THEORY OF COMPACT RINGS

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In his paper K. Asano [1, Theorem 3][1] proved that a commutative ring with the identity element satisfying the ascending chain condition for ideals is a direct sum of a finite number of Dedekind domains and completely primary uni-serial rings if there exists no ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$ for any maximal prime ideal $\mathfrak{p}$. In the theory of topological rings, as I. Kaplansky showed, the compactness assumption plays a rôle similar to that of the finiteness assumption for abstract rings.

In this note we shall investigate the structure of a compact ring having no one-sided ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$ for any maximal open ideal $\mathfrak{p}$ and prove several theorems similar to Asano's. In case of a compact ring, however, the ring need not be commutative but has only to satisfy the condition that a product of arbitrary two maximal open left (or right) ideals in the ring is commutative. Further, in this case, the number of direct summands is not always finite. It is finite if (and only if) the ring is a compact Q-ring.

§ 1. Algebraic Preliminaries.

In this section R denotes a non-commutative associative ring with the identity element 1. Then we shall prove

Lemma 1. *Let $\mathfrak{p}(\neq R)$ be a prime ideal in R such that the residue class ring $\hat{R}=R/\mathfrak{p}$ satisfies the descending chain condition for left (right) ideals. Then $\hat{R}$ is a simple ring and hence $\mathfrak{p}$ is a maximal ideal in R.*

Proof. Let $\hat{\mathfrak{n}}$ be a nilpotent two-sided ideal in $\hat{R}$ with the nilpotency index s. Then there is an ideal $\mathfrak{n}$ in R such that $\mathfrak{n}/\mathfrak{p}=\hat{\mathfrak{n}}$ and $\mathfrak{n}^s \subseteqq \mathfrak{p}$. Since $\mathfrak{p}$ is prime $\mathfrak{n} \subseteqq \mathfrak{p}$, i. e. $\hat{\mathfrak{n}}$ is the null ideal in $\hat{R}$. This shows that $\hat{R}$ is a semi-simple ring. If $\hat{R}$ is not simple then there exist two-sided ideals $\hat{\mathfrak{a}}$, $\hat{\mathfrak{b}}$ in $\hat{R}$ different from the null ideal such that $\hat{\mathfrak{a}}\hat{\mathfrak{b}}$ is the null ideal. This means that there are two-sided ideals $\mathfrak{a}$, $\mathfrak{b}$ in R with $\mathfrak{a}\mathfrak{b} \subseteqq \mathfrak{p}$ such that $\mathfrak{a}/\mathfrak{p}=\hat{\mathfrak{a}}$ and $\mathfrak{b}/\mathfrak{p}=\hat{\mathfrak{b}}$. Hence, either $\mathfrak{a}$ or $\mathfrak{b}$ is equal to $\mathfrak{p}$ but this is a contradiction. Therefore, $\hat{R}$ is a simple ring; that is, $\mathfrak{p}$ is a maximal ideal in R.

Lemma 2. *Let in R the product of any two maximal left (or*

1) Numbers in brackets refer to the bibliography at the end of this paper.

right) ideals be commutative and let $\mathfrak{p}$ *be a prime ideal such that the residue class ring* $\hat{R}$ *of* R *by* $\mathfrak{p}$ *satisfies the descending chain condition. Then* $\hat{R}$ *is a division ring and, therefore, there exist no proper left and right ideal containing* $\mathfrak{p}$.

Proof. By Lemma 1, $\hat{R}$ is a simple ring. If $\hat{R}$ is no division ring $\hat{R}$ is represented as a direct sum of a finite number of minimal left ideals $\hat{\mathfrak{l}}_i$ $(i=1, 2, \cdots, n)$ in $\hat{R}$. Then $\hat{\mathfrak{m}}_i = (\hat{\mathfrak{l}}_1, \hat{\mathfrak{l}}_2, \cdots, \hat{\mathfrak{l}}_{i-1}, \hat{\mathfrak{l}}_{i+1}, \cdots, \hat{\mathfrak{l}}_n)$ $(i=1, 2, \cdots, n)$ are maximal left ideals in $\hat{R}$ and $\bigcap_{i=1}^{n} \hat{\mathfrak{m}}_i$ is the null ideal in $\hat{R}$. Hence there exist left ideals $\mathfrak{m}_i$ $(i=1, 2, \cdots, n)$ in R containing $\mathfrak{p}$ such that $\mathfrak{m}_i/\mathfrak{p} = \hat{\mathfrak{m}}_i$ $(i=1, 2, \cdots, n)$ and $\bigcap_{i=1}^{n} \mathfrak{m}_i = \mathfrak{p}$. Since, by our assumption, $\mathfrak{m}_i\mathfrak{m}_j = \mathfrak{m}_j\mathfrak{m}_i$ there holds that $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n \subseteqq \bigcap_{i=1}^{n} \mathfrak{m}_i = \mathfrak{p}$. Hence we obtain $\mathfrak{m}_k = \mathfrak{p}$ for some k. But this is a contradiction.

Lemma 3. *Let* $\mathfrak{a}$ *be a two-sided ideal in* R *such that the residue class ring* $R/\mathfrak{a}$ *satisfies the ascending chain condition for two-sided ideals. Then there exist prime ideals* $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n$ *containing* $\mathfrak{a}$ *such that* $\mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_n \subseteqq \mathfrak{a}$.

Proof. Suppose that the lemma is not true for $\mathfrak{a}$. Then $\mathfrak{a}$ is obviously not prime. Therefore, there exist two-sided ideals $\mathfrak{a}_1, \mathfrak{a}'_1$ such that $\mathfrak{a}_1\mathfrak{a}'_1 \subseteqq \mathfrak{a}$ and $\mathfrak{a}_1 \not\subseteqq \mathfrak{a}$, $\mathfrak{a}'_1 \not\subseteqq \mathfrak{a}$. Since $(\mathfrak{a}_1, \mathfrak{a})(\mathfrak{a}'_1, \mathfrak{a}) \subseteqq \mathfrak{a}$ and $(\mathfrak{a}_1, \mathfrak{a}) \supsetneqq \mathfrak{a}$, $(\mathfrak{a}'_1, \mathfrak{a}) \supsetneqq \mathfrak{a}$, we may suppose from the first that $\mathfrak{a}_1 \supsetneqq \mathfrak{a}$, $\mathfrak{a}'_1 \supsetneqq \mathfrak{a}$. If the lemma is true for both $\mathfrak{a}_1, \mathfrak{a}'_1$ then it is also true for $\mathfrak{a}$ because $\mathfrak{a}_1\mathfrak{a}'_1 \subseteqq \mathfrak{a}$. Hence, there exists an ideal, say $\mathfrak{a}_1$, for which the lemma is not true. Thus, we may construct an infinite ascending chain of two-sided ideals $\mathfrak{a} \subsetneqq \mathfrak{a}_1 \subsetneqq \mathfrak{a}_2 \subsetneqq \cdots$ in which the lemma is not true for each $\mathfrak{a}_i$ $(i \geqq 1)$ but this contradicts the ascending chain condition.

