NOTE ON COMPACT MANIFOLDS WITH NON-SYMMETRIC METRIC CONNECTIONS

Tominosuke Ōtsuki

Introduction. S. Bochner and K. Yano [3], [4][1] investigated grobal properties of compact manifolds with non-symmetric metric connections by means of pseudo-harmonic and pseudo-Killing tensor fields. The errors in [3], owing to the omission of torsion of the spaces, were corrected in [4]. T. Suguri [5] discussed also the spaces.

In this note, we shall give some remarks on spaces with non-symmetric metric connections with regards to the torsions of the spaces.

§1. Let S_n be an n-dimensional manifold on which there is given a positive definite metric

$$ds^2 = g_{ij}dx^i dx^j \qquad\qquad (i, j = 1, 2, \cdots\cdots, n)[2]$$

and a metric connection E^i_{jk} in local coordinates (x^i).

From the assumption, we have

$$(1) \qquad g_{ij|k} \equiv \frac{\partial g_{ij}}{\partial x^k} - g_{sj}E^s_{ik} - g_{is}E^s_{jk} = 0$$

where the solidus denotes covariant differentiation with respect to E^i_{jk}. From (1), we get

$$(2) \qquad \frac{\partial \sqrt{g}}{\partial x^i} = \sqrt{g}\, E^k_{ki}.$$

Define the torsion tensor of S_n by

$$(3) \qquad S_{ij}{}^k = \frac{1}{2}(E^k_{ij} - E^k_{ji}).$$

Now, for a scalar field φ on S_n, define an exterior form of degree $n-1$ by

$$(4) \qquad \Omega = g^{ih}(\varphi_{|h} + 2\varphi S_{hk}{}^k)d\sigma_i,$$

where

---

1) Numbers in brackets refer to the list of references at the end of the paper.
2) The summation convention of tensor analysis is used throughout.

$$d\sigma_i = (-1)^{i+1}\sqrt{g}\, dx^1 \wedge \cdots\cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots\cdots \wedge dx^n .$$

We get by (1), (2), (3)

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\{\sqrt{g}\, g^{ih}(\varphi_{|h} + 2\varphi S_{hk}{}^k)\}$$

$$= E^r_{ri}g^{ih}(\varphi_{|h} + 2\varphi S_{hk}{}^k)$$

$$\quad - (g^{ri}E^i_{ri} + g^{ir}E^h_{ri})(\varphi_{|h} + 2\varphi S_{hk}{}^k)$$

$$\quad + g^{ih}\Big(\frac{\partial \varphi_{|h}}{\partial x^i} + 2\varphi_{|i}S_{hk}{}^k + 2\varphi\frac{\partial S_{hk}{}^k}{\partial x^i}\Big)$$

$$= E^r_{ri}g^{ih}(\varphi_{|h} + 2\varphi S_{hk}{}^k)$$

$$\quad - E^r_{ir}g^{ih}(\varphi_{|h} + 2\varphi S_{hk}{}^k) - g^{ir}E^h_{ri}(\varphi_{|h} + 2\varphi S_{hk}{}^k)$$

$$\quad + g^{ih}(\varphi_{|h|i} + E^k_{hi}\varphi_{|k} + 2\varphi_{|i}S_{hk}{}^k + 2\varphi S_{hk}{}^k{}_{|i} + 2\varphi E^r_{hi}S_{rk}{}^k)$$

$$= \varDelta\varphi + 2\varphi g^{ih}(S_{hk}{}^k{}_{|i} - 2S_{hk}{}^k S_{ir}{}^r),$$

where we put

$$\varDelta\varphi = g^{ih}\varphi_{|i|h} .$$

Accordingly, let D be a bounded domain on S_n with a regular boundary, then we have the following formula

$$(5) \qquad \int_D \varDelta\varphi\, d\sigma = -2\int_D \varphi(S^i{}_k{}^k{}_{|i} - 2S^i{}_k{}^k S_{ir}{}^r)d\sigma + \int_{\partial D}\varOmega,$$

where

$$d\sigma = \sqrt{g}\, dx^1 \wedge \cdots\cdots \wedge dx^n .$$

Especially, if S_n is compact, we have

$$(6) \qquad \int_{S_n}\varDelta\varphi\, d\sigma = -2\int_{S_n}\varphi\,(S^i{}_k{}^k{}_{|i} - 2S^i{}_k{}^k S_{ir}{}^r)d\sigma .$$

