{A}(p)'$.

(7.3). *The following three conditions are equivalent.* (*a*). p *is elementary.* (*b*). *The representative group* $\mathfrak{G}(p)$ *is irreducible, that is, there is no proper closed linear sub-space of* $L^2(p)$, *invariant under the operator group* $\mathfrak{G}(p)$ *other than the* 0-*space.* (*c*). *The algebra* $\mathfrak{A}(p)$ *is irreducible.*

Hereafter we shall denote by p a fixed normalized positive definite function, and by $\mathfrak{A}$ the algebra $\mathfrak{A}(p)$ abbreviately. If u is a state on $\mathfrak{A}$, $u^\#$ denotes the linear functional on $L(\mathfrak{G})$ such that

$$(7.4). \qquad (f, u^\#) = u(P_f) \text{ for every } f \in L(\mathfrak{G}).$$

Then $u^\#$ satisfies $|(f, u^\#)| = |u(P_f)| \leq |u(I)| \| P_f \| \leq u(I) |f|_1$ and $(f^* \circ f, u^\#) = u(P_f^* P_f) \geqq 0$. $u^\#$ belongs to $M(\mathfrak{G})$ and is positive definite. $u \to u^\#$ mapps the set $\mathfrak{S}$ of all normalized states on $\mathfrak{A}$ into the set $\mathfrak{P}$ of all positive definite functions p on $\mathfrak{G}$ such that $p(e) \leqq 1$.

Lemma 7.1. $u \to u^\#$ *is a homeomorphism between* $\mathfrak{S}$ *and a compact sub-space* $\mathfrak{S}^\#$ *of* $\mathfrak{P}$.

In fact, if u and v are two elements in $\mathfrak{S}$, then there exists at least one $\alpha I + P_f \in \mathfrak{A}$ such that $u(\alpha I + P_f) \neq v(\alpha I + F_f)$. Since $u(I) = v(I) = 1$, we have $(f, u^\#) = u(P_f) \neq v(P_f) = (f, v^\#)$ and $u^\# \neq v^\#$. Then $u \to u^\#$ is a one-to-one and continuous mapping of the compact space $\mathfrak{S}$ on a compact sub-space $\mathfrak{S}^\#$ of $\mathfrak{P}$.

If $u \in \mathfrak{S}$, the canonical representation for the algebra $\mathfrak{A}$ in $L^2(u)$, and the cyclic unitary representation for $\mathfrak{G}$ on the unitary group $\mathfrak{G}(u^\#)$ on $L^2(u^\#)$ is considered. There are some relations between these representations. The representative unitary operator for an element $a \in \mathfrak{G}$ in $\mathfrak{G}(u^\#)$ is denoted by U_a, and the representative operator $\int f(a) U_a da$ on $L^2(u^\#)$ for an $f \in L(\mathfrak{G})$ is denoted by U_f. We must distinguish U_f to the operator $(P_f)_u$ which is the representative operator for P_f in $L^2(u)$. Notice that

$$(U_f \mathfrak{u}^\#, U_g \mathfrak{u}^\#) = (g^* f, u^\#) = u(P_g^* P_f) = ((P_f)_u \mathfrak{u}, (P_g)_u \mathfrak{u}).$$

Then $U_f \mathfrak{u}^\# \to (P_f)_u \mathfrak{u}$ is an isometric mapping of $L^2(u^\#)$ in $L^2(u)$. We can now embed $L^2(u^\#)$ in a sub-space of $L^2(u)$ identifying every $U_f \mathfrak{u}^\#$ and $(P_f)_u \mathfrak{u}$. Then

$$(P_f)_u(U_g\mathfrak{u}^\#) = (P_f)_u(P_g)_u\mathfrak{u} = U_f(U_g\mathfrak{u}^\#),$$

for every $f, g \in L(\mathfrak{G})$. The operator $(P_f)_u$ coincides with U_f in the space $L^2(u^\#)$. Especially, $u^\# \in L^2(u^\#)$, $(P_f)_u\mathfrak{u}^\# = U_f\mathfrak{u}^\# = (P_f)_u\mathfrak{u}$ and

$$(\mathfrak{u} - \mathfrak{u}^\#, U_f\mathfrak{u}^\#) = (\mathfrak{u} - \mathfrak{u}^\#, (P_f)_u\mathfrak{u}) = ((P_f)_u^*(\mathfrak{u} - \mathfrak{u}^\#), \mathfrak{u}) = 0.$$

Then $\mathfrak{u} - \mathfrak{u}^\#$ is orthogonal to $L^2(u^\#)$.

Lemma 7.2. *If $u \in \mathfrak{S}$, theu $L^2(u)$ contains $L^2(u^\#)$. $\mathfrak{u} - \mathfrak{u}^\#$ is orthogonal to $L^2(u^\#)$, and every element x in $L^2(u)$ is written in a unique way as*

$$x = \alpha(\mathfrak{u} - \mathfrak{u}^\#) + y, \quad y \in L^2(u^\#).$$

If $f \in L(\mathfrak{G})$, we have $(P_f)_u(\mathfrak{u} - \mathfrak{u}^\#) = 0$ and $(P_f)_u y = U_f y$ for every $y \in L^2(u^\#)$.

Notice that $\|\mathfrak{u} - \mathfrak{u}^\#\|^2 = 1 - \mathfrak{u}^\#(e)$, then

Lemma 7.3. *$L^2(u)$ coincides with $L^2(u^\#)$ if and only if $u^\#(e) = u(I) = 1$. If $u^\#(e) = 1$, then the algebra $\mathfrak{A}(u)$ coincides with the algebra $\mathfrak{A}(u^\#)$. If u belongs to $E(\mathfrak{A})$, either $u^\# = 0$, or $u^\#(e) = 1$ and $u^\# \in \mathfrak{E}$. Conversely if $u \in \mathfrak{S}$ and $u^\# \in \mathfrak{E}$, then $u \in E(\mathfrak{A})$.*

Proof. The projection E of $L^2(u^\#)$ in $L^2(u)$ belongs to $\mathfrak{A}(u)'$. Let u be an element in $E(\mathfrak{A})$, then $\mathfrak{A}(u)' = (\alpha I_u)$, and either $E = 0$ or $E = I_u$. If $E = 0$, then $u^\#$ vanishes. If $E = I_u$, then $L^2(u) = L^2(u^\#)$, $\mathfrak{u} = \mathfrak{u}^\#$, $u^\#(e) = 1$, $\mathfrak{A}(u^\#) = \mathfrak{A}(u)$, and the algebra $\mathfrak{A}(u^\#)$ is irreducible. So that $u^\#$ is elementary and belongs to $\mathfrak{E}$.

Conversely, let u be an element in $\mathfrak{S}$ such that $u^\# \in \mathfrak{E}$, then $u^\#(e) = u(I) = 1$, and $L^2(u)$ coincides with $L^2(u^\#)$, simultaneously $\mathfrak{A}(u)$ coincides with $\mathfrak{A}(u^\#)$, and is irreducible. Hence u belongsto $E(\mathfrak{A})$.

The homeomorphism $u \to u^\#$ induces a one-to-one correspondence $\mu \to \mu^\#$ between regular measures μ on $\mathfrak{S}$ and regular measures $\mu^\#$ on $\mathfrak{S}^\#$, The measure $\mu^\#$ is defined by $\mu^\#(\mathfrak{X}^\#) = \mu(\mathfrak{X})$ for every Borel set $\mathfrak{X}$, where $\mathfrak{X}^\#$ is the image of $\mathfrak{X}$ by the mapping $u \to u^\#$. Let σ be a regular measure on $\mathfrak{S}$, if φ is an element in $M(\sigma^\#)$, the function $\varphi^\#$ on $\mathfrak{S}$ defined by $\varphi^\#(u) = \varphi(u^\#)$ belongs to $M(\sigma)$. $\varphi \to \varphi^\#$ is an algebraic isomorphism between $M(\sigma^\#)$ and $M(\sigma)$, which satisfies

$$\int \varphi^\#(u)d\sigma(u) = \int \varphi(u)d\sigma^\#(u). \tag{7.5}$$

If σ is of total mass 1, then its Fourier transform $\hat{\sigma}=\int u d\sigma(u)$ is a normalized state on $\mathfrak{A}$, and for every $f\in L(\mathfrak{G})$,

$$(f, \hat{\sigma}^{\#}) = \hat{\sigma}(P_f) = \int u(P_f)d\sigma(u) = \int (f, u^{\#})d\sigma(u) = \int (f, v)d\sigma^{\#}(v).$$

That is,

$$(7.6). \qquad (f, (\hat{\sigma})^{\#}) = \int (f, v)d\sigma^{\#}(v) \quad \text{for every} \quad f\in L(\mathfrak{G}).$$

Now assume that $\sigma^{\#}(e)=1$, then by Lemma 2. 2, $\sigma^{\#}$ vanishes out-side of $\mathfrak{N}$, and the Fourier transform $\widehat{(\sigma^{\#})}$ of $\sigma^{\#}$ is $(\hat{\sigma})^{\#}$.

Lemma 7.4. *Let σ be a regular measure on $\mathfrak{S}$ with total mass 1 such that $\hat{\sigma}^{\#}(e)=1$. Then $\sigma^{\#}$ vanishes out-side of $\mathfrak{N}$, and $(\hat{\sigma})^{\#}$ is the Fourier transform $\widehat{(\sigma^{\#})}$ of $\sigma^{\#}$.*

We now assert a converse of Lemma 7. 4.

Lemma 7.5. *Let π be a regular measure on $\mathfrak{N}$ with total mass 1. If the Fourier transform $\hat{\pi}$ of π is contained in $\mathfrak{S}^{\#}$, then π vanishes out-side of $\mathfrak{S}^{\#}$, and there exists a uniquely determined regular measure σ on $\mathfrak{S}$ such that $\pi=\sigma^{\#}$.*

Proof. By the assumption that $\hat{\pi}$ belongs to $\mathfrak{S}^{\#}$, there exists a normalized state t on A such that $t^{\#}=\hat{\pi}$. Then $t^{\#}(e)=\hat{\pi}(e)=\int d\pi(u)=1$, By Lemma 7. 2 the space $L^2(t)$ coincides to $L^2(t^{\#})=L^2(\hat{\pi})$, and the algebra $\mathfrak{A}(t)$ coincides to $\mathfrak{A}(\hat{\pi})$. Extend the measure π as a regular measure on $\mathfrak{P}$ which vanishes out-side of $\mathfrak{N}$, and let $\mathfrak{D}$ denote the carier of this extended measure. If f is a non-negative continuous function on $\mathfrak{P}$ such that $\int f d\pi(\lambda)=1$, and if F_f is its relative Fourier transform, then

$$\hat{\pi}_f(a) = (P_a F_f \mathfrak{t}^{\#}, \mathfrak{t}^{\#}) = (P_a F_f \mathfrak{t}, \mathfrak{t}).$$

On the other hand, the state t_f on $\mathfrak{A}$ defined by $t_f(A)=(A_t F_f \mathfrak{t}, \mathfrak{t})$ belongs to $\mathfrak{S}$, and $t_f^{\#}=\hat{\pi}_f$.

Let v be a fixed point in $\mathfrak{D}$, and $\mathfrak{U}$ be an open neighbourhood of v. Then we can choose a non-negative continuous function $f_{\mathfrak{U}}$ on $\mathfrak{D}$ which vanishes out-side of $\mathfrak{U}$, and which satisfies $\int f_{\mathfrak{U}} d\pi(u) = 1$. The system of all those regular measures $\pi_{f_{\mathfrak{U}}}$[1] converges weakly to the point mass δ_v at v, which distributes its total mass 1 at the point v. Now we have

$$\begin{aligned}(g, \hat{\pi}_{f_{\mathfrak{U}}}) &= \int g(a)da \int u(a) f_{\mathfrak{U}}(u) d\pi(u) \\ &= \int f_{\mathfrak{U}}(u) d\pi(u) \int u(a) g(a) da \\ &= \int (u, g) f_{\mathfrak{U}}(u) d\pi(u).\end{aligned}$$

then $\hat{\pi}_{f_{\mathfrak{U}}}$ is a sequence in $M(\mathfrak{G})$ which converges weakly to v. Notice that $u \to u^{\#}$ is a homeomorphism, and that $t_{f_{\mathfrak{U}}}{}^{\#} = \hat{\pi}_{f_{\mathfrak{U}}}$, then $t_{f_{\mathfrak{U}}}$ converges weakly to a state u in $\mathfrak{S}$ such that $u^{\#} = v$. Hence the carrier $\mathfrak{D}$ of π is contained in the image $\mathfrak{S}^{\#}$ of $\mathfrak{S}$.