Lemma 4. *Let* $\mathfrak{p}$ *be a prime ideal in* R *such that the residue class ring* $\hat{R}$ of R *by* $\mathfrak{p}$ *is a division ring and there exists no proper one-sided ideal between* $\mathfrak{p}$ *and* $\mathfrak{p}^2$. *Then there exists no proper one-sided ideal between* R *and* $\mathfrak{p}^m$ $(m \geqq 0)$ *other than powers of* $\mathfrak{p}$; *that is,* $R/\mathfrak{p}^m$ *is a uni-serial ring.*

Proof. The lemma is trivial when $\mathfrak{p} = \mathfrak{p}^2$. Therefore, we assume $\mathfrak{p} \neq \mathfrak{p}^2$. Then $\mathfrak{p}$ has an element p with $p \notin \mathfrak{p}^2$. By assumption, the left and right ideals generated by p and $\mathfrak{p}^2$ are equal to $\mathfrak{p}$. Hence, for any element ξ in R there exists ξ' in R such that

$$\xi p \equiv p\xi' \mod \mathfrak{p}^2.$$

By using the above congruence relation, we can easily show that, for any integer $i \geqq 1$, $\mathfrak{p}^i$ is at once a right and left ideal generated by p^i and $\mathfrak{p}^{i+1}$. Let now $\mathfrak{a} \neq R$ be a left ideal between R and $\mathfrak{p}^m$. Then we can determine the ideal $\mathfrak{p}^s$ $(s \leqq m)$ such that $\mathfrak{a} \supseteqq \mathfrak{p}^s$ but $\mathfrak{a} \not\supseteqq \mathfrak{p}^{s-1}$ where we put $R = \mathfrak{p}^0$. If $\mathfrak{a} \neq \mathfrak{p}^s$ there is in $\mathfrak{a}$ an element $\alpha \in \mathfrak{p}^i$ $(i < s)$ such that $\alpha \notin \mathfrak{p}^{i+1}$. Hence, for some $\xi \not\equiv 0 \bmod \mathfrak{p}$ in R, there holds

$$\alpha \equiv \xi\, p^i \mod \mathfrak{p}^{i+1}.$$

Since $\hat{R}$ is a division ring there exists some $\eta \in R$ such that $\xi\eta \equiv 1 \bmod \mathfrak{p}$; that is,

$$\eta\alpha \equiv p^i \mod \mathfrak{p}^{i+1},$$

so that we obtain

$$p^{s-i-1}\eta\alpha \equiv p^{s-1} \mod \mathfrak{p}^s.$$

Hence, it follows immediately

$$\mathfrak{a} \supseteqq (p^{s-i-1}\eta\alpha,\ \mathfrak{p}^s) \supseteqq (p^{s-1},\ \mathfrak{p}^s) = \mathfrak{p}^{s-1},$$

where $(p^{s-i-1}\eta\alpha,\ \mathfrak{p}^s)$ and $(p^{s-1},\ \mathfrak{p}^s)$ denote left ideals in R. But this gives a contradiction, and therefore $\mathfrak{a} = \mathfrak{p}^s$.

Similarly we can prove that a right ideal between R and $\mathfrak{p}^m$ is a power of $\mathfrak{p}$.

Since in Lemma 4 $\mathfrak{p}^i$ is at once a left and right ideal generated by p^i and $\mathfrak{p}^{i+1}$, we obtain easily the following:

Corollary. *Let $\mathfrak{p}$ be a prime ideal in R satisfying the assumptions in Lemma 4. Then, if $\mathfrak{p}^i \neq \mathfrak{p}^{i+1}$ $(i \geqq 1)$, the residue class ring $\mathfrak{p}^i/\mathfrak{p}^{i+1}$ is a cyclic left and right $R/\mathfrak{p}$-module with the generating element $\{p^i\}$ where $\{p^i\}$ denotes the residue class in $\mathfrak{p}^i/\mathfrak{p}^{i+1}$ containing p^i.*

§ 2. **Some properties of compact rings.** By a topological ring R we mean an associative ring which is also a Hausdorff space such that, for $a, b \in R$, $a - b$ and ab are continuous functions of a and b. If $a^n \to 0$ then a is called a *topological nilpotent element* or in short a *nilpotent element.* If $a^n = 0$ for some positive integer n, then a is termed an *algebraic nilpotent element.* An algebraic nilpotent element is clearly a topological nilpotent element. An ideal (left, right or two-sided) $\mathfrak{a}$ of R with the property $\mathfrak{a}^n \to 0$ is termed a *topological nilpotent ideal* and an *algebraic nilpotent ideal* is an ideal $\mathfrak{a}$ such that $\mathfrak{a}^n = 0$ for some integer n. An ideal which consists entirely of topological (algebraic) nilpotent elements is said to be a *topological* (*algebraic*) *nilideal.* By the *radical* of a ring we always mean the *Jacobson radical*, i. e., the set-theoretical join of all qnasi-regular right (or equivalently left) ideals[1]. A topological ring is called a *Q-ring* if the totality of quasi-regular elements is open[2].

Throughout this section, we denote by $\mathfrak{O}$ a *compact ring with the identity element* 1.

Lemma 5. *$\mathfrak{O}$ is totally disconnected and has a complete system of compact open ideal (in this note an ideal means a two-sided ideal) neighborhood of* 0[3].

Lemma 6. *The radical of $\mathfrak{O}$ is topologically nilpotent and every one-sided topological nilideal is contained in the radical, hence every one-sided topological nilideal of $\mathfrak{O}$ is topologically nilpotent*[4].

Lemma 7. *Every open prime ideal $\mathfrak{p}$ of $\mathfrak{O}$ is maximal.*

Proof. Since $\mathfrak{p}$ is open (and so $\mathfrak{p}$ is also closed) the residue class ring $\mathfrak{O}/\mathfrak{p}$ is discrete. Furthermore, $\mathfrak{O}/\mathfrak{p}$ is compact as a continuous image of a compact set $\mathfrak{O}$, hence $\mathfrak{O}/\mathfrak{p}$ is a finite ring. Therefore, by Lemma 1, $\mathfrak{p}$ is maximal.

Lemma 8. *Every open ideal $\mathfrak{a}$ of $\mathfrak{O}$ contains a product of a finite number of maximal open prime ideals.*

Proof. Since $\mathfrak{a}$ is an open ideal and $\mathfrak{O}$ is compact the residue class ring $\mathfrak{O}/\mathfrak{a}$ is finite so that $\mathfrak{O}/\mathfrak{a}$ statisfies the ascending chain condition. Hence, by Lemma 3, there exist prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ in $\mathfrak{O}$ containing $\mathfrak{a}$ such that $\mathfrak{p}_1 \cdots \mathfrak{p}_s \subseteqq \mathfrak{a}$. Since every $\mathfrak{p}_i \, (1 \leqq i \leqq s)$ contains the open ideal $\mathfrak{a}$, $\mathfrak{p}_i$ is also open and so maximal.

Lemma 9. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be both left (or right) ideal in $\mathfrak{O}$ and $\mathfrak{b}$ be closed. If $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{O}$ then $(\bigcap_{n=1}^{\infty} \mathfrak{a}^n, \bigcap_{n=1}^{\infty} \mathfrak{b}^n) = \mathfrak{O}$.*

Proof. By the assumption $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{O}$ there exist elements a in $\mathfrak{a}$ and b_1 in $\mathfrak{b}$ such that $a + b_1 = 1$ and so $a^2 = 1 - 2b_1 - b_1^2 = 1 - b_2$, where $b_2 = 2b_1 - b_1^2 \in \mathfrak{b}$. In general $a^n = 1 - b_n$ for $b_n \in \mathfrak{b}$. Let $A = \{a^n;\ n = 1, 2, \cdots\}$, $A_\nu = \{a^i;\ i \geqq \nu\}$ and $D(A) = \bigcap_{\nu=1}^{\infty} \bar{A}_\nu$, where $\bar{A}$'s mean topological closures of A's, then $D(A)$ is not empty and is a commutative multiplicative group with the property $A \cdot D(A)$[5] $\subseteqq D(A)$, $D(A) \cdot A \subseteqq$

1) Cf. [8] and [6].
2) Cf. [8].
3) Cf. [8, Theorem 8, Lemma 9 and Lemma 10], [9] and [10].
4) Cf. [8, Theorem 14, Theorem 15 and Corollary].
5) Let M, N be two subsets of $\mathfrak{O}$ we denote by $M \cdot N$ the set of all elements of the form ab, where a in M and b in N.

