THEORY OF COMPACT RINGS II

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§ 1. **Introduction.** In his paper [1][1] K. Asano investigated the theory of semi-primary rings and gave some results concerning these rings. A ring R is said to be *semi-primary* if the radical N of R is algebraically nilpotent (i. e. $N^n = (0)$ for some positive integer n) and the residue class ring R/N is semi-simple.

In the theory of topological rings compact rings play a rôle similar to semi-primary rings in that of abstract rings.

We shall prove in this paper several theorems concerning compact rings which are similar to Asano's and determine the structure of compact rings satisfying certain conditions.

From these results, the condition II of [§3; 10][2] can be replaced by the condition:

II′. *There is no open left (or right) ideal between* $\mathfrak{p}$ *and* $\mathfrak{p}^2$ *for any maximal open prime ideal* $\mathfrak{p}$.

Therefore, under the condition I of [§3; 10], the condition II of [§3; 10] and the condition II′ are equivalent to each other. This means that if there is no open left ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$ then there is no open right ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$ and conversely.

In §2 of this paper we shall give some preliminary results concerning compact rings with identity. We shall consider, in §3, compact completely primary rings and prove three theorems (Theorems 3.1, 3.2 and 3.3) which will be used in the later sections. §4 is devoted to studies of compact primary rings. In the last section, §5, we shall determine the structure of compact rings which satisfy some conditions.

Most of theorems of this paper may be proved using the fact that "a compact ring with identity is an inverse limit of finite rings" (see [11]), but we shall not use this result and prove directly.

§ 2. **Preliminary propositions.** Throughout this paper we will use the terminology of Kaplansky [5, 6 and 7]. For example the *radical* of a ring always means the Jacobson radical (see [4]) and *nilpotence*

1) Numbers in brackets refer to the bibliography at the end of this paper.

2) Cf. [10; Theorem 7].

means topological nilpotence (see [10]).

In this section we denote by $\mathfrak{O}$ a compact ring with identity. Then the following two lemmas are well-known[1]:

Lemma 2.1. *$\mathfrak{O}$ is totally disconnected and has a fundamental system of compact open ideal neighborhoods of* 0.

Lemma 2.2. *The radical of $\mathfrak{O}$ is a nilpotent ideal and every one-sided nilideal is contained in the radical.*

Now we prove the following two lemmas which are known results for abstract rings satisfying the minimum condition for one-sided ideals, and the proof of Lemma 2.4 is the same as in the case of abstract rings.[2]

Lemma 2.3. *If $\mathfrak{a}$ is an ideal of $\mathfrak{O}$ such that $\mathfrak{a} = \mathfrak{O}a = b\mathfrak{O}$ then $\mathfrak{a} = a\mathfrak{O} = \mathfrak{O}b$.*

Proof. From the assumption $\mathfrak{O}a = b\mathfrak{O}$ there exist elements x and y of $\mathfrak{O}$ such that $xa = b$ and $by = a$. Then $a = xay$ and $a = x^n a y^n$ for any positive integer n. If we denote by $I'(x)$ the closure of positive powers of x, and by $K(x)$ the set of cluster points of the set of powers of x then $K(x)$ is known to be a (topological) group with the property $I'(x)\cdot K(x)^{3)} \subset K(x)$ and $K(x)\cdot I'(x) \subset K(x)$ ([8] and [9]; these results depend only on the compactness of $I'(x)$). Let e be the identity of the group $K(x)$ then $a = ea$. Hence $b = xa = x(ea) = (xe)a$.

Since xe belongs to the group $K(x)$, if we denote by z an inverse element of xe in $K(x)$, then

$$zb = z(xe)a = ea = a.$$

Therefore, $\mathfrak{O}a = \mathfrak{O}zb \subset \mathfrak{O}b$. As $\mathfrak{a} = \mathfrak{O}a$ is an ideal and $b \in \mathfrak{a}$ it is clear that $\mathfrak{O}b \subset \mathfrak{O}a$. Hence we get $\mathfrak{O}a = \mathfrak{O}b$. Similarly we can obtain $a\mathfrak{O} = b\mathfrak{O}$.

