COMPACT MOB WITH A UNIQUE LEFT UNIT

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A. D. Wallace proposed in his paper [1][1] the following problem:

If a compact connected mob has a unique left unit, is this also a right unit?[2]

By a mob we mean a Hausdorff semigroup according to him. We have already given counter examples to this problem without proof [2]. In this paper we shall discuss the structure of a compact mob which has a unique left unit but has not a right unit.

Let S be a compact mob having a unique left unit e which is not a right unit.

Lemma 1. *Se is a compact proper submob with a two-sided unit, and S is homomorphic onto Se.*

Proof. Consider a mapping f of S to Se: $f(x)=xe$. Then f is continuous, and, since e is a left unit,

$$f(x)f(y)=(xe)(ye)=x(ey)e=(xy)e=f(xy).$$

Hence f is a homomorphism of S onto Se. Since S is compact and Se is an image of S under f, Se is also a compact mob. Taking any $x\in Se$, $x=ye$ for some $y\in S$, $xe=(ye)e=y(ee)=ye=x$, whence a left unit e is also a right unit of Se. Suppose that $Se=S$, it is concluded that e is a right unit of S, contradicting to the assumption. Therefore Se is a proper submob. Thus the lemma has been completely proved.

We remark that each element of Se is fixed under f.

Lemma 2. *The inverse image of e under the homomorphism f of S to Se is composed of only one e.*

Proof. Let x be an element of S such that $f(x)=xe=e$. Then, for any $y\in S$, $xy=x(ey)=(xe)y=ey=y$. It follows that x is a left unit. According to the uniqueness of left unit, we have $x=e$.

From Lemmas 1 and 2 we have easily the following theorem:

1) Numbers in brackets refer to the references at the end of the paper.

2) We correct the misprint in the paper [1], p. 499, the 2nd line, as follows: read "compact connected mob" for "compact mob."

Theorem 1. *S is decomposed into the class sum of T_a, $S=\sum_{a\in Se} T_a$ such that*

(1) *a is only one element of Se which is contained in T_a,*
(2) *T_e is composed of only one e,*
(3) *$T_aT_b \subset T_{ab}$ where $a, b \in Se$.*

Now let $G=Se$ and let Φ be a set of mappings $\varphi_a(a\in G)$ of S into S defined as $\varphi_a(x)=ax$. Then Φ and f satisfy the following conditions.

(C_1) f is a continuous idempotent mapping of S onto G, and only one e is mapped to e by f,

(C_2) the correspondence $a\to\varphi_a$ is an algebraic[1] homomorphism of G to Φ,

(C_3) when $\varphi_a(x)$ is considered as an image of (a, x), φ_a is a continuous mapping of $G\times S$ into S,

(C_4) $\varphi_a(e)=a$ for every $a\in G$,

(C_5) $\varphi_e(x)=x$ for every $x\in S$,

(C_6) $\varphi_a f=f\varphi_a$ for every $a\in G$.

On the other hand, it can be shown that these conditions characterize S.

Theorem 2. *Let S be a compact set and let G be a proper subset of S as well as a compact mob with a two-sided unit e. If a mapping f of S onto G and a set Φ of mappings $\varphi_a(a\in G)$ of S into S are given such that the conditions (C_1)$\sim$(C_6) are satisfied, then we can construct a compact mob with a unique left unit e which is not a right unit, so that S is the extension of G and S is homomorphic to G. Moreover G is isomorphic to Φ.*

Denote by $a\cdot b$ the given product of a and b in G. Let us define a product xy of x and y in S as follows:

$$xy=\varphi_{f(x)}(y).$$

At first we shall prove the following Lemmas 3 and 4.

Lemma 3. *f is a homomorphism of S onto G with respect to the new multiplication, and maps each element of G to itself.*

Proof. According to (C_1), for any $a\in G$, as there is $x\in S$ such that $f(x)=a$, we have $f(a)=f(f(x))=f^2(x)=f(x)=a$.

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1) By an algebraic homomorphism we mean a mapping which preserves product. We require no continuity of it.

By (C_6), $f(xy)=f(\varphi_{f(x)}(y))=\varphi_{f(x)}(f(y))=\varphi_{ff(x)}(f(y))=f(x)f(y)$.

Lemma 4. *In G the new multiplication coincides with the former one: $ab=a\cdot b$ for $a, b\in G$.*

Proof. By (C_2) and (C_4), $\varphi_a\varphi_b(e)=\varphi_{a\cdot b}(e)=a\cdot b$. On the other hand, by (C_4) and Lemma 3, $\varphi_a(\varphi_b(e))=\varphi_a(b)=\varphi_{f(a)}(b)=ab$, whence $a\cdot b=ab$.