$D(A)$[1]. If we denote by e the identity element of the multiplicative group $D(A)$, then $ea \in D(A)\cdot A \subseteq D(A)$. Hence ea has the inverse element $(ea)^{-1}$ in $D(A)$, i.e., $e = (ea)^{-1}(ea)$. At the same time, since $\mathfrak{a}$ is a left ideal, $e = (ea)^{-1}(ea) = [(ea^{-1}e]a \in \mathfrak{a}$; that is, $\mathfrak{a}$ contains an idempotent e. Let $U(e)$ be any neighborhood of e. Then, since $e \in D(A) = \bigcap_{i=1}^{\infty} \bar{A}_i$, there is an $a^{\nu} \in U(e)$ and so $b_\nu = 1 - a^{\nu} \in 1 - U(e)$. This shows that $f = 1 - e \in \mathfrak{b}$ because $\mathfrak{b}$ is a closed ideal. Moreover, $f^2 = (1-e)^2 = 1 - e = f$ is an idempotent and so $f \in \bigcap_{\nu=1}^{\infty} \mathfrak{b}^{\nu}$, therefore $(\bigcap_{\nu=1}^{\infty} \mathfrak{a}^{\nu}, \bigcap_{\nu=1}^{\infty} \mathfrak{b}^{\nu}) \ni e + f = 1$. (In case $\mathfrak{a}$ and $\mathfrak{b}$ are right ideals the proof is similar.)

Let $\mathfrak{a}$ be a left (right or two-sided) ideal of $\mathfrak{O}$. Then we may easily show that the topological closure $\bar{\mathfrak{a}}$ of $\mathfrak{a}$ is also a closed left (right or two-sided) ideal. Furthermore, we obtain $\overline{\mathfrak{a}\mathfrak{b}} \supseteqq \bar{\mathfrak{a}}\,\bar{\mathfrak{b}}$ for ideals $\mathfrak{a}$, $\mathfrak{b}$ in $\mathfrak{O}$ where $\bar{\mathfrak{a}}$, $\bar{\mathfrak{b}}$ denote the topological closures of $\mathfrak{a}$, $\mathfrak{b}$ respectively.

Let now $\{\mathfrak{p}_\lambda;\ \lambda \in \Lambda\}$ be the family of all maximal open prime ideals in $\mathfrak{O}$, where Λ is an index system and $\mathfrak{p}_\lambda \neq \mathfrak{p}_\mu$ for $\lambda \neq \mu$ $(\lambda, \mu \in \Lambda)$. Put $\bigcap_{\nu=1}^{\infty} \overline{\mathfrak{p}_\lambda^{\nu}} = \mathfrak{q}_\lambda$. Since all $\mathfrak{p}_\lambda$'s are open and so closed, we obtain from Lemma 9 $(\bigcap_{\nu=1}^{\infty} \mathfrak{p}_\lambda^{\nu}, \bigcap_{\nu=1}^{\infty} \mathfrak{p}_\mu^{\nu}) = \mathfrak{O}$ for $\lambda \neq \mu$, hence $(\mathfrak{q}_\lambda, \mathfrak{q}_\mu) = \mathfrak{O}$. Thus for any finite number of $\mathfrak{q}_\lambda$'s, say $\mathfrak{q}_{\lambda_1}, \cdots, \mathfrak{q}_{\lambda_n}$, we have

$$(\mathfrak{q}_{\lambda_i}, \mathfrak{q}_{\lambda_1} \cap \cdots \cap \mathfrak{q}_{\lambda_{i-1}} \cap \mathfrak{q}_{\lambda_{i+1}} \cap \cdots \cap \mathfrak{q}_{\lambda_n}) = \mathfrak{O}$$

for $i = 1, 2, \cdots, n$.

Lemma 10. *If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathfrak{O}$ such that $\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$, then $\overline{\mathfrak{a}\mathfrak{b}} \supseteqq (\bar{\mathfrak{a}}\,\bar{\mathfrak{b}}, \bar{\mathfrak{b}}\,\bar{\mathfrak{a}})$.*

Proof. Since $\overline{\mathfrak{a}\,\mathfrak{b}} \supseteqq \bar{\mathfrak{a}}\,\bar{\mathfrak{b}}$, $\overline{\mathfrak{a}\,\mathfrak{b}} = \overline{\mathfrak{b}\,\mathfrak{a}} \supseteqq \bar{\mathfrak{b}}\,\bar{\mathfrak{a}}$. We obtain therefore $\overline{\mathfrak{a}\,\mathfrak{b}} \supseteqq (\bar{\mathfrak{a}}\,\bar{\mathfrak{b}}, \bar{\mathfrak{b}}\,\bar{\mathfrak{a}})$.

Lemma 11. *If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in $\mathfrak{O}$ such that $\mathfrak{a}\,\mathfrak{b} = \mathfrak{b}\,\mathfrak{a}$ and $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{O}$, then $\overline{\mathfrak{a}\,\mathfrak{b}} = \bar{\mathfrak{a}} \cap \bar{\mathfrak{b}}$.*

Proof. First of all, we prove that $\bar{\mathfrak{a}} \cap \bar{\mathfrak{b}} = (\bar{\mathfrak{a}}\,\bar{\mathfrak{b}}, \bar{\mathfrak{b}}\,\bar{\mathfrak{a}})$. In fact, it is clear that $\bar{\mathfrak{a}} \cap \bar{\mathfrak{b}} \supseteqq \bar{\mathfrak{a}}\,\bar{\mathfrak{b}}$, $\bar{\mathfrak{a}} \cap \bar{\mathfrak{b}} \supseteqq \bar{\mathfrak{b}}\,\bar{\mathfrak{a}}$ and so $\bar{\mathfrak{a}} \cap \bar{\mathfrak{b}} \supseteqq (\bar{\mathfrak{a}}\,\bar{\mathfrak{b}}, \bar{\mathfrak{b}}\,\bar{\mathfrak{a}})$. Conversely, $\bar{\mathfrak{a}} \cap \bar{\mathfrak{b}} = (\bar{\mathfrak{a}} \cap \bar{\mathfrak{b}})\,\mathfrak{O} = (\bar{\mathfrak{a}} \cap \bar{\mathfrak{b}})(\bar{\mathfrak{a}}, \bar{\mathfrak{b}}) = ((\bar{\mathfrak{a}} \cap \bar{\mathfrak{b}})\bar{\mathfrak{a}}, (\bar{\mathfrak{a}} \cap \bar{\mathfrak{b}})\bar{\mathfrak{b}}) \subseteqq (\bar{\mathfrak{b}}\,\bar{\mathfrak{a}}, \bar{\mathfrak{a}}\,\bar{\mathfrak{b}})$. It follows from the above and Lemma 10 that

$$\bar{\mathfrak{a}} \cap \bar{\mathfrak{b}} = (\bar{\mathfrak{a}}\,\bar{\mathfrak{b}}, \bar{\mathfrak{b}}\,\bar{\mathfrak{a}}) \subseteqq \overline{\mathfrak{a}\,\mathfrak{b}}.$$

As $\bar{\mathfrak{a}} \cap \bar{\mathfrak{b}} \supseteqq \overline{\mathfrak{a}\mathfrak{b}}$, we obtain $\bar{\mathfrak{a}} \cap \bar{\mathfrak{b}} = \overline{\mathfrak{a}\,\mathfrak{b}}$.