Corollary. *A necessary and sufficient condition for an element a of $\mathfrak{O}$ to be a unit of $\mathfrak{O}$ is that a has a left (or right) inverse.*

Lemma 2.4. *If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathfrak{O}$ such that $\mathfrak{a} = \mathfrak{O}a$, $\mathfrak{b} = \mathfrak{O}b$ and $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{O}$ then $\mathfrak{ab} = \mathfrak{ba}$.*

Proof. By the assumption $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{O}$ there is an element a' such

1) Cf. [5].

2) Cf. [1] and [2; Chapter 2].

3) If A, B are two subsets of $\mathfrak{O}$ then A•B denotes the set of all elements of $\mathfrak{O}$ of the form ab, where $a \in A$, $b \in B$.

that $a'a \equiv 1$ (b). Since $\mathfrak{b} = \mathfrak{O}b$ is a closed ideal the residue class ring $\mathfrak{O}/\mathfrak{b}$ is also a compact topological ring. Therefore, by Corollary of Lemma 2.3 we have $aa' \equiv 1$ (b).

On the other hand, since $ab \in \mathfrak{a} = \mathfrak{O}a$, there is an element b' so that $ab = b'a$. Hence

$$b' \equiv b'aa' \equiv aba' \equiv 0 \text{ (b)}.$$

This shows that $\mathfrak{ab} = \mathfrak{O}ab = \mathfrak{O}b'a \subset \mathfrak{ba}$. Similarly we get $\mathfrak{ba} \subset \mathfrak{ab}$ and so $\mathfrak{ab} = \mathfrak{ba}$.

Let us consider now maximal open prime ideals of $\mathfrak{O}$. As the same notations with [10] let $\{\mathfrak{p}_\lambda ; \lambda \in \Lambda\}$ be the set of all maximal open prime ideals of $\mathfrak{O}$ and put $\bigcap_{n=1}^{\infty}\overline{\mathfrak{p}_\lambda^n} = \mathfrak{q}_\lambda$, moreover we assume $\mathfrak{p}_\lambda \neq \mathfrak{p}_\mu$ if $\lambda \neq \mu$. Then we have proved in [10] that $(\mathfrak{q}_\lambda, \mathfrak{q}_\mu) = \mathfrak{O}$ for $\lambda \neq \mu$.

Lemma 2.5. *For any $\lambda \in \Lambda$ we have $\mathfrak{q}_\lambda = \overline{\mathfrak{q}_\lambda^2}$ and hence $\mathfrak{q}_\lambda = \overline{\mathfrak{q}_\lambda^n}$ for $n = 1, 2, \cdots$.*

Proof. Suppose $\mathfrak{q}_\lambda \neq \overline{\mathfrak{q}_\lambda^2}$; then there exists an element $z \in \mathfrak{q}_\lambda$ which does not belong to $\overline{\mathfrak{q}_\lambda^2}$. Then since $\overline{\mathfrak{q}_\lambda^2}$ is compact there exists a compact open ideal neighborhood V of 0 such that $z \notin \overline{\mathfrak{q}_\lambda^2} + V = U$. Since $\mathfrak{q}_\lambda$ is compact and U is an open ideal containing $\mathfrak{q}_\lambda^2$, there is an open ideal W containing $\mathfrak{q}_\lambda$ such that $W^2 \subset U$. Since $\mathfrak{p}_\lambda, \overline{\mathfrak{p}_\lambda^2}, \cdots$ is a descending chain of compact sets whose intersection is equal to $\mathfrak{q}_\lambda$, we can find a positive integer m so that $\overline{\mathfrak{p}_\lambda^m} \subset W$. Then

$$U = \bar{U} \supset \overline{W}^2 \supset \overline{\overline{\mathfrak{p}_\lambda^m}\overline{\mathfrak{p}_\lambda^m}} \supset \overline{\mathfrak{p}_\lambda^{2m}} \supset \mathfrak{q}_\lambda \ni z.$$

Therefore we have arrived at a contradiction. Thus $\mathfrak{q}_\lambda = \overline{\mathfrak{q}_\lambda^2}$.

