

ASSOCIATED RIEMANNIAN MANIFOLDS AND MOTIONS

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In this paper, we shall investigate the associated Riemannian manifold V_N ($N = n(n+1)/2$) with a Riemannian manifold V_n that a Riemannian metric which is naturally derived from the one of V_n is given on the bundle space of the associated principal fibre bundle of V_n and the motions on V_n in connection with V_N .

In §§1—3, we shall calculate the parameters of the connection and the curvature forms of V_N with respect to the canonical frames. In §4, we shall investigate the conditions in order that V_N becomes an Einstein space. In §§5, 6, we shall give the equations of geodesics in V_N and an elementary exposition of the relations between the Levi-Civita connection of V_n and the associated Riemannian manifold V_N with V_n . In §§7, 8, we shall show that a mapping derived from the differential mapping of a motion of V_n becomes a motion of V_N and investigate the properties of motions of V_n by means of such motions of V_N . In §9, we shall investigate sequences of motions of V_n and prove that under a suitable condition, we can derive a differentiable tangent vector field of motion from a sequence of motions which is a geometrical treatment of such sequences of motions of V_n investigated in [6, 7]. Lastly in §10, we shall give an elementary exposition of holonomy groups of V_n .

§ 1. Definitions

Let V_n be an n -dimensional Riemannian manifold and let $\mathfrak{B} = \{B, p, V_m, O_m, O_n\}$ be the associated principal fibre bundle of the tangent fibre bundle of V_n as a differentiable manifold with a Riemannian metric, that is

- i) $p : B \longrightarrow V_n$ be the projection,
- ii) for any point $x \in V_m$, the fibre $p^{-1}(x) = O_n(x)$ is homeomorphic to the n -dimensional orthogonal group O_n ,

and

- iii) $O_n(x) \ni b = \{x, e_1, \dots, e_n\}$ is an orthonormal frame at x such that $e_i, i = 1, 2, \dots, n$, are unit tangent vectors to V_n at x and mutually orthogonal.

Let be given the line element of V_n by

$$(1, 1) \quad ds^2 = \sum \omega_i \omega_i$$

and the equations of structure of V_n with respect to b are

$$(1, 2) \quad \begin{cases} d\omega_i = \sum \omega_k \wedge \omega_{ki}, \\ d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \\ \omega_{ij} = -\omega_{ji} \end{cases}$$

$$(1, 3) \quad \Omega_{ij} = \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l,$$

where R_{ijkl} are the components of curvature tensor of V_n .

Now, define a Riemannian manifold V_N of dimension $N = n(n+1)/2$ whose underlying manifold is B and whose line element is given by

$$(1, 4) \quad ds_N^2 = \sum \omega_i \omega_i + \rho^2 \sum_{i < j} \omega_{ij} \omega_{ij}, \\ \rho = \text{constant} \neq 0.$$

We shall represent (1, 4) by local coordinates in V_n . Let $x^1, x^2, \dots, x^n$ be local coordinates of a neighborhood in V_n on which the line element of V_n is written as

$$(1, 1') \quad ds^2 = \sum g_{ij}(x) dx^i dx^j.$$

Let $X_i, i = 1, 2, \dots, n$, be tangent vectors such as

$$X_i = \partial / \partial x^i = y_i^k e_k, \\ g_{ij} = X_i \cdot X_j = \sum y_i^k y_j^k.$$

In matrix notations, putting

$$(1, 5) \quad G = (g_{ij}), \quad Y = (y_i^k),$$

then we have

$$G = YY',$$

where Y' denotes the transposed matrix of Y . If we put

$$\pi = (\pi_i^j), \quad \pi_i^j = \sum \{ \begin{smallmatrix} j \\ ik \end{smallmatrix} \} dx^k,$$

where $\{ \begin{smallmatrix} j \\ ik \end{smallmatrix} \}$'s are the Christoffel symbols made by g_{ij} , then we have, as is well known,

$$dG = \pi G + G\pi',$$

and

$$\omega = Y^{-1}\pi Y - Y^{-1}dY, \quad \omega = (\omega_{ij}).$$

Hence we have

$$\begin{aligned}
\sum_{i,j} \omega_{ij} \omega_{ij} &= -\frac{1}{2} \text{Trace } \omega \omega \\
&= -\frac{1}{2} \text{Trace } (Y^{-1} \pi \pi Y - Y^{-1} \pi dY - Y^{-1} dY Y^{-1} \pi Y \\
&\quad + Y^{-1} dY Y^{-1} dY) \\
&= -\frac{1}{2} \text{Trace } (\pi \pi - 2dY Y^{-1} \pi + dY Y^{-1} dY Y^{-1}).
\end{aligned}$$

Accordingly, (4) can be written as

$$\begin{aligned}
(1, 4') \quad ds_N^2 &= \sum g_{ij}(x) dx^i dx^j \\
&\quad - \frac{1}{2} \rho^2 \text{Trace } (\pi \pi - 2dY Y^{-1} \pi + dY Y^{-1} dY Y^{-1}),
\end{aligned}$$

where

$$Y = (y_i^k), \quad \pi = (\sum \{ \frac{k}{ih} \} dx^h), \quad G = (g_{ij}), \quad G = YY'.$$

§ 2. **Parameters of the connection of Levi-Civita of V_N .** In this section, we shall determine the parameters of the connection of Levi-Civita of V_N with respect to orthonormal frames of V_N . According to the ordinary method, let us put

$$\begin{aligned}
(2, 1) \quad \left\{ \begin{aligned} d\omega_i &= \sum \omega_k \wedge \theta_{ki} + \rho \sum_{k < h} \omega_{kh} \wedge \theta_{kh; i} \\ &= \sum \omega_k \wedge \omega_{ki}, \\ d\omega_{ij} &= \frac{1}{\rho} \sum \omega_k \wedge \theta_{k; ij} + \sum_{k < h} \omega_{kh} \wedge \theta_{kh; ij} \\ &= \sum \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l, \end{aligned} \right.
\end{aligned}$$

$$(2, 2) \quad \theta_{ij} = -\theta_{ji}, \quad \theta_{kh; i} = -\theta_{i; kh}, \quad \theta_{ij; kh} = -\theta_{kh; ij}.$$

As is well known, we can solve (2, 1), (2, 2) with respect to θ_{ij} , $\theta_{kh; i}$, $\theta_{ij; kh}$ and shall obtain a unique solution. We get from (2, 1), (2, 2) the equations¹⁾

$$\begin{aligned}
(2, 3) \quad \left\{ \begin{aligned} \omega_k \wedge (\theta_{ki} - \omega_{ki}) + \omega_{kh} \wedge \frac{1}{2} \rho \gamma_{kh; i} &= 0, \\ \omega_k \wedge (\frac{1}{\rho} \theta_{k; ij} - \frac{1}{2} R_{ijkl} \omega_l) \\ &\quad + \omega_{kh} \wedge (\frac{1}{2} \theta_{kh; ij} + \delta_{ih} \omega_{kj}) = 0. \end{aligned} \right.
\end{aligned}$$

From the first of the above equations, we may put

$$\begin{aligned}
(2, 4) \quad \theta_{ki} &= \omega_{ki} + A_{kilh} \omega_h + \frac{1}{2} B_{kthj} \omega_{hj}, \\
\rho \gamma_{kh; i} &= B_{jikh} \omega_j + \frac{1}{2} C_{kthj} \omega_{hj},
\end{aligned}$$

1) In the following we shall use the summation convention.

where A 's, B 's and C 's satisfy the following relations :

$$(2, 5) \quad \begin{cases} A_{kih} = A_{hik} = -A_{khi}, \\ B_{kihj} = -B_{ikhj} = -B_{kjhi}, \\ C_{khitjl} = -C_{hkitjl} = -C_{khilj} = C_{jlitkh}. \end{cases}$$

The first of (2, 5) yields

$$(2, 6) \quad A_{kth} = 0.$$

Substituting (2, 4) into the second of (2, 3), we get

$$\begin{aligned} -\omega_k \wedge \left\{ \frac{1}{\rho^2} B_{hkitj} \omega_h + \frac{1}{2\rho^2} C_{ijklm} \omega_{lm} + \frac{1}{2} R_{ijkh} \omega_h \right\} \\ + \frac{1}{4} \omega_{kh} \wedge (2J_{kh;ij} + \partial_{ih} \omega_{kj} - \partial_{ik} \omega_{hj} - \partial_{jh} \omega_{ki} \\ + \partial_{jk} \omega_{hi}) = 0, \end{aligned}$$

from which we may put

$$(2, 7) \quad \begin{aligned} \frac{1}{\rho^2} B_{hkitj} \omega_h + \frac{1}{2\rho^2} C_{ijklm} \omega_{lm} + \frac{1}{2} R_{ijkh} \omega_h \\ = D_{ijklh} \omega_h + \frac{1}{2} E_{ijkhl} \omega_{hl}, \end{aligned}$$

$$(2, 8) \quad \begin{aligned} J_{kh;ij} + \frac{1}{2} (\partial_{ih} \omega_{kj} - \partial_{ik} \omega_{hj} - \partial_{jh} \omega_{ki} + \partial_{jk} \omega_{hi}) \\ = -E_{ijklh} \omega_l - \frac{1}{2} F_{khitjlm} \omega_{lm}, \end{aligned}$$

where D 's, E 's and F 's satisfy the following relations :

$$(2, 9) \quad \begin{cases} D_{ijkh} = D_{ijhk}, \\ E_{ijkhl} = -E_{ijklh} = -E_{jikhl}, \\ F_{khitjlm} = -F_{hkitjlm} = -F_{khijlmt} = -F_{khitjml} = F_{tmijlkh}. \end{cases}$$

Since ω_i, ω_{jk} are linearly independent, we get from (2, 7) the relations

$$(2, 10) \quad D_{ijkh} = \frac{1}{\rho^2} B_{hkitj} + \frac{1}{2} R_{ijkh},$$

$$(2, 11) \quad E_{ijkhl} = \frac{1}{\rho^2} C_{ijklh}.$$

By (2, 5) and (2, 9), we get easily

$$(2, 12) \quad D_{ijkh} = 0,$$

accordingly

$$(2, 13) \quad B_{hkitj} = -\frac{1}{2} \rho^2 R_{ijkh} = \frac{1}{2} \rho^2 R_{hkitj}.$$

On the other hand, from (2, 2) we get

$$(2, 14) \quad E_{ijkl} = -E_{khlij},$$

$$(2, 15) \quad F_{khlijm} = -F_{ijklm}.$$

By (2, 11) and (2, 5) we get

$$E_{ijkl} = E_{khlij}.$$

This equation and (2, 14) follow the relation

$$(2, 16) \quad E_{ijkl} = C_{ijkl} = 0.$$

Analogously, we get easily from (2, 9), (2, 15)

$$(2, 17) \quad F_{ijklm} = 0.$$

Thus, we obtain the solution of (2, 1) under the condition (2, 2) as follows:

$$(2, 18) \quad \begin{cases} \theta_{ij} = \omega_{ij} + \frac{1}{4} \rho^2 R_{ijkh} \omega_{kh}, \\ \theta_{ij;k} = -\frac{1}{2} \rho R_{ijkh} \omega_{kh}, \\ \theta_{ij;kh} = \frac{1}{2} (\partial_{ik} \omega_{jh} + \partial_{jh} \omega_{ik} - \partial_{jk} \omega_{ih} - \partial_{ih} \omega_{jk}). \end{cases}$$

§ 3. Curvature forms in V_N . Making use of (2, 1), (2, 18), we shall calculate the curvature forms of V_N .

