NOTE ON THE ISOMETRIC IMBEDDING OF COMPACT RIEMANNIAN MANIFOLDS IN EUCLIDEAN SPACES

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In this note, we shall state some remarks on the isometric imbedding of compact Riemannian manifolds in Euclidean spaces in connection with the works of Shiin-Shen Chern and N. H. Kuiper [1][1)] and the author [2]. Theorem 1 in [2] as follows is fundamental in our considerations.

Theorem A. *Let M be a compact Riemannian manifold of dimension n and with the property that at every point there is a q-dimensional linear subspace in the tangent space along whose plane elements the sectional curvatures are non positive. Then M can not be isometrically imbedded in an Euclidean space of dimension* $n+q-1$.

§ **1.** Let M be a Riemannian manifold of dimension n whose line element is given by

$$ds^2 = \sum g_{ij}(x)dx^i dx^j \tag{1}$$

in local coordinates $x^1, x^2, \cdots\cdots, x^n$. Let us put

$$\sum g_{ij}(x)dx^i dx^j = \sum \omega_i(x, dx)\omega_i(x, dx), \tag{2}$$

$$\begin{cases} d\omega_i = \sum \omega_j \wedge \omega_{ji}, \\ d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij} \end{cases} \tag{3}$$

where Ω_{ij} are the curvature forms of M as is well known. Let $k(p)$, $p \in M$, be the minimum number of linear differential forms in terms of which the curvature forms of M at p can be expressed. According to [1], $n - k(p)$ is called the *index of nullity* at p. Let us put

$$k(M) = \max_{p \in M} k(p)$$

A generalized Tompkins' theorem as follows was proved by means of algebraic methods in [1], [3].

1) Numbers in brackets refer to the list of references at the end of the paper.

Theorem B. *A compact Riemannian manifold M of dimension n can not be isometrically imbedded in an Euclidean space of dimension $2n - k(M) - 1$.*

We shall give more detailed results than the above theorem.

§ **2.** As a preliminary we establish the following lemmas. Let R_{ijkh} be real number such that

$$R_{ijkh} = - R_{jikh} = - R_{ijhk} = R_{khij} \tag{4}$$

and let us consider the form

$$R(x, y) = R_{ijkh} x^i y^j x^k y^h \text{ [1]} \tag{5}$$

of real $2n$ variable x^i, y^i. Let m be the maximum of dimensions of linear subspaces L of the n-dimensional real vector space such that for any $x, y \in L$

$$R(x, y) \leqslant 0.$$

Let k be the rank of the system of linear equations

$$R_{ijkh} x^h = 0, \quad i, j, k = 1, 2, \cdots\cdots, n, \tag{6}$$

in x^i and L_0 be the linear space of the solutions of (6). We have $\dim L_0 = n - k$ and

$$R(x, y) = 0, \quad x \in L_n, \ y \in L_0. \tag{7}$$

Lemma 1. *If a linear subspace L of L_n has the property that for any two $x, y \in L$, $R(x, y) \leqslant 0$, then $L \cup L_0$*[2] *has the same property.*

Proof. For any $x, y \in L \cup L_0$, we may put

$$x = a_1 x_1 + b_1 y_1, \quad y = a_2 x_2 + b_2 y_2,$$
$$x_1, x_1 \in L, \quad y_1, y_2 \in L_0.$$

By means of (7), we get easily

$$R(x, y) = (a_1 a_2)^2 R(x_1, x_2) \leqslant 0.$$

Since any 1-dimensional subspace has the above property, we get

1) The summation convention of tensor analysis is used in the following.

2) We denote by $L \smile L_0$ the linear space spanned by the elements of L and L_0.

easily the following lemma.

Lemma 2. *If* $k > 0$, *then* $n - k + 1 \leqslant m$.

Now, let us assume that (4) is of the form

$$R_{ijkh} = H_{ik}H_{jh} - H_{ih}H_{jk}, \quad H_{ij} = H_{ji}. \tag{8}$$

Accordingly we have

$$R(x, y) = \Psi(x, x)\Psi(y, y) - (\Psi(x, y))^2, \tag{9}$$

where $\Psi(x, y) = H_{ij}x^i y^j$. From (9) we may consider that $-R(x, y)$ is a generalized discriminant of the quadratic equation $H_{ij}x^i x^j = 0$ in n variables x^i. We can easily see that

$$k = \text{rank } (H_{ij}). \tag{10}$$

Lemma 3. *If* R_{ijkh} *is of the form* (8), *then* $m = n - \frac{k + \rho}{2} + 1$ *where* $\rho = |$*signature of* $\Psi(x, x)|$.

Proof. As stated above, the system of linear equatious

$$R_{ijkh}x^h = 0, \quad i, j, k = 1, 2, \cdots\cdots, n$$

is clearly equivalent to the system of linear equations

$$H_{ij}x^j = 0, \quad i = 1, 2, \cdots\cdots, n.$$

Let us suppose that $k = n$. If $\Psi(x, x) = H_{ij}x^i x^j$ is definite, then for any linearly independent vectors x, y, we have $R(x, y) = \Psi(x, x)\Psi(y, y) - (\Psi(x, y))^2 > 0$ as is well known. Hence $m = 1$. Since $\rho = n$, the above stated relation holds good.