Theorem 1. *On a compact space S_n with a non-symmetric metric connection, in order that for any scalar field φ, we have*

$$\int_{S_n}\varDelta\varphi\, d\sigma = 0,$$

it is necessary and sufficient that

$$(7) \qquad S \equiv S^i{}_k{}^k{}_{|i} - 2S^i{}_k{}^k S_{ir}{}^r = 0.$$

Let V_n be the Riemann space with line element

$$ds^2 = g_{ij}dx^i dx^j .$$

Then we have easily the relation

(8) $$E_{jk}^i = \{_{jk}^i\} + S_{jk}{}^i - S^i{}_{jk} - S^i{}_{kj},$$

where $\{_{jk}^i\}$'s are the Christoffel symbolds made by g_{ij}. Accordingly, we get

$$
\begin{aligned}
S &= S^i{}_k{}^k{}_{|i} - 2S^i{}_k{}^k S_{ir}{}^r \\
&= S^i{}_k{}^k{}_{,i} + S^h{}_k{}^k (S_{hi}{}^i - S^i{}_{hi} - S^i{}_{ih}) - 2S^i{}_k{}^k S_{ir}{}^r,
\end{aligned}
$$

that is

$$S = S^i{}_k{}^k{}_{,i},$$

where the comma denotes covariant differentiation of V_n.

For a given compact Riemann space with line element

$$ds^2 = g_{ij}dx^i dx^j,$$

if we have a tensor field on V_n, $S_{ij}{}^k = -S_{ji}{}^k \not\equiv 0$ such that every where $S \equiv S^i{}_k{}^k{}_{,i} = 0$, then we can obtain a space S_n with a non-symmetric metric connection on which for any scalar field φ, we have

$$\int_{S_n} \Delta\varphi \, d\sigma = 0.$$

If S_{ijk} is skew-symmetric, $S = 0$ always holds good.

Let φ_i be a covariant vector field on V_n and put

(10) $$S_{ij}{}^k = \delta_i^k \psi_j - \delta_j^k \psi_i .$$

Then we have

$$
\begin{aligned}
S_{ik}{}^k &= -(n-1)\psi_i, \\
S &= -(n-1)\psi_{i,j}g^{ij}.
\end{aligned}
$$

Accordingly, in this case, (7) becomes

(11) $$g^{ij}\psi_{i,j} = 0,$$

that is, the differential form $\psi_i dx^i$ of degree 1 is co-exact. In other words, the $n-1$-cochain corresponding to $\psi_i dx^i$ is a cocycle. According to de Rahm's theorem, there exists always a vector field ψ_i such that $\psi_i \not\equiv 0$, $g^{ij}\psi_{i,j} = 0$.

Thus we see that *for any compact Riemann space V_n, there exists a space S_n with a non-symmetric metric connection E_{ji}^k such that*

$$S_{ik}{}^k \overset{\neq}{=} 0, \qquad g^{ij} S_{ik}{}^k{}_{,j} = 0.$$

§2. Let

$$\varphi = \frac{1}{p!} \varphi_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

be an exterior differential form of degree p on a compact space S_n with a non-symmetric metric connection, where $\varphi_{i_1 \cdots i_p}$ are a skew-symmetric tensor field over S_n. We define a pseudo-exterior differentiation $\hat{d}$ by

(12) $$\hat{d}\varphi = \frac{1}{(p+1)!} \varphi_{i_1 \cdots i_{p+1}} dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}},$$

where

(13) $$\varphi_{i_1 \cdots i_{p+1}} = (-1)^p \left\{ \varphi_{i_1 \cdots i_p | i_{p+1}} - \sum_{s=1}^{p} \varphi_{i_1 \cdots i_{s-1} i_{p+1} i_{s+1} \cdots i_p | i_s} \right\}.$$

We define also a pseudo-codifferentiation $\hat{\delta}$ by

(14) $$\hat{\delta}\varphi = \frac{-1}{(p-1)!} g^{jk} \varphi_{j i_1 \cdots i_{p-1} | k} dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}}.$$

According to S. Bochner [3], if

(15) $$\hat{d}\varphi = 0,$$
(16) $$\hat{\delta}\varphi = 0,$$

then we call φ *pseudo-harmonic*.

