SOME REMARKS ON π-REGULAR RINGS OF BOUNDED INDEX

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Recently, G. Azumaya introduced the concepts of right, left, and strong π-regularities of elements in a ring and obtained a sufficient condition that a right (left) π-regular element is strongly π-regular. Previously, R. Arens and I. Kaplansky ([1])[1] studied rings in which all elements are right π-regular, and one of them continued his consideration in [6]. In this note, we shall prove several additional properties of π-regular rings of bounded index. As is well-known, Neumann's regularity is preserved under the construction of the complete matrix ring ([3, Lemma 2]). But the complete matrix ring over a strongly regular ring is not strongly regular except trivial cases. Our principal aim is to show the following fact: The $r \times r$ complete matrix ring over a π-regular ring of bounded index is also π-regular and of bounded index.

§1 is preparations of subsequent sections and contains some definitions and fundamental results without proofs. §2 deals with nil-rings of bounded index, and in §3 our principal theorem will be shown. In §4 we shall consider a ring of bounded index, and under this assumption several properties of the unique maximal π-regular ideal which are similar to those of the unique maximal regular ideal as in [3] will be investigated.

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1. Definitions and fundamental results. Let a be an element of a ring A. If there exists an element x such that $axa = a$, we say that a is *regular*. If there exists x such that $a^2x = a(xa^2 = a)$, a is said to be *right* (*left*) *regular*, and in case a is right as well as left regular a is *strongly regular*. A *regular* (*right regular, left regular, strongly regular*) *ring* will mean a ring in which all elements are regular (right regular, left regular, strongly regular)[2].

Now we introduce the following definitions. Let a be an element of a ring A. If some power of a is regular (right regular, left regular,

1) Numbers in brackets refer to the references cited at the end of this paper.
2) Cf. [10].

strongly regular), then we say that a is *π-regular* (*right π-regular, left π-regular, strongly π-regular*). And a *π-regular* (*right π-regular, left π-regular, strongly π-regular*) *ring* will be defined in the obvious way. A two-sided ideal in A is said to be a *π-regular* (*strongly π-regular*) *ideal* if each element of the ideal is π-regular (strongly π-regular).

As is easily verified, a is right (left) π-regular if and only if there exist an element x and an integer n such that $a^{n+1}x = a^n (xa^{n+1} = a^n)$. For any strongly π-regular element a of a ring A, the least integer n for which there holds $a^{n+1}x = a^n$ with some x is called the *π-index* of a. And the least upper bound of all π-indices of strongly π-regular elements is denoted as the *π-index* of A. On the other hand, the least upper bound of all indices of nilpotent elements of A is called the *index* of A. And in case the index of A is finite we say that the ring is of bounded index. Every nilpotent element is strongly π-regular, and moreover the index coincides with the π-index for such element.

G. Azumaya proved the following theorem[1]:

Thoerem 1. *Let A be a ring of bounded index. If a is right (left) π-regular then it is strongly π-regular. And moreover, there exists an element x such that $ax = xa$ and $a^{n+1}x = a^n$, where n is the index of A.*

The proof was completed by making use of the elementary method, and as corollaries following results are obtained.

Corollary 1. *For any ring A the π-index of A coincides with the index of A.*

Corollary 2. *A right (left) π-regular element in a ring without nonzero nilpotent elements is strongly regular.*[2]

In §§**2**-**4**, we shall restrict our attentions to the case of bounded index. Hence, in this case, right π-regularity, left π-regularity, and strong π-regularity are equivalent to each other. Further, the following theorem was given in [2] ([2, Theorem 5]):

Theorem 2. *Under the assumption that A is of bounded index, the following four conditions are equivalent to each other*:

i) *A is π-regular,*

ii) *A is right π-regular,*

iii) *A is left π-regular,*

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1) See [2].

2) Cf. [4] or [6, p. 7].

iv) *A is strongly π-regular.*

The next theorem proved by R. Arens and I. Kaplansky is fundamental ([1, Theorem 3.1]):

Theorem 3. *Let A be a ring in which for any a there exists an x such that $a^{n+1}x = a^n$ (with n independent of a). Then the radical*[1] *of A is a nil-ideal with index of nilpotency at most n. If A is primitive, it is a matrix ring of degree at most n over a division ring.*

An extremal example of π-regular rings of bounded index is a nil-ring of bounded index, which will be considered in the next section.

