ON FIBRES OF FIBRE SPACES WHOSE TOTAL SPACE IS CONTRACTIBLE

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1. In this note, we consider a fibre space (E, F, B, p), where E is the total space, B the base space, the map $p: E \rightarrow B$ the projection, and $F = p^{-1}(b_0)$ the fibre over a point $b_0 \in B$, *in the sense of Serre*, (i. e., being assumed the truth of the covering homotopy theorem for maps of finite polyhedra). The purpose of this note is to prove the following theorem.

Theorem 1. *In a fibre space (E, F, B, p) such that the total space E is a CW-complex and the fibre F is a locally finite CW-complex*[1], *we assume that E is contractible to a vertex $x_0 \in F$ in itself (with x_0 stationary throughout the contraction). Then F is a homotopy-associative H-space having a (two-sided) inversion*[2].

2. As the first step of proofs of Theorem 1, we prove the following theorem, which is an analogy of the result of E. H. Spanier and J. H. C. Whitehead [4, Theorem (1.1)].

Theorem 2. *If the fibre space (E, F, B, p) satisfies the hypotheses of Theorem 1, then there exists a (continuous) map $\bar{\mu}$ of $E \times F$ into E having the following properties:*

(1) $p \circ \bar{\mu}(u, x) = p(u)$ *for every* $u \in E$ *and* $x \in F$.

(2) $\bar{\mu} \mid F \times F = \mu$ *is an H-structure of F having x_0 as an unit element.*

(3) *This H-structure μ has a left inversion*[2].

Proof of (1) *and* (2). We notice that $E \times F$ and

1) For definitions, Cf. [6], § 5, p. 223.

2) A space X is an H-space (has an H-structure) if there is a multiplication μ in X, i. e. a map $\mu: X \times X \rightarrow X$, such that $\mu(x_0, x) = \mu(x, x_0) = x$ for some x_0 (called an unit) and every $x \in X$. If two maps $(x, y, z) \rightarrow \mu(x, \mu(y, z))$ and $(x, y, z) \rightarrow \mu(\mu(x, y), z)$ of $X \times X \times X$ into X are homotopic each other, we say μ is homotopy-associative. μ has a (two-sided) inversion, if there exists a map $\sigma: X \rightarrow X$ such that the maps $x \rightarrow \mu(\sigma(x), x)$ and $x \rightarrow \mu(x, \sigma(x))$ of X into X are homotopic to the constant map $x \rightarrow x_0$ respectively. If only one of these maps has this property, we say σ is an one-sided (left or right) inversion of μ.

$$E \vee F = (E \times x_0) \cup (x_0 \times F) \quad \text{and} \quad F \vee F = (F \times x_0) \cup (x_0 \times F)$$

are CW-complexes, because E and F are CW-complexes and the latter is locally finite[3].

Let $k_t: (E, x_0) \to (E, x_0)$ $(0 \leqslant t \leqslant 1)$ be the homotopy between $k_1 =$ the identity map and $k_0: E \to x_0$ the constant map, which gives a contraction of E; and we define the map $g_0: E \times F \to E$ by $g_0(u, x) = x$. and the homotopy $g'_t: E \vee F \to E\,(0 \leqslant t \leqslant 1)$ by

$$\begin{cases} g'_t(u, x_0) = k_t(u), \\ g'_t(x_0, x) = x, \end{cases} \qquad \text{for } u \in E \text{ and } x \in F.$$

Then, there is a homotopy $g_t: E \times F \to E$ $(0 \leqslant t \leqslant 1)$ of g_0 which is an extension of g'_t,[4] and it holds relations $g_1(u, x_0) = u$ and $g_1(x_0, x) = x$. Therefore the map $p \circ g_1: E \times F \to B$ satisfies $p \circ g_1(u, x) = p(u)$ for $(u, x) \in E \vee F$.