We have

$$\begin{aligned} \Omega_{ij} &= d\theta_{ij} - \theta_{ik} \wedge \theta_{kj} - \frac{1}{2} \theta_{i;hk} \wedge \theta_{kh;j} \\ &= d\omega_{ij} + \frac{1}{4} \rho^2 d(R_{ijkh} \omega_{kh}) \\ &\quad - (\omega_{ik} + \frac{1}{4} \rho^2 R_{iklm} \omega_{lm}) \wedge (\omega_{kj} + \frac{1}{4} \rho^2 R_{kjst} \omega_{st}) \\ &\quad + \frac{1}{8} \rho^2 R_{khiu} \omega_i \wedge R_{khjm} \omega_m \\ &= \Omega_{ij} + \frac{1}{4} \rho^2 (R_{ijkh, l} \omega_l + R_{ijkh} \omega_{il} + R_{ilkh} \omega_{jl} \\ &\quad + R_{ijlh} \omega_{kl} + R_{ijkl} \omega_{hl}) \wedge \omega_{kh} \\ &\quad + \frac{1}{4} \rho^2 R_{ijkh} (\Omega_{kh} + \omega_{kl} \wedge \omega_{lh}) \\ &\quad - \frac{1}{4} \rho^2 (R_{iklm} \omega_{lm} \wedge \omega_{kj} + R_{kjst} \omega_{ik} \wedge \omega_{st}) \\ &\quad - \frac{1}{16} \rho^2 R_{iklm} R_{kjst} \omega_{lm} \wedge \omega_{st} \\ &\quad + \frac{1}{8} \rho^2 R_{khiu} R_{khjm} \omega_i \wedge \omega_m, \end{aligned}$$

that is

$$\begin{aligned}
 (3, 1) \quad H_{ij} = & \varrho_{ij} + \frac{1}{4} \rho^2 R_{ijkh} \varrho_{kh} + \frac{1}{8} \rho^3 R_{khl} R_{khjm} \omega_l \wedge \omega_m \\
 & + \frac{1}{4} \rho^2 R_{ijkh, l} \omega_l \wedge \omega_{kh} \\
 & - \frac{1}{4} \rho^2 (R_{ijkh} \omega_{kl} \wedge \omega_{lh} + \frac{1}{4} \rho^2 R_{iklm} R_{kjst} \omega_{lm} \wedge \omega_{st}),
 \end{aligned}$$

where a comma denotes the covariant differentiation of V_a . We have

$$\begin{aligned}
 H_{i; jk} = & d\varrho_{i; jk} - \theta_{ih} \wedge \theta_{h; jk} - \frac{1}{2} \theta_{zhl} \wedge \theta_{hl; jk} \\
 = & \frac{1}{2} \rho d(R_{jkil} \omega_h) \\
 & - (\omega_{lh} + \frac{1}{4} \rho^2 R_{ihlm} \omega_{lm}) \wedge \left(\frac{1}{2} \rho R_{jkhs} \omega_s \right) \\
 & - \frac{1}{8} \rho R_{hli} \omega_s \wedge (\delta_{hj} \omega_{ik} + \delta_{ik} \omega_{hj} - \delta_{hk} \omega_{lj} - \delta_{lj} \omega_{hk}) \\
 = & \frac{1}{2} \rho (R_{jkil, l} \omega_l + R_{likh} \omega_{jl} + R_{lilh} \omega_{kl} + R_{jklh} \omega_{il} + R_{jkil} \omega_{hl}) \wedge \omega_h \\
 & + \frac{1}{2} \rho R_{jkil} \omega_l \wedge \omega_{lh} + \frac{1}{2} \rho R_{jkhs} \omega_s \wedge \omega_{lh} + \frac{1}{8} \rho^3 R_{ihlm} R_{jkhs} \omega_s \wedge \omega_{lm} \\
 & - \frac{1}{8} \rho (R_{jhis} \omega_s \wedge \omega_{ik} + R_{hkis} \omega_s \wedge \omega_{hj} - R_{khis} \omega_s \wedge \omega_{lj} \\
 & \quad - R_{hjis} \omega_s \wedge \omega_{hk}) \\
 = & \frac{1}{2} \rho R_{jkil, l} \omega_l \wedge \omega_h \\
 & + \frac{1}{4} \rho (R_{jilh} \omega_h \wedge \omega_{lk} + R_{hkil} \omega_l \wedge \omega_{hj}) \\
 & + \frac{1}{8} \rho^3 R_{ihlm} R_{jkhs} \omega_s \wedge \omega_{lm}.
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 R_{jkil, l} \omega_l \wedge \omega_h = & - (R_{jkhil, l} \omega_l \wedge \omega_h + R_{jkil, h} \omega_l \wedge \omega_h) \\
 = & R_{jkhil, l} \omega_h \wedge \omega_l + R_{jkil, h} \omega_l \wedge \omega_h,
 \end{aligned}$$

that is,

$$2R_{jkil, l} \omega_l \wedge \omega_h = R_{jkhil, l} \omega_h \wedge \omega_l.$$

Thus we obtain

$$\begin{aligned}
 (3, 2) \quad H_{i; jk} = & \frac{1}{4} \rho R_{jkil, l} \omega_l \wedge \omega_h \\
 & + \frac{1}{4} \rho (R_{ihjl} \omega_h \wedge \omega_{lk} - R_{ihkl} \omega_h \wedge \omega_{lj}) \\
 & + \frac{1}{8} \rho^3 R_{ihlm} R_{jkhs} \omega_s \wedge \omega_{lm}.
 \end{aligned}$$

Finally, we have

$$\begin{aligned}
 H_{ij;kh} &= d\theta_{ij;kh} - \theta_{ij;l} \wedge \theta_{li;kh} - \frac{1}{2} \theta_{ij;lm} \wedge \theta_{lm;kh} \\
 &= \frac{1}{2} \{ \partial_{ik}(\Omega_{jh} + \omega_{jl} \wedge \omega_{lh}) + \partial_{jh}(\Omega_{ik} + \omega_{il} \wedge \omega_{lk}) \\
 &\quad - \partial_{lh}(\Omega_{jk} + \omega_{jl} \wedge \omega_{lk}) - \partial_{jk}(\Omega_{lh} + \omega_{il} \wedge \omega_{lh}) \\
 &\quad + \frac{1}{4} \rho^2 R_{ijklm} R_{khlis} \omega_m \wedge \omega_s \\
 &\quad - \frac{1}{8} (\partial_{il} \omega_{jm} + \partial_{jm} \omega_{il} - \partial_{lm} \omega_{jl} - \partial_{jl} \omega_{lm}) \\
 &\quad \times (\partial_{ik} \omega_{mh} + \partial_{mh} \omega_{ik} - \partial_{lh} \omega_{mk} - \partial_{mk} \omega_{lh}),
 \end{aligned}$$

that is

$$\begin{aligned}
 H_{ij;kh} &= \frac{1}{2} (\partial_{ik} \Omega_{jh} + \partial_{jh} \Omega_{ik} - \partial_{lh} \Omega_{jk} - \partial_{jk} \Omega_{lh}) \\
 (3, 3) \quad &+ \frac{1}{4} \rho^2 R_{ijklm} R_{khlis} \omega_m \wedge \omega_s \\
 &+ \frac{1}{4} (\partial_{ik} \omega_{jl} \wedge \omega_{lh} + \partial_{jh} \omega_{il} \wedge \omega_{lk} - \partial_{lh} \omega_{jl} \wedge \omega_{lk} \\
 &\quad - \partial_{jk} \omega_{il} \wedge \omega_{lh}).
 \end{aligned}$$

Now, let us put

$$(3, 4) \quad H_{AB} = \frac{1}{2} P_{ABCD} \hat{\omega}_C \wedge \hat{\omega}_D,$$

where

$$\begin{aligned}
 A, B, \dots &= i, j, [ij], \dots, \\
 \hat{\omega}_i &= \omega_i, \quad \hat{\omega}_{[ij]} = \rho \omega_{ij}
 \end{aligned}$$

then P_{ABCD} are components of the Riemann-Cristoffel tensor of V_N with respect to the orthonormal frames which are derived from the ones of V_n .