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1) Cf. [13, Lemma 3] and [14, Theorem 1].

Lemma 12. *If* $\mathfrak{p}_\lambda \mathfrak{p}_\mu = \mathfrak{p}_\mu \mathfrak{p}_\lambda$ *for any* $\lambda, \mu \in \Lambda$, *then* $\bigcap_{\lambda\in\Lambda} \mathfrak{q}_\lambda = (0)$.

Proof. We show that the assumption $\bigcap_{\lambda\in\Lambda} \mathfrak{q}_\lambda \neq (0)$ leads us to a contradiction. In fact, if $\bigcap_{\lambda\in\Lambda} \mathfrak{q}_\lambda \neq (0)$ we can find an element c of $\mathfrak{O}$ and a compact open ideal neighborhood U of 0 such that $c \in \bigcap_{\lambda\in\Lambda} \mathfrak{q}_\lambda$, $c \notin U$. Since U is an open ideal, by Lemma 8, U contains a product of a finite number of $\mathfrak{p}_\lambda$'s:

$$U \supseteqq \mathfrak{p}_{\lambda_1}^{r_1} \mathfrak{p}_{\lambda_2}^{r_2} \cdots \mathfrak{p}_{\lambda_s}^{r_s},$$

where $\mathfrak{p}_{\lambda_1}, \cdots, \mathfrak{p}_{\lambda_s}$ are distinct maximal open prime ideals and $r_i \geqq 1$ for $i = 1, 2, \cdots, s$. As U is closed we have

$$U \supseteqq \overline{\mathfrak{p}_{\lambda_1}^{r_1} \mathfrak{p}_{\lambda_2}^{r_2} \cdots \mathfrak{p}_{\lambda_s}^{r_s}}.$$

From the fact $(\mathfrak{p}_{\lambda_i}, \mathfrak{p}_{\lambda_j}) = \mathfrak{O}$ $(i \neq j)$ and Lemma 11 we conclude that

$$U \supseteqq \overline{\mathfrak{p}_{\lambda_1}^{r_1}} \cap \overline{\mathfrak{p}_{\lambda_2}^{r_2}} \cap \cdots \overline{\mathfrak{p}_{\lambda_s}^{r_s}} \supseteqq \mathfrak{q}_{\lambda_1} \cap \mathfrak{q}_{\lambda_2} \cap \cdots \cap \mathfrak{q}_{\lambda_s} \supseteqq \bigcap_{\lambda\in\Lambda} \mathfrak{q}_\lambda \ni c.$$

This is a contradiction to the assumption $c \notin U$. Thus the lemma is proved.

As $\mathfrak{q}_\lambda$ is a closed ideal of $\mathfrak{O}$ the residue class ring $\hat{\mathfrak{O}}_\lambda = \mathfrak{O}/\mathfrak{q}_\lambda$ is also a compact topological ring. We denote by $\hat{\mathfrak{O}}$ the Cartesian direct sum of $\{\hat{\mathfrak{O}}_\lambda, \lambda \in \Lambda\}$. (It is well-known that $\hat{\mathfrak{O}}$ is compact by the weak topology.)

We shall prove, under the assumption $\mathfrak{p}_\lambda \mathfrak{p}_\mu = \mathfrak{p}_\mu \mathfrak{p}_\lambda$ for any $\lambda, \mu \in \Lambda$, that $\mathfrak{O}$ is isomorphic and homeomorphic to $\hat{\mathfrak{O}}$.

Let φ_λ be the natural homomorphism of $\mathfrak{O}$ onto $\hat{\mathfrak{O}}_\lambda$, and for any element a of $\mathfrak{O}$ let us put $\varphi(a) = \{\varphi_\lambda(a)\}$. Then it is clear that φ is a homomorphic and continuous mapping from $\mathfrak{O}$ into $\hat{\mathfrak{O}}$. Let $a \neq 0$ be an element of $\mathfrak{O}$, then since $\bigcap_{\lambda\in\Lambda} \mathfrak{q}_\lambda = (0)$, $a \notin \mathfrak{q}_\tau$ for some $\tau \in \Lambda$. Hence $\varphi_\tau(a)$ is a non-zero element of $\hat{\mathfrak{O}}_\tau$ and $\varphi(a)$ is a non-zero element of $\hat{\mathfrak{O}}$. This shows that φ is univalent. Finally, let $\{\hat{a}_\lambda\}$ be any element of $\hat{\mathfrak{O}}$, where each $\hat{a}_\lambda$ is the λ-component of $\{\hat{a}_\lambda\}$. If we put $\varphi_\lambda^{-1}(\hat{a}_\lambda) = M_\lambda$, then M_λ's are closed subsets of $\mathfrak{O}$, and we can express $M_\lambda = a_\lambda + \mathfrak{q}_\lambda$, $a_\lambda \in M_\lambda$. Taking any finite number of M_λ's, say $M_{\lambda_1}, \cdots, M_{\lambda_n}$, we shall show $\bigcap_{i=1}^{n} M_{\lambda_i} \neq \phi$ (the empty set). As $(\mathfrak{q}_{\lambda_j}, \mathfrak{d}_{\lambda_j}) = \mathfrak{O}$ for $j = 1, 2, \cdots, n$, where $\mathfrak{d}_{\lambda_j} = \mathfrak{q}_{\lambda_1} \cap \cdots \cap \mathfrak{q}_{\lambda_{j-1}} \cap \mathfrak{q}_{\lambda_{j+1}} \cap \cdots \cap \mathfrak{q}_{\lambda_n}$) we can write

$$a_{\lambda_i} = b_{\lambda_i} + c_{\lambda_i},\ b_{\lambda_i} \in \mathfrak{q}_{\lambda_i},\ c_{\lambda_i} \in \mathfrak{d}_{\lambda_i}.$$

Then $c_{\lambda_1} + \cdots + c_{\lambda_n} \in \bigcap_{i=1}^{n} M_{\lambda_i}$. For, if $c_{\lambda_1} + \cdots + c_{\lambda_n} \notin M_{\lambda_k} = a_{\lambda_k} + \mathfrak{q}_{\lambda_k}$

for some k, then $c_{\lambda_1}+\cdots+c_{\lambda_{k-1}}+c_{\lambda_{k+1}}+\cdots+c_{\lambda_n}\notin(a_{\lambda_k}-c_{\lambda_k})+\mathfrak{q}_{\lambda_k}=b_{\lambda_k}+\mathfrak{q}_{\lambda_k}=\mathfrak{q}_{\lambda_k}$. On the other hand, since $c_{\lambda_i}\in\mathfrak{q}_{\lambda_k}$ for $i\neq k$ we have arrived at a contradiction. Hence $\{M_\lambda\}$ has the finite intersection property and since $\mathfrak{O}$ is compact we have $\bigcap_{\lambda\in\Lambda}M_\lambda\neq\phi$. Choose an element a in $\bigcap_{\lambda\in\Lambda}M_\lambda$, then it is clear that $\varphi(a)=\{\hat{a}_\lambda\}$. This shows that φ is an onto mapping. Since $\mathfrak{O}$ and $\hat{\mathfrak{O}}$ are compact, the univalent continuous mapping φ is a homeomorphism. Thus $\mathfrak{O}$ and $\hat{\mathfrak{O}}$ are isomorphic and homeomorphic each other.