Let $h': E \times E \times I \to B$ $(I = [1, 2])$ be defined by

$$h'(u, x, t) = \begin{cases} p \circ g_{3-2t}(u, x) & \text{for } 1 \leqslant t \leqslant 3/2, \\ p \circ k_{2t-3}(u) & \text{for } 3/2 \leqslant t \leqslant 2. \end{cases}$$

Then $h'(u, x, 1) = p \circ g_1(u, x)$, $h'(x_0 \times F \times I) = b_0$ and $h'(u, x, 2) = p(u)$. Also $h' | (E \vee F) \times I$ is homotopic, relative $(E \times x_0 \times 1) \cup (E \times x_0 \times 2) \cup (x_0 \times F \times I)$, to the map $h: (E \vee F) \times I \to B$ such that $h(u, x, t) = p(u)$. This homotopy can be extended, first to a homotopy of $(E \times F \times 1) \cup (E \times F \times 2) \cup (E \vee F) \times I$ as stationary on $(E \times F \times 1) \cup (E \times F \times 2)$, and then to a homotopy of $E \times F \times I$ into B (by applying the homotopy extension theorem for CW-complex). The last map $h: E \times F \times I \to B$ defines the homotopy $h_t: E \times F \to B\,(1 \leqslant t \leqslant 2)$ which satisfies $h_1 = p \circ g_1$ and

$$h_t(u, x) = p(u), \qquad \text{for} \begin{cases} t = 2 \text{ and } (u, x) \in E \times F, \\ 1 \leqslant t \leqslant 2 \text{ and } (u, x) \in E \vee F. \end{cases}$$

Because (E, F, B, p) is a fibre space in the sense of Serre and $E \times F$ is a CW-complex, we can apply the covering homotopy theorem due to I. M. James and J. H. C. Whitehead [2, Theorem 5.1], and hence there

3) Cf. [6], § 5, property (H), p. 227.

4) This is a consequence of the homotopy extension theorem for CW-complexes, cf. [6], § 5, property (J), p. 228.

exists a homotopy $g_t : E \times F \to E$ $(1 \leqslant t \leqslant 2)$ such that $p \circ g_t = h_t$ and stationary with h_t. It follows immediately that the map $\bar{\mu} = g_2$ satisfies the properties (1) and (2).

3. *Proof of* (3). As (E, F, B, p) is a fibre space, it follows evidently that $(E \times F, F \times F, B, p')$ is also a fibre space, where $p' : E \times F \to B$ is defined by $p'(u, x) = p(u)$ and $F \times F = p'^{-1}(b_0)$ is the fibre over $b_0 \in B$.

Making use of $\bar{\mu}$, let $\bar{\varphi} : E \times F \to E \times F$ be defined by

$$\bar{\varphi}(u, x) = (\bar{\mu}(u, x), x) \qquad \text{for } u \in E \text{ and } x \in F,$$

and set $\varphi = \bar{\varphi} \mid F \times F$. Then, by property (1), $p' \circ \bar{\varphi}(u, x) = p'(u, x)$ for every $u \in E$ and $x \in F$. Therefore $\bar{\varphi}$ induces an endomorphism of the homotopy sequence of the fibre space $(E \times F, F \times F, B, p')$:

$$\begin{array}{ccccccccc} \cdots \to & \pi_{n+1}(B) & \to & \pi_n(F \times F) & \to & \pi_n(E \times F) & \to & \pi_n(B) & \to \cdots \\ & \downarrow i_* & & \downarrow \varphi_* & & \downarrow \bar{\varphi}_* & & \downarrow i_* & \\ \cdots \to & \pi_{n+1}(B) & \to & \pi_n(F \times F) & \to & \pi_n(E \times F) & \to & \pi_n(B) & \to \cdots \end{array}$$

where i_* is the induced isomorphism of the identity map $i : B \to B$. Because E is contractible to x_0, the map $q : E \times F \to x_0 \times F$ such that $q(u, x) = (x_0, x)$ induces an isomorphism of $\pi_n(E \times F)$ onto $\pi_n(x_0 \times F)$, and hence $\bar{\varphi}_*$ is an isomorphism onto, by $q = q \circ \bar{\varphi}$. Therefore, by five lemma, it is immediately seen that φ_* is an isomorphism onto; and this shows that the map $\varphi : F \times F \to F \times F$ is a homotopy equivalence, as $F \times F$ is a CW-complex[5]. We denote by $\theta : F \times F \to F \times F$ a homotopy inverse of φ, i. e., a map such that $\theta \circ \varphi$ and $\varphi \circ \theta$ are homotopic to the identity map, respectively.