In the first place, we get from (3, 1) the equations as follow.

$$\begin{aligned}
 (3, 5) \quad P_{ijklh} &= R_{ijklh} + \frac{1}{4} \rho^2 R_{ijklm} R_{lmkh} \\
 &\quad + \frac{1}{8} \rho^2 (R_{lmik} R_{lmjh} - R_{lmih} R_{lmjk}),
 \end{aligned}$$

$$(3, 6) \quad P_{i[j[kh]]} = \frac{1}{2} \rho R_{ijklh}.$$

Since

$$- \frac{1}{4} \rho^2 R_{ijklh} \omega_{kl} \wedge \omega_{lh} - \frac{1}{16} \rho^4 R_{iklm} R_{kjlst} \omega_{lm} \wedge \omega_{st}$$

$$\begin{aligned}
&= \frac{1}{8} \rho^2 (R_{ijml} \partial_{kh} - R_{ijkh} \partial_{ml}) \omega_{kl} \wedge \omega_{mh} \\
&\quad + \frac{1}{32} \rho^4 (R_{iskl} R_{jshm} - R_{ismh} R_{jskl}) \omega_{kl} \wedge \omega_{mh},
\end{aligned}$$

we have

$$\begin{aligned}
(3, 7) \quad P_{i[jkl][m]h} &= \frac{1}{2} (R_{ijklm} \partial_{th} - R_{ijlm} \partial_{kh} - R_{ijkh} \partial_{lm} + R_{ijlh} \partial_{km}) \\
&\quad + \frac{1}{4} \rho^2 (R_{iskl} R_{jshm} - R_{ismh} R_{jskl}).
\end{aligned}$$

Since we have

$$\begin{aligned}
&\frac{1}{4} \rho (R_{ihjl} \omega_h \wedge \omega_{lk} - R_{ihkl} \omega_h \wedge \omega_{lj}) \\
&\quad + \frac{1}{8} \rho^3 R_{ihlm} R_{jkhs} \omega_s \wedge \omega_{lm} \\
&= \frac{1}{4} \rho (R_{ihjl} \partial_{km} - R_{ihkl} \partial_{jm}) \omega_h \wedge \omega_{lm} \\
&\quad + \frac{1}{8} \rho^3 R_{ihlm} R_{jkhs} \omega_h \wedge \omega_{lm},
\end{aligned}$$

we get from (3, 1)

$$\begin{aligned}
(3, 8) \quad P_{i[jk]h[lm]} &= \frac{1}{4} (R_{ihjlm} \partial_{kh} - R_{ihkl} \partial_{jm} - R_{ihjm} \partial_{kl} + R_{ihkm} \partial_{jl}) \\
&\quad - \frac{1}{4} \rho^2 R_{ihlm} R_{sjkh},
\end{aligned}$$

and

$$(3, 9) \quad P_{i[jk][lm][st]} = 0.$$

Lastly, since

$$\begin{aligned}
&\frac{1}{4} \{ \partial_{ik} \omega_{jl} \wedge \omega_{lh} + \partial_{jh} \omega_{il} \wedge \omega_{lk} - \partial_{ih} \omega_{jl} \wedge \omega_{lk} - \partial_{jk} \omega_{il} \wedge \omega_{lh} \} \\
&= \frac{1}{16} \{ \partial_{ik} \partial_{jl} \partial_{mh}^{st} - \partial_{ik} \partial_{jm} \partial_{lh}^{st} + \partial_{jh} \partial_{il} \partial_{mk}^{st} - \partial_{jh} \partial_{lm} \partial_{ik}^{st} \\
&\quad - \partial_{ih} \partial_{jl} \partial_{mk}^{st} + \partial_{ih} \partial_{jm} \partial_{lk}^{st} - \partial_{jk} \partial_{il} \partial_{mh}^{st} + \partial_{jk} \partial_{lm} \partial_{lh}^{st} \} \omega_{lm} \wedge \omega_{st} \\
&= \frac{1}{16} \{ \partial_{ij} \partial_{mh}^{st} - \partial_{ij} \partial_{lh}^{st} + \partial_{ij} \partial_{mk}^{st} - \partial_{ij} \partial_{lk}^{st} \} \omega_{lm} \wedge \omega_{st} \\
&= \frac{1}{32} \{ \partial_{ij} \partial_{mh}^{st} - \partial_{ij} \partial_{lh}^{st} + \partial_{ij} \partial_{mk}^{st} - \partial_{ij} \partial_{lk}^{st} \\
&\quad - \partial_{ij} \partial_{th}^{lm} + \partial_{ij} \partial_{sh}^{lm} - \partial_{ij} \partial_{tk}^{lm} + \partial_{ij} \partial_{sk}^{lm} \} \omega_{lm} \wedge \omega_{st}, \quad (1)
\end{aligned}$$

we have

1) Where δ_{ij}^{kh} denote the generalized Kronecker's δ .

$$\begin{aligned}
(3, 10) \quad & P_{[ij][kh][lm][st]} \\
&= \frac{1}{4} \{ \delta_{ij}^{ks} \delta_{lm}^{ht} - \delta_{ij}^{kt} \delta_{lm}^{hs} - \delta_{ij}^{hs} \delta_{lm}^{kt} + \delta_{ij}^{ht} \delta_{lm}^{ks} \\
&\quad - \delta_{ij}^{kl} \delta_{st}^{hm} + \delta_{ij}^{km} \delta_{st}^{hl} + \delta_{ij}^{hl} \delta_{st}^{km} - \delta_{ij}^{hm} \delta_{st}^{kl} \}
\end{aligned}$$

(3, 10) shows that V_N can not always become flat.

Now, we shall calculate components of the Ricci curvature tensor with respect to the canonical orthonormal frames derived from those of V_n from (3, 5)–(3, 10).

Let us put

$$(3, 11) \quad P_{AB} = \sum_C P_{ACBC}.$$

We have

$$\begin{aligned}
P_{ij} &= \sum_k P_{ikjk} + \frac{1}{2} \sum_{h,k} P_{i[hk]j[hk]} \\
&= R_{ikjk} + \frac{1}{4} \rho^2 R_{iklm} R_{lmjk} \\
&\quad + \frac{1}{8} \rho^2 (R_{lmij} R_{lmkk} - R_{lmik} R_{lmkj}) \\
&\quad + \frac{1}{8} (R_{ijhh} \delta_{kk} - R_{ijkh} \delta_{hk} - R_{ijhk} \delta_{kh} - R_{ijkk} \delta_{hh}) \\
&\quad - \frac{1}{8} \rho^2 R_{sthk} R_{stjk},
\end{aligned}$$

that is

$$(3, 12) \quad P_{ij} = R_{ij} + \frac{1}{4} \rho^2 R_{iklm} R_{lmjk}.$$

Nextly, we have

$$\begin{aligned}
P_{i[jk]} &= P_{ih[jk]h} + \frac{1}{2} P_{i[lm][jk][lm]} \\
&= P_{ih[jk]h} = -\frac{1}{2} \rho R_{ihjk,h} \\
&= \frac{1}{2} \rho (R_{ihkh,j} + R_{ihhj,k}),
\end{aligned}$$

that is

$$(3, 13) \quad P_{i[jk]} = -\frac{1}{2} \rho (R_{ij,k} - R_{ik,j}).$$

Then, we have

$$\begin{aligned}
P_{[ij][kh]} &= P_{[ij]l[kh]l} + \frac{1}{2} P_{[ij][st][kh][st]} \\
&= -\frac{1}{4} \rho^2 R_{ijlm} R_{lmkh} \\
&\quad + \frac{1}{8} \{ \delta_{ij}^{\delta s} \delta_{kh}^{\delta t} - \delta_{ij}^{\delta t} \delta_{kh}^{\delta s} - \delta_{ij}^{\delta s} \delta_{kh}^{\delta t} + \delta_{ij}^{\delta t} \delta_{kh}^{\delta s} \\
&\quad - \delta_{ij}^{\delta k} \delta_{st}^{\delta h} + \delta_{ij}^{\delta h} \delta_{st}^{\delta k} + \delta_{ij}^{\delta k} \delta_{st}^{\delta h} - \delta_{ij}^{\delta h} \delta_{st}^{\delta k} \} \\
&= -\frac{1}{4} \rho^2 R_{ijlm} R_{lmkh} - \frac{1}{2} (n-2) \delta_{ij}^{kh},
\end{aligned}$$

hence

$$(3, 14) \quad P_{[ij][kh]} = -\frac{1}{4} \rho^2 R_{ijlm} R_{lmkh} - \frac{1}{2} (n-2) \delta_{ij}^{kh}.$$

Lastly we get from (3, 12), (3, 14) the scalar curvature of V_N as follows.

$$\begin{aligned}
P &= P_{ii} + \frac{1}{2} P_{[ij][ij]} \\
&= R_{ii} + \frac{1}{4} \rho^2 R_{iklm} R_{iklm} - \frac{1}{8} \rho^2 R_{ijlm} R_{ijlm} - \frac{n(n-1)(n-2)}{4},
\end{aligned}$$

that is

$$(3, 15) \quad P = -\frac{1}{4} n(n-1)(n-2) + R + \frac{1}{8} \rho^2 R_{ijlm} R_{ijlm}.$$

§ 4. **Some special cases.** In this section, we shall consider the spaces V_n whose associated Riemannian spaces V_N are Einstein spaces, that is

$$(4, 1) \quad P_{AB} = \frac{P}{N} \delta_{AB}.$$

These equations are written by (3, 12)–(3, 14) as

$$(4, 2) \quad R_{ij,k} - R_{ik,j} = 0,$$

$$(4, 3) \quad R_{ij} + \frac{1}{4} \rho^2 R_{iklm} R_{jklm} = \frac{P}{N} \delta_{ij},$$

$$(4, 4) \quad -\frac{(n-2)}{2} \delta_{ij}^{kh} - \frac{1}{4} \rho^2 R_{ijlm} R_{lmkh} = \frac{P}{N} \delta_{ij}^{kh}.$$

By contraction, we get from (4, 3), (4, 4), (3, 15)

$$\begin{aligned}
R + \frac{1}{4} \rho^2 R_{ijkl} R_{ijkl} &= \frac{P}{N} n \\
&= \frac{2}{n+1} \left\{ -\frac{n(n-1)(n-2)}{4} + R + \frac{1}{8} \rho^2 R_{ijkl} R_{ijkl} \right\}
\end{aligned}$$

and

$$\begin{aligned}
-\frac{1}{2} n(n-1)(n-2) - \frac{1}{4} \rho^2 R_{ijkl} R_{ijkl} &= \frac{P}{N} n(n-1) \\
&= \frac{2(n-1)}{n+1} \left\{ -\frac{n(n-1)(n-2)}{4} + R + \frac{1}{8} \rho^2 R_{ijkl} R_{ijkl} \right\},
\end{aligned}$$

that is

$$(4, 5) \quad (n-1)R + \frac{1}{4} n \rho^2 R_{ijkl} R_{ijkl} = -\frac{1}{2} n(n-1)(n-2),$$

which shows that (4, 3) and (4, 4) are linearly dependent.