In the next place, let us assume that $\Psi(x, x)$ is not definite. Taking a suitable base of L_n, we may put

$$\begin{aligned} &H_{\alpha\alpha} = 1, \quad \alpha = 1, 2, \cdots\cdots r, \\ &H_{\lambda\lambda} = -1, \quad \lambda = r + 1, \cdots\cdots, n, \quad 0 < r < n, \\ &H_{ij} = 0, \quad i \neq j, i, j = 1, 2, \cdots\cdots, n, \end{aligned}$$

Let L_r, L_{n-r} be the subspaces of L_n given by $x^\lambda = 0$, $\lambda = r + 1, \cdots\cdots, n$; $x^\alpha = 0$, $\alpha = 1, 2, \cdots\cdots, r$, respectively, for which we have $L_r \cup L_{n-r} = L_n$ and $L_r \cap L_{n-r} = 0$.

Let L be a linear subspace such that $R(x, y) \leqslant 0, x, y \in L$ and dim $L \geqslant 2$. Since the quadratic form $\Psi(x, x)$ is definite on L_r and L_{n-r}, it follows that

(11) $$\dim L \cap L_r \leqslant 1, \quad \dim L \cap L_{n-r} \leqslant 1.$$

Let $\pi_r : L_n \to L_r$ and $\pi_{n-r} : L_n \to L_{n-r}$ be the projections, then it must be

(12) $$\dim \pi_r(L), \ \dim \pi_{n-r}(L) \geqslant \dim L - 1,$$

for otherwise we get easily $\dim L \cap L_{n-r} \geqslant 2$ or $\dim L \cap L_r \geqslant 2$ which contradicts to (11). Accordingly we get from (12)

$$\dim L \leqslant n - r + 1, \ r + 1.$$

Now, we may assume that $n - r \leqslant r$. Let us take $\xi_A = (\xi_A^\alpha) \in L_r$ and $\eta_A = (\eta_A^\lambda) \in L_{n-r}$, $A = 1, 2, \cdots\cdots, n - r$ such that

$$\textstyle\sum_\alpha \xi_A^\alpha \xi_B^\alpha = \sum_\lambda \eta_A^\lambda \eta_B^\lambda = \delta_{AB} \text{ }^{1)}$$

Then we can define linearly independent vectors $\zeta_1, \cdots\cdots, \zeta_{n-r+1}$ of L_n by

$$\zeta_A = (\xi_A^\alpha, \eta_A^\lambda), \ A = 1, 2 \cdots\cdots, n - r - 1,$$
$$\zeta_{n-r} = (\xi_{n-r}^\alpha, 0), \quad \zeta_{n-r+1} = (0, \eta_{n-r}^\lambda).$$

Hence we have

(13) $$\Psi(\zeta_A, \zeta_B) = \Psi(\zeta_A, \zeta_{n-r}) = \Psi(\zeta_A, \zeta_{n-r+1}) = \Psi(\zeta_{n-r}, \zeta_{n-r+1}) = 0,$$
$$\Psi(\zeta_{n-r}, \zeta_{n-r}) = - \Psi(\zeta_{n-r+1}, \zeta_{n-r+1}) = 1$$
$$A, B = 1, 2, \cdots\cdots, n - r - 1.$$

Let L be the space spanned by $\zeta_1, \cdots\cdots, \zeta_{n-r+1}$. For any two vectors x, y of L

$$x = \sum_{A=1}^{n-r+1} u_A \zeta_A, \quad y = \sum_{A=1}^{n-r+1} v_A \zeta_A,$$

we get from (13)

$$\Psi(x, x) = u_{n-r}^2 - u_{n-r+1}^2,$$
$$\Psi(y, y) = v_{n-r}^2 - v_{n-r+1}^2,$$
$$\Psi(x, y) = u_{n-r} v_{n-r} - u_{n-r+1} v_{n-r+1},$$

hence

1) $\delta_{AB} = 1$ if $A = B$ and $\delta_{AB} = 0$ if $A \neq B$.

$$R(x, y) = -(u_{n-r}v_{n-r-1} - u_{n-r-1}v_{n-r})^2 \leqslant 0.$$

Thus we see that $m = n - r + 1$ in this case. Since $\rho = r - (n - r)$ by definition, we get

$$m = \frac{n - \rho}{2} + 1. \tag{14}$$

In general case, we have by virture of Lemma 1 the relation

$$m = (n - k) + (\frac{k - \rho}{2} + 1) = n - \frac{k + \rho}{2} + 1.$$

§ **3.** Now we shall proceed geometrical considerations. For an n-dimensional Riemannian manifold M, $k(M) = 0$ is equivalent to that M is locally Euclidean. Accordingly we get from Lemma 2 and Theorem A a more detailed theorem than Theorem B in § 2.