We now define maps σ and τ of F into F by

$$\begin{cases} \sigma(x) = q_1 \circ \theta(x_0, x), \\ \tau(x) = q_2 \circ \theta(x_0, x), \end{cases} \qquad \text{for } x \in F,$$

where q_1 is the projection of $F \times F$ onto F of the first factor, and q_2 onto F of the second factor. As φ can be written by $\varphi(w) = (\mu(q_1(w), q_2(w)), q_2(w))$ for $w \in F \times F$, we have $\varphi \circ \theta \circ j(x) = (\mu(\sigma(x), \tau(x)), \tau(x))$ where $j : F \to x_0 \times F$ is $j(x) = (x_0, x)$. Because $\varphi \circ \theta$ is homotopic to the identity map, let $i_t : F \times F \to F \times F$ be a homotopy between $i_0 = \varphi \circ \theta$ and $i_1 =$ the identity map; then $q_1 \circ i_t \circ j : F \to F$ defines a homo-

5) This is a consequence of Theorem 1 of [6, §1, p. 215].

topy between maps $x \to \mu(\sigma(x), \tau(x))$ and $x \to x_0$ of F into F, and also $q_2 \circ i_t \circ j : F \to F$ between maps $x \to \tau(x)$ and $x \to x$. Combining these properties, it is seen that the map $x \to \mu(\sigma(x), x)$ of F into F is homotopic to the map $x \to \mu(\sigma(x), \tau(x))$ and hence to the constant map $x \to x_0$. This show that σ is an inversion of μ and so (3) is valid; and hence proofs of Theorem 2 is completed.

4. As the second step, we prove the following theorem, which is an modification of results of H. Samelson [3, Theorem 1] and [5, Theorem 1].

Theorem 3. *If the fibre space* (E, F, B, p) *satisfies:*

(1) *There exists a map* $\bar{\mu} : E \times F \to E$, *having properties* (1), (2) *and* (3) *of Theorem* 2.

(2) E *is contractible to* x_0 *(with* x_0 *stationary).*

Then there exists an H-homomorpism[6] f, *which is also a weak homotopy equivalence*[7], *of* F *into the space* $\Lambda(B)$ *of loops in* B *with the base point* b_0.

If F *is a locally finite CW-complex in addition, the H-structure* μ *of* F *is homotopy-associative, and also has a (two-sided) inversion.*

Proof. The existence of such map f can be shown by applying [3, Proposition 1] and [5, Theorem 1].

Because f induces isomorphisms of all homotopy groups of F and $\Lambda(B)$, two maps of a CW-complex into F are homotopic if and only if two composed maps of these maps and f are homotopic each other[8]. Therefore, if we assume F is a locally finite CW-complex, the homotopy-associativity of F, i. e., the fact that two maps $(x, y, z) \to \mu(x, \mu(y, z))$ and $(x, y, z) \to \mu(\mu(x, y), z)$ of $F \times F \times F$ into F are homotopic, is an immediate consequence of the fact that f is an H-homomorphism and the

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6) For H-spaces X and Y with multiplication μ and μ' respectively, we say a map $f : X \to Y$ is an H-homomorphism, if two maps $(x_1, x_2) \to f \circ \mu(x_1, x_2)$ and $(x_1, x_2) \to \mu'(f(x_1), f(x_2))$ of $X \times X$ into Y are homotopic. If μ and μ' have inversions σ and σ' respectively, we assume in addition that two maps $x \to f \circ \sigma(x)$ and $x \to \sigma' \circ f(x)$ of X into Y are homotopic each other.

For the may f of this theorem, the multiplication of F is $\mu = \bar{\mu} | F \times F$ and that of the H-space $\Lambda(B)$ is the natural multiplication (composition of loops).

7) This means that f induces isomorphisms of all the homotopy groups of F and $\Lambda(B)$.

8) This property is easily proved, for example, by making use of Theorem 1.7 of [1, Chapter VII, p. 98].

H-space $\Lambda(B)$ of loops in B with natural multiplication (composition of loops) is homotopy-associative.

As the consequence of the homotopy-associativity of μ, it is easy to show that the left inversion σ of μ is also the right inversion, as follows. As the map $x \to \mu(\sigma(x), x)$ is homotopic to the constant map $x \to x_0 =$ an unit, the map $x \to \sigma \circ \sigma(x)$ of F into F to the map $x \to \mu(\sigma \circ \sigma(x), \mu(\sigma(x), x))$, and the latter to the map $x \to \mu(\mu(\sigma \circ \sigma(x), \sigma(x)), x)$, and so to the identity map $x \to x$. Therefore the map $x \to \mu(x, \sigma(x))$ is homotopic to the map $x \to \mu(\sigma \circ \sigma(x), \sigma(x))$, and hence to the constant map $x \to x_0$ of F into F. This proves σ is also a right inversion of μ, and proofs of Theorem 3 is completed.

Combining Theorems 2 and 3, we have Theorem 1 immediately.

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