Now, we get by (3, 15) and (4, 5) the equation

$$(4, 6) \quad P = \frac{(n+1)}{2n} R - \frac{(n+1)(n-1)(n-2)}{4}$$

Substituting (4, 6) into (4, 3) and (4, 4), we have

$$(4, 3') \quad R_{ij} + \frac{1}{4} \rho^2 R_{iklm} R_{jklm} = \left(\frac{1}{n^2} R - \frac{(n-1)(n-2)}{2n} \right) \delta_{ij},$$

$$(4, 7) \quad -\frac{1}{4} \rho^2 R_{ijkl} R_{klm} = \left(\frac{1}{n^2} R + \frac{n-2}{2n} \right) \delta_{ij}^{kl}.$$

By contraction, we get from (4, 7)

$$(4, 8) \quad -\frac{1}{4} \rho^2 R_{iklm} R_{jklm} = \left(\frac{n-1}{n^2} R + \frac{(n-1)(n-2)}{2n} \right) \delta_{ij},$$

hence this and (4, 3') follow the equation

$$(4, 9) \quad R_{ij} = \frac{1}{n} R \delta_{ij}.$$

(4, 9) shows that if V_N is an Einstein space, then V_n is also an Einstein space.

If $n > 2$ and V_n is an Einstein space, then, as is well known, (4, 2) is automatically satisfied. If $n = 2$, since $R_{ij} = \frac{1}{2} R \delta_{ij}$, (4, 2) becomes

$$R_{,1} = R_{,2} = 0,$$

that is $R = \text{constant}$. But, if V_N for V_2 is an Einstein space, then, by

means of (4, 6), R is constant because $N > 2$. Then we obtain the theorem.

Theorem 1. *Let V_n be an n -dimensional Riemannian space and V_N be the Riemannian space of dimension $N = \frac{1}{2}n(n+1)$ associated with V_n . In order that V_N be an Einstein space, it is necessary and sufficient that V_n be an Einstein space and satisfy the equation*

$$-\frac{1}{4}\rho^2 R_{ij}{}^{lm} R_{lm}{}^{kh} = \frac{1}{n} \left(\frac{R}{n} + \frac{n-2}{2} \right) \delta_{ij}^{kh}.$$

Proof. The necessity of the conditions is evident by the arguments above. We shall prove the sufficiency.

Since V_n is an Einstein space, (4, 2) is clearly satisfied for $n > 2$. When $n = 2$, (4, 2) is equivalent to $R = \text{constant}$ but it can be derived from (4, 7).

We get from (3, 15), (4, 7)

$$\begin{aligned} \frac{P}{N} &= \frac{2}{n(n+1)} \left\{ -\frac{n(n-1)(n-2)}{4} + R - \frac{1}{2} \left(\frac{n-1}{n} R \right. \right. \\ &\quad \left. \left. + \frac{(n-1)(n-2)}{2} \right) \right\} \\ &= -\frac{(n-1)(n-2)}{2n} + \frac{R}{n^2}. \end{aligned}$$

On the other hand, we get from (4, 7)

$$\begin{aligned} &-\frac{n-2}{2} \delta_{ij}^{kh} - \frac{1}{4} \rho^2 R_{ijlm} R_{lmkh} \\ &= -\frac{n-2}{2} \delta_{ij}^{kh} + \left(\frac{1}{n^2} R + \frac{n-2}{2n} \right) \delta_{ij}^{kh} \\ &= \left(-\frac{(n-1)(n-2)}{2n} + \frac{R}{n^2} \right) \delta_{ij}^{kh} = \frac{P}{N} \delta_{ij}^{kh}. \end{aligned}$$

Analogously we have

$$\begin{aligned} &R_{ij} + \frac{1}{4} \rho^2 R_{iklm} R_{jklm} \\ &= \frac{R}{n} \delta_{ij} - \left(\frac{n-1}{n^2} R + \frac{(n-1)(n-2)}{2n} \right) \delta_{ij} = \frac{P}{N} \delta_{ij}. \end{aligned}$$

Thus we see that the system of equations (4, 2), (4, 8) and (4, 4) is equivalent to the one of (4, 7) and (4, 9).

Now, let V_n be a space of constant curvature, that is whose curvature tensor satisfies the equations

$$R_{ijkh} = -K(\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk}),$$

$$K = \text{constant}$$

with respect to orthonormal frames. Since

$$R_{ij} = -(n-1)K_{ij}, \quad R = -n(n-1)K,$$

$$\frac{1}{n^2}R + \frac{n-2}{2n} = -\frac{n-1}{n}K + \frac{n-2}{2n}$$

and

$$-\frac{1}{4}\rho^2 R_{ij}{}^{lm} R_{lm}{}^{kh} = -\frac{1}{2}\rho^2 K^2 \delta_{ij}{}^{kh}$$

it follows that if ρ is a constant such that

$$(4, 10) \quad \rho^2 K^2 = \frac{2(n-1)}{n}K - \frac{n-2}{n},$$

then V_N becomes an Einstein space. Thus we have a corollary.

Corollary. *An n -dimensional Riemann space of constant curvature K has an Einstein space as its associated Riemann space if and only if $K > \frac{n-2}{2(n-1)}$ (or $-R > \frac{n(n-2)}{2}$).*

Lastly, we shall consider the special case $n = 3$. Putting

$$(4, 11) \quad \begin{aligned} R_{223} &= -K_{11}, & R_{312} &= -K_{23} = -K_{32}, \\ R_{313} &= -K_{22}, & R_{123} &= -K_{31} = -K_{13}, \\ R_{121} &= -K_{33}, & R_{331} &= -K_{12} = -K_{21}, \end{aligned}$$

we have

$$(4, 12) \quad \begin{cases} R_{11} = -K_{22} - K_{33}, & R_{23} = K_{32} \\ R_{22} = -K_{33} - K_{11}, & R_{31} = K_{13}, \\ R_{33} = -K_{11} - K_{22}, & R_{12} = K_{21} \end{cases}$$

and

$$(4, 13) \quad R = -2 \sum_{i=1}^3 K_{ii}.$$

Then, (4, 9) becomes

$$(4, 14) \quad \begin{aligned} K_{11} &= K_{22} = K_{33} = \kappa, \\ K_{23} &= K_{31} = K_{12} = 0. \end{aligned}$$

Since

$$\begin{aligned} R_{2ilm}R_{lm2i} &= 2\Sigma K_{ii}K_{ii} = 2\kappa^2, \\ R_{2ilm}R_{lm2i} &= 2\Sigma K_{ij}K_{ji} = 0, \\ \text{etc.,} \end{aligned}$$

(4, 7) becomes

$$(4, 15) \quad \frac{1}{2}\rho^2\kappa^2 = \frac{2}{3}\kappa - \frac{1}{6}$$

Since $R = -6\kappa$, (4, 15) yields the following corollary.

Corollary. *A 3-dimensional Einstein space has an Einstein space as its associated Riemann space if and only if $-R > \frac{3}{2}$.*

Remark. (4, 7) follows that $R + \frac{1}{2}n(n-2) < 0$.

§ 5. Geodesics in V_N . The differential equations of geodesics in V_N are

$$\begin{aligned} & \frac{d\omega_i + \theta_{ki}\omega_k + \frac{1}{2}\rho^2\gamma_{kh;i}\omega_{kh}}{\omega_i} \\ &= \frac{\rho d\omega_{ij} + \theta_{k;ij}\omega_k + \frac{1}{2}\rho^2\gamma_{kh;ij}\omega_{kh}}{\rho\omega_{ij}} \end{aligned}$$

Since we have by means of (2, 18)

$$\begin{aligned} & d\omega_i + \theta_{ki}\omega_k + \frac{1}{2}\rho^2\gamma_{kh;i}\omega_{kh} \\ &= d\omega_i + (\omega_{ki} + \frac{1}{4}\rho^2 R_{klih}\omega_{lh})\omega_k - \frac{1}{4}\rho^2 R_{khlj}\omega_j\omega_{kh} \\ &= d\omega_i + \omega_{ki}\omega_k - \frac{1}{2}\rho^2 R_{ijkh}\omega_j\omega_{kh}, \end{aligned}$$

and

$$\begin{aligned} & \rho d\omega_{ij} + \theta_{k;ij}\omega_k + \frac{1}{2}\rho^2\gamma_{kh;ij}\omega_{kh} \\ &= \rho d\omega_{ij} + \frac{1}{2}\rho R_{ijkh}\omega_k\omega_h + \frac{1}{4}(\partial_{ki}\omega_{hj} + \partial_{hj}\omega_{ki} - \partial_{kj}\omega_{hi} - \partial_{hi}\omega_{kj})\omega_{kh} \\ &= \rho d\omega_{ij}, \end{aligned}$$

the equations of geodesics in V_N become

$$(5, 1) \quad \frac{d\omega_i + \omega_{kl}\omega_k - \frac{1}{2}\rho^2 R_{ijkh}\omega_l\omega_{kh}}{\omega_i} = \frac{d\omega_{ij}}{\omega_{ij}}$$

Let $\bar{C}$ be a geodesic in V_N and C be its image in V_n by the projection $p: V_N \rightarrow V_n$. Let τ, s be the arclengths of $\bar{C}, C$ respectively. Then, (5, 1) is written as

$$\begin{cases} \frac{d\omega_i}{d\tau^2} + \frac{\omega_{kl}}{d\tau} \frac{\omega_k}{d\tau} - \frac{1}{2}\rho^2 R_{ijkh} \frac{\omega_j}{d\tau} \frac{\omega_{kh}}{d\tau} = 0, \\ \frac{d\omega_{ij}}{d\tau^2} = 0. \end{cases}$$