Lemma 13. *Each* $\hat{\mathfrak{O}}_\lambda=\mathfrak{O}/\mathfrak{q}_\lambda$ *is a primary ring.*

Proof. We denote by $\hat{0}_\lambda$ the zero element of $\hat{\mathfrak{O}}_\lambda$ and let $\hat{\mathfrak{p}}_\lambda=\mathfrak{p}_\lambda/\mathfrak{q}_\lambda$ (since $\mathfrak{p}_\lambda$ is closed $\mathfrak{p}_\lambda\supseteq\mathfrak{q}_\lambda$), then $\bigcap_{n=1}^{\infty}\hat{\mathfrak{p}}_\lambda^n=(\hat{0}_\lambda)$. If $\hat{\mathfrak{p}}_\lambda$ is not a topological nilideal, it must contain a non-zero idempotent[1], which contradicts the fact $\bigcap_{n=1}^{\infty}\hat{\mathfrak{p}}_\lambda^n=(\hat{0}_\lambda)$. Hence $\hat{\mathfrak{p}}_\lambda$ is a topological nilideal and is a topological nilpotent ideal from Lemma 6. Moreover, $\hat{\mathfrak{O}}_\lambda/\hat{\mathfrak{p}}_\lambda\cong\mathfrak{O}/\mathfrak{p}_\lambda$ (finite simple) and hence $\hat{\mathfrak{O}}_\lambda$ is a primary ring with the radical $\hat{\mathfrak{p}}_\lambda$. (Note that $\hat{\mathfrak{p}}_\lambda$ is the unique maximal two-sided ideal in $\hat{\mathfrak{O}}_\lambda$, i. e., every two-side ideal properly contained in $\hat{\mathfrak{O}}_\lambda$ is contained in $\hat{\mathfrak{p}}_\lambda$.)

It follows from the above we have the following theorem:

Theorem 1. *A compact ring* $\mathfrak{O}$ *with the identity element* 1 *in which the product of any two maximal open prime ideals is commutative is isomorphic and homeomorphic to a Cartesian direct sum of compact primary rings.*

Theorem 2. *In Theorem* 1, *if* $\mathfrak{O}$ *is a Q-ring it is isomorphic and homeomorphic to a Cartesian direct sum of a finite number of compact primary rings and conversely.*

Proof. Let R be the radical of $\mathfrak{O}$, then by Lemma 6, R is topologically nilpotent. Hence $R^{n(\lambda)}\subseteq\mathfrak{p}_\lambda$ for some positive integer $n(\lambda)$. As $\mathfrak{p}_\lambda$ is prime $R\subseteq\mathfrak{p}_\lambda$; that is, every $\mathfrak{p}_\lambda$ contains R. At the same time R is open[2], therefore the number of maximal open prime ideals is finite. Hence the number of the direct summands is finite. The converse is trivial.

Lemma 14. *If in* $\mathfrak{O}$ *the product of any two maximal open left (or right) ideals is commutative, then products of maximal open prime*

1) Cf. [14, Theorem 1].
2) Cf. [8, Theorem 7 and Lemma 10].

ideals are commutative and $\mathfrak{O}/\mathfrak{p}$ is a finite field, where $\mathfrak{p}$ is any maximal open prime ideal.

Proof. Since $\mathfrak{p}$ is an open prime ideal $\mathfrak{O}/\mathfrak{p}$ is a finite ring so that, by Lemma 2, $\mathfrak{O}/\mathfrak{p}$ is a finite division ring. By the well-known theorem of Wedderburn, it is a finite field. Clearly $\mathfrak{p}$ is a maximal left ideal in $\mathfrak{O}$ and therefore the product of any maximal open prime ideals is commutative.

As a direct consequence of Theorem 1 and Lemma 4 we have the following theorem:

Theorem 3. *A compact ring $\mathfrak{O}$ with the identity element 1 in which the product of any two maximal open left (or right) ideals is commutative is isomorphic and homeomorphic to a Cartesian direct sum of compact completely primary rings, and the number of the direct summands is finite if and only if $\mathfrak{O}$ is a Q-ring.*

§ 3. Structure of compact rings satisfying special conditions.

In this section we consider the structure of a compact ring $\mathfrak{O}$ with the identity element 1 satisfying the following conditions:

I. *The product of any two maximal open left (or right) ideals in $\mathfrak{O}$ is commutative.*

II. *There exists no one-sided open ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$ for any maximal open prime ideal $\mathfrak{p}$ in $\mathfrak{O}$.*

Then from Theorem 3 and the condition **I** $\mathfrak{O}$ is a Cartesian direct sum of compact completely primary rings $\mathfrak{O}_\lambda$'s and the number of the direct summands is finite if and only if $\mathfrak{O}$ is a Q-ring.

Throughout this section we use the same notations as in § 2. Then by the condition **II** we have the following lemma:

Lemma 15. *If, for any maximal open prime ideal $\mathfrak{p}$ in $\mathfrak{O}$, $\mathfrak{p}^2$ is not open in $\mathfrak{O}$ then $\mathfrak{p} = \bigcap_{n=1}^{\infty} \overline{\mathfrak{p}^n}$, hence $\mathfrak{O}/\mathfrak{q}$ is a finite field, where $\mathfrak{q} = \bigcap_{n=1}^{\infty} \overline{\mathfrak{p}^n}$.*

Proof. Let $\{U_\alpha\}$ be a complete system of compact open ideal neighborhoods of 0 in $\mathfrak{O}$. Then, for any $U_\alpha(\neq \mathfrak{O}) \subseteqq \mathfrak{p}$, $(\mathfrak{p}^2, U_\alpha)$ is an open ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$, hence by the condition **II** either $(\mathfrak{p}^2, U_\alpha) = \mathfrak{p}$ or $(\mathfrak{p}^2, U_\alpha) = \mathfrak{p}^2$.

If $\mathfrak{p}^2$ is not open then $(\mathfrak{p}^2, U_\alpha) = \mathfrak{p}$ for every $U_\alpha \subseteqq \mathfrak{p}$. This shows that $\mathfrak{p}^2$ is everywhere dense in $\mathfrak{p}$, so $\overline{\mathfrak{p}^2} = \mathfrak{p}$.

Now we assume that, for some interger $n(> 1)$, $\overline{\mathfrak{p}^n} = \mathfrak{p}$. Then $\overline{\mathfrak{p}^{n+1}} \supseteqq \overline{\mathfrak{p}^n}\,\mathfrak{p} = \mathfrak{p}^2$, hence $\overline{\mathfrak{p}^{n+1}} = \overline{\mathfrak{p}^2} = \mathfrak{p}$. Therefore, if $\mathfrak{p}^2$ is not open we have $\mathfrak{q} = \mathfrak{p}$

and $\mathfrak{O}/\mathfrak{q} = \mathfrak{O}/\mathfrak{p}$ is a finite field.