2. Nil-rings of bounded index. It is the well-known result of Levitzki that a nil-ring of bounded index is semi-nilpotent, or equivalently, a finitely generated nil-ring of bounded index is nilpotent ([7, p. 1035]). From this fact we shall readily obtain the following theorem:

Theorem 4. *Let A be a nil-ring of bounded index generated by $a_1, \cdots\cdots, a_g$. Then $A^{f(n,g)} = 0$, where n is the index of A and $f(n, g)$ is a positive integer depending solely on n and g.*

Although the proof by making use of Levitzki's result is easy (see Remark below), we shall, for caution's sake, give here a direct proof.

Proof. As $a^n = 0$ for all a in A, there holds $\sum_{(i_1,\cdots\cdots,i_n)} x_{i_1}\cdots\cdots x_{i_n} = 0$ for any $x_1, \cdots\cdots, x_n$ in A, where $(i_1, \cdots\cdots, i_n)$ runs over all permutations of $1, \cdots\cdots, n$[2].

Let a be an arbitrary element and let m be an integer greater than $[n/2]$. We consider here the $2m+1$ subrings A_i which are defined as follows[3]:

$$A_{2j-1} = a^{m-j+1}Ra^{j-1} \qquad (j = 1, 2, \cdots\cdots, m+1),$$
$$A_{2j} = a^{m-j+1}Ra^{j} \qquad (j = 1, 2, \cdots\cdots, m),$$

To be easily verified, if $s > t$ then $A_sA_t \subseteqq Ra^{m+1}R$. Hence we obtain $A_{i_1}A_{i_2}\cdots\cdots A_{i_n} \subseteqq Ra^{m+1}R$ for each $(i_1, \cdots\cdots, i_n) \neq (1, \cdots\cdots, n)$. Accordingly, by the remark stated at the beginning of this proof, we have $A_1A_2\cdots\cdots A_n \subseteqq Ra^{m+1}R$, whence $\{(a^m)\}^{n+1} \subseteqq Ra^{m+1}R$, where (a^m) de-

1) The terms "radical", "primitive ideals", and "semi-simple" will be in the sense of Jacobson [5].

2) Cf. [8, Lemma 3].

3) Cf. [8, Proof of Theorem 1].

notes the two-sided ideal generated by a^m. Hence if $m=[n/2]+1$, then $\{(a^m)\}^{(n+1)^m}=0$.

For $n=1$ our assertion is clear. For $n=2$, as $xy=-yx$ for any x, y in A, we can take $g+1$ for $f(2, g)$.

Let $n>2$ and assume the validity of our assertion for $m<n$. Now we set $m=[n/2]+1$. Let F be the ring freely generated by g non-commutative indeterminates $x_1, \cdots\cdots, x_g$, and let I_F be the two-sided ideal in F generated by all m-th powers of the elements of F. As the quotient ring F/I_F is a nil-ring of bounded index $m(<n)$ generated by g elements, by our induction hypothesis $F^{f(m,g)} \subseteqq I_F$. We consider here the set $G_F=\{x_{i_1} \cdots\cdots x_{i_{f(m,g)}} \mid x_{i_j}$ is some x_k $(k=1, \cdots\cdots, g)\}$, and denote by $p(m, g)$ the least number of ideals of the form (y^m) such that the ideal sum of them contains G_F. As A is canonically homomorphic to F, the set $G=\{a_{i_1} \cdots\cdots a_{i_{f(m,g)}} \mid a_{i_j}$ is some a_k $(k=1, \cdots\cdots, g)\}$ is contained in some ideal B in A which can be represented as the sum of at most $p(m, g)$ ideals of the form (b^m). Hence, we obtain $B^{p(m,g)\cdot(n+1)^m}=0$, whence $a_{i_1} \cdots\cdots a_{i_{f(n,g)}}=0$, where a_{i_j} is some a_k and $f(n, g)=f(m, g)\cdot p(m, g)\cdot(n+1)^m$. This completes our induction.

Corollary. *Let A be a nil-ring of bounded index. Then the $r \times r$ complete matrix ring $(A)_r$ over A is a nil-ring of bounded index too.*

Proof. Let $a^n=0$ for all a in A. For any element $\alpha=[a_{ij}]$ in $(A)_r$, we consider the subring B generated by a_{ij} $(i, j=1, \cdots\cdots, r)$. Then, by Theorem 4, $B^{f(n, r^2)}=0$. Our assertion follows from this fact at once.

Remark. In Theorem 4, A is a homomorphic image of F/I_F (n in place of m). Hence we can take, by Levitzki's result, the nilpotency index of F/I_F for $f(n, g)$.

In case A is a nil-algebra of bounded index n over a field K of characteristic 0 (not necessarily with a finite generating system), M. Nagata proposed that $A^{f(n)}=0$ with some integer $f(n)$ depending solely on n. On the other hand, in case the characteristic of K is $p \neq 0$, he presented a non-nilpotent, nil-algebra of bounded index ([9]).