Hence, we have

$$(5, 2) \quad \begin{cases} \frac{d\omega_i}{d\tau^2} + \frac{\omega_{kl}}{d\tau} \frac{\omega_k}{d\tau} = \frac{1}{2}\rho^2 R_{ijkh} \frac{\omega_j}{d\tau} c_{kh}, \\ \omega_{ij} = c_{ij}d\tau, \end{cases}$$

where $c_{ij} = -c_{ji}$ are constants. In a local coordinate neighborhood (x^λ) , if we put

$$(5, 3) \quad dx^\lambda = y_i^\lambda \omega_i, \quad g^{\lambda\mu} = y_i^\lambda y_i^\mu,$$

then since we have

$$\omega_{ik} y_k^\lambda = y_i^\mu \{_{\mu\nu}^\lambda \} dx^\nu + dy_i^\lambda,$$

where $\{_{\mu\nu}^\lambda \}$'s are the Christoffel symbols made by $g_{\lambda\mu}$, (5, 2) is written as

$$(5, 2') \quad \begin{cases} \frac{D}{d\tau} \frac{dx^\lambda}{d\tau} = \frac{1}{2}\rho^2 R_{\mu\nu\omega}^\lambda \frac{dx^\mu}{d\tau} y_i^\nu y_j^\omega c_{ij}, \\ \frac{Dy_i^\lambda}{d\tau} = c_{ij} y_j^\lambda, \end{cases}$$

where $\{_{\mu\nu}^\lambda \}$'s are the Christoffel symbols made by $g_{\lambda\mu}$ and D denotes the covariant differential in V_n . From (5, 2), we see that

$$\frac{ds}{d\tau} = k = \text{constant} \quad 0 \leq k \leq 1.$$

This equation shows that a geodesic in V_N has a constant angle with the field l' of n -dimensional horizontal tangent subspaces $l'_b \subset T_b(B)$, $b \in B$, which will be defined in §6. By means of (1, 4), we must have

$$(5, 3) \quad 1 = k^2 + \frac{1}{2} \rho^2 c_{ij} c_{ij}.$$

In the case $k = 0$, C is clearly a point curve, hence $\bar{C}$ is the image of a one parameter subgroup of O_n by an admissible mapping of the fibre bundle $\mathfrak{B}$. In the case $k = 1$, we have $c_{ij} = 0$ by the above equation, hence C is a geodesic in V_n and the points of $\bar{C}$ are the parallel displaced orthonormal frames of V_n along C . For an example, if V is a 2-sphere, then C is a circle on the sphere.

§ 6. The Levi-Civita's connection of V_n and its explanation in V_N .

According to §1, let $\mathfrak{B} = \{B, p, V_n, O_n, O_n\}$ be the associated principal fibre bundle of V_n . Any point $b \in B$ is represented as

$$(6, 1) \quad b = (x(b), e_i(b))$$

where $x(b) = p(b)$ and $e_i(b)$, $i = 1, 2, \dots, n$, are unit tangent vectors to V_n at $x(b)$ and orthonal each others.

Let $v_i(b)$, $v_{ij}(b)$, $i < j$, $i, j = 1, 2, \dots, n$, be tangent vectors to B dual to $\omega_i(b)$, $\omega_{ij}(b)$, $i < j$, $i, j = 1, 2, \dots, n$.

In the following, for a differentiable mapping f of a differentiable manifold X into another differentiable manifold Y , we shall denote the differential mapping of f by $f_*: T(X) \rightarrow T(Y)$ and the dual mapping of f_* by $f^*: T^*(Y) \rightarrow T^*(X)$, where $T(X)$, $T(Y)$ ($T^*(X)$, $T^*(Y)$) are the spaces of tangent (cotangent) vectors to X , Y respectively.

Since $\omega_i(b) = p^* \omega_i(b)$, where $\omega_i(b)$ in the right-hand side is regarded as a cotangent vector to V_n at $x(b)$ such that $\delta_{ij} = \langle \omega_i(b), e_j(b) \rangle$, we have $\langle \omega_i(b), p_* v_j(b) \rangle = \langle \omega_i(b), v_j(b) \rangle = \delta_{ij}$, hence

$$(6, 2) \quad p_* v_i(b) = e_i(b).$$

Analogously we have

$$(6, 3) \quad p_* v_{ij}(b) = 0.$$

For any $\alpha = ((a_i^j(\alpha))) \in O_n$, we denoted the right translation corresponding to α by $r(\alpha)$ which is defined by

$$(6, 4) \quad r(\alpha)(b) = (x(b), a_i^j(\alpha) e_j(b)),$$

where $((a_i^j(\alpha)))$ is an n -dimensional orthogonal matrix. Since we have

$$a_i^j(\alpha_2 \alpha_1) = a_k^j(\alpha_2) a_i^k(\alpha_1), \quad \alpha_1, \alpha_2 \in O_n$$

it follows that

$$(6, 5) \quad r(\alpha_1) \circ r(\alpha_2) = r(\alpha_2 \alpha_1).$$

Now, we shall consider $(r(\alpha))^*$. Let $b = f(x) = (x, e_i(x))$ be a differentiable local cross-section of $\mathfrak{B}$ defined on a neighborhood U in V_α . Let us put

$$(6, 6) \quad e_i(b) = y_i^k(b) e_k(x), \quad x = p(b),$$

then we can consider $x(b)$, $y_i^j(b)$ as local coordinates of the point b . Let us put

$$(6, 7) \quad \theta_k(x) = f^* \omega_k(b), \quad \theta_{kh}(x) = f^* \omega_{kh}(b),$$

then we have the equations

$$(6, 8) \quad \begin{cases} \omega_i(b) = z_k^i(b) \theta_k(x), \\ \omega_{ij}(b) = y_i^k(b) z_k^j(b) \theta_{kh}(x) + z_k^j(b) dy_i^k(b) \end{cases}$$

in the coordinates $x(b)$, $y_i^j(b)$, where $y_i^k(b) z_k^j(b) = \delta_i^j$.

Since we have from (6, 4) (6, 6)

$$r(\alpha)(b) = (x, a_i^j(\alpha) y_j^k(b) e_k(x)), \quad x = p(b),$$

we get

$$(6, 9) \quad \begin{aligned} r(\alpha)^* \omega_i(r(\alpha)(b)) &= a_j^i(\alpha^{-1}) \omega_j(b) = a_j^i(\alpha) \omega_j(b), \\ r(\alpha)^* \omega_{ij}(r(\alpha)(b)) &= a_i^k(\alpha) a_j^h(\alpha) \omega_{kh}(b). \end{aligned}$$

Accordingly, we get

$$(6, 10) \quad \begin{cases} r(\alpha)_* v_j(b) = a_j^i(\alpha) v_i(r(\alpha)(b)), \\ r(\alpha)_* v_{kh}(b) = a_i^k(\alpha) a_j^h(\alpha) v_{ij}(r(\alpha)(b)). \end{cases}$$

Now, let be I'_b the tangent subspace to B at b spanned by $v_i(b)$, ..., $v_n(b)$ which define a differentiable field I' on V_α . By (6, 2), (6, 10) it follows that

$$(6, 11) \quad \begin{cases} r(\alpha)_* I'_b = I'_{r(\alpha)(b)}, & \alpha \in O_\alpha, \\ p_* I'_b = T_{p(b)}(V_\alpha). \end{cases}$$

Let $\mu_b: T_b(B) \rightarrow T_b(O_\alpha(x))$, $p(b) = x$, be the projection defined by

$$(6, 12) \quad \mu_b(\sum v_i v_i(b) + \sum_{i < k} v_{ij} v_{ij}(b)) = \sum_{i < j} v_{ij} v_{ij}(b)$$

and denote also by b the admissible mapping $O_n \rightarrow O_n(x)$ defined by $b(\alpha) = r(a)(b)$, $\alpha \in O_n$. Putting $\pi_b = (b_*)^{-1} \mu_b: T_b(B) \rightarrow T_e(O_n)$, $e =$ the identity of O_n , we obtain a $L(O_n)$ -valued differential from π defined on B by $\pi|T_b(B) = \pi_b$. Then it follows from (6, 11) that $r(a)^* \pi = ad(\alpha^{-1}) \pi$.

Let $\iota_x: O_n(x) \rightarrow B$ be the imbedding mapping then we get from (6, 8)

$$(6, 13) \quad b^* \iota_x^* \omega_{ij}(b(a)) = a_i^j(\alpha^{-1}) da_i^k(\alpha) = a_j^k(\alpha) da_i^k(\alpha) = \hat{\omega}_{ij}(\alpha),$$

which are left invariant differential forms on O_n .

Let $(\hat{v}_{ij}(a))$ be the tangent vector fields on O_n dual to $(\hat{\omega}_{ij}(a))$. Since

$$\begin{aligned} \langle \omega_{ij}(b), (\iota_x b)_* \hat{v}_{kh}(e) \rangle &= \langle \hat{\omega}_{ij}(e), \hat{v}_{kh}(e) \rangle, \\ \langle \omega_i(b), (\iota_x b)_* \hat{v}_{kh}(e) \rangle &= \langle \iota_x^* \omega_i(b), b_* \hat{v}_{kh}(e) \rangle \\ &= \langle 0, b_* \hat{v}_{kh}(e) \rangle = 0, \end{aligned}$$

we have

$$(6, 14) \quad (\iota_x b)_* \hat{v}_{ij}(e) = v_{ij}(b).$$

Let $\iota: p^{-1}(U) \rightarrow B$ be the imbedding mapping and define a mappings $\rho: U \rightarrow U \times O_n$, $\phi: U \times O_n \rightarrow B$ by

$$\begin{aligned} \rho(x) &= x \times e, \\ \phi(x, \alpha) &= (x, a_i^j(\alpha) e_j(f(x))). \end{aligned}$$

Then we have a $L(O_n)$ -valued differential from $\theta = (\iota \phi \rho)^*$ on U . Since $(\iota \phi \rho)(x) = (x, e_i(f(x)))$, we get from (6, 8)

$$\begin{aligned} (\iota \phi \rho)^* \omega_i(f(x)) &= \theta_i(x), \\ (\iota \phi \rho)^* \omega_{ij}(f(x)) &= \theta_{ij}(x) = I_{ik}^j(x) \theta_k(x) \end{aligned}$$

hence

$$(\iota \phi \rho)_* e_k(f(x)) = v_k(f(x)) + \frac{1}{2} I_{ik}^j(x) v_{ij}(f(x)).$$

Accordingly, we have by (6, 14)

$$\begin{aligned} \langle \hat{v}, e_k(x) \rangle &= \langle \pi, (\iota \phi \rho)_* e_k(x) \rangle \\ &= \langle \pi, v_k(f(x)) + \frac{1}{2} I_{ik}^j(x) v_{ij}(f(x)) \rangle \\ &= \frac{1}{2} \langle \pi, I_{ik}^j(x) v_{ij}(f(x)) \rangle = \frac{1}{2} I_{ik}^j(x) \hat{v}_{ij}(e). \end{aligned}$$

On the other hand, we can define canonically a $L(O_n)$ -valued differential form θ on U from θ_{ij} by

$$\theta(v) = \langle \theta^i_j(x), v \rangle \hat{v}_{ij}(e), \quad v \in T_x(V_n)$$

this shows that

$$\theta = \hat{\theta}.$$

That is, the parameters $\hat{\theta}$ on $U \subset V_n$ derived from the connection in the sense of C. Ehresmann [3] defined by the field of tangent subspaces T_b by the local cross section $f: U \rightarrow B$ are the parameters θ on U of the Levi-Civita connection of V_n with respect to the field of orthonormal frames defined by f .