Now, we define a *generalized Laplacian operator* on S_n by

(17) $$\hat{\varDelta} = -(\hat{d}\hat{\delta} + \hat{\delta}\hat{d}).$$

For any differential form of degree p on S_n, we have

$$(\hat{\varDelta}\varphi)_{i_1 \cdots i_p} = -(\hat{d}\hat{\delta}\varphi)_{i_1 \cdots i_p} - (\hat{\delta}\hat{d}\varphi)_{i_1 \cdots i_p}$$

$$= -\sum_{s=1}^{p} (-1)^{s-1} (\hat{\delta}\varphi)_{i_1 \cdots \hat{i}_s \cdots i_p | i_s} + g^{jk}(\hat{d}\varphi)_{j i_1 \cdots i_p | k}$$

$$= \sum_{s=1}^{p} (-1)^{s-1} g^{jk} \varphi_{j i_1 \cdots \hat{i}_s \cdots i_p | k | i_s} + g^{jk} \varphi_{i_1 \cdots i_p | j | k}$$

$$\quad + \sum_{s=1}^{p} (-1)^s g^{jk} \varphi_{j i_1 \cdots \hat{i}_s \cdots i_p | i_s | k}$$

$$= g^{jk} \varphi_{i_1 \cdots i_p | j | k} + \sum_{s=1}^{p} g^{jk} (\varphi_{i_1 \cdots i_{s-1} j \cdots i_p | k | i_s} - \varphi_{i_1 \cdots i_{s-1} j \cdots i_p | i_s | k}),$$

where $\overset{\wedge}{i_s}$ denotes the omission of the index i_s.

Let

$$E^i_{j\,kh} = \frac{\partial E^i_{jh}}{\partial x^k} - \frac{\partial E^i_{jk}}{\partial x^h} - E^r_{jk}E^i_{rh} + E^r_{jh}E^i_{rk},$$

$$E_{jk} = E^h_{j\,kh}$$

be the components of the curvature tensor, Ricci tensor of S_n respectively. Then, by means of a well known formula, we can write the above equation as follows.

$$
\begin{aligned}
(\overset{\wedge}{\Delta}\varphi)_{i_1\cdots i_p} &= g^{jk}\varphi_{i_1\cdots i_p\mid j\mid k} \\
&+ \sum_{s=1}^{p} g^{jk}(E^h_{j\,k i_s}\varphi_{i_1\cdots i_{s-1}h\cdots i_p} \\
&+ \sum_{t<s} E^h_{i_t\,k i_s}\varphi_{i_1\cdots i_{t-1}h\cdots i_{s-1}j\cdots i_p} \\
&+ \sum_{s<t} E^h_{i_t\,k i_s}\varphi_{i_1\cdots i_{s-1}j\cdots i_{t-1}h\cdots i_p} \\
&- 2S_{k i_s}{}^h\varphi_{i_1\cdots i_{s-1}j\cdots i_p\mid h}) \\
&= g^{jk}\varphi_{j_1\cdots i_p\mid j\mid k} + \sum_{s=1}^{p} E^h_{i_s}\varphi_{i_1\cdots i_{s-1}h\cdots i_p} \\
&+ \sum_{s<t}(E^h_{i_s i_t}{}^j - E^h_{i_t i_s}{}^j)\varphi_{i_1\cdots i_{s-1}h\cdots i_{t-1}j\cdots i_p} \\
&+ 2\sum_{s=1}^{p} S_{i_s}{}^{jk}\varphi_{i_1\cdots i_{s-1}j\cdots i_p\mid h}.
\end{aligned}
$$

(18)

Accordingly, for a pseudo-harmonic tensor field φ, we have

$$
\begin{aligned}
&g^{jk}\varphi_{i_1\cdots j_p\mid j\mid k} + \sum_{s=1}^{p} E^h_{i_s}\varphi_{i_1\cdots i_{s-1}h\cdots i_p} \\
&+ \sum_{s<t}(E^h_{i_s i_t}{}^k - E^h_{i_t i_s}{}^k)\varphi_{i_1\cdots i_{s-1}h\cdots i_{t-1}k\cdots i_p} \\
&+ 2\sum_{s=1}^{p} S_{i_s}{}^{hk}\varphi_{i_1\cdots i_{s-1}h\cdots i_p\mid k} = 0.
\end{aligned}
$$

(19)

If S_n is a compact space such that $S_{ik}{}^k = 0$, especially a compact Riemann space, in order that φ be pseudo-harmonic, it is necessary and sufficient that (19) hold good for φ [4].

In the following, we shall investigate the same problem without any restriction for $S_{ij}{}^k$.