Let σ denote the regular measure in $\mathfrak{S}$ such that $\sigma(\mathfrak{X}) = \pi(\mathfrak{X}^{\#})$ for every Borel set $\mathfrak{X}$ in $\mathfrak{S}$, then $\sigma^{\#}$ coincides with π.

We now apply Lemma 7.4 and 7.5 to the state q on $\mathfrak{A}$ defined by $q(A) = (A\mathfrak{p}, \mathfrak{p})$, then it satisfies $q^{\#} = p$. If σ is a regular measure on $E(\mathfrak{A})$ whose Fourier transform is q, then $\sigma^{\#}$ is a regular measure on $\mathfrak{R}$ which vanishes out-side of $\mathfrak{E}$, whose Fourier transform is p. Conversely, if π is a regular measure on $\mathfrak{E}$ whose Fourier transform is p, then there exists a uniquely determined regular measure σ on $\mathfrak{S}$ such that $\sigma^{\#} = \pi$, whose Fourier transform is q.

Theorem 15. *$\sigma \to \sigma^{\#}$ is a one-to-one correspondence between the system of all regular measures on $E(\mathfrak{A})$ whose Fourier transform is q, and the system of all regular measures on $\mathfrak{E}$ whose Fourier transform is p.*

Theorem 16. *Let σ be a regular non-negative measure on $E(\mathfrak{A})$ whose Fourier transform is q, then the Fourier transform of $\sigma^{\#}$ is p. Let $\varphi \to F_\varphi$ denote the Fourier transform on $M(\sigma^{\#})$ relative to the measure $\sigma^{\#}$, and let $\varphi^{\#} \to F_{\varphi^{\#}}$ denote the Fourier transform on $M(\sigma)$ relative to the measure σ. Then F_φ coincides with $F_{\varphi^{\#}}$ for every $\varphi \in M(\sigma^{\#})$, and $\mathfrak{F}(\sigma^{\#})$ coincides with $\mathfrak{F}(\sigma)$.*

Proof. The correspondence $\sigma \to \sigma^{\#}$ between regular measures σ in $E(\mathfrak{A})$ and regular measures $\sigma^{\#}$ in $\mathfrak{E}$ induces the correspondence between their Fourier transforms $\hat{\sigma} \to (\hat{\sigma})^{\#} = \widehat{(\sigma^{\#})}$. Especially, if $\sigma_{\varphi^{\#}}$ is a measure

1) $\pi_{f_{\mathfrak{U}}}(A) = \int_A f_{\mathfrak{U}}\, d\pi$.

relative to $\varphi \in M(\sigma^{\#})$, then $(\sigma_{(\varphi^{\#})})^{\#} = (\sigma^{\#})_{\varphi}$. Notice that

$$\hat{\sigma}_{\varphi^{\#}}(A) = (AF_{\varphi^{\#}}\mathfrak{p}, \mathfrak{p}) \quad \text{for every } A \in A(p)$$

and

$$\widehat{(\sigma^{\#})}(a) = (P_a F_{\varphi}\mathfrak{p}, \mathfrak{p}) \quad \text{for every } a \in \mathfrak{G},$$

where $F_{\varphi^{\#}}$ and F_{φ} belong to $\mathfrak{A}(p)' = \mathfrak{G}(p)'$. By Lemma 7. 4 $(\widehat{\sigma_{(\varphi^{\#})}})^{\#}$ is the Fourier transform $\widehat{(\sigma^{\#})}_{\varphi}$ of $(\sigma^{\#})_{\varphi}$. Then

$$\int f(a)\,(P_a F_{\varphi^{\#}}\mathfrak{p}, \mathfrak{p})\,da = (P_f F_{\varphi^{\#}}\mathfrak{p}, \mathfrak{p}) = \widehat{\sigma_{\varphi^{\#}}}(P_f) = (f, \widehat{\sigma^{\#}}_{\varphi})$$
$$= \int f(a)\,(P_a F_{\varphi}\mathfrak{p}, \mathfrak{p})\,da.$$

This proves $(P_a F_{\varphi}\mathfrak{p}, \mathfrak{p}) = (P_a F_{\varphi}\mathfrak{p}, \mathfrak{p})$ and $F_{\varphi} = F_{\varphi^{\#}}$.

Hence the range $\mathfrak{F}(\sigma^{\#})$ of the map $\varphi \rightarrow F_{\varphi}$ coincides with the range $\mathfrak{F}(\varphi)$ of the map $\varphi^{\#} \rightarrow F_{\varphi^{\#}}$.

Theorem 17. *Let π be a bounded non-negative regular measure on $\mathfrak{E}$. If one of the following four conditions are satisfied, all other conditions are necessarily satisfied.*

(1). *π is diagonal.*

(2). *The Fourier transform $\varphi \rightarrow F_{\varphi}$ relative to π is an algebraic isomorphism.*

(3). *Let $L^2(\pi)$ denote the Hilbert space of all measurable and square summable functions on $\mathfrak{E}$, then for every $\varphi \in L^2(\pi)$ we can choose a sequence $\{f_n\}$ in $L(\mathfrak{G})$ such that*

$$\lim \int |\widehat{f_n}(\lambda) - \varphi(\lambda)|^2\, d\,\pi(\lambda) = 0$$

and

$$\lim \int \widehat{(f_n{}^* f_n)}(\lambda) d\pi(\lambda) = \int |\varphi(\lambda)|^2\, d\pi(\lambda).$$

(4). *Let p denote the Fourier transform of π, and let σ denote the regular measure on $E(\mathfrak{A}(p))$ such that $\sigma^{\#} = \pi$ and that $q(A) = (A\mathfrak{p}, \mathfrak{p}) = \hat{\pi}(A)$ for every $A \in \mathfrak{A}(p)$. Then σ is diagonal on $E(\mathfrak{A})$.*

Proof. The next Lemma asserts a characterization of the orthogonality relation between positive definite functions on $\mathfrak{G}$.

Lemma 7.6. *In order that two positive definite functions q and r be mutually orthogonal, it is necessary and sufficient that, putting $p = q + r$, the operator K in $\mathfrak{G}(p)'$ determined by $q(a) = (P_a K \mathfrak{p}, \mathfrak{p})$ be a projection.*

Proof. If q and r are mutually orthogonal positive definite functions, $u(a) = (P_a K(I - K)\mathfrak{p}, \mathfrak{p})$ is a positive definite function such that $q - u$ and $r - u$ are positive definite. This implies $u = 0$ and $K(I - K) = 0$. Then K should be a projection.

Conversely, assume that q and r be two positive definite functions such that the operator K is a projection. If v is a positive definite function such that $q - v$ and $r - v$ are positive definite, the operator V determined by $v(a) = (P_a V \mathfrak{p}, \mathfrak{p})$ is a definite Hermitian such that $K - V$ and $I - K - V$ are positive definite. This implies $V = 0$ and $v = 0$. Hence q and r are mutually orthogonal.

To prove the Theorem, it is sufficient to show that the three conditions (1), (2) and (3) are respectively equivalent to the last condition (4).

Proof that the condition (1) *is equivalent to the condition* (2).

Let π be a measure on $\mathfrak{E}$ with $\pi(\mathfrak{E}) = 1$, whose Fourier transform is p. and let σ be the measure on $E(\mathfrak{A})$ such that $\sigma^{\#} = \pi$ and $q = \hat{\sigma}$, then $q^{\#} = p$. Let $\varphi \to F_\varphi$ be the Fourier transform relative to π, and let φ denote the characteristic function on a Borel set in $\mathfrak{E}$. π is diagonal if and only if all those F_φ (φ are characteristic functions on Borel sets in $\mathfrak{E}$) are projections.

On the other hand, the measure σ is diagonal if and only if all those F_φ (φ are characteristic functions on Borel sets in $E(\mathfrak{A})$) are projections. The correspondence $\varphi \to \varphi^{\#}$ between $M(\pi)$ and $M(\sigma)$ induces the one-to-one correspondence between the set of all characteristic functions φ on those sets $\mathfrak{X}$ measurable by σ, and the system of all those characteristic functions $\varphi^{\#}$ on sub-sets $\mathfrak{X}^{\#}$ in $\mathfrak{E}$ measurable by π. Because $F_\varphi = F_{\varphi^{\#}}$ is satisfied for every $\varphi \in M(\pi)$, those sets (F_φ : *φ are characteristic functions on Borel sets in $\mathfrak{E}$*) and ($F_{\varphi^{\#}}$: *$\varphi^{\#}$ are characteristic functions on Borel sets in $E(\mathfrak{A})$*) are coincident with each other. Therefore π is diagonal if and only if σ is diagonal.

Proof that the condition (2) *is equivalent to the condition* (4).

The measure σ on $E(\mathfrak{A})$ is diagonal if and only if the relative Fourier transform $\varphi^{\#} \to F_{\varphi^{\#}}$ between $M(\sigma)$ and $\mathfrak{F}(\sigma)$ are algebraicly isomorphic. Notice that $\varphi \to \varphi^{\#}$ is an algebraic isomorphism between $M(\pi)$ and $M(\sigma)$. Then σ is diagonal if and only if the relative Fourier transform

$\varphi \rightarrow F_\varphi$ between $M(\pi)$ and $\mathfrak{F}(\pi)$ is an algebraic isomorphism.

Proof that the condition (3) *is equivalent to the condition* (4).

Let π be a non-negative regular measure on $\mathfrak{E}$ which satisfies the condition (3). If φ is a function in $L^2(\pi)$, there exists a sequence $\{f_n\}$ in $L(\mathfrak{G})$ such that

$$\lim \int |\hat{f}_n(\lambda) - \varphi(\lambda)|^2 d\pi(\lambda) = 0,$$

and

$$\lim \int \widehat{(f_n{}^*f_n)}(\lambda) d\pi(\lambda) = \int |\varphi(\lambda)|^2 d\, \pi(\lambda).$$

Notice that $\hat{f}_n(\lambda^\#) = \overline{\lambda(L_{\bar{f}_n})}$ and $\widehat{(f_n{}^*f_n)}(\lambda) = \overline{\lambda(L_{\bar{f}_n}{}^*L_{\bar{f}_n})}$ for every $\lambda \in E(\mathfrak{A})$, then we have

$$\lim \int |\overline{\lambda(L_{\bar{f}_n})} - \overline{\varphi^\#(\lambda)}|^2 d\sigma(\lambda) = 0$$

and

$$\lim \int \lambda(L_{\bar{f}_n}{}^*L_{\bar{f}_n}) d\sigma(\lambda) = \int |\overline{\varphi^\#(\lambda)}|^2 d\sigma(\lambda),$$

$\{L_{f_n}\}$ is an approximative image of the Fourier transform of $\varphi^\#$. Then the measure σ satisfies the condition (5. 4) in Theorem 13, and is a diagonal measure on $E(\mathfrak{A})$.

Conversely, assume that σ be a diagonal measure on $E(\mathfrak{A})$. The set $(P_f\mathfrak{p}: f \in L(\mathfrak{G}))$ is everywhere dense in $L^2(p) = L^2(q)$. If φ is a bounded measurable function in $M(\pi)$, then $\varphi^\#$ is a bounded measurable function in $E(\mathfrak{A})$, and we can choose a sequence $L_{f_n}\mathfrak{p}$ which converges to $F_\varphi \mathfrak{p} = F_{\varphi^\#}\mathfrak{p}$. By Lemma 4. 2, we have

$$\lim \int |\lambda(L_{f_n}) - \varphi^\#(\lambda)|^2 d\sigma(\lambda) = 0,$$

and

$$\lim \int \lambda(L_{f_n}{}^*L_{f_n}) d\sigma(\lambda) = \int |\varphi^\#(\lambda)|^2 d\sigma(\lambda).$$

Therefore

$$\lim \int |(\hat{\bar{f}}_n)(\lambda) - \overline{\varphi(\lambda)}|^2 \, d\pi(\lambda) = 0$$

and

$$\lim \int (\widehat{\bar{f}_n{}^*\bar{f}_n})(\lambda) d\pi(\lambda) = \int |\varphi(\lambda)|^2 \, d\pi(\lambda).$$

Hence the measure π satisfies the condtion (3) in the Theorem. The equivalency of the conditions (3) and (4) is thus proved. This concludes the Theorem.

The main theorems 1, 2, 3 and 4 in the Introduction are now easily shown. Theorem 1 is evident, then we shall prove the rest.