§ 7. Motions of V_N derived from motions of V_n . Let f be a motion of V_n , that is a homeomorphism onto itself such that $(f(x_1), f(x_2)) = (x_1, x_2)$, $x_1, x_2 \in V_n$, where (x_1, x_2) denotes the Riemannian distance in V_n between x_1 and x_2 . As is well known, f is differentiable. Furthermore we have

$$(7, 1) \quad (f_* X_1) \cdot (f_* X_2) = X_1 \cdot X_2, \quad X_1, X_2 \in T_x(V_n),$$

where $X_1 \cdot X_2$ denotes the inner product of X_1 and X_2 . Accordingly, we can define a differentiable homeomorphism $\bar{f} = \chi(f)$ on B by

$$(7, 2) \quad \bar{f}(b) = (f(x(b)), f_* e_i(b)), \quad b \in B.$$

Since $p\bar{f} = f p$, we have $p_* f_* v_i(b) = f_* p_* v_i(b) = f_* e_i(b) = e_i(\bar{f}(b))$ by (6, 2), (7, 2), hence $\bar{f}_* v_i(b) = v_i(\bar{f}(b)) +$ a linear combinations of $v_{ij}(\bar{f}(b))$ and $\bar{f}^* \omega_i(\bar{f}(b)) = \omega_i(B) +$ a linear combination of $\omega_{ij}(b)$. On the other hand, since we can consider $\omega_i(b)$ as differential forms in V_n , we obtain

$$(7, 3) \quad \bar{f}^* \omega_i(\bar{f}(b)) = \omega_i(b).$$

Furthermore, we get from (1, 2), (7, 3)

$$\begin{aligned} d\omega_i &= \sum \omega_k \wedge \bar{f}^* \omega_{ki}, \\ \bar{f}^* \omega_{ij} &= -\bar{f}^* \omega_{ji}, \end{aligned}$$

hence we have

$$(7, 4) \quad \bar{f}^* \omega_{ij}(\bar{f}(b)) = \omega_{ij}(b).$$

Thus we obtain the following theorem.

Theorem 2. *If f is a motion of V_n , then the transformation $\bar{f} = \chi(f)$ derived from f by (7, 2) is also a motion of V_N and*

$$(7, 5) \quad \chi(f_1 \circ f_2) = \chi(f_1) \circ \chi(f_2).$$

Now, denoting the group of motions of V_n by $M(V_n)$, we get easily from (6, 9) the following theorem.

Theorem 3. *Any right translation of B is a motion of V_N and commutes with $\chi(f)$, $f \in M(V_n)$.*

It is sufficient to prove the second part of the theorem.

For $\alpha \in O_n$, $f \in M(V_n)$, $b \in B$, we have

$$\begin{aligned} r(\alpha)(\chi(f)(b)) &= r(\alpha)((f(x(b)), f_*e_i(b))) \\ &= (f(x(b)), a_i^j(\alpha) f_*e_j(b)) \\ &= (f(x(b)), f_*(a_i^j(\alpha)e_j(b))) \\ &= \chi(f)((x)(b), a_i^j(x)e_j(b)) \\ &= \chi(f)(r(\alpha)(b)). \end{aligned}$$

Hence we have the relation

$$(7, 6) \quad r(\alpha) \circ \chi(f) = \chi(f) \circ r(\alpha).$$

We see also easily that

$$(7, 7) \quad r(O_n) \cap \chi(M(V_n)) = 1,$$

where 1 denotes the identity transformation.

§ 8. **Some mappings on V_N .** Now, let S^{n-1} be the $(n-1)$ -dimensional unit sphere: $\sum w^i w^i = 1$ in an n -dimensional Euclidean space R^n . For any complete Riemannian manifold V_n , we define a mapping $\Phi: B \times S^{n-1} \times R \rightarrow B$ as follows:

For $b \in B$, $w = (w^1, \dots, w^n) \in S^{n-1}$, $s \in R$, let $\gamma(b, w, s)$ be the geodesic arc in V_n starting at $p(b) = x(b)$ whose tangent unit vector at $x(b)$ is $w^i e_i(b)$ and whose length is s . Let $F(b, w, s)$ be the end point of $\gamma(b, w, s)$. By parallel displacing $e_i(b)$ along this geodesic, we get a curve $\bar{\gamma}(b, w, s)$ in B whose points are these frames, hence $p(\bar{\gamma}(b, w, s)) = \gamma(b, w, s)$. Let $\Phi(b, w, s)$ be the end point of $\bar{\gamma}(b, w, s)$.

The mapping Φ is clearly differentiable and have the following properties:

$$(8, 1) \quad p(\Phi(b, w, s)) = F(b, w, s).$$

$$(8, 2) \quad r(\alpha) \circ \Phi(b, w, s) = \Phi(r(\alpha)b, \alpha^{-1}w, s), \quad \alpha \in O_n,$$

$$(8, 3) \quad F(b, w, s) = F(r(\alpha)b, \alpha^{-1}w, s),$$

$$(8, 4) \quad \Phi(b, w, 0) = b.$$

Furthermore, since any motion of V_n preserves geodesics and parallel displaced vector fields, it follows that

$$(8, 5) \quad \begin{aligned} \bar{f}(\Phi(b, w, s)) &= \Phi(\bar{f}(b), w, s), \\ \bar{f} &= \chi(f), \quad f \in M(V_n), \end{aligned}$$

and by $p \circ \bar{f} = f \circ p$ and (8, 2),

$$(8, 6) \quad f \circ F(b, w, s) = F(\bar{f}(b), w, s).$$

Let $\{f_m\}$, $m = 1, 2, \dots$, be a sequence of motions on V_n . For a fixed point $b_0 \in B$, we suppose that $\bar{f}_m(b_0)$, $\bar{f}_m = \chi(f_m)$, converge to a point b'_0 . For any point $b \in B$, we can take an $\alpha \in O_n$, a $w \in S^{n-1}$ and a real number s such that $b = r(\alpha)(\Phi(b_0, w, s))$. Hence, by (7, 5), (8, 5), we get

$$\begin{aligned} \lim \bar{f}_m(b) &= \lim \bar{f}_m(r(\alpha)(\Phi(b_0, w, s))) \\ &= \lim r(\alpha)(\bar{f}_m(\Phi(b_0, w, s))) \\ &= r(\alpha)(\lim \Phi(\bar{f}_m(b_0), w, s)), \end{aligned}$$

that is

$$(8, 7) \quad \lim \bar{f}_m(b) = r(\alpha)(\Phi(b'_0, w, s)).$$

Thus we can define a limiting map $\hat{f}: B \rightarrow B$ by

$$(8, 8) \quad \hat{f}(b) = \lim \chi(f_m)(b), \quad b \in B,$$

which is clearly a motion of V_n . By the above equation, we get easily

$$(8, 9) \quad \hat{f} \circ r(\alpha) = r(\alpha) \circ \hat{f}, \quad \alpha \in O_n.$$

Furthermore, since we have $p(\hat{f}(b)) = \lim f_m(p(b))$, we get a limiting map $f: V_n \rightarrow V_n$ by

$$f(x) = \lim f_m(x), \quad x \in V_n,$$

such that $f \in M(V_n)$ and

$$f \circ p = p \circ \hat{f}.$$

Now, we have by (8, 1), (8, 7) the relation

$$\begin{aligned} f(x) &= F(b'_0, w, s), \\ x &= \hat{f}(b), \quad b = r(\alpha)(\Phi(b_0, w, s)). \end{aligned}$$

On the other hand, we get from (8, 5) the equation

$$\begin{aligned} p(\bar{f}(\Phi(b_0, w, s))) &= f(p(\Phi(b_0, w, s))) = f(x) \\ &= p(\Phi(\bar{f}(b_0), w, s)) = F(\bar{f}(b_0), w, s), \\ \bar{f} &= \chi(f), \end{aligned}$$

hence

$$\begin{aligned} F(\dot{f}(b_0), w, s) &= F(\bar{f}(b_0), w, s) \\ w &\in S^{n-1}, \quad s \in R. \end{aligned}$$

It follows that $\dot{f}(b_0) = \bar{f}(b_0)$ and by (8, 5), (7, 5)

$$\begin{aligned} \dot{f}(b) &= r(\cdot)(\Phi(\bar{f}(b_0), w, s)) \\ &= r(\cdot)(\bar{f}(\Phi(b_0, w, s))) = \bar{f}(r(\cdot)(\Phi(b_0, w, s))) \\ &= \dot{f}(b), \end{aligned}$$

that is

$$(8, 10) \quad \lim \chi(f_m)(b) = \chi(\lim f_m)(b), \quad b \in B$$

For any V_n which is not complete, we can carry the same argument by means of a finite number of points of B such as b_0 . Thus, we obtain.