The Proof of Theorem 2. If we define xy as above mentioned, it is proved that the product is associative by use of Lemmas 3 and 4. In fact $x(yz)=\varphi_{f(x)}(yz)=\varphi_{f(x)}(\varphi_{f(y)}(z))=\varphi_{f(x)\cdot f(y)}(z)=\varphi_{f(x)f(y)}(z)=\varphi_{f(xy)}(z)$ $=(xy)z$. The continuity of multiplication is clear by (C_1) and (C_3). From (C_5), it follows that e is a left unit. Its uniqueness is proved as follows. Let c be a left unit of S, and let x be an inverse image of $u\in G$ under $f: f(x)=u$. From $cx=x$, we have $f(c)u=u$ for every $u\in G$; $f(c)$ coincides with a two-sided unit of G, i. e. $f(c)=e$. The condition (C_1) makes it hold that $c=e$. Next we shall prove that $f(x)=xe$. By (C_1) and the definition of the multiplication, we have $f(x)=f(x)\cdot e=f(x)e=\varphi_{f(f(x))}(e)=\varphi_{f(x)}(e)=xe$. In particular, for $a\in G$, $f(a)=a$. Since G is a proper subset, $G\ni xe\neq x$ for $x\in S-G$. This shows that a unique left unit e is not a right unit of S. Thus S is a compact mob having a unique left unit but no right unit. Of course S is homomorphic to G by Lemma 1. The proof of one-to-one correspondence of $a\to\varphi_a$ is clear by the following.

$$a\neq b,\quad \varphi_a(e)=a\neq b=\varphi_b(e);\ \text{hence}\quad \varphi_a\neq\varphi_b.$$

Thus the proof of the theorem has been completely finished.

Now we shall investigate whether G is unipotent or not.

Lemma 5. *Let X be a compact unipotent mob, an idempotent of which is e. If $eX=X$, then X is a group*[1].

Proof. Let x be any element of X. Since Xx is a compact submob of X, it contains e [2], in other words, $zx=e$ for some $z\in X$. Of course e is a left unit of X. Hence X is a group.

Theorem 3. *Let S be a compact mob with a unique left unit e which is not a right unit. Then Se contains an idempotent beside e.*

1) The proof of Lemma 5 is similar as that of Lemma 1 in [4] or Lemma 2 (2') in [5]. (Readers should remark the supplement to [5], Kōdai Math. Sem. Rep., No. 3, 1954, p. 96.)

2) See Lemma 4 in [3].

Proof. At first we shall prove that S contains an idempotent different from e. Suppose that S is unipotent. Since e is a left unit, from Lemma 5 follows that S is a group and so e is a right unit of S at the same time. This conflicts with the assumption. Therefore S contains an idempotent different from e. Let a be an idempotent beside e. According to Lemma 2, $ae \neq e$; and it is proved that ae is an idempotent: $(ae)(ae) = a(ea)e = (aa)e = ae$. Hence Se contains an idempotent ae distinct from e.

Finally we give examples of S.

Example 1. Finite semigroups. (See [6].)

(1) $S = \{a, b, c, d\}$, $G = \{a, d\}$.

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	b	c	d

$$f = \begin{pmatrix} abcd \\ aaad \end{pmatrix}, \qquad \varphi_a(x) = a, \qquad \varphi_d(x) = x.$$

(2) $S = \{a, b, c, d\}$, $G = \{a, b, d\}$.

	a	b	c	d
a	a	b	a	a
b	a	b	a	b
c	a	b	a	a
d	a	b	c	d

$$f = \begin{pmatrix} abcd \\ abad \end{pmatrix}. \qquad \varphi_a = \begin{pmatrix} abcd \\ abaa \end{pmatrix}, \quad \varphi_b = \begin{pmatrix} abcd \\ abab \end{pmatrix}, \quad \varphi_d = \begin{pmatrix} abcd \\ abcd \end{pmatrix}.$$

In particular, we give examples of a connected S.

Example 2. $S = \{(x, y);\ 0 \leqq x \leqq y \leqq 1\}$, $G = \{x;\ 0 \leqq x \leqq 1\}$. The multiplication and the topology in G are given as usual, and S is considered to contain a subset corresponding one by one to G: $(x, x) \leftrightarrow x$; f and Φ are defined as

$$f((x, y)) = x, \quad \varphi_a((x, y)) = (ax, ay) \text{ for every } a \in G.$$

Then the example is equivalent to Example 1 in the previous paper [2].

Example 3. Let us consider Example 2 in [2], the symbols in which are used also here.

$$S = A \cup B, \ G = A, \ f(x) = \begin{cases} x, & \text{if } x \in A, \\ 0, & \text{if } x \in B, \end{cases};$$

$$\text{for } a \in A, \ \varphi_a(x) = \begin{cases} ax, & \text{if } x \in A, \\ x, & \text{if } x \in B. \end{cases}$$

References

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