Proof of Theorem 2. Let π be a diagonal measure on $\mathfrak{E}$, p be its Fourier transform, and σ be the diagonal measure on $E(\mathfrak{A}(p))$ such that $\sigma^{\#} = \pi$. If two sequences $\{f_n\}$ and $\{g_n\}$ in $L(\mathfrak{G})$ are approximative images of the Fourier transforms of functions φ and ψ in $L^2(\pi)$ respectively, as we already observed in the proof of Theorem 17, $L_{\bar{f}_n}$ and $L_{\bar{g}_n}$ are approximative images of functions $\overline{\varphi}^{\#}$ and $\overline{\psi}^{\#}$ respectively. By Theorem 10, we have

$$\lim \int |\lambda(L_{\bar{g}_n}{}^* L_{\bar{f}_n}) - \overline{\varphi^{\#}(\lambda)}\psi^{\#}(\lambda)| \, d\pi(\lambda) = 0.$$

Notice that $\overline{\lambda(L_{\bar{g}_n}{}^* L_{\bar{f}_n})} = \lambda^{\#}(g_n{}^* f_n)$, then

$$\lim \int |(\widehat{g_n{}^* f_n})(\lambda) - \varphi(\lambda)\overline{\psi(\lambda)}| \, d\pi(\lambda) = 0.$$

This is the Parseval's equalty in Theorem 2.

Proof of the Theorem 3. If π is a diagonal measure on $\mathfrak{E}$, by Theorem 17, the relative Fourier transform $\varphi \to F_\varphi$ is an algebraic isomorphism. Therefore it is sufficient to show that $\mathfrak{F}(\pi)$ is a maximal abelian self-adjoint sub-algebra of $\mathfrak{G}(p)'$. Let p be the Fourier transform of π, and let σ be the diagonal measure on $E(\mathfrak{A}(p))$ such that

$\sigma^{\#} = \pi$. By Theorem 11, $\mathfrak{F}(\sigma) = \mathfrak{F}(\pi)$ is a maximal an abelian self-adjoint sub-algebra of $\mathfrak{A}(\sigma)' = \mathfrak{A}(p)' = \mathfrak{G}(p)'$.

Proof of Theorem 4. Let p be an arbitrary normalized positive definite function on $\mathfrak{G}$. Then the state q on $\mathfrak{A} = \mathfrak{A}(p)$ such that $q(A) = (A\mathfrak{p}, \mathfrak{p})$ is normalized. Further $L^2(p) = L^2(q)$, $\mathfrak{A}(q) = \mathfrak{A}$ and $\mathfrak{p} = \mathfrak{q}$ satisfies the conditions of the canonical representation of $\mathfrak{A}$.

According to Theorem 17, the system of all diagonal measures on $\mathfrak{E}$ whose Fourier transform is p, and the system of all diagonal measures on $E(\mathfrak{A})$ whose Fourier transform is q is one-to-one correspondent by the correspondence $\sigma \rightarrow \sigma^{\#}$. And we have always $\mathfrak{F}(\sigma) = \mathfrak{F}(\sigma^{\#})$. Given each maximal abelian self-adjoint sub-algebra $\mathfrak{R}$ of $\mathfrak{A}(p)' = \mathfrak{A}(p)' = \mathfrak{A}(q)'$, a diagonal measure σ *on* $E(\mathfrak{A})$ whose Fourier transform is q, and the range $\mathfrak{F}(\sigma)$ of which relative Fourier transform is $\mathfrak{R}$, exists and is uniquely determined. Then the diagonal measure π on $\mathfrak{E}$ whose Fourier transform is p, and the range $\mathfrak{F}(\pi)$ of which relative Fourier transform is $\mathfrak{R}$, exists and is uniquely determined.

Chapter 8. The Plancherel's theorem on unimodular groups.

A locally compact group $\mathfrak{G}$ is called *unimodular* if $\mathfrak{G}$ has a two-sided invariant (Haar) measure μ. Consider a fixed separable unimodular group $\mathfrak{G}$, and let $L^2(\mathfrak{G})$ denote the Hilbert space of all measurable and square-summable functions on $\mathfrak{G}$. Let $\mathfrak{E}$ denote the space of all elementary positive definite functions on $\mathfrak{G}$, on which the Pontrjagin's topology is defined. A Borel measure τ on $\mathfrak{E}$ is called *fundamental* if the following three conditions are satisfied.

(8.1). *$\mathfrak{E}$ is a sum of a set T of measure-0, and countable system of compact sub-sets with finite masses.*

(8.2). *If f is a function on $\mathfrak{G}$ summable and square summable by the Haar measure on $\mathfrak{G}$, then f satisfies a Plancherel's equality*

$$\int |f(a)|^2 \, da = \iiint \lambda(a^{-1}b)\overline{f(a)}\, f(b)\; da\, db\, d\tau(\lambda).$$

(8.3). *If X is a Borel set in $\mathfrak{E}$ with a finite mass, we denote by $\hat{\varphi}_X$ the function on $\mathfrak{G}$ such that*

$$\hat{\varphi}_X(a) = \int_X \overline{\lambda(a)}\, d\tau(\lambda).$$

If X and Y are mutually disjoint Borel sets in $\mathfrak{E}$ with finite masses, then

$$\int \hat{\varphi}_X(a)\overline{\hat{\varphi}_Y(a)}\, da = 0.$$

If τ is a Borel measure on $\mathfrak{E}$, we denote by $L^2(\tau)$ the Hilbert space of all functions on $\mathfrak{E}$ measurable and square summable by τ.

Theorem 18. *If τ is a fundamental measure on $\mathfrak{G}$, then τ possesses the following properties.*

(8.4). *If $f \in L^2(\mathfrak{G})$, the Fourier transform $\hat{f}$ of f:*

$$\hat{f}(\lambda) = \underset{\substack{H\to\mathfrak{G}\\ \mu(H)<\infty}}{\text{l.i.m.}} \int_H \lambda(a) f(a)\, da \ ^{1)}$$

exists. $\hat{f}(\lambda)$ is a function on $\mathfrak{G}$ and belongs to $L^2(\tau)$.

(8.5). *If $f \in L^2(\mathfrak{G})$, the inversion law*

$$f(a) = \underset{\substack{F\to\mathfrak{G}\\ \tau(F)<\infty}}{\text{l.i.m.}} \int_F \hat{f}_{a^{-1}}(\lambda)\, d\tau(\lambda) \ ^{2)}$$

is satisfied, where f_a denotes the translation of f;

$$f_a(b) = f(a^{-1}b).$$

(8.6). *If $\varphi \in L^2(\tau)$, then the Fourier transform $\hat{\varphi}$ of φ:*

$$\hat{\varphi}(a) = \underset{\substack{F\to\mathfrak{G}\\ \tau(F)<\infty}}{\text{l.i.m.}} \int_F \overline{\lambda(a)}\varphi(\lambda) d\tau(\lambda) \ ^{3)}$$

exists. $\varphi(a)$ is a function on $\mathfrak{G}$ and belongs to $L^2(\mathfrak{G})$.

(8.7). *If φ and ψ are elements of $L^2(\tau)$, then*

$$\int \hat{\varphi}(a)\overline{\hat{\psi}(a)}da = \int \varphi(\lambda)\overline{\psi(\lambda)}d\tau(\lambda)$$

and

$$\int \hat{\varphi}(a)\hat{\psi}(a^{-1}b)\, da = \int \overline{\lambda(a)}\varphi(\lambda)\psi(\lambda)\, d\tau(\lambda).$$

(8.8). *If φ is an element of $L^2(\tau)$, then $\hat{\hat{\varphi}} = \varphi$. If f is an*

---

1), 2) and 3). l.i.m. is the abbreviation of the *limit in the mean.* For instance, (8.4) implies that, given any positive number ε, we can choose a Borel sub-set H of $\mathfrak{G}$ with a finite mass such that every Borel set K with a finite mass which contains H satisfies

$$\int |\hat{f}(\lambda) - \int_K \lambda(a)f(a)\, da|^2\, d\tau(\lambda) < \varepsilon.$$

element of $L^2(\mathfrak{G})$, *then* $\hat{\hat{\hat{f}}}=\hat{f}$. *The operator* $f\rightarrow\hat{\hat{f}}$ *defined on* $L^2(\mathfrak{G})$ *is a projection operator.*

(8.9). *If* f *is an element of* $L^2(\mathfrak{G})$, *then given any positive number* ε, *we can choose a* $\varphi\in L^2(\tau)$, $\psi_1,\ldots.\psi_n\in L^2(\tau)$ *and* $a_1,\ldots.a_n\in\mathfrak{G}$ *such that*

$$\int|f(x)-\hat{\varphi}(x)-\sum_{i=1}^{n}(\hat{\psi}_i(a_ix)-\hat{\psi}_i(xa_i))|^2dx<\varepsilon.$$

A closed linear sub-space $\mathfrak{M}$ of $L^2(\mathfrak{G})$ is called *a maximal abelian sub-system* of $L^2(\mathfrak{G})$ if $\mathfrak{M}$ satisfies the next three conditions.

(8.10). $f\in\mathfrak{M}$ implies $f^*\in\mathfrak{M}$, where f^* denotes the function in $L^2(\mathfrak{G})$ such that $f^*(a)=\overline{f(a^{-1})}$.

(8.11). If $f, g\in\mathfrak{M}$, then

$$\int f(a)g(a^{-1}b)\,da=\int g(a)f(a^{-1}b)\,da.$$

(8.12). If f is an element of $L^2(\mathfrak{G})$, then given any positive number ε, we can choose $\varphi\in\mathfrak{M}$, $\psi_1,\ldots.\psi_n\in\mathfrak{M}$ and $a_1,\ldots.a_n\in\mathfrak{G}$ such that

$$\int|f(x)-\hat{\varphi}(x)-\sum(\hat{\psi}_i(a_ix)-\hat{\psi}_i(xa_i))|^2\,dx<\varepsilon.$$

Theorem 19. *If* τ *is a fundamental measure on* $\mathfrak{C}$, *the Fourier transform* $\varphi\rightarrow\hat{\varphi}$ *maps* $L^2(\tau)$ *on a maximal abelian sub-system of* $L^2(\mathfrak{G})$. *Conversely, if* $\mathfrak{M}$ *is a maximal abelian sub-system of* $L^2(\mathfrak{G})$, *a fundamental measure* τ *on* $\mathfrak{C}$ *which maps* $L^2(\tau)$ *on* $\mathfrak{M}$ *exists. The correspondence between fundamental measures and maximal abelian sub-systems of* $L^2(\mathfrak{G})$ *is one-to-one.*

Proof of the Theorem 18.

Let $L(\mathfrak{G})$ denote the set of all summable functions on $\mathfrak{G}$, and $L(\mathfrak{G})\cap L^2(\mathfrak{G})$ denote the set of all summable and square summable functions on $\mathfrak{G}$. The Plancherel's equality (8.1) implies

(8.13). $(f, g)=\int f(a)\overline{g(a)}\,da=\iiint\lambda(a^{-1}b)\overline{g(a)}f(b)\,da\,db\,d\tau(\lambda)$ *for every* $f, g\in L(\mathfrak{G})\cap L^2(\mathfrak{G})$.

A *faithfull unitary representation* of $\mathfrak{G}$ on $L^2(\mathfrak{G})$: $a\in\mathfrak{G}\rightarrow L_a$ is determined by

$$L_af=f_a\ (f_a(b)=f(a^{-1}b), f\in L^2(\mathfrak{G})).$$

If $f \in L^2(\mathfrak{G})$, the integral operator L_f is defined by

$$(L_f x)(a) = \int f(b)x(b^{-1}a)\,db \quad (x \in L^2(\mathfrak{G})).$$

Lemma 8.1. *If φ is a bounded function on $\mathfrak{E}$ measurable by τ, then there exists a bounded linear operator K_φ on $L^2(\mathfrak{G})$ which satisfies*

$$(K_\varphi f, g) = \iiint \lambda(a^{-1}b)\overline{g(a)}f(b)\,da\,db\,\varphi(\lambda)\,d\tau(\lambda)$$

for every $f, g \in L(\mathfrak{G}) \cap L^2(\mathfrak{G})$. K_φ commutes to every L_a $(a \in G)$ and to every L_f $(f \in L(\mathfrak{G}))$.