Theorem 4. *Let V_n be a Riemannian manifold and let $\{f_m\}$, $m = 1, 2, \dots$, be a sequence of motions of V_n . Then the sequence $\{\chi(f_m)\}$ is simultaneously convergent or do not convergent at every point of B . In the first case, we have*

$$\lim \chi(f_m)(b) = \chi(\lim f_m)(b), \quad b \in B.$$

In the next place, we suppose that for a sequence $\{f_m\}$ of motions of V_n , $\lim f_m(x_0) = x'_0$. For a subsequence $\{f_{m_\lambda}\}$ of $\{f_m\}$, we may suppose that $\lim \bar{f}_{m_\lambda}(b_0) = b'_0$, where b_0 is a fixed element in $p^{-1}(x_0)$. Then, by means of the above theorem there exists a $f \in M(V_n)$ such that $f(x) = \lim_{\lambda \rightarrow \infty} f_{m_\lambda}(x)$ and $\chi(f)(b) = \lim_{\lambda \rightarrow \infty} \chi(f_{m_\lambda})(b)$, $x \in V_n$, $b \in B$. Accordingly, we see that if $\lim f_m(x) = f(x)$, then $\lim \chi(f_m)(b) = \chi(f)(b)$. Thus, we obtain

Theorem 5. *For any V_n , $\chi: M(V_n) \rightarrow M(V_N)$ is continuous in the sense of weakly convergence, that is, if $\lim f_m(x) = f(x)$, then $\lim \chi(f_m)(b) = \chi(f)(b)$, $x \in V_n$, $b \in V_N$.*

§ 9. **Tangent vector fields over V_n derived from sequences of motions of V_n .** For the sake of simplicity, let V_n be a complete Rie-

mannian manifold. For any $f \in M(V_n)$, $x \in V_n$, since we have by Theorem 3

$$\begin{aligned} (b, \chi(f)(b)) &= (r(\alpha)(b), r(\alpha)(\chi(f)(b))) \\ &= (r(\alpha)(b), \chi(f)(r(\alpha)(b))), \\ b &\in p^{-1}(x), \quad \alpha \in O_n, \end{aligned}$$

we define a function $u_f: V_n \rightarrow R$ by

$$(9, 1) \quad u_f(x) = (b, \chi(f)(b)), \quad b \in p^{-1}(x),$$

which is differentiable. If $f \neq 1$, then everywhere $u_f(x) \neq 0$ by (8, 5).

Now, let be given a sequence $\{f_m\}$, $m = 1, 2, \dots$, of motions of V_n which are mutually distinct and weakly converge to the identity transformation. For simplicity, we put

$$u_m(x) = u_{f_m}(x), \quad x \in V_n.$$

By Theorem 5, we have

$$\lim_{m \rightarrow \infty} \chi(f_m)(b) = b, \quad b \in B.$$

Then we define a tangent vector field τ over V_n by

$$(9, 2) \quad \tau(x)(h) = \lim_{m \rightarrow \infty} \frac{(f_m^* h)(x) - h(x)}{u_m(x)},$$

where $x \in V_n$ and h is any differentiable function defined on an open neighborhood of x . We shall show that $\tau(x)$ can be defined by the right hand side of (9, 2) and τ is a differentiable tangent vector field over V_n .

Now, we define a differentiable function by

$$(9, 3) \quad w(b, b', w, s) = (\Phi(b, w, s), \Phi(b', w, s)), \\ b, b' \in B, w \in S^{n-1}, s \in R.$$

By Theorem 3, (8, 2), we get

$$(9, 4) \quad w(b, b', w, s) = w(r(\alpha)(b), r(\alpha)(b'), \alpha^{-1}w, s) \quad \alpha \in O_n.$$

For any point x_0 , we can take a spherical neighborhood U_{x_0} such that for a fixed $b_0 \in p^{-1}(x_0)$, $F(b_0, w, s)$ gives a geodesic polar coordinate system on U_{x_0} . Then we can define a function u by

$$(9, 5) \quad u(b_0, b_1, x) = w(b_0, b_1, w, s),$$

where $x \in U_{p(b_0)}$, $x = F(b_0, w, s)$, $b_1 \in B$. By (8, 3), (8, 4), (9, 4), we get a relation as

$$(9, 6) \quad u(b_0, b_1, x) = u(r(\alpha)(b_0), r(\alpha)(b_1), x).$$

For a motion f on V_m , we have

$$\begin{aligned} u_f(F(b_0, w, s)) &= (\Phi(b_0, w, s), \chi(f)(\Phi(b_0, w, s))) \\ &= (\Phi(b_0, w, s), \Phi(\chi(f)(b_0), w, s)) \\ &= w(b_0, \chi(f)(b_0), w, s), \end{aligned}$$

hence

$$(9, 7) \quad u_f(x) = u(b_0, \chi(f)(b_0), x), \quad x \in U_{p(b_0)}, \quad b_0 \in B.$$

For a fixed $w \in S^{n-1}$, a fixed $s \in R$, the differentiable mapping $\Phi_{w,s}: B \rightarrow B$ by

$$(9, 8) \quad \Phi_{w,s}(b) = \Phi(b, w, s), \quad b \in B$$

is a differentiable homeomorphism on B and it is evident from the definition of Φ that

$$(9, 9) \quad \begin{cases} \Phi_{-w,s} = \Phi_{w,-s}, \\ \Phi_{w,s} \Phi_{-w,s} = 1. \end{cases}$$

Accordingly, if for a point $b_0 \in B$, the tangent unit vectors to elementary geodesic arcs $\gamma(b_0, \bar{f}_m(b_0))$, $\bar{f}_m = \chi(f_m)$, from b_0 to $\bar{f}_m(b_0)$ at b_0 converge to a tangent vector to B at b_0 , then for any point $b \in B$, the same is true. Furthermore, since for the function $u(b_0, b_1, x)$ which is differentiable with respect to $b_0, b_1 \in B$, $x \in U_{p(b_0)}$, we have

$$u(b_0, b_1, x_0) = (b_0, b_1), \quad x_0 = p(b_0),$$

we can take an open neighborhood of x_0 , $U'_{x_0} \subset \bar{U}'_{x_0} \subset U_{x_0}$ such that

$$(9, 10) \quad \lim_{m \rightarrow \infty} \frac{u(b_0, b_m, x)}{(b_0, b_m)} = \lim_{m \rightarrow \infty} \frac{U_m(x)}{(b_0, b_m)} \neq 0, \\ b_m = \bar{f}_m(b_0), \quad x \in U'_{x_0}.$$

In U_{x_0} , we get from the equation $x = F(b_0, w, s)$ the inverse mapping

$$w = w(x), \quad s = s(x), \quad x \neq x_0,$$

which are differentiable. Then, we have by (8, 6)

$$(9, 11) \quad f_m(x) = F(b_m, w(x), s(x)).$$

Therefore, we have by (9, 10), (9, 11)

$$\begin{aligned} & \lim_{m \rightarrow \infty} \frac{(f_m^* h)(x) - h(x)}{u_m(x)} \\ &= \lim_{m \rightarrow \infty} \frac{\frac{h(F(b_m, w(x), s(x))) - h(x)}{(b_0, b_m)}}{\frac{u(b_0, b_m, x)}{(b_0, b_m)}} \end{aligned}$$

This equation shows that $\gamma(x)$ is defined and differentiable on $U'_{x_0} - x_0$. We have proved that we can define a vector field γ over V_n by (9, 2) and it is differentiable on it.

On the other hand, from the above consideration, we can define a differentiable scalar field σ_{x_0} over V_n for a point $x_0 \in V_n$ by

$$(9, 12) \quad \sigma_{x_0}(x) = \lim_{m \rightarrow \infty} \frac{u_m(x)}{(b_0, \chi(f_m)(b_0))}, \quad x \in V_n, \quad b_0 \in p^{-1}(x_0),$$

which is everywhere positive. Then, we can define a differentiable covariant vector field τ over V_n as follows: in local coordinates $x^1, \dots, x^n$ on $U \subset V_n$

$$(9, 13) \quad \tau_i(x) = \frac{\partial}{\partial x^i} \log \sigma_{x_0}(x) \left(= \frac{\partial}{\partial x^i} \log \sigma_{x_1}(x) \right) \\ x_0, x_1 \in V_n, \quad x \in U,$$

which does not depend on the point x_0 .

Now, in the coordinate neighborhood U'_{x_0} , for sufficiently large m , we must have

$$(9, 14) \quad g_{ij}(x) = g_{kh}(f_m(x)) \frac{\partial f_m^* x^k}{\partial x^i}(x) \frac{\partial f_m^* x^h}{\partial x^j}(x)$$

where $g_{ij}(x)$ are the components of the fundamental tensor of V_n with respect to the coordinates $x^1, \dots, x^n$. Taking a suitable neighborhood W of b_0 in V_n , we can consider differentiable functions $H_{ij}(b, x)$ defined on $W \times U'_{x_0}$ by

$$(9, 15) \quad \begin{aligned} & H_{ij}(b, x) = -g_{ij}(x) \\ & + g_{kh}(F(b, w(x), s(x))) \frac{\partial}{\partial x^i} x^k(F(b, w(x), s(x))) \times \\ & \times \frac{\partial}{\partial x^j} x^h(F(b, w(x), s(x))), \quad b \in W, \quad x \in U'_{x_0}, \end{aligned}$$

where we use $\frac{\partial}{\partial x^i}$ conventionally but there will be no confusion.

By (9, 11), (9, 14), we have for sufficiently large m the equation

$$H_{ij}(b_m, x) = 0.$$

Then we get easily the equation

$$\begin{aligned} 0 &= \lim_{m \rightarrow \infty} \frac{H_{ij}(b_m, x)}{u_m(x)} \\ &= \gamma_{i,j}(x) + \gamma_{j,i}(x) + \gamma_i(x)\tau_j(x) + \gamma_j(x)\tau_i(x), \end{aligned}$$

that is

$$(9, 16) \quad \gamma_{i,j}(x) + \gamma_{j,i}(x) + \gamma_i(x)\tau_j(x) + \gamma_j(x)\tau_i(x) = 0,$$

where $\gamma_i(x) = g_{ij}(x)\gamma^j(x)$ and a comma denotes the covariant differentiation of V_n . This relations are clearly true on any coordinate neighborhood since the fields γ , τ are defined on V_n and do not depend on the point x_0 .

If we define a differentiable contravariant vector field ξ by

$$\xi = \sigma_{x_0}\gamma,$$

that is

$$(9, 17) \quad \xi(x)(h) = \lim_{m \rightarrow \infty} \frac{(f_m^*h)(x) - h(x)}{(b_0, \chi(f_m)(b_0))},$$

where $x \in V_n$, h is any differentiable function defined on an open neighborhood of x and b_0 is a fixed point of B .

Then we get by (9, 13), (9, 16)

$$\xi_{i,j}(x) + \xi_{j,i}(x) = 0$$

in any local coordinate neighborhood. This is the equation of Killing.

Since we can omit, in the above consideration, *the condition that V_n is complete as in § 8 by means of a finite number of points in B such as b_0 , we obtain the classical theorem [6].*

Theorem 6. *Let V_n be a Riemannian manifold and let $\{f_m\}$ be a sequence of motions of V_n which are mutually distinct and weakly converge to the identity transformation. If the tangent unit vector to elementary geodesic arcs $\gamma(b_0, \chi(f_m)(b_0))$ from b_0 to $\chi(f_m)(b_0)$ at a fixed point $b_0 \in B$ converge to a tangent vector, then we can obtain a*

differentiable tangent vector field which represents an infinitesimal transformation of motion by (9, 17).

