A NOTE ON MATRIX RINGS

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Let R be a ring with an identity. Often, R can be represented as the total matrix ring over an m-irreducible ring (Definition 1). For example, a ring with minimum condition and a homogeneous π-regular ring ([4, Theorem 4.2] or [5, Theorem 5.6])[1] possess this property. For such rings, it seems to be of interest to investigate the problem concerning the uniqueness of the representations as total matrix rings over m-irreducible rings. In this note, we shall examine this problem for a special kind of rings.

Throughout the note, R will be a ring with an identity, and the terms "radical", "primitive ideal" will be in Jacobson's sense ([3]).

We shall begin our course by setting the following definition:

Definition 1. A ring R is said to be *m-irreducible* if R is not representable as the total $n \times n$ matrix ring over any ring with $n > 1$.

A P-ring ([6, Definition 4.1]) is m-irreducible. In general, if a non-zero homomorphic image of R is m-irreducible, then so is R.

Definition 2. Let N be the radical of a ring R. If R/N is a (finite or infinite) complete direct sum of total matrix rings over division rings: $\sum_\sigma^c (\bar{D}_\sigma)_{n_\sigma}$, then R is said to be *weakly semi-primary.*

Lemma 1. *If R is a weakly semi-primary ring, then*

i) *$(R)_n$ is weakly semi-primary,*

ii) *eRe is also weakly semi-primary, where e is an idempotent.*

Proof. i) As is well-known, the radical of $(R)_n$ is $(N)_n$. Hence $R/N = \sum_\sigma^c (\bar{D}_\sigma)_{n_\sigma}$ implies that $(R)_n/(N)_n \cong \sum_\sigma^c (\bar{D}_\sigma)_{nn_\sigma}$, that is, $(R)_n$ is weakly semi-primary.

ii) The radical of eRe is $eNe = eRe \cap N$ ([4, Theorem 3.1]). Hence, $eRe/eNe \cong \bar{e}\bar{R}\bar{e} = \sum_\sigma^c \bar{e}(\bar{D}_\sigma)_{n_\sigma}\bar{e} = \sum_\sigma^c \bar{e}_\sigma(\bar{D}_\sigma)_{n_\sigma}\bar{e}_\sigma$, where $^-$ will mean the residue class modulo N, and where $\bar{e}_\sigma$ will be the σ-component of $\bar{e}$. Clearly $\bar{e}_\sigma(\bar{D}_\sigma)_{n_\sigma}\bar{e}_\sigma$ is simple and satisfies the minimum condition. These facts show that eRe is weakly semi-primary.

The following is an immediate consequence of Lemma 1:

Corollary. *The total $n \times n$ matrix ring $(R)_n$ is weakly semi-primary if and only if R is so.*

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1) Numbers in brackets refer to the references cited at the end of this note.

Lemma 2. *Let* $\mathfrak{M} = \mathfrak{m}_1 \oplus \cdots\cdots \oplus \mathfrak{m}_n$
$= \mathfrak{n}_1 \oplus \cdots\cdots \oplus \mathfrak{n}_n$
be two direct decompositions of a module $\mathfrak{M}$ *with an operator domain* Ω *into* n *isomorphic submodules. If the* Ω*-endomorphism ring of* $\mathfrak{m}_1$ *is weakly semi-primary, then* $\mathfrak{m}_1 \cong \mathfrak{n}_1$.

Proof.[1] Let R be the Ω-endomorphism ring of $\mathfrak{M}$, and let e_i and f_j be the projections onto $\mathfrak{m}_i$ and $\mathfrak{n}_j$ respectively. Then $R = Re_1 \oplus \cdots\cdots \oplus Re_n$ $(\cong (e_1 Re_1)_n)$, whence R is weakly semi-primary by Corollary to Lemma 1. Here we set $R/N = \sum_{\sigma}^{c} (\bar{D}_\sigma)_{n_\sigma}$, N the radical of R. Next $\bar{e}_i$ and $\bar{f}_j$ will denote the residue classes of e_i and f_j modulo N respectively. Then,

$$\bar{e}_1 + \cdots\cdots + \bar{e}_n = \bar{f}_1 + \cdots\cdots + \bar{f}_n = \bar{1}.$$

To be easily seen, $\bar{e}_1$ is isomorphic to $\bar{f}_1$ modulo the primitive ideal $\sum_{\sigma \neq \tau}^{c} (\bar{D}_\sigma)_{n_\sigma}$ for each τ. Hence $\bar{e}_1$ is isomorphic to $\bar{f}_1$, whence e_1 is isomorphic to f_1, that is, $\mathfrak{m}_1$ is isomorphic to $\mathfrak{n}_1$.

Considering R itself and the totality of left-multiplications of elements of R as $\mathfrak{M}$ and Ω in Lemma 2 respectively, we obtain the next:

Corollary. *Let* $R = (R_1)_n = (R_2)_n$. *If* R *is weakly semi-primary, then* $R_1 \cong R_2$.

Definition 3. A weakly semi-primary ring is called a *semi-primary ring* if the radical of the ring is nil.

Now we are going to establish our principal theorem which can be considered as a generalization of the familiar structure theorem of Wedderburn.

Theorem. *Let* R *be a semi-primary ring*: $R/N = \sum_{\sigma}^{c} (\bar{D}_\sigma)_{n_\sigma}$, N *the radical, and let* n *be the greatest common divisor of* n_σ*'s. Then,*

i) *R is the total $n \times n$ matrix ring over an m-irreducible ring,*

ii) *$R = (R_1)_n = (R_2)_n$ implies that $n_1 = n_2$ and $R_1 \cong R_2$, where R_1 and R_2 are m-irreducible.*

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1) The present proof is essentially same with that of [7, Theorem 46.3].

2) Let S be a ring and M be its radical. Two idempotents e and f are called *isomorphic* (*in* S) if there exist two elements a and b such that $ab = e$ and $ba = f$. e and f are isomorphic if and only if the left [right] ideals Se [eS] and Sf [fS] are operator isomorphic ([2, pp. 527 - 528] or [7, Theorem 10.2]). If e and f are isomorphic modulo M, then e and f are virtually isomorphic ([7, Theorem 18.12]). In case S is the endomorphism ring of a module $\mathfrak{N}$, the isomorphism of e with f means the isomorphism of $\mathfrak{N}e$ with $\mathfrak{N}f$ ([7, p. 230]).

Proof. It is almost trivial that $R/N = (T)_n$ with a ring T. Moreover, as N is nil, we obtain that $R = (T_0)_n$ with some T_0 ([1, Theorem 1.21]). Clearly T_0 is m-irreducible. For, if not, n would not be the greatest common divisor of n_σ's. If $R = (R_1)_{n_1}$, R_1 m-irreducible, then n_1 divides n. If $n_1 < n$, then $(R_1)_{n_1} \cong ((T_0)_{n/n_1})_{n_1}$ implies that $(T_0)_{n/n_1} \cong R_1$ by Corollary to Lemma 2. But it contradicts with the m-irreducibility of R_1. Hence, $n_1 = n$ and $R_1 \cong T_0$.

Remark. Let R be an FI_1-ring of which all the primitive images have the equal degree n. Then R is a total $n \times n$ matrix ring over a P-ring. If $R = (B)_m$, B m-irreducible, then $m = n$.[1]

References

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1) Cf. [5, Theorem 3.2] and [5, Theorem 5.6].