Proof. Notice that

$\iint \lambda(a^{-1}b)\overline{f(a)}f(b)\,da\,db \geqq 0$ for $\lambda \in \mathfrak{E}$ and $f \in L(\mathfrak{G}) \cap L^2(\mathfrak{G})$. A bilinear form $(f, g)_\varphi$ in $L(\mathfrak{G}) \cap L^2(\mathfrak{G})$:

$$(f, g)_\varphi = \iiint \lambda(a^{-1}b)\overline{g(a)}f(b)\,da\,db\,\varphi(\lambda)\,d\tau(\lambda)$$

satisfies the inequality

$$|\varphi|_\infty (f, f) \geqq (f, f)_\varphi \geqq -|\varphi|_\infty (f, f),$$

so that a bounded linear operator K_φ on $L^2(\mathfrak{G})$ is determined by

$$(K_\varphi f, g) = (f, g)_\varphi \quad \text{for every } f, g \in L(\mathfrak{G}) \cap L^2(\mathfrak{G}).$$

K_φ is definite if φ is non-negative and real. The norm of K_φ does not exceed $|\varphi|_\infty$. K_φ commutes to every $L_a (a \in \mathfrak{G})$, because, for $f, g \in L(\mathfrak{G})$ $L^2(\mathfrak{G})$,

$$\begin{aligned}(K_\varphi L_a f, g) &= (K_\varphi f, L_a^{-1} g)\\ &= \iiint \lambda(b^{-1}a^{-1}c)\overline{g(b)}f(c)\,db\,dc\,\varphi(\lambda)\,d\tau(\lambda).\end{aligned}$$

If $f \in L(\mathfrak{G})$, then L_f is a strong integral $L_f = \int f(a)L_a\,da$, and K_φ commutes to L_f.

Lemma 8.2. *If φ is a bounded, summable, real and non-negative function on $\mathfrak{E}$, then its Fourier transform*

$$\varphi(a) = \int \lambda(a) \varphi(\lambda)\, d\tau(\lambda)$$

is a bounded, continuous, positive, definite and square-summable function on $\mathfrak{G}$. K_φ *is an integral operator*

$$(K_\varphi f)(a) = \int \hat{\varphi}(b^{-1}a) f(b)\, db \ (f \in L^2(\mathfrak{G})).$$

There exists an element $\mathfrak{p}$ *in* $L^2(\mathfrak{G})$ *such that*

$$\varphi(a) = \int \overline{\mathfrak{p}(a^{-1}b)}\, \mathfrak{p}(b)\, db$$

Proof. Let τ_φ denote a Borel measure on $\mathfrak{E}$:

$$\tau_\varphi(X) = \int_X \varphi(\lambda)\, d\tau(\lambda).$$

τ_φ is a bounded regular measure in the sense of Definition 2. 1. then by Theorem 7, $\overline{\hat{\varphi}(a)} = \int \lambda(a)\varphi(\lambda) d\tau(\lambda)$ is a continuous positive definite function on $\mathfrak{G}$.
And

$$\begin{aligned}(K_\varphi f, g) &= \iiint \lambda(a^{-1}b)\overline{g(a)} f(b)\, da\, db\, \varphi(\lambda)\, d\tau(\lambda)\\ &= \iint f(b)\hat{\varphi}(b^{-1}a)\overline{g(a)}\, da\, db.\end{aligned}$$

Hence

$$(K_\varphi f)(a) = \int f(b)\, \hat{\varphi}(b^{-1}a)\, db.$$

$p(a) = \hat{\varphi}(a)$ is a continuous positive definite function, and the corresponding cyclic unitary representation for $\mathfrak{G}$ is dotermined as follows. There exists a Hilbert space $L^2(p)$, a strongly continuous unitary representation $a \in \mathfrak{G} \to P_a$ on $L^2(p)$ and a cyclic element $\mathfrak{p}$ in $L^2(p)$ such that $\varphi(a) = (P_a\mathfrak{p}, \mathfrak{p})$.

Then for $f, g \in L(\mathfrak{G}) \cap L^2(\mathfrak{G})$,

$$\begin{aligned}(K_\varphi^{\frac{1}{2}} f, K_\varphi^{\frac{1}{2}} g) = (K_\varphi f, g) &= \iint \overline{\hat{\varphi}(a^{-1}b) g(a)} f(b)\, da\, db\\ &= (P_f\mathfrak{p}, P_g\mathfrak{p}),\end{aligned}$$

where $P_f=\int f(a)\,P_a\,da$. Now $P_f\mathfrak{p}\to K_\varphi^{-\frac{1}{2}}f$ is an isometric mapping of $L^2(p)$ in a sub-space of $L^2(\mathfrak{G})$. We can therefore embedd the space $L^2(p)$ as a sub-space of $L^2(\mathfrak{G})$ identifying every $P_f\mathfrak{p}$ and every $K_\varphi^{-\frac{1}{2}}f$. By Lemma 8. 1 K_φ and $K_\varphi^{-\frac{1}{2}}$commute to every $L_f(f\in L(\mathfrak{G}))$, then

$$L_fP_g\mathfrak{p}=L_fK_\varphi^{-\frac{1}{2}}g=K_\varphi^{-\frac{1}{2}}(f\circ g)=P_fP_g\mathfrak{p}.$$

This means that L_f and P_f represent the same operator on $L^2(p)$. Especially $L_f\mathfrak{p}=P_f\mathfrak{p}$, and

$$\int f(a)\overline{\hat{\varphi}(a)}\,da=(P_f\mathfrak{p},\mathfrak{p})=(L_f\mathfrak{p},\mathfrak{p})$$
$$=\iint f(a)\mathfrak{p}(a^{-1}b)\overline{\mathfrak{p}(b)}\,da\,db.$$

That is,

$$\hat{\varphi}(a)=\int\overline{\mathfrak{p}(a^{-1}b)}\mathfrak{p}(b)\,db.$$

Similarly

$$\int f(a)\overline{\hat{\varphi}(a)}\,da=(L_f\mathfrak{p},\mathfrak{p})=(K_\varphi^{-\frac{1}{2}}f,\mathfrak{p})$$
$$=(f,K_\varphi^{-\frac{1}{2}}\mathfrak{p})=\int f(a)\,\overline{(K_\varphi^{-\frac{1}{2}}\mathfrak{p})\,(a)}\,da.$$

Thus $\hat{\varphi}$ coincides to $K_\varphi^{-\frac{1}{2}}\mathfrak{p}$ and belongs to $L^2(\mathfrak{G})$.

Lemma 8.3. *If φ and ψ are two bounded non-negative functions on $\mathfrak{E}$ summable by τ such that*

$$\int\hat{\varphi}(a)\,\overline{\hat{\psi}(a)}\,da=0,$$

then we have

$$\int\hat{\varphi}(b)\hat{\psi}(b^{-1}a)\,db=0.$$

$\overline{\hat{\varphi}}$ and $\overline{\hat{\psi}}$ are mutually orthogonal as a pair of positive definite functions, i.e. there is no positive definite function $r\neq 0$ such that $\overline{\hat{\varphi}}-r$ and $\overline{\hat{\psi}}-r$ are positive definite.

Proof. Let $\mathfrak{p}$ be an element in $L^2(\mathfrak{G})$ such that $\hat{\psi}(a)=\int \overline{\mathfrak{p}(a^{-1}b)}\mathfrak{p}(b)db$, then

$$\int \hat{\varphi}(a)\overline{\hat{\psi}(a)}\,da=\iint\hat{\varphi}(a)\mathfrak{p}^*(ba^{-1})\overline{\mathfrak{p}^*(b)}\,db\,da$$
$$=\iint \mathfrak{p}^*(a)\hat{\varphi}(a^{-1}b)\overline{\mathfrak{p}^*(b)}\,da\,db$$
$$=(K_\varphi\mathfrak{p}^*)(a)\overline{\mathfrak{p}^*(a)}\,da=\int|(K_\varphi^{\frac{1}{2}}\mathfrak{p}^*)\,(a)|^2\,da=0.$$

Then $K_\varphi\mathfrak{p}^*=K_\varphi^{\frac{1}{2}}(K_\varphi^{\frac{1}{2}}\mathfrak{p}^*)=0$ and

$$\int\varphi(b)\psi(b^{-1}a)\,db=\iint\varphi(b)\mathfrak{p}(a^{-1}bc)\mathfrak{p}(c)\,db\,dc$$
$$=\int\varphi(bc^{-1})\overline{\mathfrak{p}(a^{-1}b)}\mathfrak{p}(c)\,db\,dc=\int\overline{K_\varphi\mathfrak{p}^*(b)}\mathfrak{p}^*(ba)\,db=0.$$

Those $\bar{\hat{\varphi}}(a)$ and $\bar{\hat{\psi}}(a)$ are mutually orthogonal positive definite functions. If they are not mutually orthogonal, there is a positive definite function $r\neq 0$ such that $\bar{\hat{\varphi}}-r$ and $\bar{\hat{\psi}}-r$ are positive definite. Since

$$0\leqq\iint r(a^{-1}b)\,\overline{f(a)}\,f(b)\,da\,db\leqq\iint\overline{\varphi(a^{-1}b)\overline{f(a)}}f(b)\,da\,db\leqq|\varphi|_\infty(f,f)$$

for every f in $L(\mathfrak{G})\cap L^2(\mathfrak{G})$, there exists a definite Hermitian K_r on $L^2(\mathfrak{G})$ such that

$$\iint r(a^{-1}b)\overline{g(a)}\,f(b)\,da\,db=(K_rf,\,g).$$

Operators K_r, $K_\varphi-K_r$ and $K_\psi-K_r$ are simultaneously definite Hermitians, whereas $K_\varphi K_\psi=0$. Then $K_r=0$ and $r=0$, which contradicts to the assumption that $r\neq 0$. Hence $\bar{\hat{\varphi}}$ and $\bar{\hat{\psi}}$ are mutually orthogonal.

By Lemma 8. 3 and the condition (8. 3), we have immediately

Lemma 8.4. *If A and B are mutually disjoint Borel sets in $\mathfrak{F}$ with finite masses, then $\hat{\varphi}_A$ and $\hat{\varphi}_B$ are mutually orthogonal positive definite functions such that*

$$\int\hat{\varphi}_A(b)\varphi_B(b^{-1}a)\,db=0\,.$$

Lemma 8.5. *If A is a Borel set in $\mathfrak{F}$ with a finite mass, then* *the operator* $K_A = K_{\varphi_A}$ *is a projection operator, and*

$$\overline{\hat{\varphi}_A(a^{-1})} = \hat{\varphi}_A(a) = \int \varphi_A(b)\varphi_A(b^{-1}a)\, db.$$

Proof. Let $\{X_i\}$ be an assending sequence of Borel sets in $\mathfrak{E}$ with finite masses such that $\tau(\mathfrak{E} - (\sum_i X_i)) = 0$, and let A_n denote the common part of X_n and A. Put $\varphi_n = \varphi_{X_n}$, $\psi_n = \varphi_{A_n}$, $K_n = K_{\varphi_n}$ and $T_n = K_{\psi_n}$. Then by Lemma 8. 4,

$$K_n K_A = T_n K_A \quad (n = 1, 2, \ldots\ldots).$$

The norms of those operators K_n and T_n are not larger than 1., $\{K_n\}$ and $\{T_n\}$ converge weakly to the identity I and the operator K_A respectively. In fact, if f, g are elements in $L(\mathfrak{G}) \cap L^2(\mathfrak{G})$ then

$$\begin{aligned}(K_n f, g) &= \iiint_{X_n} \lambda(a^{-1}b)g(a)f(b)\, da\, db\, d\tau(\lambda) \\ &\rightarrow \iiint_E \lambda(a^{-1}b)g(a)f(b)\, da\, db\, d\tau(\lambda) = (f, g),\end{aligned}$$

and

$$\begin{aligned}(T_n f, g) &= \iiint_{A_n} \lambda(a^{-1}b)g(a)f(b)\, da\, db\, d\tau(\lambda) \\ &\rightarrow \iiint_A \lambda(a^{-1}b)g(a)f(b)\, da\, db\, d\tau(\lambda) = (K_A f, g)\end{aligned}$$

as $n \rightarrow \infty$. So that $K_A = K_A K_A = K_A^*$, and K_A is a projection, that is,

$$\hat{\varphi}_A(a) = \int \hat{\varphi}_A(b)\hat{\varphi}_A(b^{-1}a)\, db.$$

Lemma 8.6. *Let A be a Borel set in $\mathfrak{E}$ with a finite mass. If φ and ψ are functions in $L^2(\tau)$ which vanish out-side of A, then*

$$\int \hat{\varphi}(b)\hat{\psi}(b^{-1}a)\, db = \int \overline{\lambda(a)}\varphi(\lambda)\psi(\lambda)\, d\tau(\lambda)$$

and

$$\int \hat{\varphi}(a)\overline{\psi(a)}\, da = \int \varphi(\lambda)\overline{\psi(\lambda)}\, d\tau(\lambda).$$