For any two exterior differential forms φ, ψ of degree p and a bounded domain D with a regular boundary, we define an *inner product* of φ and ψ on D by

(20)
$$(\varphi,\psi)_D = \frac{1}{p!}\int_D \varphi^{i_1\cdots i_p}\psi_{i_1\cdots i_p}\,d\sigma.$$

If S_n is compact and $D = S_n$, we simply write

$$(\varphi, \psi)_{S_n} = (\varphi, \psi).$$

Let φ and ψ be exterior differential forms of degree $p-1$ and p respectively. Define a differential form of degree 1 by

(21) $$\varphi \lrcorner \psi = \frac{1}{(p-1)!} \varphi^{i_1 \cdots i_{p-1}} \psi_{i i_1 \cdots i_{p-1}} dx^i.$$

We call $\varphi \lrcorner \psi$ the *left inner product* of φ and ψ, and we define analogously $\varphi \lrcorner \psi$ for φ and ψ of degrees p and $q (p \leq q)$.

Then, we have

$$\frac{(\partial \sqrt{g} (\varphi \lrcorner \psi)^i)}{\sqrt{g} \, \partial x^i} = (\varphi \lrcorner \psi)^i{}_{|i} - 2 S_{ik}{}^k (\varphi \lrcorner \psi)^i$$

$$= \frac{1}{(p-1)!} (\varphi_{i_1 \cdots i_{p-1} | i} \psi^{i i_1 \cdots i_{p-1}} + \varphi_{i_1 \cdots i_{p-1}} \psi^{i i_1 \cdots i_{p-1}}{}_{|i}$$

$$- 2 S_{ik}{}^k (\varphi \lrcorner \psi)^i$$

$$= \frac{1}{p!} (\hat{d}\varphi)_{i_1 \cdots i_p} \psi^{i_1 \cdots i_p} - \frac{1}{(p-1)!} \varphi_{i_1 \cdots i_{p-1}} (\hat{\delta}\psi)^{i_1 \cdots i_{p-1}}$$

$$- 2 S_{ik}{}^k (\varphi \lrcorner \psi)^i.$$

Hence we have

$$(\hat{d}\varphi, \psi)_D - (\varphi, \hat{\delta}\psi)_D = 2 \int_D S^{ik}{}_k (\varphi \lrcorner \psi)_i \, d\sigma + \int_{\partial D} (\varphi \lrcorner \psi)^i \, d\sigma_i.$$

Using a differential form π of degree 1 defined by

(22) $$\pi = S_{ik}{}^k dx^i,$$

we obtain the following formula

(23) $$(\hat{d}\varphi, \psi)_D - (\varphi, \hat{\delta}\psi)_D = 2(\pi, \varphi \lrcorner \psi)_D + \int_{\partial D} (\varphi \lrcorner \psi)^i \, d\sigma_i$$

For any exterior differential form φ of degree p, we obtain from (23)

$$(\hat{d}\varphi, \hat{d}\varphi)_D - (\varphi, \hat{\delta}\hat{d}\varphi)_D = 2(\pi, \varphi \lrcorner \hat{d}\varphi)_D + \int_{\partial D} (\varphi \lrcorner \hat{d}\varphi)^i \, d\sigma_i$$

$$(\hat{d}\hat{\delta}\varphi, \varphi)_D - (\hat{\delta}\varphi, \hat{\delta}\varphi)_D = 2(\pi, \hat{\delta}\varphi \lrcorner \varphi)_D + \int_{\partial D} (\hat{\delta}\varphi \lrcorner \varphi)^i \, d\sigma_i.$$

Hence we have the formula

$$(\hat{d}\varphi, \hat{d}\varphi)_D + (\hat{\delta}\varphi, \hat{\delta}\varphi)_D + (\varphi, \hat{\varDelta}\varphi)_D$$
$$= 2(\pi, \varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi)_D + \int_{\partial D} (\varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi)^t d\sigma_i .$$

Let S_n be compact, putting $D = S_n$ we get

$$(24) \qquad (\hat{d}\varphi, \hat{d}\varphi) + (\hat{\delta}\varphi, \hat{\delta}\varphi) + (\varphi. \hat{\varDelta}\varphi) = 2(\pi, \varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi).$$

From (24), we see that on a compact S_n, the system of equations

$$(19) \qquad\qquad\qquad \hat{\varDelta}\varphi = 0,$$

$$(25) \qquad\qquad (\pi, \varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi) = 0$$

is equivalent to the one

$$(15) \qquad\qquad\qquad \hat{d}\varphi = 0,$$

$$(16) \qquad\qquad\qquad \hat{\delta}\varphi = 0.$$

Thus we have a conclusion.