Let us assume $\mathfrak{q}_\lambda = \overline{\mathfrak{q}_\lambda^n}$ and prove $\mathfrak{q}_\lambda = \overline{\mathfrak{q}_\lambda^{n+1}}$. Since $\overline{\mathfrak{q}_\lambda^{n+1}} \supset \overline{\mathfrak{q}_\lambda^n}\mathfrak{q}_\lambda = \mathfrak{q}_\lambda\mathfrak{q}_\lambda = \mathfrak{q}_\lambda^2$ we get $\overline{\mathfrak{q}_\lambda^{n+1}} \supset \overline{\mathfrak{q}_\lambda^2} = \mathfrak{q}_\lambda$. This shows that $\overline{\mathfrak{q}_\lambda^{n+1}} = \mathfrak{q}_\lambda$.

Now we give the following two lemmas without proofs, because the lemmas can be proved by using the methods of [10].

Lemma 2.6. *If $\mathfrak{q}_\lambda\mathfrak{q}_\mu = \mathfrak{q}_\mu\mathfrak{q}_\lambda$ for any $\lambda, \mu \in \Lambda$ then $\bigcap_{\lambda\in\Lambda}\mathfrak{q}_\lambda = (0)$ and hence $\mathfrak{O}$ is decomposed into a complete direct sum of compact primary rings.*

Lemma 2.7. *Under the assumption of Lemma 2.6 let $\mathfrak{O} = \sum_\lambda \mathfrak{O}_\lambda$ be an expression of $\mathfrak{O}$ as a complete direct sum of compact primary rings $\mathfrak{O}_\lambda$'s.*

If $\mathfrak{a}$ is a closed left (right or two-sided) ideal of $\mathfrak{O}$ then $\mathfrak{a}$ can be written in the form

$$\mathfrak{a} = \sum_\lambda \mathfrak{a}_\lambda,$$

where $\mathfrak{a}_\lambda$ *is a closed left (right or two-sided) ideal of* $\mathfrak{O}_\lambda$. ($\sum_\lambda$ *denotes complete direct sum.*)

Using the above lemmas we get the following theorem:

Theorem 2.1. *Let* $\mathfrak{O}$ *be a compact ring with identity. Then the following four conditions are equivalent:*

(1) $\mathfrak{O}$ *is a complete direct sum of compact primary rings.*

(2) *A product of any two maximal open prime ideals is commutative, i.e.* $\mathfrak{p}_\lambda\mathfrak{p}_\mu = \mathfrak{p}_\mu\mathfrak{p}_\lambda$ *for any* $\lambda, \mu \in \Lambda$.

(3) $\mathfrak{q}_\lambda\mathfrak{q}_\mu = \mathfrak{q}_\mu\mathfrak{q}_\lambda$ *for any* $\lambda, \mu \in \Lambda$.

(4) *An ideal* $\mathfrak{a}$ *of* $\mathfrak{O}$ *with the property* $\mathfrak{a} = \overline{\mathfrak{a}^2}$ *has an identity as a ring.*

Proof. From Lemma 2.7 it is obvious that (1) implies (2), (1) implies (3) and (1) implies (4). We have proved in Theorem 1 of [10] that (2) implies (1); and Lemma 2.6 shows (3) implies (1).

We shall prove (4) implies (3). By Lemma 2.5 every $\mathfrak{q}_\lambda$ has the property $\mathfrak{q}_\lambda = \overline{\mathfrak{q}_\lambda^2}$, therefore $\mathfrak{q}_\lambda$ has an identity as a ring. Let e_λ be an identity element of $\mathfrak{q}_\lambda$ then $\mathfrak{q}_\lambda = \mathfrak{O}e_\lambda$. Since $(\mathfrak{q}_\lambda, \mathfrak{q}_\mu) = \mathfrak{O}$ for $\lambda \neq \mu$ we obtain, by Lemma 2.4, $\mathfrak{q}_\lambda\mathfrak{q}_\mu = \mathfrak{q}_\mu\mathfrak{q}_\lambda$. In case $\lambda = \mu$, $\mathfrak{q}_\lambda\mathfrak{q}_\mu = \mathfrak{q}_\mu\mathfrak{q}_\lambda$ is obvious. Thus the theorem has been completely proved.

§ 3. **Completely primary rings.**[1] In this section we denote by $\mathfrak{O}$ *a compact completely primary ring with a radical* $\mathfrak{p}$.

Lemma 3.1. *If there is no left ideal between* $\mathfrak{p}$ *and* $\mathfrak{p}^2$ *then every power of* $\mathfrak{p}$ *is open, hence the set of powers of* $\mathfrak{p}$ *forms a fundamental system of ideal neighborhoods of* 0.