Proof. Define a Borel measure τ_A on $\mathfrak{E}$ by $\tau_A(X) = \tau(X \cap A)$. τ_A

is a regular measure on $\mathfrak{G}$, and $p(a)=\overline{\hat{\varphi}_A(a)}=\int \lambda(a)\,d\tau_A(\lambda)$ is a continuous positive definite function. If B is a Borel set contained in A, then by Lemma 8. 3 $(\overline{\hat{\varphi}_B(a)}=\int_B \lambda(a)\,d\tau_A(\lambda)$ and $\overline{\hat{\varphi}_{A-B}(a)}=\int_{(A-B)} \lambda(a)\,d\tau_A(\lambda)$ are mutually orthogonal positive definite functions. Therefore τ_A is a diagonal measure on $\mathfrak{G}$. By Lemma 8. 5,

$$p(a)=\overline{\hat{\varphi}_A(a)}=\int \hat{\varphi}_A(b)\hat{\varphi}_A(b^{-1}a)\,db=(L_a\hat{\varphi}_A,\hat{\varphi}_A).$$

The smallest closed linear sub-space $L^2(p)$ which contains all $L_a\hat{\varphi}_A$ $(a\in\mathfrak{G})$ is the range of the projection operator K_A, and the group of those unitary operators L_a restricted on the space $L^2(p)$ is the cyclic unitary representation group which corresponds to the positive definite function p. If φ is a real, bounded and measurable function on $\mathfrak{G}$ which vanishes out-side of A, then we can choose a constant γ such that $\gamma K_A\pm K_\varphi$ are positive definite. K_A is a projection, then $K_\varphi K_A=K_A K_\varphi=K_\varphi$ and $\hat{\varphi}(a)=$ $=\int \hat{\varphi}(b)\hat{\varphi}_A(b^{-1}a)\,db=\int \hat{\varphi}_A(b)\hat{\varphi}(b^{-1}a)db=(K_\varphi\hat{\varphi}_A)\,(a)$. The operator K_φ restricted on $L^2(p)$ commutes to every operator L_a restricted on $L^2(p)$, and satisfies

$$\begin{aligned}(L_aK_\varphi\hat{\varphi}_A,\hat{\varphi}_A)&=(K_\varphi\hat{\varphi}_A, L_a{}^{-1}\hat{\varphi}_A L_a{}^{-1}\varphi_A)\\&=(\varphi, L_a{}^{-1}\hat{\varphi}_A)\\&=\int \hat{\varphi}(b)\overline{\hat{\varphi}_A(ab)}\,db=\int \hat{\varphi}(b)\hat{\varphi}_A(b^{-1}a^{-1})\,db\\&=\hat{\varphi}(a^{-1})=\int \lambda(a)\varphi(\lambda)d\tau_A(\lambda).\end{aligned}$$

Then $\varphi\in M(\tau_A)\to K_\varphi$ is the Fourier transform relative to the measure τ_A. τ_A is a diagonal measure, and by Theorem 3 $\varphi\in M(\tau_A)\to K_\varphi$ is an algebraic isomorphism. If φ, ψ are bounded measurable functions which vanish out-side of A, we have $K_{\varphi\psi}=K_\varphi K_\psi$ and $\widehat{\varphi\psi}=\hat{\varphi}\circ\hat{\psi}$. That is,

$$\int \overline{\lambda(a)}\varphi(\lambda)\psi(\lambda)\,d\tau(\lambda)=\int \hat{\varphi}(b)\hat{\psi}(b^{-1}a)\,db.$$

Notice that $\hat{\psi}(a)=\overline{(\hat{\psi}a^{-1})}=\int \overline{\lambda(a)\psi(\lambda)}\,d\tau_A(\lambda)$,

then

$$\int \varphi(\lambda)\overline{\psi(\lambda)}\,d\tau(\lambda)=\int \hat{\varphi}(a)\overline{\hat{\psi}(a)}\,da.$$

Finally, if φ and ψ are two arbitrary elements in $L^2(\tau)$ which vanishes out-side of A, we can choose two sequences $\{\varphi_n\}$ and $\{\psi_n\}$ of bounded measurable functions on $\mathfrak{C}$ which vanish out-side of A, and which satisfy $\int |\varphi_n(\lambda) - \varphi(\lambda)|^2 \, d\tau(\lambda) \to 0$, $\int |\varphi_n(\lambda) - \psi(\lambda)|^2 \, d\tau(\lambda) \to 0$ as $n \to \infty$. Then

$\hat{\varphi}_n(a) = \int \overline{\lambda(a)}\varphi_n(\lambda) \, d\tau(\lambda) \to \hat{\varphi}(a) = \int \overline{\lambda(a)}\varphi(\lambda) \, d\tau(\lambda)$ and $\hat{\psi}_n(a) \to \hat{\psi}(a)$ uniformly on $\mathfrak{G}$ as $n \to \infty$.

Therefore

$$\int |\hat{\varphi}_n(a) - \hat{\varphi}(a)|^2 \, da = \int |\varphi_n(\lambda) - \varphi(\lambda)|^2 \, d\tau(\lambda) \to 0,$$

$$\int |\hat{\psi}_n(a) - \hat{\psi}(a)|^2 \, da = \int |\psi_n(\lambda) - \psi(\lambda)|^2 \, d(\lambda) \to 0;$$

$$\int \hat{\varphi}(a)\overline{\hat{\psi}(a)} \, da = \int \varphi(\lambda)\overline{\psi(\lambda)} \, d\tau(\lambda).$$

and

$$\int \hat{\varphi}(b)\hat{\varphi}(b^{-1}a) \, db = \int \overline{\lambda(a)}\varphi(\lambda)\psi(\lambda) \, d\tau(\lambda).$$

We are now in the position to prove (8. 4)... (8. 9) in the Theorem 18. The system of all elements in $L^2(\tau)$ which vanish out-side of certain Borel sets in $\mathfrak{C}$ with finite masses, is everywhere dense in $L^2(\tau)$, and for every element ψ in this system the Fourier transform $\psi(a) = \int \overline{\lambda(a)}\psi(\lambda) d\tau(\lambda)$ is determined. This transform $\psi \to \hat{\psi}$ is isometric, and extended to an isometric transform of $L^2(\tau)$ on a closed linear sub-space $\mathfrak{M}$ of $L^2(\mathfrak{G})$. Let ψ be an element in $L^2(\tau)$. For each Borel set F in $\mathfrak{C}$ with a finite mass the function ψ_F on $\mathfrak{C}$: $\psi_F(\lambda) = \psi(\lambda)$ for $\lambda \in F$ and $\psi_F(\lambda) = 0$ out-side of F: is defined, and satisfies $\psi = \underset{F \to \mathfrak{C}, \tau(F) < \infty}{\text{l.i.m.}} \psi_F$. That is, for every positive number ε, there exists a Borel set F_0 in $\mathfrak{C}$ with a finite mass such that every Borel set F with $F \supseteqq F_0$ and with a finite mass satisfies $\int |\psi(\lambda) - \psi_F(\lambda)|^2 \, d\tau(\lambda) < \varepsilon$. Then the Fourier transform of ψ:

$$\hat{\psi}(a) = \underset{F \to \mathfrak{C}}{\text{l.i.m.}} \, \hat{\psi}_F(a) = \underset{F \to \mathfrak{C}}{\text{l.i.m.}} \int_F \overline{\lambda(a)}\psi(\lambda) \, d\tau(\lambda)$$

exists.

Notice that, if φ, ψ belong to $L^2(\tau)$,

$$\int_F \overline{\lambda(a)}\varphi(\lambda)\psi(\lambda)\, d\tau(\lambda) = \int \hat{\varphi}_F(b)\hat{\psi}_F(b^{-1}a)\, db$$
$$\to \int \hat{\varphi}(b)\hat{\psi}(b^{-1}a)\, db$$

and

$$\int_F \varphi(\lambda)\psi(\lambda)\, d\tau(\lambda) = \int \hat{\varphi}_F(a)\overline{\hat{\psi}_F(a)}\, da$$
$$\to \int \hat{\varphi}(a)\overline{\hat{\psi}(a)}\, da$$

uniformly on $\mathfrak{G}$ as $F \to \mathfrak{F}$. Then we have the two equality in (8. 10).

Proof of (8. 7). The Fourier transform $\hat{f}(\lambda)$ of an $f \in L(\mathfrak{G}) \cap L^2(\mathfrak{G})$: $\hat{f}(\lambda) = \int \lambda(a) f(a)\, da$ is a bounded continuous function on $\mathfrak{G}$, and if φ is a function in $L^2(\tau)$ which vanishes out-side of a certain Borel sub-set of $\mathfrak{F}$ with a finite mass, then f satisfies

$$\int \hat{f}(\lambda)\overline{\varphi(\lambda)}\, d\tau(\lambda) = \iint f(a)\lambda(a)\overline{\varphi(\lambda)}\, d\tau(\lambda)\, da$$
$$= \int f(a)\overline{\hat{\varphi}(a)}\, da.$$

Denote by P the projection operator on the set $\mathfrak{M} = (\hat{\varphi} : \varphi \in L^2(\tau))$, and ψ be an element of $L^2(\tau)$ such that $Pf = \hat{\psi}$. Then

$$\int \hat{f}(\lambda)\overline{\varphi(\lambda)}\, d\tau(\lambda) = \int f(a)\overline{\hat{\varphi}(a)}\, da = \int \hat{\psi}(a)\overline{\hat{\varphi}(a)}\, da$$
$$= \int \psi(\lambda)\overline{\varphi(\lambda)}\, d\tau(\lambda)$$

$\hat{f}$ coincides with ψ and belongs to $L^2(\tau)$. The transform $f \to \hat{\hat{f}}$ is bounded and linear on $L(\mathfrak{G}) \cap L^2(\mathfrak{G})$, which is extended to a bounded linear transform of $L^2(\mathfrak{G})$ in $L^2(\tau)$. If f is an element in $L^2(\mathfrak{G})$ and H is a Borel sub-set of $\mathfrak{G}$ with a finite mass, a function f_H on $\mathfrak{G}$ is defined by $f_H(a) = f(a)$ on H and $f_H(a) = 0$ out-side of H. f_H converges to f by the topology of $L^2(\mathfrak{G})$ as $H \to \mathfrak{G}$. Then $\hat{f}_H$ converges to $\hat{f}$ by the topolgy of $L^2(\tau)$ as $H \to \mathfrak{G}$. That is, the Fourier transform of f:

$$\hat{f} = \underset{\substack{H\to\mathfrak{G} \\ \mu(H)<\infty}}{\text{l.i.m.}}\ \hat{f}_H = \underset{\substack{H\to\mathfrak{C} \\ \mu(H)<\infty}}{\text{l.i.m.}} \int_H \lambda(a) f(a)\, da$$

exists and belongs to $L^2(\mathfrak{G})$. Here

$$\hat{\hat{f}} = \underset{\substack{H\to\mathfrak{G} \\ \mu(H)<\infty}}{\text{l.i.m.}}\ \hat{\hat{f}}_H = \underset{\substack{H\to\mathfrak{G} \\ \mu(H)<\infty}}{\text{l.i.m.}}\ P(f_H) = Pf.$$

P is the projection operator on the space $(\hat{\varphi} : \varphi \in L^2(\tau))$. From this, we have $\hat{\hat{\varphi}} = \varphi$ for every $\varphi \in L^2(\tau)$. In fact, if $f \in L^2(\mathfrak{G})$, then

$$(\hat{\hat{f}}, \hat{\psi}) = (Pf, \hat{\psi}) = (f, \hat{\psi}) \text{ for every } \psi \in L^2(\tau).$$

Especially, if $\varphi \in L^2(\tau)$, then $\hat{\varphi} \in L^2(\mathfrak{G})$ and

$$(\varphi, \psi) = (\hat{\varphi}, \hat{\psi}) = (\hat{\hat{\varphi}}, \hat{\psi}) = (\hat{\hat{\varphi}}, \psi)$$

for every $\psi \in L^2(\tau)$. This means $\varphi = \hat{\hat{\varphi}}$.

We next show the invertion formula (8. 4).