Theorem 2. *On a compact space S_n with a non-symmetric metric connection, in order that a exterior differential form φ be pseudo-harmonic, it is sufficient that*

$$(26) \qquad\qquad\qquad \hat{\varDelta}\varphi = 0,$$
$$S_{ik}{}^k(\varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi)^t = 0.$$

§3. In this section, we shall deduce some grobal results from (23), (24) on a compact space S_n.

Let S_n be compact and φ be any exterior differential form of degree p. By (12), (14), (21) we have

$$(\varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi)_i$$
$$= \frac{1}{p!} \varphi^{i_1 \cdots i_p}\Big(\varphi_{i_1 \cdots i_p | i} - \sum_{s=1}^{p} (-1)^{s-1}\varphi_{i i_1 \cdots \hat{i}_s \cdots i_p | i_s}\Big)$$
$$+ \frac{1}{(p-1)!} \varphi^{j i_1 \cdots i_{p-1}}{}_{|j}\varphi_{i i_1 \cdots i_{p-1}}$$
$$= \frac{1}{2}(\varphi \lrcorner \varphi)_{|i} + \frac{1}{(p-1)!}(\varphi^{j i_1 \cdots i_{p-1}}{}_{|j}\varphi_{i i_1 \cdots i_{p-1}}$$
$$- \varphi^{j i_1 \cdots i_{p-1}}\varphi_{i i_1 \cdots i_{p-1} | j}).$$

Hence we have

$$2(\pi, \varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi) = (\pi, \hat{d}(\varphi \lrcorner \varphi))$$
$$+ \frac{2}{(p-1)!}\int \varphi^{j i_1 \cdots i_{p-1}}{}_{|j}\varphi_{i i_1 \cdots i_{p-1}}S^{ik}{}_k d\sigma$$

$$- \frac{2}{(p-1)!} \int \varphi^{j i_1 \cdots i_{p-1}} \varphi_{i i_1 \cdots i_{p-1} | j} S^{i k}{}_k d\sigma$$

$$= (\pi, \hat{d}(\varphi \lrcorner \varphi)) + 2(-1)^p (\hat{\delta}\varphi, \pi \lrcorner \varphi)$$

$$- \frac{2}{(p-1)!} \int \varphi^{j i_1 \cdots i_{p-1}} (\varphi_{i i_1 \cdots i_{p-1}} S^{i k}{}_k)_{|j} d\sigma$$

$$+ \frac{2}{(p-1)!} \int \varphi^{j i_1 \cdots i_{p-1}} \varphi_{i i_1 \cdots i_{p-1}} S^{i k}{}_{k | j} d\sigma$$

$$= (\pi, \hat{d}(\varphi \lrcorner \varphi)) + 2(-1)^p (\hat{\delta}\varphi, \pi \lrcorner \varphi)$$

$$+ 2(-1)^p (\varphi, \hat{d}(\pi \lrcorner \varphi)) + \frac{2}{(p-1)!} \int \varphi^{j i_1 \cdots i_{p-1}} \varphi_{i i_1 \cdots i_{p-1}} S^{i k}{}_{k | j} d\sigma.$$

By virtue of (7), (23), the last side of the equation above is written as

$$= (\hat{\delta}\pi, \varphi \lrcorner \varphi) + 2(\pi, (\varphi \lrcorner \varphi)\pi)$$

$$+ 2(-1)^p \{(\hat{\delta}\varphi, \pi \lrcorner \varphi) + (\varphi, \hat{d}(\pi \lrcorner \varphi)\}$$

$$+ \frac{2}{(p-1)!} \int \varphi^{j i_1 \cdots i_{p-1}} \varphi_{i i_1 \cdots i_{p-1}} S^{i k}{}_{k | j} d\sigma$$

$$= -(S, \varphi \lrcorner \varphi) + 4(-1)^p \{(\hat{\delta}\varphi, \pi \lrcorner \varphi) + (\pi, (\pi \lrcorner \varphi) \lrcorner \varphi)\}$$

$$+ \frac{2}{(p-1)!} \int \varphi^{j i_1 \cdots i_{p-1}} \varphi_{i i_1 \cdots i_{p-1}} S^{i k}{}_{k | j} d\sigma,$$