Proof. If $\mathfrak{p} = \mathfrak{p}^2$ then, since $\mathfrak{p}$ is nilpotent by Lemma 2.2, $\mathfrak{p}$ must coincide with 0. In this case $\mathfrak{O} \cong \mathfrak{O}/\mathfrak{p} = K$ is a finite field and our lemma is true.

We suppose that $\mathfrak{p} \neq \mathfrak{p}^2$ and let p be an element of $\mathfrak{p}$ which does not belong to $\mathfrak{p}^2$. Then by the assumption that there is no left ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$, we get $\mathfrak{p} = \mathfrak{O}p + \mathfrak{p}^2$. Therefore $\mathfrak{p}^n = \mathfrak{O}p^n + \mathfrak{p}^{n+1}$ for any positive integer n. Then $\mathfrak{p}^n/\mathfrak{p}^{n-1}$ is a cyclic left $K(= \mathfrak{O}/\mathfrak{p})$-module.

1) A ring R is said to be *primary* if R has an identity and the residue class ring R/N of R modulo its radical N is a simple ring, and R is *completely primary* if R is primary and R/N is a division ring.

Since K is a field there is no left ideal between $\mathfrak{p}^n$ and $\mathfrak{p}^{n+1}$ for $n = 1, 2, \cdots$. Suppose $\mathfrak{p}^k$ is open but $\mathfrak{p}^{k+1}$ is not open (we put $\mathfrak{O} = \mathfrak{p}^0$). Then for any ideal neighborhood V which is contained in $\mathfrak{p}^k$, $\mathfrak{p}^{k+1} + V$ is an open ideal between $\mathfrak{p}^k$ and $\mathfrak{p}^{k+1}$. Hence $\mathfrak{p}^{k+1} + V$ must coincide with $\mathfrak{p}^k$; $\mathfrak{p}^{k+1} + V = \mathfrak{p}^k$. This equality holds for any ideal neighborhood $V \subset \mathfrak{p}^k$. This means that $\mathfrak{p}^{k+1}$ is everywhere dense in $\mathfrak{p}^k$, i.e. $\overline{\mathfrak{p}^{k+1}} = \mathfrak{p}^k$. Then for any positive integer $m \geqq k$ we have $\overline{\mathfrak{p}^m} = \mathfrak{p}^k$. Since $\mathfrak{p}$ is nilpotent $\mathfrak{p}^k$ must be equal to zero. Then $\mathfrak{O} \cong \mathfrak{O}/\mathfrak{p}^k$ is a finite set and any subset of $\mathfrak{O}$ is open, which is a contradiction to the assumption $\mathfrak{p}^{k+1}$ is not open. Hence any power of $\mathfrak{p}$ is open.

Lemma 3.2. *If there is no left ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$ then $\mathfrak{O}$ has no proper left ideal*[1] *other than powers of $\mathfrak{p}$.*

Proof. First we shall prove this lemma for open left ideals. Let $\mathfrak{a}$ be an arbitrary open proper left ideal of $\mathfrak{O}$. Then since $\mathfrak{p}$ is nilpotent there exists a positive integer s such that

$$\mathfrak{p}^s \subset \mathfrak{a}, \quad \mathfrak{p}^{s-1} \not\subset \mathfrak{a},$$

where, since $\mathfrak{p}$ is open by Lemma 3.1 and since $\mathfrak{O}/\mathfrak{p}$ is a field, $s \geqq 1$. To prove $\mathfrak{a} = \mathfrak{p}^s$, suppose $\mathfrak{p}^s \subsetneqq \mathfrak{a}$. Then there is an element a of $\mathfrak{a}$ which does not belong to $\mathfrak{p}^s$. Since a is a non-zero element of $\mathfrak{O}$ and $\mathfrak{p}$ is nilpotent there exists a positive integer i such that $a \in \mathfrak{p}^i$ but $a \notin \mathfrak{p}^{i+1}$, where $i < s$. If $\mathfrak{p} = \mathfrak{p}^2$ then by the same argument as in above lemma our lemma is true. Therefore we suppose $\mathfrak{p} \neq \mathfrak{p}^2$ and let p be an element of $\mathfrak{p}$ not belonging to $\mathfrak{p}^2$. Then by Lemma 3.1 for any positive integer n we have $\mathfrak{p}^n = \mathfrak{O}p^n + \mathfrak{p}^{n+1}$. Hence, for some $\xi \not\equiv 0 \bmod \mathfrak{p}$ in $\mathfrak{O}$, we have