§ 10. $\Phi(b, \xi)$ and holonomy groups. In this paragraph, we shall investigate the automorphisms on V_N which are generalizations of $\Phi(b, w, s)$ in § 8 and the holonomy group of V_n .

Let W be the set of piecewise differentiable arcs parameterized with arclengths in an n -dimensional Euclidean space. We shall classify the elements of W as follows: $W \ni \gamma_a: 0 \leq s \leq l_a \rightarrow R^n$, $a = 1, 2$, are $\gamma_1 \sim \gamma_2$, (1) if there exists a translation such that $\gamma_2 = \psi \circ \gamma_1$, (2) if for some $k, c \geq 0$ such that $0 \leq k - c \leq k + c \leq l_1 = l_2 + 2c$, and we have

$$\begin{aligned} \gamma_1(s) &= \gamma_2(s) & \text{for } 0 \leq s \leq k - c, \\ \gamma_1(s) &= \gamma_2(2k - s) & \text{for } k - c \leq s \leq k, \\ \gamma_1(s) &= \gamma_2(s - 2c) & \text{for } k - c \leq s \leq l_1 \end{aligned}$$

or (3) if there exists a relation between γ_1 and γ_2 exchanged γ_1 and γ_2 in (2).

Let $\mathfrak{W}$ be the set of equivalent classes of W by the above equivalent relation.

For $\gamma_1, \gamma_2 \in W$ such that the end point of γ_1 is the starting point of γ_2 , $\gamma = \gamma_1 \gamma_2$ is usually defined by

$$\gamma(s) = \begin{cases} \gamma_1(s) & 0 \leq s \leq l_1, \\ \gamma_2(s - l_1) & l_1 \leq s \leq l_1 + l_2. \end{cases}$$

We define multiplication in $\mathfrak{W}$ as follows:

$\xi_1, \xi_2 \in \mathfrak{W}$, we take $\gamma_1 \in \xi_1, \gamma_2 \in \xi_2$ such that the end point of γ_1 is the starting point of γ_2 and we denote the class containing $\gamma_1 \gamma_2$ by $\xi_1 \cdot \xi_2$. Clearly $\xi_1 \cdot \xi_2$ does not depend on the choice of $\gamma_1 \in \xi_1$ and $\gamma_2 \in \xi_2$.

We can easily prove that $\mathfrak{W}$ is a group with respect to this multiplication. $\mathfrak{W}$ contains the n -dimensional translation group $\mathfrak{T}_n$ of R^n as a subgroup.

We define a homomorphism $\sigma: \mathfrak{W} \rightarrow \mathfrak{T}_n$ as follows: For any $\xi \in \mathfrak{W}$, let γ be a representative with the minimum length in ξ , and let $\sigma(\gamma)$ be the translation corresponding to the sensed segment from the starting point to the end point of γ . We can easily see that $\sigma(\gamma)$ does not depend on the choice of γ and σ is a homomorphism onto. Let $\mathfrak{W}_0$ be the kernel of σ . We obtain easily the relations.

$$\begin{aligned}
 (10, 1) \quad & \mathfrak{W}_0 \cap \mathfrak{I}_n = 1, \\
 (10, 2) \quad & \mathfrak{W} = \mathfrak{W}_0 \cdot \mathfrak{I}_n = \mathfrak{I}_n \cdot \mathfrak{W}_0, \\
 (10, 3) \quad & \sigma(\mathfrak{I}_n) = \mathfrak{I}_n.
 \end{aligned}$$

Now, for any $\xi \in \mathfrak{W}$, we define a homeomorphism $\eta_\xi: V_N \rightarrow V_N$ as follows: Let $\gamma \in \xi$ be a representative with its end point at the origin O of R^n . For any point $b \in V_N$, we take a curve C in V_n and a curve $\bar{C}$ in B such that

- (i) $p(\bar{C}) = C$,
- (ii) *the points of $\bar{C}$ are the parallel displaced frames along C .*
- (iii) *the point b is the end point of $\bar{C}$,*
- (iv) *by the linear mapping $I_b: T_{p(b)}(V_n) \rightarrow R_n$, $I_b(e_i(b)) = w_i$, the tangent unit vector C at $p(b)$ is transformed to the tangent unit vector to γ at O , where w_i is the i -th unit vector at O of R^n ,*

and

- (v) *the developement of C on R^n so that the condition in (iv) is satisfied at $p(b)$ is γ .*

As is well known, for γ and b , C , $\bar{C}$ are uniquely determined under these conditions (i)–(v).

Let b' be the starting point of $\bar{C}$ which depends only on ξ, b and put $b' = \eta_\xi(b)$. η_ξ is clearly a homeomorphism on V_N and from the above definition it follows that

$$(10, 4) \quad \eta_{\xi_1} \circ \eta_{\xi_2} = \eta_{\xi_1 \cdot \xi_2}, \quad \xi_1, \xi_2 \in \mathfrak{W}$$

$$(10, 5) \quad \eta(x) \circ \eta_\xi = \eta_{\alpha^{-1}(\xi)} \circ \eta(x), \quad \alpha \in O_n.$$

The set $\mathfrak{R}$ of all the $\eta_\xi, \xi \in \mathfrak{W}$, is a group of automorphism on V_N and the correspondence $\eta: \mathfrak{W} \rightarrow \mathfrak{R}$ by $\eta(\cdot) = \eta_\xi$ is a homomorphism by (10, 4).

For any $w \in S^{n-1}$, $s \in R$, we get easily the relation

$$(10, 6) \quad \eta_{sw}(b) = \phi(b, w, -s) = \phi(b, -w, s).$$

By means of (10, 2), putting

$$\mathfrak{R}_0 = \eta(W_0), \quad S_n = \eta(T_n),$$

k_0 is an invariant subgroup of k and

$$(10, 7) \quad \mathfrak{R} = \mathfrak{R}_0 \cdot S_n = S_n \cdot \mathfrak{R}_0.$$

Now, for a fixed point $x \in V_n$, let $\mathcal{Q}_x$ be the set of piecewise differen-

table closed curves in V_x starting and ending at x and parameterized with arclength. Classifying the elements of $\mathcal{Q}_x$ by the equivalent relation (2) which was used when we derived $\mathfrak{B}$ from W in the beginning of this paragraph, we define a group II_x with multiplication by the usual method in it.

For any $b \in p^{-1}(x)$, $C \in \mathcal{Q}_x$, we obtain $\bar{C} \subset B$, $\gamma \subset R^n$ such that C , $\bar{C}$, γ have the above mentioned properties (i)–(v). Then, let $\psi_C : O_n(x) \rightarrow O_n(x)$ be defined by

$$(10, 8) \quad \psi_C(b) = \psi_\xi(b), \quad \xi = \xi(C, b)$$

where ξ denotes the class containing γ depending on b and C . Since by a right translation $r(\alpha)$, $\alpha \in O_n$, a system $\{C, \bar{C}, \xi\}$ is transformed to $\{C, r(\alpha)(\bar{C}), \alpha^{-1}(\xi)\}$, we get

$$(10, 9) \quad r(\alpha) \circ \psi_C = \psi_C \circ r(\alpha).$$

By definition, we get easily

$$(10, 10) \quad \psi_{C_1} \circ \psi_{C_2} = \psi_{C_1 \cdot C_2}.$$

Since ψ_C depends only on the element in II_x containing C , it defines a homomorphism of II_x onto a group of automorphisms on $O_n(x)$ by means of (10, 10). For any $b \in O_n(x)$, $C \in \xi \in II_x$, let $\beta_b(\xi)$ be defined by

$$\psi_C(b) = r(\beta_b(\xi))b,$$

then for any $\xi_1, \xi_2 \in II_x$, we have by (11, 9), (11, 10)

$$\begin{aligned} r(\beta_b(\xi_1)\beta_b(\xi_2))b &= r(\beta_b(\xi_2)(r(\beta_b(\xi_1))b)) \\ &= r(\beta_b(\xi_2))\psi_{C_1}(b) = \psi_{C_1}(r(\beta_b(\xi_2))b) \\ &= \psi_{C_1}(\psi_{C_2}(b)) = \psi_{C_1 \cdot C_2}(b) \\ &= r(\beta_b(\xi_1\xi_2))b, \quad C_1 \in \xi_1, C_2 \in \xi_2, \end{aligned}$$

hence

$$(10, 10) \quad \beta_b(\xi_1)\beta_b(\xi_2) = \beta_b(\xi_1 \cdot \xi_2)$$

The transformation $\beta_b : II_x \rightarrow O_n$ is a homomorphism. For $b_1 = r(\alpha_1)b$, we have

$$\begin{aligned} \psi_C(b_1) &= r(\beta_{b_1}(\xi))b_1 = r(\beta_{b_1}(\xi))(r(\alpha_1)b) \\ &= r(\alpha_1\beta_{b_1}(\xi))b \end{aligned}$$

$$\begin{aligned}
&= r(\alpha_1) \psi_c(b) = r(\alpha_1) (r(\beta_b(\zeta))b) \\
&= r(\beta_b(\zeta) \alpha_1) b,
\end{aligned}$$

hence

$$(10, 11) \quad \beta_{r(\alpha)b}(\zeta) = \alpha^{-1} \beta_b(\zeta) \alpha, \quad \zeta \in H_x, \quad \alpha \in O_x.$$

$H_{x,b} = \beta_b(H_x)$ is the holonomy group of V_x at x with respect to b .

With regards to $\xi(C, b)$, we get analogously the formulas:

$$(10, 12) \quad \xi(C, r(\alpha)b) = \alpha^{-1}(\xi(C, b)),$$

$$\begin{aligned}
(10, 13) \quad \xi(C_1 C_2, b) &= \xi(C_1, \psi_{c_2}(b)) \cdot \xi(C_2, b) \\
&= (\beta_b(\zeta_2^{-1}) (\xi(C_1, b))) \cdot \xi(C_2, b), \\
C_a &\in \zeta_a \in H_x, \quad a = 1, 2.
\end{aligned}$$

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