Theorem 1. *A compact locally non Euclidean Riemannian manifold M of dimension n can not be isometrically imbedded in an Euclidean space of dimension $2n - k(M)$.*

Now, let M be an n-dimonsional Riemannian manifold whose curvature forms

$$\Omega_{ij} = \frac{1}{2} \Sigma\ R_{ijkh}\omega_k \wedge \omega_h \tag{15}$$

are of the form (8). If $k(p) \geqslant 3$, $p \in M$, then H_{ij} is determined uniquely save for signs as is well known. Accordingly the absolute value of signature of the quadratic form $H_{ij}x^i x^j$ is an invariant of M at p. We denote this by $\rho(p)$ and put $\rho(M) = \max_{p \in M} \rho(p)$. Furthermore if $k(p) \leqslant 4$, then the field of H_{ij} satisfies the Codazzi's equation

$$H_{ij,k} - H_{ik,j} = 0. \tag{16}$$

Since (8) is the Gauss equations, M can be locally isometrically imbedded in an Euclidean space of dimension $n + 1$. By means of Lemma 3 and Theorem A, we obtain the following theorem.

Theorem 2. *Let M be a compact Riemannian manifold of dimension n on which there exists a symmetric tensor field H_{ij} such that*

$$R_{ijkh} = H_{ik}H_{jh} - H_{ih}H_{jk}.$$

Then M can not be isometrically imbedded in an Euclidean space of dimension $2n - \frac{1}{2}(k(M) + \rho(M))$. [1]

This theorem shows that even though M is of imbedding class 1, that is it can be locally isometrically imbedded in an Euclidean space of dimension $n+1$, it does not so in the large and shows negatively an order of imbeddability of M into Euclidean spaces.

Lastly we consider the case $k(M) = 2$. Then there exists a skew-symmetric tensor S_{ij} such that

$$R_{ijkh} = \sigma S_{ij} S_{kh}, \tag{17}$$

that is M is a space of separated curvature. Thus we get generally the following theorem.

Theorem 3. *A compact Riemannian manifold of dimension n with separated curvature and with non-positive scalar curvature can not be isometrically imbedded in an Euclidean space of dimension $2n-1$.*

§ **4.** In this section, we shall investigate especially compact Riemannian manifolds of dimension 3. By means of orthonormal frames, we put

$$\begin{aligned} R_{2323} &= K_{11}, & R_{3112} &= K_{23} = K_{32}, \\ R_{3131} &= K_{22}, & R_{1223} &= K_{31} = K_{13}, \\ R_{1212} &= K_{33}, & R_{2331} &= K_{12} = K_{21} \end{aligned}$$

and

$$x^2y^3 - x^3y^2 = v^1, \quad x^3y^1 - x^1y^3 = v^2, \quad x^1y^2 - x^2y^1 = v^3.$$

Then we have easily the equations

$$R_{ij} = R_i{}^k{}_{jk} = \frac{1}{2} R\delta_{ij} - K_{ij}, \quad R = R_i{}^i \tag{18}$$

and

$$R(x, y) = \frac{1}{2} R \sum v^i v^i - R_{ij} v^i v^j. \tag{19}$$

1) We put $\rho(M) = \max_{p \in M} \rho(p)$, $\rho(p) = |\text{signature of } (H_{ij}(p))|$.

Making use of frames such that $R_{ij}=0,\ i\neq j,$ we get

$$(20)\qquad 2R(x, y) = (-R_{11}+R_{22}+R_{33})v^1v^1+(R_{11}-R_{22}+R_{33})v^2v^2 + (R_{11}+R_{22}-R_{33})v^3v^3.$$

In order that $m=3$, it is necessary and sufficient that $R_{11}, R_{22}, R_{33}\leqslant 0$ and $|R_{11}|, |R_{22}|, |R_{33}|$ are the lengths of the sides of a triangle including the case in which the triangle degenerates. In order that $m=1$, it is necessary and sufficient that $R_{11}, R_{22}, R_{33}>0$ and R_{11}, R_{22}, R_{33} are the lengths of the sides of a triangle. Thus we obtain the following theorems.

Theorem 4. *A compact Riemannian manifold of dimension* 3 *with the property that at every point its Ricci tensor R_{ij} is negative semi-definite and the absolute values of its eigen values are the lengths of the three sides of a triangle (including the case in which the triangle degenerates), can not be isometrically imbedded in an Euclidean space of dimension* 5.

Corollary. *A compact Einstein space of dimension* 3 *with non-positive scalar curvature at every point can not be isometrically imbedded in an Euclidean space of dimension* 5.

Theorem 5. *A compact Riemannian manifold of dimension* 3 *with the property that there exists no point at which its Ricci tensor R_{ij} is positive definite and its eigen values are the lengths of the three sides of a triangle, can not be isometrically imbedded in an Euclidean space of dimension* 4.

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