that is

$$2(\pi, \varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi)$$

$$= -(S, \varphi \lrcorner \varphi) + 4(-1)^p (\hat{\delta}\varphi, \pi \lrcorner \varphi)$$

$$+ \frac{2}{(p-1)!} \int \varphi^{j i_1 \cdots i_{p-1}} \varphi_{i i_1 \cdots i_{p-1}} S^{i k}{}_{k | j} d\sigma$$

$$+ \frac{(-1)^p 4}{(p-1)!} \int S_{i k}{}^k S^{j h}{}_h \varphi_{i_1 \cdots i_{p-1} j} \varphi^{i i_1 \cdots i_{p-1}} d\sigma.$$

Define a symmetric tensor of order 2 by

(27) $$S_{ij} = \frac{1}{2}(S_{ik}{}^k{}_{|j} + S_{jk}{}^k{}_{|i}) - 2 S_{ik}{}^k S_{jk}{}^h.$$

Then, we have easily

(28) $$S = g^{ij} S_{ij}.$$

Making use of S_{ij}, we obtain a formula on a compact S_n as

$$(\hat{d}\varphi, \ \hat{d}\varphi) + (\hat{\delta}\varphi, \ \hat{\delta}\varphi) + (\varphi, \ \varDelta\varphi)$$

$$(29) \qquad = -(S, \ \varphi \lrcorner \varphi) + (-1)^p 4(\hat{\delta}\varphi, \ \pi \lrcorner \varphi)$$

$$+ \frac{2}{(p-1)!} \int S_{ij}\varphi^{ii_1 \cdots i_{p-1}}\varphi^j_{i_1 \cdots i_{p-1}} d\sigma \ .$$

Accordingly we obtain from (29) the theorem.

Theorem 3. *On any compact space S_n with a non-symmetric metric connection, for any pseudo-harmonic field φ, we have*

$$(30) \qquad \frac{1}{p!} \int S\varphi^{i_1 \cdots i_p}\varphi_{i_1 \cdots i_p} d\sigma \ = \ \frac{2}{(p-1)!} \int S_{ij}\varphi^{ii_1 \cdots i_{p-1}}\varphi^j_{i_1 \cdots i_{p-1}} d\sigma.$$

Define a symmetric tensor of order 2 by

$$L_{ij} = S_{ij} - \frac{1}{2p}g_{ij}S$$

$$(31)$$

$$= \frac{1}{2}(S_{ik}{}^k{}_{|j} - S_{ik}{}^k{}_{|i}) - 2S_{ik}{}^k S_{jh}{}^h - \frac{1}{2p}g_{ij}(S^{hk}{}_{k|h} - 2S^{hk}{}_k S_{hr}{}^r).$$

Then (30) is written as

$$(30') \qquad \int_{S_n} L_{ij}\varphi^{ii_1 \cdots i_{p-1}}\varphi^j_{i_1 \cdots i_{p-1}} d\sigma \ = \ 0.$$

Since we have

$$L = g^{ij}L_{ij} = \left(1 - \frac{n}{2p}\right)S,$$

if $n \neq 2p$ *and L_{ij} is positive definite or negative definite, then there exists no pseudo-harmonic field of degree p on S_n. If L_{ij} is positive semi-definite or negative semi-definite, then any pseudo-harmonic field φ of degree p must satisfy*

$$(32) \qquad L_{ij}\varphi^{ii_1 \cdots i_{p-1}}\varphi^j_{i_1 \cdots i_{p-1}} = 0.$$

In the theory of Bochner and Yano [3], [4], the argument in the existence of pseudo-harmonic tensor fields holds good for the spaces such that $S = 0$. But, for the spaces S_n such that $S \neq 0$, in order to perform the anologous argument to the case $S = 0$, we can also make use of the tensor L_{ij}.