From the conditions **I, II** and Lemma 15 each $\hat{\mathfrak{O}}_\lambda$ is a compact completely primary ring having no one-sided ideal between $\hat{\mathfrak{p}}_\lambda$ and $\hat{\mathfrak{p}}_\lambda^2$ for the radical $\hat{\mathfrak{p}}_\lambda$ or a finite field.

Now, then we consider the structure of a compact ring $\mathfrak{O}'$ with the following properties:

1. *$\mathfrak{O}'$ is completely primary but is not a field.*
2. *There is no one-sided ideal between $\mathfrak{p}'$ and $\mathfrak{p}'^2$ for the radical $\mathfrak{p}'$.*

By Lemma 4, we obtain the following

Lemma 16. *There exists no one-sided ideal between $\mathfrak{O}'$ and $\mathfrak{p}'^m$ other than powers of $\mathfrak{p}'$; that is, $\mathfrak{O}'/\mathfrak{p}'^m$ is a uni-serial ring.*

Lemma 17. i) *$\mathfrak{p}'^i$ $(i = 1, 2, \cdots)$ form a complete system of neighborhoods of 0 so that $\mathfrak{p}' \neq \mathfrak{p}'^2$.*

ii) $\mathfrak{p}' \neq (0)$.

iii) *$\mathfrak{p}'$ is the unique maximal open ideal of $\mathfrak{O}'$.*

Proof. i) Let $\mathfrak{a}$ be an open ideal. Then there exists an integer ν such that $\mathfrak{p}'^\nu \subseteq \mathfrak{a}$; this means, by Lemma 16, that $\mathfrak{a}$ is a power of $\mathfrak{p}'$. Therefore, every ideal neighborhood of 0 is a power of $\mathfrak{p}'$. Let now $\mathfrak{p}'^m \neq (0)$ for some m. Then there exists an element $c (\neq 0) \in \mathfrak{p}'^m$ and an ideal neighborhood U of 0 such that $c \notin U$. Since $U = \mathfrak{p}'^\nu$ for some ν and $c \in \mathfrak{p}'^m$ it must be $\mathfrak{p}'^m \supsetneqq \mathfrak{p}'^\nu$; that is, $\mathfrak{p}'^m$ is an open ideal of $\mathfrak{O}'$. Hence $\mathfrak{p}'^i$ $(i = 1, 2, \cdots)$ form a complete system of neighborhoods of 0. As $\bigcap_{i=1}^{\infty} \mathfrak{p}'^i = (0)$, $\mathfrak{p}' \neq \mathfrak{p}'^2$.

ii) If $\mathfrak{p}' = (0)$ then $\mathfrak{O}'$ is obviously a discrete field.

iii) Since $\mathfrak{O}'$ is completely primary $\mathfrak{p}'$ is a maximal open ideal in $\mathfrak{O}'$. As every open ideal is, by i) and Lemma 16, a power of $\mathfrak{p}'$, $\mathfrak{p}'$ is the unique maximal open ideal of $\mathfrak{O}'$.

If, for some integer m, $\mathfrak{p}'^m = (0)$ then $\mathfrak{O}'$ is discrete and finite. Hence we obtain.

Theorem 4. *If in $\mathfrak{O}'$ $\mathfrak{p}'^m = (0)$, for some integer m, then $\mathfrak{O}'$ is a completely primary uni-serial finite ring.*

Lemma 18. *Every proper one-sided ideal*[1] *(open or not) in $\mathfrak{O}'$ coincides with some $\mathfrak{p}'^m$; that is, there exists no proper one-sided ideal in $\mathfrak{O}'$ other than $\mathfrak{p}', \mathfrak{p}'^2, \mathfrak{p}'^3, \cdots$.*

Proof. Let $\mathfrak{l} (\neq \mathfrak{O}', (0))$ be any left ideal of $\mathfrak{O}'$, then there is an integer $m (\gtrsim 0)$ such that $\mathfrak{p}'^m \supseteqq \mathfrak{l}$, $\mathfrak{p}'^{m+1} \not\supseteqq \mathfrak{l}$. Choose an element c of $\mathfrak{l}$

1) A proper ideal means an ideal $\neq \mathfrak{O}'$, (0).

not contained in $\mathfrak{p}'^{m+1}$. Then for any $\nu \geqq m$, $(\mathfrak{O}'c, \mathfrak{p}'^{\nu})$ is an open left ideal contained in $\mathfrak{p}'^{m}$ but not in $\mathfrak{p}'^{m+1}$. Therefore, by Lemma 16, $(\mathfrak{O}'c, \mathfrak{p}'^{\nu})$ must coincide with $\mathfrak{p}'^{m}$. Since $\{\mathfrak{p}'^{n};\ n=1, 2, \cdots\}$ forms a neighborhood system of 0, $\mathfrak{O}'c$ is everywhere dense in $\mathfrak{p}'^{m}$. Furthermore $\mathfrak{O}'c$ is compact and closed, so $\mathfrak{O}'c = \mathfrak{p}'^{m}$, whence $\mathfrak{l} = \mathfrak{p}'^{m}$. In case of right ideals, the proof is similar.

Theorem 5. *$\mathfrak{O}'$ is an integral domain if and only if $\mathfrak{p}'^{m} \neq (0)$ for every integer m.*

Proof. If $\mathfrak{p}'^{m} = (0)$ for some integer m then $\mathfrak{O}'$ has to have a zero divisor.

Now we assume that for every m $\mathfrak{p}'^{m} \neq (0)$. Since, by Lemma 17, $\mathfrak{p}' \neq \mathfrak{p}'^{2}$ there exists an element π contained in $\mathfrak{p}'$ but not in $\mathfrak{p}'^{2}$. Then by Lemma 18, $\mathfrak{O}'\pi = \mathfrak{p}' = \pi\mathfrak{O}'$ and by induction $\mathfrak{O}'\pi^{\nu} = \mathfrak{p}'^{\nu} = \pi^{\nu}\mathfrak{O}'$. If a and b are non-zero elements of $\mathfrak{O}'$, then there exist integers ν and μ such that $a \in \mathfrak{p}'^{\nu}$, $a \notin \mathfrak{p}'^{\nu+1}$, $b \in \mathfrak{p}'^{\mu}$, $b \notin \mathfrak{p}'^{\mu+1}$ and hence we can express $a = u\pi^{\nu}$, $b = v\pi^{\mu}$, where u, v must be units[1] of $\mathfrak{O}'$. ($\mathfrak{p}'^{0}$ means $\mathfrak{O}'$ and π^{0} means 1.) Therefore $ab = u\pi^{\nu}v\pi^{\mu} = uv_1 \cdot \pi^{\nu+\mu}$, where v_1 is a unit with the property $v_1\pi^{\nu} = \pi^{\nu}v$. If $ab = 0$ then $\pi^{\nu+\mu} = 0$ and so $\mathfrak{p}'^{\nu+\mu} = \mathfrak{O}'\pi^{\nu+\mu} = 0$, which contradicts $\mathfrak{p}'^{m} \neq 0$ for every m. Hence $ab \neq 0$ and $\mathfrak{O}'$ is an integral domain.

Let $\mathfrak{S}$ be a totally disconnected locally compact (t. d. l. c.) division ring. An element $a \neq 0$ of $\mathfrak{S}$ is termed *divergent* if a^{-1} is a topological nilpotent element and a is called a *neutral element* if a is neither divergent nor nilpotent.