If F is a Borel sub-set of $\mathfrak{C}$ with a finite mass, then the operator K_F on $L^2(\mathfrak{G})$:

$$(K_F f)(a) = \int f(b) \hat{\varphi}_F(b^{-1}a)\, db$$

is a projection operator, and converges strongly to the identity as $F \to \mathfrak{C}$. Then for every $f \in L^2(\mathfrak{G})$,

$$f(a) = \underset{F\to\mathfrak{C}}{\text{l.i.m.}}\ (K_F f)(a) = \underset{F\to\mathfrak{C}}{\text{l.i.m.}} \int \hat{\varphi}_F(b^{-1}a) f(b)\, db$$

$$= \underset{F\to\mathfrak{C}}{\text{l.i.m.}} \int \overline{\hat{\varphi}_F(b)}\, f(ab)\, db$$

$$= \underset{F\to\mathfrak{C}}{\text{l.i.m.}} \int_F \hat{f}_{a^{-1}}(\lambda)\, d\tau(\lambda).$$

Proof of (8. 12).

Lemma 8.7. *Let f be a function in $L^2(\mathfrak{G})$ under the condition of*

(8.14). $\int f(a)\hat{\varphi}(a^{-1}b)\,da = \int \hat{\varphi}(a)f(a^{-1}b)\,da$ *for every* $\varphi \in \mathfrak{M}$.

Then f is a Fourier transform of a suitable $\psi \in L^2(\tau)$.

Proof. If f in $L^2(\mathfrak{G})$ is under the condition of (8.14), then $f_1 = \frac{1}{2}(f_1 + f^*)$ and $f_2 = \frac{1}{2i}(f - f^*)$ are also under the condition of (8.14), and satisfy $f_1^* = f_1$, $f_2^* = f_2$. Since $f = f_1 + if_2$, it is sufficient to prove the Lemma assuming that f is under the conditions of (8.14) and

(8.15). $f = f^*$.

If g is a function in $L^2(\mathfrak{G})$, we denote by R_g the closed operator in $L^2(\mathfrak{G})$:

$$(R_g h)(a) = \int g(b)g(b^{-1}a)\,db \quad \text{for every } h \in \mathfrak{D}(R_f).$$

$\mathfrak{D}(R_f)$ denotes the set of all h in $L^2(\mathfrak{G})$ such that

$$\int \left| \int h(b)g(b^{-1}a)\,db \right|^2 da < \infty.$$

The adjoint operator of R_g is the operator R_{g*}. In fact, if h and k are two elements in $L^2(\mathfrak{G})$ such that $(x, k) = (R_g x, h)$ for every $x \in \mathfrak{D}(R_g)$, then $h \in \mathfrak{D}(R_{g*})$ and $k = R_{g*}h$.

f satisfies the conditions (8.14) and (8.15), then R_f is self-adjoint and commutes to every K_φ as φ are bounded and square summable functions on $\mathfrak{C}$. R_f has the spectral resolution $R_f = \int \lambda\, d\, E(\lambda)$, where every $E(\lambda)$ commutes to every K_φ (as φ is bounded and in $L^2(\tau)$).

f is the limit of the sequence $(f_n : f_n = E(n)f - E(-n-0)f)$ in $L^2(\mathfrak{G})$, whereas $R_{f_n} = (E(n) - E(-n-0))R_f = \int_{-n}^{n} \lambda\, dE(\lambda)$ commutes to every K_φ. Then f_n satisfies the conditions (8.14) and (8.15). If every f_n is the Fourier transform of a certain φ_n in $L^2(\tau)$, then $f = \text{l.i.m. } f_n$ is the Fourier transform of l.i.m. $\varphi_n = \varphi$ in $L^2(\tau)$. Therefore it is sufficient to prove the Lemma assuming that f in $L^2(\mathfrak{G})$ is under the conditions of (8.14), (8.15) and

(8.16) R_f is bounded linear.

Assume that an element f in $L^2(\mathfrak{G})$ satisfies the three above conditions. For each Borel set F in $\mathfrak{C}$ with a finite mass, we put $f_F = K_F f$

$=\int f(b)\hat{\varphi}_F(b^{-1}a)\,db$, then $\int f_F(b)\hat{\varphi}_F(b^{-1}a)\,db=f_F(a)$ and $K_F R_{f_F}=R_{f_F}K_F$ $=R_{f_F}$.

If every f_F is the Fourier transform of a certain φ_F in $L^2(\tau)$, then $f=\underset{F\to\mathfrak{E}}{\text{l.i.m.}}\, f_F$ is the Fourier transform of $\varphi=\underset{F\to\mathfrak{E}}{\text{l.i.m.}}\,\varphi_F$ in $L^2(\tau)$, therefore it is sufficient to prove the Lemma assuming that the considered element f in $L^2(\mathfrak{G})$ satisfies the four conditions (8.14), (8.15), (8.16)and

(8.17). $R_f K_F=K_F R_f=R_f$ for a certain Borel set F in $\mathfrak{E}$ with a finite mass.

If F is a Borel set in the condition (8.17), as we already observed, the regular measure τ_F on $\mathfrak{E}:\tau_F(X)=\tau(F\cap X)$ is a diagonal measure, and $p(a)=\overline{\hat{\varphi}_F(a)}=(L_a\varphi_F,\varphi_F)=\int\lambda(a)\,d\tau_F(\lambda)$ is a continuous positive definite function. The Hilbert space $L^2(p)$ of the cyclic unitary representation for $\mathfrak{G}$ which corresponds to $p(a)$ is the range of the projection K_F. If φ is a bounded function on $\mathfrak{E}$ measurable by τ and which vanishes out-side of F, then its Fourier transform relative to the measure τ_F is the restriction of the operator K_φ on the space $L^2(p)$. Then the set of all bounded operators K_φ restricted on the space $L^2(p)$ consists of a maximal abelian sub-algebra of the commutor $\mathfrak{G}(p)'$ of $\mathfrak{G}(p)$.

By the assumption f satisfies the four conditions (8.14).... (8.17), then the restriction of the operator R_f on $L^2(p)$ commutes to every operators L_a and K_φ restricted on $L^2(p)$, and consequently coincides with the restriction of a certain K_φ such that ψ is bounded and vanishes outside of F. Those operators K_ψ and R_f are coincident with each other on $L^2(\mathfrak{G})$; f and $\hat{\psi}$ are coincident with each other. This establishes the Lemma 8.7.

We shall now prove (8.12) of Theorem 18.

For each $a\in\mathfrak{G}$ and $f\in L^2(\mathfrak{G})$, let $\Gamma_a f$ denote the function in $L^2(\mathfrak{G})$: $(\Gamma_a f)(b)=f(ab)-f(ba)$. The set $\mathfrak{M}=(\hat{\varphi}:\varphi\in L^2(\tau))$ is a closed linear sub-space of $L^2(\mathfrak{G})$, and by Lemma 8.7 an element f in $L^2(\mathfrak{G})$ belengs to $\mathfrak{M}$ if and only if f is orthogonal to every $\Gamma_a\hat{\varphi}$ $(a\in\mathfrak{G},\hat{\varphi}\in\mathfrak{M})$. Then the orthogonal component of the space $\mathfrak{M}$ in $L^2(\mathfrak{G})$ is spaned by all those $\Gamma_a\hat{\varphi}$, and the smallest linear set which contains all $\mathfrak{M}$ and all $\Gamma_a\hat{\varphi}$ $(a\in\mathfrak{G}$ $\hat{\varphi}\in\mathfrak{M})$ is everywhere dense in $L^2(\mathfrak{G})$. This concludes (8.12). Thus Theorem 18 is completed.

An auxiliary result for self-adjoint operators.

Let A and B denote two self-adjoint operators on a Hilbert space $\mathfrak{H}$

with the respective spectral resolutions $A = \int \lambda dE(\lambda)$ and $B = \int \mu dF(\mu)$. We assume that A and B commute[1] to each other, then every pairs $E(\lambda)$ and $F(\mu)$ commute to each others[2].

A complex resolution of the identity $\{G(z)\}$ is the system of all those projections $G(z)$ defined for every complex number $z = \lambda + i\mu$ as $G(z) = E(\lambda)F(\mu)$.

If φ is a real Baire's function on the complex number field $\mathfrak{K}$, which is bounded in each open circular in $\mathfrak{K}$ with finite radius, then a self-adjoint operator $\varphi(A + iB) = \int_K \varphi(z)dG(z)$ is determined by

$$(\varphi(A + iB)x, y) = \int \varphi(z)\, d\, (E(z)x, y)$$

for every x, y in the domain $\mathfrak{D}(\varphi(A + iB))$, where the domain $\mathfrak{D}(\varphi(A + iB))$ of the operator is the set of all those x in $\mathfrak{H}$ with

$$\int |\varphi(z)|^2\, d\, \| G(z)\, x \|^2 < \infty,$$

and integrals $\int \varphi(z)\, d\, (G(z)x, y)$ are Lebesgue-Radon-Stieltjes integrals[3] on $\mathfrak{K}$. If φ is uniformly bounded on $\mathfrak{K}$, $\varphi(A + iB)$ is a bounded linear operator. Clearly we have $A = \int \lambda dG(\lambda + i\mu)$ and $B = \int \mu dG(\lambda + i\mu)$, then we put $A - B = \int (\lambda - \mu)dG(\lambda + i\mu)$. $A - B$ is a self-adjoint operator.

Lemma 8.8. *Let g be an element in the domain $\mathfrak{D}(A - B)$ with $(A - B)g = 0$, then $E(\lambda)g = F(\lambda)g$ for every real number λ.*

Proof. Let φ be a real function on the real number field $\mathfrak{R}$, which is bounded and continuous together with its derivative $\varphi'(\lambda)$. Then the function ψ on $\mathfrak{K}$:

$$\psi(\lambda + i\mu) = (\varphi(\lambda) - \varphi(\mu))\,(\lambda - \mu) \quad \text{for } \lambda \neq \mu,$$

$$\psi(\lambda + i\mu) = \varphi'(\lambda) \qquad\qquad \text{for } \lambda = \mu$$

1) [7]. P. 301. Def. 8.2.

2) M.H. Stone [7]. P. 314. Def. 8.5.

3) M.H. Stone [7] P.P. 312—313

is a uniformly bounded continuous function on $\mathfrak{K}$, and, if g is an element in $\mathfrak{D}(A-B)$ with $(A-B)g=0$, then

$$\|(A-B)g\|^2=\int|(\lambda-\mu)|^2\,d\,\|G(\lambda+i\mu)g\|^2=0$$

and

$$\|\varphi(A)g-\varphi(B)g\|^2=\int|\varphi(\lambda)-\varphi(\mu)|^2\,d\,\|G(\lambda+i\mu)g\|^2$$
$$=\int\psi(\lambda+i\mu)^2(\lambda-\mu)^2\,d\,\|G(\lambda+i\mu)g\|^2=0.$$

where $\varphi(A)=\int\varphi(\lambda)\,dE(\lambda)$ and $\varphi(B)=\int\varphi(\lambda)\,dF(\lambda)$ are bounded Hermitian operators on $\mathfrak{H}$. Hence

Sub-lemma 1. *If φ is a function on $\mathfrak{R}$ bounded and continuous together with its derivative, then $\varphi(A)g=\varphi(B)g$.*

If $\{\varphi_n\}$ is a uniformly bounded sequence of Baire's functions on $\mathfrak{R}$ with $\varphi_n(\lambda)\to\varphi(\lambda)$ at each point in $\mathfrak{R}$, then $\varphi_n(A)g$ converges to $\varphi(A)g$. In fact,

$$\|\varphi_n(A)g-\varphi(A)g\|^2=\int|\varphi_n(\lambda)-\varphi(\lambda)|^2\,d\,\|E(\lambda)g\|^2\to 0$$

as $n\to\infty$. Then

Sub-lemma 2. *If $\{\varphi_n\}$ is a sequence of uniformly bounded Baire's functions on $\mathfrak{R}$ with $\varphi_n(\lambda)\to\varphi(\lambda)$ at each point in $\mathfrak{R}$, and if $\varphi_n(A)g=\varphi_n(B)g$ for $n=1,\ 2,\ldots,$ then we have $\varphi(A)g=\varphi(B)g$.*

By Sub-lemma 1 and Sub-lemma 2, we have $\varphi(A)g=\varphi(B)g$ for every bounded Baire's function φ on $\mathfrak{R}$. Let φ_λ be the characteristic function on the interval $(-\infty,\lambda)$, then

$$E(\lambda)g=\varphi_\lambda(A)g=\varphi_\lambda(B)g=F(\lambda)g.$$

Thus Lemma 8. 8 is proved.