3. Matrix rings over π-regular rings of bounded index. A π-regular ring A is *homogeneous* if it is semi-simple and such that for every primitive ideal P, A/P is an $n \times n$ complete matrix ring over a division ring (n independent of P)[1].

1) See [6, p. 67] and [6, Theorem 2.3].

In Theorem 4.2 of [6], I. Kaplansky announced that a homogeneous π-regular ring with an identity is a matrix ring over a strongly regular ring. The converse of this fact is also true, but we have more generally the following:

Lemma 1. *Let A be regular and of bounded index. Then the $r \times r$ complete matrix ring $(A)_r$ is also regular and of bounded index.*

Proof. As $(A)_r$ is regular ([3, Lemma 2]), it only remains to prove that $(A)_r$ is of bounded index. Let P be an arbitrary primitive ideal in A. By [6, Theorem 2.3] (or by Theorems 2 and 3), $(A)_r/(P)_r \cong (D)_{mr}$, where D is a division ring and $m \leq n =$ the index of A. $(A)_r$ is a subdirect sum of complete matrix rings of degree at most nr over division rings, for $(A)_r = (A)_r/ \cap (P)_r$, where P runs over all primitive ideals in A. Hence $(A)_r$ is of bounded index.

Lemma 2. *If a is right π-regular modulo a π-regular ideal I of bounded index, then a is actually right π-regular. In particular, if A/I is π-regular and of bounded index, then A is π-regular and of bounded index too.*

Proof. Let m be the index of I. Then, for some element x, y and for some integer n, we obtain $(a^{mn+1}x - a^n)^{m+1}y = (a^{mn+1} - a^n)^m$. Hence $a^{mn+1}z = a^{mn}$ with some z.

Lemma 3. *If A is a homogeneous π-regular ring of bounded index, then so is the $r \times r$ complete matrix ring $(A)_r$.*

Proof. Let Z be the center of A. Then AZ is regular (and is of bounded index) and A/AZ is a nil-ring of bounded index ([6, Theorem 4.3]). Hence $(AZ)_r$ and $(A/AZ)_r$ are π-regular and of bounded index by Lemma 1 and Corollary to Theorem 4 respectively. As $(A)_r/(AZ)_r \cong (A/AZ)_r$, our assertion is an immediate consequence of Lemma 2.

Corollary. *If A is a semi-simple π-regular ring of bounded index, then so is $(A)_r$.*

Proof. Let $I_\alpha = \cap P$, where P runs over all primitive ideals such that A/P is isomorphic to some $m \times m$ complete matrix ring over a division ring with $m \leqq \alpha$. If the index of A is n, then it is clear that $I_n = 0$. We may prove our assertion by making use of the induction with respect to the number i such that $I_i \neq 0$ and $I_{i+1} = 0$.

If $i = 0$, A is strongly regular. Hence our assertion is true. We assume now that it is true for $j < i$. As, to be easily seen, I_i is homogeneous[1], by Lemma 3, $(I_i)_r$ is π-regular and of bounded index.

1) See [6, Theorem 3.1 and p. 67].

On the other hand, by the induction hypothesis, $(A/I_i)_r$ is also π-regular and of bounded index. Hence our induction is completed by means of Lemma 2.

Now we can prove readily the following principal theorem:

Theorem 5. *The $r\times r$ complete matrix ring $(A)_r$ over a π-regular ring of bounded index is π-regular and of bounded index too.*

Proof. Let N be the radical of A. Since $(A)_r/(N)_r \cong (A/N)_r$, our theorem is an immediate consequence of Theorem 3, Corollary to Theorem 4, and Corollary to Lemma 3.

4. π-regular ideals in a ring of bounded index. We begin this section with the following lemma.

Lemma 4. *Let A be a ring of bounded index. Then there exists the unique maximal (strongly) π-regular ideal.*

Proof. We set $\Pi(A)=\{a\in A \mid (a) \text{ is } \pi\text{-regular}\}$. Clearly, $\Pi(A)$ is closed under the right as well as left multiplications by the elements of A. Now, if a_1, a_2 are in $\Pi(A)$, then $(a_1-a_2)\subseteqq(a_1)+(a_2)$. Hence, for any element $a\in(a_1-a_2)$, we have $a=u_1+u_2$, where $u_i\in(a_i)$ $(i=1,2)$. As $u_1^{n+1}r=u_1^n$ with some $r\in A$ and some integer n, it follows that $a^{n+1}r-a^n=(u_1+u_2)^{n+1}r-(u_1+u_2)^n=u_1^{n+1}r-u_1^n+u_2^*=u_2^*$ with some $u_2^*\in(a_2)$. By Lemma 2, a is π-regular. Thus $\Pi(A)$ defined above is a two-sided ideal.