Then the set $\mathfrak{O}^*$ of all non-divergent elements of $\mathfrak{S}$ forms a compact open integral domain and the set $\mathfrak{p}^*$ of all topological nilpotent elements forms a maximal open ideal of $\mathfrak{O}^*$ such that $\mathfrak{O}^*/\mathfrak{p}^* = K^*$ is a finite field. And every element x of $\mathfrak{S}$ contained in $\mathfrak{O}^*$ or $x^{-1} \in \mathfrak{p}^*$: that is,

$$\mathfrak{S} = \mathfrak{O}^* \cup \mathfrak{p}^{**-1},$$

where $\mathfrak{p}^{**}$ is the set of all non-zero elements of $\mathfrak{p}^*$. Such an $\mathfrak{O}^*$ is called a *maximal compact open order* of $\mathfrak{S}$[11].

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1) An element u is said to be a unit if there exist u', u'' such that $u'u = uu'' = 1$. In the above case u is not contained in $\mathfrak{p}'$ and $\mathfrak{O}'/\mathfrak{p}'$ is a field, hence there exists u' such that $u'u \equiv 1$ $(\mathfrak{p}')$. It follows that $(\mathfrak{O}'u'u, \mathfrak{p}') = \mathfrak{O}'$ and, by Lemma 9, $(\mathfrak{O}'u'u, \bigcap_{n=1}^{\infty}\mathfrak{p}'^{n}) = \mathfrak{O}'$, that is, $(\mathfrak{O}'u'u, 0) = \mathfrak{O}'$, whence $\mathfrak{O}'u'u = \mathfrak{O}'$. Analogously there exists u'' such that $uu''\mathfrak{O}' = \mathfrak{O}'$. Hence u is a unit.

11) Cf. [5].

A division ring $\mathfrak{k}$ is said to be a *semi-topological division ring*, if $\mathfrak{k}$ is a Hausdorff-space and mappings $(x, y) \rightarrow x + y$, $(x, y) \rightarrow xy$ from the product space $\mathfrak{k} \times \mathfrak{k}$ into $\mathfrak{k}$ are continuous. Then we have the following two lemmas.

Lemma 19. *A locally compact semi-topological division ring is a topological division ring.*

This lemma has been proved in [12, Theorem 7].

Lemma 20. *Let K be an abstract division ring and $\sum_0$ a family of subsets of K with the following properties*:

(i) $\bigcap_{U \in \Sigma_0} U = (0)$,

(ii) *if $V, W \in \sum_0$ then there exists U in $\sum_0$ such that $U \subseteqq V \cap W$,*

(iii) *if $V \in \sum_0$, $a \in V$ then there exists U in $\sum_0$ such that $a + U \subseteqq V$,*

(iv) *if $V \in \sum_0$ then there exists U in $\sum_0$ such that $U - U \subseteqq V$,*

(v) *if $V \in \sum_0$ then there exist U_1 and U_2 in $\sum_0$ such that $U_1 U_2 \subseteqq V$,*

(vi) *for any $V \in \sum_0$ and a of K there exists U_1 and U_2 in $\sum_0$ so that $V \supseteqq a U_1$, $V \supseteqq U_2 a$.*

Then we can introduce a topology in K, by which K becomes a semi-topological division ring having $\sum_0$ as a complete system of neighborhoods of 0.

The proof is clear.

Theorem 6. *A compact integral domain $\mathfrak{O}$ with the radical $\mathfrak{p} \neq (0)$ in which there is no one-sided ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$ is a maximal compact open order of a t. d. l. c. division ring.*

Proof. Since $\mathfrak{O}$ is a compact integral domain it is a completely primary ring ([8], Theorem 19) but not a field, because $\mathfrak{p} \neq (0)$. Thus, by Lemmas 17, 18 and Theorem 4, $\mathfrak{O}$ has no proper one-sided ideal other than $\mathfrak{p}, \mathfrak{p}^2, \cdots$ and, for every integer m, $\mathfrak{p}^m \neq (0)$. And every element $a \neq 0$ of $\mathfrak{O}$ can be written in the form $a = u\pi^\nu = \pi^\nu u'$ for some fixed $\pi \in \mathfrak{p}$, where u and u' are units of $\mathfrak{O}$.

If a and b are two non-zero elements of $\mathfrak{O}$, then $\mathfrak{O}a$ and $\mathfrak{O}b$ are non-zero left ideals of $\mathfrak{O}$, say $\mathfrak{O}a = \mathfrak{p}^\nu$, $\mathfrak{O}b = \mathfrak{p}^\mu$. Then $(0) \neq \mathfrak{p}^\kappa \subseteqq \mathfrak{O}a \cap \mathfrak{O}b$ for $\kappa \geqq \max(\nu, \mu)$. Hence there exist non-zero elements x, y in $\mathfrak{O}$ such that $x a = y b$. In the same way, we can find non-zero elements x', y' such that $a x' = b y'$. Therefore $\mathfrak{O}$ can be extended to the quotient division ring $\mathfrak{S}$ in the sense of an abstract ring[1]. If we take $\{\mathfrak{p}, \mathfrak{p}^2, \mathfrak{p}^3,$

1) Cf. [2], [3] and [7].

$\cdots\cdots\}$ as $\sum_0$ in Lemma 20, then $\sum_0$ satisfies the condition (i)—(vi) of Lemma 20. In fact, (i)—(v) are obvious. To prove (vi) let $x = ab^{-1}$ be any element of $\mathfrak{S}$, where a and b are contained in $\mathfrak{O}$. Then for any $\mathfrak{p}^\nu$, $\mathfrak{p}^\nu b = \mathfrak{p}^\kappa$ for some κ. As $\mathfrak{p}$ is nilpotent and $\mathfrak{p}^\kappa$ is open there is a $\mathfrak{p}^\mu$ such that $\mathfrak{p}^\mu a \subseteq \mathfrak{p}^\kappa = \mathfrak{p}^\nu b$, therefore $\mathfrak{p}^\nu \supseteq \mathfrak{p}^\mu ab^{-1} = \mathfrak{p}^\mu x$. Similarly we may prove that $x\mathfrak{p}^{\mu'} \subseteq \mathfrak{p}^\nu$ for some μ'. Hence $\mathfrak{S}$ becomes a semi-topological division ring having the system $\sum_0$ of neighborhoods of zero. Moreover, it is clear that the original topology of $\mathfrak{O}$ is equivalent to the relative topology induced in $\mathfrak{O}$, and hence $\mathfrak{S}$ is totally disconnected, locally compact. Since $\mathfrak{S}$ is locally compact, by Lemma 19, $\mathfrak{S}$ is a topological division ring. Thus $\mathfrak{O}$ is a compact open order of a t. d. l. c. division ring.

Now, let $x = ab^{-1}$ be any element of $\mathfrak{S}$ which is not contained in $\mathfrak{O}$, where $a, b \in \mathfrak{O}$. Then we can express $a = u\pi^\nu$, $b = v\pi^\mu$ for $\pi \in \mathfrak{p}$ but $\pi \notin \mathfrak{p}^2$, where u, v are units of $\mathfrak{O}$. Since $\mathfrak{O} \not\ni x = ab^{-1} = (u\pi^\nu)(v\pi^\mu)^{-1} = u\pi^{\nu-\mu}v^{-1}$, $u, v^{-1} \in \mathfrak{O}$, we have $\nu < \mu$. Hence $x^{-1} = v\pi^{\mu-\nu}u^{-1} \in \mathfrak{p}$ because of $v, u^{-1} \in \mathfrak{O}$ and $\pi^{\mu-\nu} \in \mathfrak{p}$. Thus, for every element x of $\mathfrak{S}$, either $x \in \mathfrak{O}$ or $x^{-1} \in \mathfrak{p}$:

$$\mathfrak{S} = \mathfrak{O} \cup \mathfrak{p}^{*-1},$$

where $\mathfrak{p}^*$ is the set of all non-zero elements of $\mathfrak{p}$.