Lemma 8.9. *If f and g are two elements in $L^2(\mathfrak{G})$ such that $f=f^*$, $g=g^*$ and $\int f(b)g(b^{-1}a)db=\int g(b)f(b^{-1}a)\,db$, then two self-adjoint operators R_f and R_g commute to each other.*

Proof. The operator $R_f : (R_f h)(a) = \int h(b) f(b^{-1}a)\, db$ for every $h \in \mathfrak{D}(R_f)$: is self-adjoint, and has the spectral resolution $R_f = \int \lambda d E(\lambda)$. $\mathfrak{D}(R_f)$ is the set of all $h \in L^2(\mathfrak{G})$ such that $\int h(b) f(b^{-1}a)\, db$ belongs to $L^2(\mathfrak{G})$.

The operator Z on $L^2(\mathfrak{G}) : h \to Zh = h^*$ is conjugate-linear, isometric and $Z^2 = I$. Then the operator $L_f = ZR_fZ$ is a self-adjoint operator. $\mathfrak{D}(L_f)$ is the set of all h in $L^2(\mathfrak{G})$ such that $\int f(b) h(b^{-1}a)\, db$ belongs to $L^2(\mathfrak{G})$. For every h in $\mathfrak{D}(L_f)$ we have $L_f h = \int f(b) h(b^{-1}a) db$. The spectral resolution of L_f is $L_f = \int \lambda d\, ZE(\lambda)Z$. Put $f_n = E(n)f - E(-n-0)f$, then $R_{f_n} = \int_{-n}^{n} \lambda\, d\, E(\lambda)$, and $L_{f_n} = Z\, R_{f_n} Z = \int_{-1}^{n} \lambda\, d\, ZE(\lambda)\, Z$. Every pairs R_{f_n} and L_{f_m} commute to each others, in fact,

$$R_{f_n} L_{f_m} h = L_{f_m} R_{f_n} h = (f_m \circ h \circ f_n)(a) \text{ for every } h \in L^2(\mathfrak{G}).$$

Every pairs $E(\lambda)$ and $ZE(\mu)Z$, consequently the pair R_f and L_f commute to each others, respectively.

Every element h in $L(\mathfrak{G}) \cap L^2(\mathfrak{G})$ belongs to $\mathfrak{D}(R_f)$, $\mathfrak{D}(L_f)$ and $\mathfrak{D}(R_f - L_f)$, and satisfies

$(R_f - L_f)h = R_f h - L_f h = \int f(b)\,(h(b^{-1}x) - h(xb^{-1}))db$. If k is an element in $\mathfrak{D}(R_f - L_f)$, for every h in $L(\mathfrak{G}) \cap L^2(\mathfrak{G})$, we have

$$\begin{aligned}((R_f - L_f)k, h) &= (k, (R_f - L_f)h) \\ &= \iint f(b)\,(k(b^{-1}a) - k(ab^{-1}))\,\overline{h(a)}\, da db.\end{aligned}$$

Then

$$(R_f - L_f)k\,(a) = \int f(b)\,(k(b^{-1}a) - k(ab^{-1}))db.$$

Conversely, if k is an element in $L^2(\mathfrak{G})$ such that $k'(a) = \int f(b)\, (k(b^{-1}a) - f(ab^{-1}))\, db$ belongs to $L^2(\mathfrak{G})$, then for every h in $\mathfrak{D}(R_f - L_f)$ we have

$$((R_f - L_f)h, k) = \iint h(a)\,(f(a^{-1}b) - f(ba^{-1}))\overline{k(a)}\, da db = (h, k').$$

$R_f - L_f$ is self-adjoint, then k belongs to $\mathfrak{D}(R_f - L_f)$ and $k' = (R_f - L_f)k$. By the assumption of the Lemma, f and g are elements in $L^2(\mathfrak{G})$ such that $f = f^*$, $g = g^*$ and $\int f(a)g(a^{-1}b)\, da = \int g(a)f(a^{-1}b)\, da$.

Then we have $g \in \mathfrak{D}(R_f - L_f)$ and $(R_f - L_f)g = 0$.

By Lemma 8.8 we have $E(\lambda)g = ZE(\lambda)Zg$ $(-\infty < \lambda < \infty)$, then

$$R_{f_n}g = \int_{-n}^{n} dE(\lambda)g = \int_{-n}^{n} \lambda d\, ZE(\lambda)Zg = L_{f_n}g.$$

That is,

$$\int g(b)f_n(b^{-1}a)\, db = \int f_n(b)g(b^{-1}a)\, db \ (n = 1, 2, \ldots).$$

Let $R_g = \int \lambda\, d\, F(\lambda)$ be the spectral resolution of R_g, and put $g_m = F(m)g - F(-m-0)g$, then analogously

$$\int g_m(b)f_n(b^{-1}a)\, da = \int f_n(b)g_m(b^{-1}a)\, db \ (n, m = 1, 2, \ldots).$$

R_{g_m} and R_{f_n} are bounded operators, and commute to each other. Hence all those paires $E(\lambda)$ and $F(\mu)$, R_f and R_g commute to each others respectively.

Proof of Theorem 19.

We consider a fixed maximal abelian sub-system $\mathfrak{M}$ of $L^2(\mathfrak{G})$. $\mathfrak{M}$ satisfies the three conditions (8.10), (8.11) and (8.12).

Putting $(\Gamma_a f)(b) = f(ab) - f(ba)$, the latter two conditions are equivalent to the following two conditions respectively.

(8.11)′. *If $f \in \mathfrak{M}$ and $a \in \mathfrak{G}$, then $\Gamma_a f$ is orthogonal to every $g \in \mathfrak{M}$.*

(8.12)′. *$L^2(\mathfrak{G})$ is the closed linear span of the sum of the space $\mathfrak{M}$ and the set $(\Gamma_a f:\ a \in \mathfrak{G},\ f \in \mathfrak{M})$.*

By (8.12)′ and (8.13)′, the orthogonal component of the space $\mathfrak{M}$ is the closed linear span of the set $(\Gamma_a f: a \in \mathfrak{G},\ f \in \mathfrak{M})$. Then

Lemma 8.10. *If g is an element in $L^2(\mathfrak{G})$ such that*

$$\int g(b)f(b^{-1}a)\, db = \int f(b)g(b^{-1}a)\, db \ \textit{for every}\ f \in \mathfrak{M},$$

then g belongs to $\mathfrak{M}$.

An element u in $L^2(\mathfrak{G})$ is called a *unit* if it satisfies $u(a) = \overline{u(a^{-1})} = \int u(b)u(b^{-1}a)db$. An element u in $L^2(\mathfrak{G})$ is a unit if and only if R_e is a projection.

Lemma 8.11. *The smallest linear set which contains all units in $\mathfrak{M}$ is everywhere dense in $\mathfrak{M}$.*

Proof. If f is an element of $\mathfrak{M}$, f is a linear combination $f = f_1 + if_2$ of those f_i in $\mathfrak{M}$ with $f_i = f_i^*$. Therefore it is sufficient to show that all those elements f in $\mathfrak{M}$ with $f = f^*$ are approximated by linear combinations of units in $\mathfrak{M}$.

If f is an element in $\mathfrak{M}$ with $f = f^*$, R_f is self-adjoint and has the spectral resolution $R_f = \int \lambda \, dE(\lambda)$.

Let α and β be those numbers such that either $\alpha \geqq \beta > 0$ or $0 > \alpha \geqq \beta$. Let $\varphi_{\alpha\beta}$ denote the function on the real number field such that $\varphi_{\alpha\beta}(\lambda) = \frac{1}{\lambda}$ for $\alpha > \lambda \geqq \beta$ and $\varphi_{\alpha\beta}(\lambda) = 0$ for either $\lambda > \alpha$ or $\beta \geqq \lambda$. Then $\int \varphi_{\alpha\beta}(\lambda) \, d\, E(\lambda)$ is a bounded operator, and $u_{\alpha\beta} = \int \varphi_{\alpha\beta}(\lambda) \, d\, E(\lambda) f$ is an element in $L^2(\mathfrak{G})$ such that $R_{u_{\alpha\beta}} = E(\alpha) - E(\beta)$, R_f commutes to every R_g as $g \in \mathfrak{M}$, then every $E(\lambda)$ commutes to every $R_g (g \in \mathfrak{M})$, and $R_{u_{\alpha\beta}}$ commutes to every R_g $(g \in \mathfrak{M})$, that is, $\int u_{\alpha\beta}(b) g(b^{-1}a) \, db = \int g(b) u_{\alpha\beta}(b^{-1}a) \, db$ for every $g \in \mathfrak{M}$. By Lemma 8. 10, $u_{\alpha\beta}$ are units in $\mathfrak{M}$. If φ is a linear combination of those functions $\varphi_{\alpha\beta}$, then $\int \varphi(\lambda) dE(\lambda) f$ is clearly a linear combination of those $u_{\alpha\beta} = \int \varphi_{\alpha\beta}(\lambda) \, d\, E(\lambda) f$. Let ψ be a function on the real number field such that $\psi(0) = 0$ and $\psi(\lambda) = 1$ for $\lambda \neq 0$. Then we can choose a uniformly bounded sequence $\psi_n(\lambda)$ of those functions, which are linear combinations of $\varphi_{\alpha\beta}$, and which converges at each point λ to the function ψ.

Now $\int \varphi_n(\lambda) \, dE(\lambda) f$ converges to $\int \psi(\lambda) \, dE(\lambda) f = f$ by the topology of $L^2(\mathfrak{G})$. Then f is approximated by the linear combinations of units in $\mathfrak{M}$, and the Lemma is concluded.

Let u be a unit in $\mathfrak{M}$, and denote by $L^2(\bar{u})$ the smallest closed linear space which contains all $L_a u$, which is the range of the projection operator R_u. The group $\mathfrak{G}(\bar{u})$ of all operators L_a restricted on the space $L^2(\bar{u})$.

consists of the unitary representation group for $\mathfrak{G}$ which corresponds to the positive definite function $\overline{u(a)} = (L_a u, u)$.

Lemma 8.12. *Let u be a unit in $\mathfrak{M}$, and let $\mathfrak{M}(u)$ denote the set of all restrictions of such operators R_f on $L^2(\bar{u})$ that f belongs to $\mathfrak{M}$ and R_f is bounded. Then $\mathfrak{M}(u)$ is a maximal abelian sub-algebra of the commutor $\mathfrak{G}(\bar{u})'$ of the unitary group $\mathfrak{G}(\bar{u})$.*

Proof. Let K be a bounded linear operator on $L^2(\bar{u})$ which commutes to every L_a in $\mathfrak{G}(\bar{u})$ and every R_f in $\mathfrak{M}(u)$. If h is a summable and square summable function in $\mathfrak{G}$, K commutes to the operator $L_h = \int h(a) L_a \, da$ and satisfies

$$KR_u h = KL_h u = L_h K u = R_g h,$$

where $g = Ku$ belongs to $L^2(\bar{u})$. Now we have $Kh = R_g h$ for every h in $L^2(\bar{u})$, then K coincides with the restriction of the operator R_g on $L^2(\bar{u})$. The operator R_g commutes to every R_k in $\mathfrak{M}$ such that R_k is bounded linear, and to every R_u such that u is a unit in $\mathfrak{M}$. Then

$$\int g(b) u(b^{-1}a) \, db = \int u(b) g(b^{-1}a) \, db \text{ for every unit } u \text{ in } \mathfrak{M}.$$

The smallest closed linear set which contains all the units in $\mathfrak{M}$ coincides with $\mathfrak{M}$, then every element k in $\mathfrak{M}$ satisfies

$$\int g(b) k(b^{-1}a) \, db = \int k(b) g(b^{-1}a) \, db,$$

By Lemma 8.10 g belongs to $\mathfrak{M}$, and k belongs to $\mathfrak{M}(u)$. Hence $\mathfrak{M}(u)$ is a maximal abelian sub-algebra of $\mathfrak{G}(\bar{u})'$.