$$a \equiv \xi p^i \pmod{\mathfrak{p}^{i+1}}$$

Since $\mathfrak{O}/\mathfrak{p}$ is a field there exists some $\eta \in \mathfrak{O}$ such that $\eta\xi \equiv 1 \bmod \mathfrak{p}$; that is, $\eta a \equiv p^i \pmod{\mathfrak{p}^{i+1}}$. Therefore we get

$$p^{s-i-1}\eta a \equiv p^{s-1} \pmod{\mathfrak{p}^s},$$

where we put $p^0 = 1$. Hence it follows immediately that

$$\mathfrak{a} \supset \mathfrak{O}p^{s-i-1}\eta a + \mathfrak{p}^s \supset \mathfrak{O}p^{s-1} + \mathfrak{p}^s = \mathfrak{p}^{s-1}.$$

But this gives a contradiction, and therefore $\mathfrak{a} = \mathfrak{p}^s$.

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1) A *proper left (right or two-sided) ideal* $\mathfrak{a}$ of $\mathfrak{O}$ is a left (right or two-sided) ideal distinct from $\mathfrak{O}$ and (O).

Now let $\mathfrak{l}$ be any proper left ideal of $\mathfrak{O}$. Suppose $\mathfrak{l} \subset \mathfrak{p}^t$, $\mathfrak{l} \not\subset \mathfrak{p}^{t+1}$ and z an element of $\mathfrak{l}$ which does not belong to $\mathfrak{p}^{t+1}$. Then for any ideal neighborhood $V \subset \mathfrak{p}^t$ (by Lemma 3.1 $\mathfrak{p}^t$ is open) $\mathfrak{O}z + V$ is an open left ideal such that $\mathfrak{p}^t \supset \mathfrak{O}z + V$, $\mathfrak{p}^{t+1} \not\supset \mathfrak{O}z + V$. This implies, by the above, $\mathfrak{O}z + V = \mathfrak{p}^t$. Since this equality holds for any ideal neighborhood $V \subset \mathfrak{p}^t$ we obtain $\mathfrak{O}z$ is everywhere dense in $\mathfrak{p}^t$, i. e. $\overline{\mathfrak{O}z} = \mathfrak{p}^t$. Hence

$$\mathfrak{p}^t = \overline{\mathfrak{O}z} = \mathfrak{O}z \subset \mathfrak{l}.$$

Thus we get $\mathfrak{l} = \mathfrak{p}^t$.

Lemma 3.2′ (Dual of Lemma 3.2). *If there is no right ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$ then $\mathfrak{O}$ has no proper right ideal other than powers of $\mathfrak{p}$.*

Lemma 3.3. *Let R be a completely primary finite ring with radical N such that $N^2 = 0$. If there is no proper left ideal of R other than N then there is no proper right ideal of R other than N.*

Proof. Since $K = R/N$ is a field, if N is a principal right ideal then R has no proper right ideal other than N. We shall prove that N is a principal right ideal.

Let $G = R \setminus N$[1] ($\neq \square$[2]) then G is obviously a multiplicative group whose identity is the same with that of R. Take any non-zero element a of N; then, since R has no proper left ideal other than N, $N = Ra$. Hence

$$N = Ra = (G \cup N)a = Ga \cup Na = Ga \cup \{0\}.$$

For any subset X of R we denote by $\mathrm{ord}(X)$ the number of elements of X. Then

$$\mathrm{ord}(N) = \mathrm{ord}(Ga) + 1,$$

because $Ga \ni 0$.

We prove now $\mathrm{ord}(Ga) = \mathrm{ord}(aG)$. For any $x, y \in G$, if $xa = ya$ then $(x - y)a = 0$ hence $x - y$ must be contained in N. Therefore $a(x - y) = 0$; this means $ax - ay = 0$ and so $ax = ay$. This implies $\mathrm{ord}(Ga) \geqq \mathrm{ord}(aG)$. Similarly we obtain $\mathrm{ord}(Ga) \leqq \mathrm{ord}(aG)$, that is, $\mathrm{ord}(Ga) = \mathrm{ord}(aG)$.