The next properties of $\Pi(A)$ are similar to those of $M(A)$ in Theorems 2, 3, and 4 of [3].

Theorem 6. *Let A be a ring of bounded index. Then*

i) $\Pi(A/\Pi(A))=0$.

ii) *If I is a two-sided ideal in A, then $\Pi(I)=I\cap\Pi(A)$.*

iii) *Let $(A)_r$ be the $r\times r$ complete matrix ring over A, then $(\Pi(A))_r$ is the unique maximal strongly π-regular ideal.*

Proof. i) It is easy to see that $A/\Pi(A)$ is of bounded index (Corollary 1 to Theorem 1 and Lemma 2). Let $\bar{b}\in\Pi(A/\Pi(A))$, where $\bar{b}$ denotes the residue class modulo $\Pi(A)$ containing b. For each element a in (b), there holds that $\bar{a}^{n+1}\bar{x}=\bar{a}^n$ with some x and n. Hence $a^{n+1}x-a^n\in\Pi(A)$. By Lemma 2, a is π-regular, whence b is contained in $\Pi(A)$.

ii) Let $a\in\Pi(I)$ and $b\in(a)$. As is easily verified by a brief computation, $b^3\in(a)'$, where $(a)'$ is the two-sided ideal generated by a in I. Hence b^3 is π-regular, accordingly b is so. Thus we have proved that $\Pi(I)\subseteqq I\cap\Pi(A)$. The converse relation is clear.

iii) We consider here the ring $A^* = (1) + A$, which consists of all pairs (n, a) with integers n and a in A, and in which the addition and the multiplication are defined as follows: $(n, a) + (n', a') = (n + n', a + a')$, $(n, a)(n', a') = (nn', na' + n'a + aa')$. Let ε_{ij} be a matrix in $(A^*)_r$ with an identity in the (i, j) position and zeros elsewhere. If $\alpha = [a_{ij}]$ is a matrix in $(A)_r$, then $\alpha_{ij} = \varepsilon_{1i}\alpha\varepsilon_{j1}$ is a matrix with a_{ij} in the $(1, 1)$ position and zeros elsewhere, and it is in $(\alpha)^*$, where $(\alpha)^*$ denotes the two-sided ideal generated by α in $(A^*)_r$.[1] For any element $a' \in (a_{ij})$, let α' be a matrix with a' in the $(1, 1)$ position and zeros elsewhere. Then α' also belongs to $(\alpha)^*$. As is easily seen, α'^3 is in (α). If (α) is a strongly π-regular ideal, then there exists an matrix $\beta = [b_{ij}] \in (A)_r$ such that $\alpha'^{3(n+1)} \cdot \beta = \alpha'^{3n}$. But this implies that $a'^{3n+3} \cdot b_{11} = a'^{3n}$, and hence a' is π-regular, that is, (a_{ij}) is a π-regular ideal. Hence, $(A)_r$ possesses the unique maximal strongly π-regular ideal, which coincides with $(\Pi(A))_r$ by Theorem 5.

References

[1] R. Arens and I. Kaplansky, Topological representation of algebras, Trans. Amer. Math. Soc. 63 (1948), 457 - 481.

[2] G. Azumaya, Strongly π-regular rings, Journ. Fac. Sci. Hokkaido Univ., 13 (1954), 34 - 39.

[3] B. Brown and N. H. McCoy, The maximal regular ideal of a ring, Proc. Amer. Math. Soc. 1 (1950), 165 - 171.

[4] A. Gertschikoff, Über Ringe, die in eine direkte Summe von Körpern zerlegbar sind, Mat. Sbornik N.S. 7 (1940), 591 - 597 (Russian with German summary).

[5] N. Jacobson, The radical and semi-simplicity for arbitrary rings, Amer. Journ. Math. 67 (1945), 300 - 320.

[6] I. Kaplansky, Topological representation of algebras, Trans. Amer. Math. Soc. 68 (1950), 62 - 75.

[7] J. Levitzki, On a problem of A. Kurosch, Bull. Amer. Math. Soc. 52 (1946), 1033 - 1035.

[8] J. Levitzki, A theorem on polynomial identities, Proc. Amer. Math. Soc. 1 (1950), 334 - 341.

[9] M. Nagata, On a problem concerning nil-algebras, Journ. Math. Soc. Japan, 4 (1954), 296 - 301.

[10] J. v. Neumann, Regular rings, Proc. Nat. Acad. Sci. U.S.A. 22 (1936), 707 - 713.

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1) We consider here A as a subring of A^*.