From Theorems 3, 4, 5 and 6 we have

Theorem 7. *Let $\mathfrak{O}$ be a compact ring with the identity element* 1 *in which the product of any two maximal open left* (*or right*) *ideals is commutative and there is no one-sided open ideals between $\mathfrak{p}$ and $\mathfrak{p}^2$ for every maximal open prime ideal $\mathfrak{p}$. Then $\mathfrak{O}$ is a Cartesian direct sum of maximal compact open orders of t. d. l. c. division rings, completely primary uni-serial finite rings and finite fields. And the number of the direct summands is finite if and only if $\mathfrak{O}$ is a Q-ring.*

Corollary. *$\mathfrak{O}$ is the same as in Theorem* 7 *and is a Q-ring then $\mathfrak{O}$ satisfies the first axiom of countability.*

Let $\mathfrak{O}$ be a compact ring with the identity element 1 in which the product of any two maximal open left ideals is commutative and $\mathfrak{p}$ a maximal open prime ideal of $\mathfrak{O}$. Then the following conditions are equivalent when $\mathfrak{p} \neq \mathfrak{p}^2$:

1) There exists no left ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$.
2) There exists no open left ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$.
3) $\mathfrak{p}/\mathfrak{p}^2$ is a cyclic $\mathfrak{O}$-left module.

4) If $\mathfrak{a}$ is a left ideal such that $\mathfrak{p} \supseteqq \mathfrak{a} \supseteqq \mathfrak{p}^2$ then there exists a left ideal $\mathfrak{a}'$ such that $\mathfrak{a} = \mathfrak{p}\mathfrak{a}'$.

5) Left ideals between $\mathfrak{p}$ and $\mathfrak{p}^2$ are totally ordered.

6) For any three left ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ with $\mathfrak{p} \supseteqq \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \supseteqq \mathfrak{p}^2$, there holds that

$$\mathfrak{a} \cap (\mathfrak{b}, \mathfrak{c}) = (\mathfrak{a} \cap \mathfrak{b}, \mathfrak{a} \cap \mathfrak{c}).$$

7) For any three left ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ with $\mathfrak{p} \supseteqq \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \supseteqq \mathfrak{p}^2$,

$$[\mathfrak{a} : (\mathfrak{b} \cap \mathfrak{c})]_l = ([\mathfrak{a} : \mathfrak{b}]_l, [\mathfrak{a} : \mathfrak{c}]_l),$$

where $[\mathfrak{a} : \mathfrak{b}]_l = \{x : x\mathfrak{b} \subseteqq \mathfrak{a}, x \in \mathfrak{O}\}$, that is, $[\mathfrak{a} : \mathfrak{b}]_l$ is the left quotient ring of $\mathfrak{a}$ by $\mathfrak{b}$.

Finally we give two theorems without proof.

Theorem 8. *Let $\mathfrak{O}$ be a compact ring with the identity element* 1 *in which the product of any two maximal open left (or right) ideals is commutative. Then $\mathfrak{O}$ is a Cartesian direct sum of maximal compact open orders of t. d. l. c. division rings, completely primary uni-serial finite rings and finite fields, when $\mathfrak{O}$ satisfies one of the following seven conditions*:

For any maximal open prime ideal $\mathfrak{p}$,

1. *there exists no one-sided ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$,*
2. *there exists no one-sided open ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$,*
3. *$\mathfrak{p}/\mathfrak{p}^2$ is a cyclic $\mathfrak{O}$-left and right module,*
4. *if $\mathfrak{a}$ is a left ideal with $\mathfrak{p} \supseteqq \mathfrak{a} \supseteqq \mathfrak{p}^2$ then there exists a left ideal $\mathfrak{a}'$ such that $\mathfrak{a} = \mathfrak{p}\mathfrak{a}'$, and if $\mathfrak{a}$ is a right ideal with $\mathfrak{p} \supseteqq \mathfrak{a} \supseteqq \mathfrak{p}^2$ then there exists a right ideal $\mathfrak{a}''$ such that $\mathfrak{a} = \mathfrak{a}''\mathfrak{p}$,*
5. *left ideals between $\mathfrak{p}$ and $\mathfrak{p}^2$ are totally ordered, and the same is true for right ideals,*
6. *for any three left ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ with $\mathfrak{p} \supseteqq \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \supseteqq \mathfrak{p}^2$,*

$$\mathfrak{a} \cap (\mathfrak{b}, \mathfrak{c}) = (\mathfrak{a} \cap \mathfrak{b}, \mathfrak{a} \cap \mathfrak{c}),$$

and the same is true for right ideals,

7. *for any three left ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ with $\mathfrak{p} \supseteqq \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \supseteqq \mathfrak{p}^2$,*

$$[\mathfrak{a} : (\mathfrak{b} \cap \mathfrak{c})]_l = ([\mathfrak{a} : \mathfrak{b}]_l, [\mathfrak{a} : \mathfrak{c}]_l),$$

and the same is true for right ideals taking right quotient ideals.

Corollary. *Let $\mathfrak{O}$ be a compact ring with the identity element* 1 *in which the product of any two maximal open left ideals is commutative. Then, if every maximal open prime ideal of $\mathfrak{O}$ is a principal* *left and right ideal, every closed ideal of $\mathfrak{O}$ is principal left and right ideal.*

Theorem 9. *Let $\mathfrak{O}$ be a compact Q-ring with the identity element* 1 *in which the product of any two maximal open left ideals is commutative. Then the following seven conditions are equivalent to each other.*

1. *$\mathfrak{O}$ is a Cartesian direct sum of a finite number of maximal compact open orders of t.d.l.c. division rings, completely primary uni-serial finite rings and finite fields.*

2. *There is no one-sided ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$ for any maximal open prime ideal $\mathfrak{p}$.*

3. *For any maximal open prime ideal $\mathfrak{p}$, $\mathfrak{p}/\mathfrak{p}^2$ is a cyclic left and right $\mathfrak{O}$-module.*

4. *Every one-sided ideal of $\mathfrak{O}$ is a principal ideal.*

5. *For left ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{a} \subseteqq \mathfrak{b}$, there exists a left ideal $\mathfrak{c}$ such that $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$, and for right ideals $\mathfrak{a}, \mathfrak{b}$ with $\mathfrak{a} \subseteqq \mathfrak{b}$, there exists a right ideal $\mathfrak{c}$ such that $\mathfrak{a} = \mathfrak{c}\mathfrak{b}$.*

6. *For any three left ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$,*

$$\mathfrak{a} \cap (\mathfrak{b}, \mathfrak{c}) = (\mathfrak{a} \cap \mathfrak{b}, \mathfrak{a} \cap \mathfrak{c}).$$

It is the same with right ideals.

7. *For any three left ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$,*

$$[\mathfrak{a} : (\mathfrak{b} \cap \mathfrak{c})]_l = ([\mathfrak{a} : \mathfrak{b}]_l, [\mathfrak{a} : \mathfrak{c}]_l).$$

It is the same with right ideals taking right quotient ideals.

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