§4. Nextly, we shall investigate the same problem for pseudo-Killing tensor fields. According to Bochner and Yano [4], we call a skew-symmetric tensor $\varphi_{i_1 \cdots i_p}$ *pseudo-Killing* if

(33) $\varphi_{i_1 \cdots i_p \mid j} = - \varphi_{i_1 \cdots i_{s-1} j i_{s-1} \cdots i_p \mid i_s}$, $s = 1, \cdots, p.$

Let $\varphi_{i_1 \cdots i_p}$ be a pseudo-Killing tensor field of degree p on S_n, then for the exterior differential form of degree p

$$\varphi = \frac{1}{p!} \varphi_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

we have clearly from (33)

$$\hat{\delta}\varphi = 0.$$

Accordingly we have

$$(\hat{d}\varphi)_{i_1 \cdots i_p} = -(\hat{\delta}\hat{d}\varphi)_{i_1 \cdots i_p}$$

$$= g^{jk} \left\{ \varphi_{i_1 \cdots i_p \mid j} - \sum_{s=1}^{p} \varphi_{i_1 \cdots i_{s-1} j \cdots i_p \mid i_s} \right\}_{\mid k}$$

$$= (p+1) g^{jk} \varphi_{i_1 \cdots i_p \mid j \mid k}.$$

Define a *linear operator* K defined for any exterior differential form ψ of degree p by

(34)

$$(K\psi)_{i_1 \cdots i_p} = g^{jk} \psi_{i_1 \cdots i_p \mid j \mid k} - \frac{1}{p} \sum_s E^h{}_{i_s} \psi_{i_1 \cdots i_{s-1} h \cdots i_p}$$

$$- \frac{1}{p} \sum_{s<t} (E^h{}_{i_s i_t}{}^j - E^h{}_{i_t i_s}{}^j) \psi_{i_1 \cdots i_{s-1} h \cdots i_{t-1} j \cdots i_p}$$

$$- \frac{2}{p} \sum_s S_{i_s}{}^{jh} \psi_{i_1 \cdots i_{s-1} j \cdots i_p \mid h}.$$

Then, for a pseudo-Killing tensor φ of degree p, we have by (18) and the equation above

$$K\varphi = 0.$$

By means of (29) and (34), for any field φ on compact space S_n, we have

$$(\hat{d}\varphi, \hat{d}\phi) + (\hat{\delta}\varphi, \hat{\delta}\varphi) - p(\varphi, K\varphi) + \frac{(p+1)}{p!} \int g^{jk} \varphi_{i_1 \cdots i_p \mid j \mid k} \varphi^{i_1 \cdots i_p} d\sigma$$

$$= (-1)^p 4(\hat{\delta}\varphi, \pi \lrcorner \varphi) + \frac{2}{(p-1)!} \int L_i{}^j \varphi^{i i_1 \cdots i_{p-1}} \varphi_{j i_1 \cdots i_{p-1}} d\sigma.$$

The left hand side of the above equation is written as

$$(\hat{d}\varphi, \hat{d}\varphi) + (\hat{\delta}\varphi, \hat{\delta}\varphi) - p(\varphi, K\varphi)$$

$$+ \frac{p+1}{2 \cdot p!} \int \{ \hat{\Delta}(\varphi_{i_1 \cdots i_p} \varphi^{i_1 \cdots i_p}) - 2\varphi_{i_1 \cdots i_p \mid j} \varphi^{i_1 \cdots i_1 \mid j} \} d\sigma.$$

Hence we have by (6) an equation

$$(\hat{d}\varphi, \hat{d}\varphi) + (\hat{\delta}\varphi, \hat{\delta}\varphi) - p(\varphi, K\varphi) - \frac{p+1}{p!} \int \varphi_{i_1 \cdots i_p | j} \varphi^{i_1 \cdots i_{p-1} | j} \, d\sigma$$

$$= (-1)^p 4(\hat{\delta}\varphi, \pi \lrcorner \varphi) + \frac{2}{(p-1)!} \int M_i{}^j \varphi^{i i_1 \cdots i_{p-1}} \varphi_{j i_1 \cdots i_{p-1}} \, d\sigma.$$

where we put

(35)
$$M_{ij} = L_{ij} + \frac{p+1}{2p} S g_{ij} = S_{ij} + \frac{1}{2} S g_{ij}.$$