By Theorem 3 the algebra $\mathfrak{M}(u)$ determines a diagonal measure τ_u on $\mathfrak{E}$ whose relative Fourier transform $\varphi \to R_{\hat{\varphi}}$ is an algebraic isomorphism between $M(\tau_u)$ and $\mathfrak{M}(u)$. If φ is a bounded and measurable function on $\mathfrak{E}$, its relative Fourier transform $R_{\hat{\varphi}}$ in $\mathfrak{M}(u)$ is determined by

$$(L_a R_{\hat{\varphi}} u, u) = \int \lambda(a) \varphi(\lambda) \, d\tau_u(\lambda).$$

We can assume that $\hat{\varphi}$ is an element of $\mathfrak{M}$ and contained in the range of the projection R_u. Then

$$\hat{\varphi}(a)=\int u(b)\hat{\varphi}(b^{-1}a)\,db=(L_a^{-1}R_{\hat{\varphi}}u,u)=\int\overline{\lambda(a)}\varphi(\lambda)\,d\tau_u(\lambda).$$

If φ and ψ are bounded functions on $\mathfrak{E}$ summable by τ_u, then $R_{\widehat{\varphi\psi}}=R_{\hat{\varphi}}R_{\hat{\psi}}$ and

$$\int\hat{\varphi}(b)\hat{\psi}(b^{-1}a)\,db=(L_a^{-1}R_{\hat{\varphi}}R_{\hat{\psi}}u,u)=(L_a^{-1}R_{\widehat{\varphi\psi}}u,u)$$
$$=\int\overline{\lambda(a)}\varphi(\lambda)\psi(\lambda)d\tau_u(\lambda).$$

Especially,

$$\int\hat{\varphi}(b)\overline{\hat{\psi}(b)}\,db=\int\varphi(\lambda)\overline{\psi(\lambda)}\,d\tau_u(\lambda).$$

Lemma 8.13. *If u_1 and u_2 are mutually orthogonal units in $\mathfrak{M}$, then there is a Borel set N in $\mathfrak{E}$ such that $\tau_{u_1}(\mathfrak{E}-N)=0$ and $\tau_{u_2}(N)=0$.*

Proof. $u=u_1+u_2$ is a unit, and the restrictions of R_{u_1} and R_{u_2} on $L^2(u)$ are mutually orthogonal projections. Then there exists a Borel set N in $\mathfrak{E}$ such that $R_{u_1}=R_{\hat{\varphi}_N}$. The measure $\tau_1:\tau_1(X)=\tau_u(X\cap N)$ is now a diagonal measure with $u_1(a)=\int\lambda(a)d\tau_1(\lambda)$, whose relative Fourier transform mapps $M(\tau_1)$ on $\mathfrak{M}(u_1)$. Then τ_1 coincides with τ_{u_1}, and τ_2 coincides with $\tau-\tau_1$. Hence $\tau_1(\mathfrak{E}-N)=0$ and $\tau_2(N)=0$.

Lemma 8.14. *There exists a system $\{u_i\}$ of mutually orthogonal units in $\mathfrak{M}$ such that $I=\sum_i R_{u_i}$.*

Proof. The family of all sub-systems of $\mathfrak{M}$ whose elements are mutually orthogonal units in $\mathfrak{M}$, is inductive, and by the Zorn's lemma there is at least one maximal system in the family. We consider such a fixed maximal system $\{u_i\}$, then there is no unit in $\mathfrak{M}$ orthogonal to every u_i other than 0. R_{u_i} are mutually orthogonal projections, and $E=\sum_i R_{u_i}$ is a projection on $L^2(\mathfrak{G})$.

If g is an element in $L^2(\mathfrak{G})$ such that R_g is bounded, then $R_{Eg}=ER_g$. In fact, for every h in $L^2(\mathfrak{G})$, we have

$$R_{Eg}h=h\circ(Eg)^{1)}=\sum_i h\circ g\circ u_i=E(R_gh).$$

1) $f\circ g$ is the convolution: $(f\circ g)(a)=\int f(b)g(b^{-1}a)\,db$

If u is a unit in $\mathfrak{M}$, every $R_{u_\iota}u = \int u(b)u_\iota(b^{-1}a)\,db$ is contained in $\mathfrak{M}$, and $Eu = \Sigma R_{u_\iota}u$ is contained in $\mathfrak{M}$. $R_{Eu} = ER_u = R_{u_\iota}R_u = R_uR_{u_\iota} = R_uE$ is a projection. Then Eu and $(I-E)u$ are units in $\mathfrak{M}$, $(I-E)u$ is orthogonal to every u_ι, and we have $(I-E)u=0$
Therefore

$$ER_u = R_uE = R_u \textit{ for every unit } u \textit{ in } \mathfrak{M}.$$

We now show that E is the identity I. Assume that E is not the identity I, then there is an element f in $L^2(\mathfrak{G})$ such that $\|f\| = (\int |f(a)|^2 da)^{\frac{1}{2}} = 1$ and $Ef=0$. We can choose a g in $L^2(\mathfrak{G})$ such that $\|f-g\| < \frac{1}{2}$ and R_g is bounded linear. Then $h=(I-E)g$ is $\neq 0$. In fact,

$$\|h\| \geqq \|(I-E)f\| - \|(I-E)(f-g)\| > \frac{1}{2}.$$

Notice that $R_h = (I-E)R_g$ is bounded and linear, then $k = R_hh^*$ belongs to $L^2(\mathfrak{G})$. It satisfies $k(a) = k^*(a)$, $Ek = ER_hh^* = R_{Eh}h^* = 0$ and $k \neq 0$. In fact, $k(a) = \int \overline{h(b)}\, h(ba)\,db$ is a continuous function on $\mathfrak{G}$ with $k(e) = \int |h(b)|^2 db = \|h\|^2 \neq 0$.

Now for every unit u in $\mathfrak{M}$, we have
$R_ek = R_uEk = 0$ and
$\int k(b)\,u(b^{-1}a)\,db = 0$. Since $u=u^*$ and $k=k^*$, we have
$\int u(b)k(b^{-1}a)\,db = 0$. $\mathfrak{M}$ is the smallest closed linear space which contains all units in $\mathfrak{M}$, then we have

$$\int f(b)k(b^{-1}a)\,db = \int k(b)f(b^{-1}a)\,db = 0$$

for every f in $\mathfrak{M}$. By Lemma 8. k and k^* should belong to $\mathfrak{M}$, therefore,

$$\int k(b^{-1})k(b^{-1}a)\,db = 0 \text{ and } k=0.$$

This contradicts to the fact $k \neq 0$, then E should be the identity I. This concludes the Lemma.

Let $\{u_i\}$ be a system of mutually orthogonal units in $\mathfrak{M}$ with $I = \sum_i R_{u_i}$, and for each u_i determine the diagonal measure $\tau_i = \tau_{u_i}$. If $i \neq j$, then there exists a Borel set N_{ij} such that $N_{ij} = \mathfrak{E} - N_{ji}$ and $\tau_i(N_{ji}) = \tau_j(N_{ij}) = 0$.

Now $\{N_i : N_i = \prod_{j=i}^{\infty} N_{ij}\}$ are mutually disjoint Borel sets in $\mathfrak{E}$ such that $\tau_i(E - N_i) = 0$ $(i = 1, 2, \ldots.)$. We define a Borel measure τ on $\mathfrak{E}$ by

$$\tau(X) = \sum_i \tau_i(X \cap N_i) \text{ for every Borel set } X.$$

We shall show that τ is a fundamental measure whose Fourier transform $\varphi \to \hat{\varphi}$ maps $L^2(\tau)$ in $\mathfrak{M}$.

Every diagonal measure τ_i is regular, and satisfies the condition 8. 1, then τ satisfies the condition 8. 1. τ satisfies the condition 8. 2. In fact, if f and g are summable and square summable functions on $\mathfrak{G}$, then

$$\int f(a)\overline{g(a)}\, da = \sum_i (R_{u_i} f, R_{u_i} g) = \sum_i \int \overline{u_i(b^{-1}a)g(b)}\, f(a)\, db\, da.$$

Notice that

$$\overline{u_i(a)} = \int \lambda(a)\, d\tau_i(\lambda) = \int_{N_i} \lambda(a)\, d\tau(\lambda),$$

then

$$\int f(a)\overline{g(a)}\, da = \iiint \lambda(b^{-1}a)\overline{g(b)} f(a)\, db\, da\, d\tau(\lambda),$$

the Plancherel's equality is satisfied.

If A and B are Borel sets in $\mathfrak{E}$ with finite masses,

$$\varphi_i(a) = \int_{A \cap N_i} \overline{\lambda(a)}\, d\tau_i(\lambda) \textit{ and } \psi_i(a) = \int_{B \cap N_i} \overline{\lambda(a)}\, d\tau_i(a)$$

are elemsnts in the range of the projection R_{u_i}. And if A and B are mutually disjoint, they are mutually orthogonal. Further $\int |\varphi_i(a)|^2\, da = \tau(A \cap N_i)$ *and* $\int |\psi_i(a)|^2\, da = \tau(B \cap N_i)$,

Then $\varphi_A(a) = \int_A \overline{\lambda(a)}\, d\tau(\lambda) = \sum \varphi_i(a)$

and

$$\varphi_B(a) = \int_B \overline{\lambda(a)}\, d\tau(a) = \Sigma\, \psi_{\rho_i}(a)$$

are mutually orthogonal elements in $L^2(\mathfrak{G})$. Thus τ satisfies the condition (8.3), and τ is a fundamental measure.

The Fourier transform $\varphi \to \hat{\varphi}$ maps $L^2(\tau)$ on $\mathfrak{M}$. In fact, if φ is a bounded function on $\mathfrak{E}$ summable by τ which vanishes out-side of a certain N_i, then

$$\varphi(a) = \int \overline{\lambda(a)}\, d\tau(\lambda) = \int \overline{\lambda(a)}\, d\tau_i(\lambda)$$

belongs to $\mathfrak{M}$. Therefore the Fourier transform $\varphi \to \hat{\varphi}$ maps $L^2(\tau)$ in a sub-space of $\mathfrak{M}$. But, the range of the Fourier transform is a maximal abelian sub-system of $L^2(\mathfrak{G})$, then it coincides with $\mathfrak{M}$.

Finally, the correspondence between fundamental measures on $\mathfrak{E}$ and maximal abelian sub-systems of $L^2(\mathfrak{G})$ is one-to-one. To prove this, it is sufficient to show that, if τ and ρ are two fundamental measures such that their respective Fourier transforms map $L^2(\tau)$ and $L^2(\rho)$ on the same maximal abelian snb-system $\mathfrak{M}$, then $\rho = \tau$.

If τ and ρ are such two fundamental measures, choose a system $\{u_i\}$ of mutually orthogonal units in $\mathfrak{M}$ such that $I = \Sigma R_{u_i}$. Then those u_i are respectively Fourier transforms:

$$u_i(a) = \int_{E_i} \overline{\lambda(a)}\, d\tau(\lambda) = \int_{F_i} \overline{\lambda(a)}\, d\rho(\lambda).$$

The measures τ_i and ρ_i are defined by $\tau_i(X) = \tau_i(X \cap E_i)$ and by $\rho_i(Y) = \rho(Y \cap F_i)$ respectively. For each fixed i, τ_i and ρ_i are diagonal measures which correspond to the same positive definite function $\overline{u_i(a)}$ and the same maximal abelian sub-algebra $\mathfrak{M}(u_i)$ of $\mathfrak{G}(\bar{u}_i)'$. By Theorem 3, τ_i and ρ_i coincide with each other, then $\tau = \sum_i \tau_i$ and $\rho = \sum_i \rho_i$ coincide with each other. This completes Theorem 19.

REFERENCES

[1] W. AMBROSE, *The L^2 system of a unimodular group.* Trans. Amer. Math. Soc. vol. 65 (1949) pp. 27—48.

[2] I. GELFAND and D. A. RAIKOV, *Irreducible representations of locally compact groups.* Rec. Mat. (Mat. Sbornik) N. S. vol. 13 (1943) pp. 301—316

[3] R. GODEMENT, *Les fonctions de type positive et la théore des groupes.* Trans. Amer. Math. Soc.. vol. 63 (1943) pp. 1—84

[4] ———, *Sur la théorie des représentations unitaires,* Ann. of Math. vol. 53

(1951) pp. 68—124.

[5] F. I. MAUTNER, *The completeness of the irreducible unitary representations of a locally compact group.* Proc Nat. Acad. Sci. vol. 34 (1948) pp. 52—74.

[6] ————, *Note on the Fourier inversion formula on groups.* Trans. Amer. Math. Soc. vol. 78 (1955) pp. 371—384.

[7] I. E. SEGAL, *Decompositihns of operator algebras*, I and II. Memoirs Amer. Math. Soc. vol. 9 (1950)

[8] M. H. STONE, *Linear transformations in Hilbert space.* Amer. Math. Soc. Coll. Pub. vol. 15 (1932)

[9] D. A. RAIKOV, *On various types of convergence of positive definite functions.* Doklady Akad. Nauk SSSR (N. S.) vol. 58 (1947) pp. 1279—1282.

[10] H. YOSHIZAWA, *On some types of conaergence of positive definite functions.* Osaka Math. J. vol. 1 (1949) pp. 90—94.

DEPARTMENT OF MATHMATICS,
OKAYAMA UNIVERSITY

(Received February 8, 1956)