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1) Let A be a set and B, C subsets of A. We denote by $B \setminus C$ the complement of C in B, that is, the set of elements of B not contained in C.

2) $\square$ denotes the empty set.

Then, from $aR = a(G \cup N) = aG \cup aN = aG \cup \{0\}$ and aG does not contain 0, we have

$$\operatorname{ord}(aR) = \operatorname{ord}(aG) + 1 = \operatorname{ord}(Ga) + 1 = \operatorname{ord}(N).$$

Since aR is contained in N we get $N = aR$.

Theorem 3.1. *If there is no left (or right) ideal of* $\mathfrak{O}$ *between* $\mathfrak{p}$ *and* $\mathfrak{p}^2$ *then there is no proper one-sided ideal of* $\mathfrak{O}$ *other than powers of* $\mathfrak{p}$.

Proof. We prove this theorem in case of left ideals. By Lemma 3.2 $\mathfrak{O}$ has no proper left ideals other than powers of $\mathfrak{p}$.

Let $R = \mathfrak{O}/\mathfrak{p}^2$ then by Lemma 3.1 $\mathfrak{p}^2$ is open and therefore R is a finite ring. We denote by N the radical of R then $N = \mathfrak{p}/\mathfrak{p}^2$ and for R, N all conditions of Lemma 3.3 are satisfied. Hence R has no proper right ideal other than N. This means there is no right ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$. Hence by Lemma 3.2′ $\mathfrak{O}$ has no proper right ideal other than powers of $\mathfrak{p}$.

From Theorem 3.1 and proofs of Lemmas 3.1, 3.2 we obtain the following theorem:

Theorem 3.2. *In* $\mathfrak{O}$ *the following twelve conditions are equivalent to each other:*

(1L) *Every left ideal of* $\mathfrak{O}$ *is a principal left ideal.*

(1R) *Every right ideal of* $\mathfrak{O}$ *is a principal right ideal.*

(1) $\mathfrak{O}$ *is a principal ideal ring, that is, if* $\mathfrak{l}, \mathfrak{r}$ *and* $\mathfrak{m}$ *are left, right and two-sided ideals, respectively, then* $\mathfrak{l} = \mathfrak{O}a, \mathfrak{r} = b\mathfrak{O}$ *and* $\mathfrak{m} = \mathfrak{O}c = c\mathfrak{O}$.

(2L) $\mathfrak{p}$ *is a principal left ideal.*

(2R) $\mathfrak{p}$ *is a principal right ideal.*

(2) $\mathfrak{p}$ *is a principal left and principal right ideal.*

(3L) *There is no left ideal between* $\mathfrak{p}$ *and* $\mathfrak{p}^2$.

(3R) *There is no right ideal between* $\mathfrak{p}$ *and* $\mathfrak{p}^2$.

(3) *There is no one-sided ideal between* $\mathfrak{p}$ *and* $\mathfrak{p}^2$.

(4L) $\mathfrak{O}$ *has no proper left ideal other than powers of* $\mathfrak{p}$.

(4R) $\mathfrak{O}$ *has no proper right ideal other than powers of* $\mathfrak{p}$.

(4) $\mathfrak{O}$ *has no proper one-sided ideal other than powers of* $\mathfrak{p}$.

From Theorem 3.2 and results of [10] we get the following:

Theorem 3.3. *If* $\mathfrak{O}$ *satisfies one of the twelve conditions of Theorem* 3.2 *then* $\mathfrak{O}$ *is*

1) *a finite field if and only if* $\mathfrak{p} = 0$,

2) *a completely primary uni-serial finite ring if and only if* $\mathfrak{p}^m = 0$ *for some positive integer* $m \geqq 2$.

3) *a maximal compact open order of a totally disconnected locally compact (t. d. l. c.) division ring*[1] *if and only if* $\mathfrak{p}^n \neq 0$ *for any positive integer* n.