Since we have

$$(\hat{d}\varphi)_{i_1 \cdots i_p j} (\hat{d}\varphi)^{i_1 \cdots i_p j} - (p+1)^2 \varphi_{i_1 \cdots i_p | j} \varphi^{i_1 \cdots i_p | j}$$

$$= (\varphi_{i_1 \cdots i_p | j} - \sum_{s=1}^{p} \varphi_{i_1 \cdots i_{s-1} j \cdots i_p | s}) (\varphi^{i_1 \cdots i_p | j} - \sum_{t=1}^{p} \varphi^{i_1 \cdots i_{t-1} j \cdots i_p | i_t})$$

$$- (p+1)^2 \varphi_{i_1 \cdots i_p | j} \varphi^{i_1 \cdots i_p | j}$$

$$= - \frac{p(p+1)}{2} (\varphi_{i_1 \cdots i_{p-1} i | j} + \varphi_{i_1 \cdots i_{p-1} j | i}) (\varphi^{i_1 \cdots i_{p-1} i | j} + \varphi^{i_1 \cdots i_{p-1} j | i}),$$

we obtain the relation

$$(\hat{d}\varphi, \hat{d}\varphi) - \frac{p+1}{p!} \int \varphi_{i_1 \cdots i_p | j} \varphi^{i_1 \cdots i_p | j} \, d\sigma$$

$$= - \frac{1}{2 \cdot (p-1)!} \int (\varphi_{i_1 \cdots i_{p-1} i | j} + \varphi_{i_1 \cdots i_{p-1} j | i})$$
$$(\varphi^{i_1 \cdots i_{p-1} i | j} + \varphi^{i_1 \cdots i_{p-1} j | i}) \, d\sigma.$$

Thus we obtain a formula from the last equation

$$(\hat{\delta}\varphi, \hat{\delta}\varphi) - p(\varphi, K\varphi)$$

$$- \frac{1}{2 \cdot (p-1)!} \int (\varphi_{i_1 \cdots i_{p-1} j | k} + \varphi_{i_1 \cdots i_{p-1} k | j})$$
$$(\varphi^{i_1 \cdots i_{p-1} j | k} + \varphi^{i_1 \cdots i_{p-1} k | j}) \, d\sigma$$

(36)

$$= (-1)^p 4(\hat{\delta}\varphi, \pi \lrcorner \varphi)$$

$$+ \frac{2}{(p-1)!} \int M_j{}^k \varphi^{j i_1 \cdots i_{p-1}} \varphi_{k i_1 \cdots i_{p-1}} \, d\sigma.$$

If φ is pseudo-Killing, we have from (36)

$$\int M_j{}^k \varphi^{j i_1 \cdots i_{p-1}} \varphi_{k i_1 \cdots i_{p-1}} \, d\sigma = 0$$

since (33), $\hat{\delta}\varphi = 0$ and $K\varphi = 0$.

Thus, we obtain the following theorem.

Theorem 4. *On a compact space S_n with a non-symmetric metric connection, for any pseudo-Killing tensor field* $\varphi_{i_1 \cdots i_p}$, *we have*

$$(37) \qquad \int M_j{}^k \varphi^{j i_1 \cdots i_{p-1}} \varphi_{k i_1 \cdots i_p} \, d\sigma \; = \; 0.$$

On the equivalent conditions (Bochner and Yano [4], Theorem 14), we obtain easily from (36) the following theorem.

Theorem 5. *On a compact space S_n with a non-symmetric metric connection, in order that a skew-symmetric tensor field* $\varphi_{i_1 \cdots i_p}$ *be pseudo-Killing, it is necessary and sufficient that*

$$\hat{\delta}\varphi \; = \; 0, \qquad K\varphi \; = \; 0$$

and (37) *hold good.*

REFERENCES

[1] S. Bochner, Vector fields and Ricci curvature. Bulletin of the Amer. Math. Soc., Vol. 52, 1946, pp. 776 – 797.

[2] ———, Curvature and Betti numbers, Annals of Math., Vol. 49, 1948, pp. 379 – 390.

3] ———, Curvature and Betti numbers II. Annals of Math., Vol. 50, 1949, pp. 77 – 93.

4] S. Bochner and K. Yano, Tensor-fields in non-symmetric connections. Annals of Math., Vol. 56, 1952, pp. 504 – 519.

5] T. Suguri, On pseudo-harmonic tensor fields. Memoirs of Fac. of Sci., Kyūshū Univ., Ser. A, Vol VII, No. 2, pp. 61 – 68.

6] E. Hopf, Elementare Bemerkungen über die Lösungen partieller Differential-gleichungen zweiter Ordnung von elliptischen Typus. Preuss. Akad. Viss. Sitzungsber., 19 (1927), pp. 149 – 152.

7] K. Kodaira, Harmonic fields in Riemannian manifolds. Annals of Math., Vol. 50, 1949, pp. 587 – 665.

8] S. S. Chern, Lecture note on differential geometry I, 1952.

Department of Mathematics,
    Okayama University