§ 4. **Primary rings.** Throughout this section $\mathfrak{O}$ denotes a *compact primary ring with radical* $\mathfrak{p}$. Then by a Kaplansky's theorem[2] $\mathfrak{O}$ is a matrix ring over a compact completely primary ring $\mathfrak{O}_0$: $\mathfrak{O} = \sum_{i,j=1}^{r} \mathfrak{O}_0 c_{ij}$, where c_{ij}'s are matrix units. If $\mathfrak{p}_0$ is the radical of $\mathfrak{O}_0$ then it is well-known that $\mathfrak{p} = \sum_{i,j=1}^{r} \mathfrak{p}_0 c_{ij}$.

Then we have the following theorem[3]:

Theorem 4.1. *In* $\mathfrak{O}$ *the following nine conditions are equivalent to each other*:

(1L) *Every left ideal of* $\mathfrak{O}$ *is a principal left ideal.*

(1R) *Every right ideal of* $\mathfrak{O}$ *is a principal right ideal.*

(1) $\mathfrak{O}$ *is a principal ideal ring.*

(2L) $\mathfrak{p}$ *is a principal left ideal.*

(2R) $\mathfrak{p}$ *is a principal right ideal.*

(2) $\mathfrak{p}$ *is a principal left and principal right ideal.*

(3L) $\mathfrak{p}/\mathfrak{p}^2$ *is a cyclic left* $\mathfrak{O}$*-module.*

(3R) $\mathfrak{p}/\mathfrak{p}^2$ *is a cyclic right* $\mathfrak{O}$*-module.*

(3) $\mathfrak{p}/\mathfrak{p}^2$ *is a cyclic left and right* $\mathfrak{O}$*-module.*

Proof. It is clear that (1L) implies (2L) and (2L) implies (3L).

We prove that (3L) implies (1L) and (1R). Since $\mathfrak{p}/\mathfrak{p}^2$ is a cyclic left $\mathfrak{O}$-module we get $\mathfrak{O}/\mathfrak{p}^2$ is a left uni-serial ring.[4] Hence $\mathfrak{O}_0/\mathfrak{p}_0^2$ is a left uni-serial ring and therefore there is no left ideal between $\mathfrak{p}_0$ and $\mathfrak{p}_0^2$.

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1) Cf. [3].

2) Cf. [5].

3) Cf. [1 and 2]. For abstract rings the same theorems with Theorem 4.1 has been completely proved by Asano, therefore we omit a detailed proof of this theorem.

4) A ring R is called *left uni-serial* if

(a) R satisfies the descending and the ascending chain condition for left ideals,

(b) R is a direct sum of a finite number of primary rings,

(c) if e is a primitive idempotent of R then a left ideal Re has a unique composition series as a left R-module.

Then by Theorem 3.2 we can conclude that any left or right ideal of $\mathfrak{O}_0$ is a principal ideal. Therefore any left or right ideal of $\mathfrak{O}$ is a principal ideal.

Similarly we can prove that (1R) $\rightarrow$ (2R) $\rightarrow$ (3R) $\rightarrow$ (1R) and (1L). Furthermore, (1L) and (1R) $\leftrightarrow$ (1), (2L) and (2R) $\leftrightarrow$ (2), and (3L) and (3R) $\leftrightarrow$ (3).

From Theorems 3.3 and 4.1 we obtain the following structure theorem:

Theorem 4.2. *If $\mathfrak{O}$ satisfies one of the nine conditions of Theorem* 4.1 *then $\mathfrak{O}$ is*

1) *a matrix ring over a finite field if and only if $\mathfrak{p} = 0$,*

2) *a matrix ring over a completely primary uni-serial finite ring if and only if $\mathfrak{p}^m = 0$ for some positive integer $m \geqq 2$,*

3) *a matrix ring over a maximal compact open order of a t.d.l.c. division ring if and only if $\mathfrak{p}^n \neq 0$ for any positive integer n.*

§ 5. **Structure of compact rings with certain conditions.** In this last section we give two structure theorems (Theorems 5.1 and 5.3) concerning compact rings with certain conditions.

First of all we give the following theorem which is an immediate consequence of Theorem 3.3 and Theorems 3, 7 of [10].

Theorem 5.1. *Let $\mathfrak{O}$ be a compact ring with identity in which a product of any two maximal open left (or right) ideals is commutative and there exists no open left (or right) ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$ for every maximal open prime ideal $\mathfrak{p}$. Then $\mathfrak{O}$ is a complete direct sum of maximal compact open orders of t.d.l.c. division rings, completely primary uni-serial finite rings and finite fields. And the number of the direct summands is finite if and only if $\mathfrak{O}$ is a Q-ring.*

Theorem 5.2. *In a compact ring $\mathfrak{O}$ with identity the following nine conditions are equivalent to each other:*

(1L) *Every closed left ideal of $\mathfrak{O}$ is a pricipal left ideal.*

(1R) *Every closed right ideal of $\mathfrak{O}$ is a principal right ideal.*

(1) *Every closed ideal of $\mathfrak{O}$ is a principal left and principal right ideal.*

(2L) *Every maximal open prime ideal of $\mathfrak{O}$ is a principal left*

ideal.

(2R) *Every maximal open prime ideal of* $\mathfrak{O}$ *is a principal right ideal.*

(2) *Every maximal open prime ideal of* $\mathfrak{O}$ *is a principal right ideal.*

(3L) *A product of any two maximal open prime ideals of* $\mathfrak{O}$ *is commutative and for any maximal open prime ideal* $\mathfrak{p}$, $\mathfrak{p}/\mathfrak{p}^2$ *is a cyclic left* $\mathfrak{O}$*-module.*

(3R) *A product of any two mvximal open prime ideals of* $\mathfrak{O}$ *is commutative and for any maximal open prime ideal* $\mathfrak{p}$, $\mathfrak{p}/\mathfrak{p}^2$ *is a cyclic right* $\mathfrak{O}$*-module.*

(3) *A product of any two maximal open prime ideals of* $\mathfrak{O}$ *is commutative and for any maximal open prime ideal* $\mathfrak{p}$, $\mathfrak{p}/\mathfrak{p}^2$ *is a cyclic left and right* $\mathfrak{O}$*-module.*

Proof. It is clear that (1L) implies (2L). We suppose (2L) holds; then, by Lemma 2.4, a product of any two maximal open prime ideals of $\mathfrak{O}$ is commutative. Moreover, since every maximal open prime ideal $\mathfrak{p}$ is a principal left ideal: $\mathfrak{O}p = \mathfrak{p}$, we get $\mathfrak{p}^2 = \mathfrak{O}p^2$. Hence $\mathfrak{p}/\mathfrak{p}^2 = \mathfrak{O}p/\mathfrak{O}p^2$ is a cyclic left $\mathfrak{O}$-module. Thus (2L) implies (3L).

We now prove that (3L) implies (1L) and (1R). From Theorem 2.1 $\mathfrak{O}$ is a complete direct sum of compact primary rings $\mathfrak{O}_\lambda$'s: $\mathfrak{O} = \sum_\lambda \mathfrak{O}_\lambda$. If $\mathfrak{a}$ is a closed left ideal of $\mathfrak{O}$ then by Lemma 2.7 $\mathfrak{a}$ can be expressed in the form $\mathfrak{a} = \sum_\lambda \mathfrak{a}_\lambda$, where each $\mathfrak{a}_\lambda$ is a closed left ideal of $\mathfrak{O}_\lambda$. Therefore to prove that (3L) implies (1L) and (1R) we may assume that $\mathfrak{O}$ is a compact primary ring. Then from Theorem 4.1 we can conclude that (3L) implies (1L) and (1R).

Similarly we can prove (1R) $\rightarrow$ (2R) $\rightarrow$ (3R) $\rightarrow$ (1R) and (1L). Moreover, we can easily get (1) $\rightarrow$ (2) $\rightarrow$ (3) $\rightarrow$ (3L) $\rightarrow$ (1)

Finally we get the following theorem as a direct consequence of Theorems 4.2 and 5.2:

Theorem 5.3. *Let* $\mathfrak{O}$ *be a compact ring with identity which satisfies one of the nine conditions of Theorem* 5.2. *Then* $\mathfrak{O}$ *is decomposed into a complete direct sum of compact primary rings* $\mathfrak{O}_\lambda$*'s:* $\mathfrak{O} = \sum_\lambda \mathfrak{O}_\lambda$, *and each compact primary ring* $\mathfrak{O}_\lambda$ *appearing in the direct summands has a type of* 1), 2) *or* 3) *of Theorem* 4.2. *Moreover the number of the direct summands is finite if and only if* $\mathfrak{O}$ *is a Q-ring.*

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